Automatica, Vol. 12, pp. 497–506. Pergamon Press, 1976. Printed in Great Britain

Minimal Stable Partial Realization*†

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An asymtotically stable minimal order realization of a partial sequence of Markov parameters is achieved by reducing the problem to a standard but minimal one in decision algebra.

Summary—In this paper two equivalent sets of necessary and sufficient conditions for the existence of an asymptotically stable partial realization are developed. Both sets are expressed as multivariable polynomial equations which may be tested for the existence of a solution in a finite number of rational steps via decision methods. Should a solution exist, it may be evaluated with the aid of polynomial factorization. The first set of conditions are based on results due to Ho and Kalman, and are useful for the case where the number of specified Markov parameters is greater than the order of the realization. For other cases, the second set of conditions which include results from a companion paper on minimal observers, require less computational effort to be tested.

1. INTRODUCTION

SINCE THIS paper is concerned with asymptotically stable partial realizations of time invariant systems using decision methods, the relevant aspects of both these topics are now reviewed.

1.1 *Asymptotically stable partial realizations*

An important modelling or realization problem is to determine a state-space model of a given system from a particular input–output description of that system. Gilbert[1] first achieved minimal order realizations of a specified transfer function matrix, but his algorithm requires that the transfer function matrix elements have distinct poles. Kalman[2] developed an algorithm to reduce the state-space of any non-minimal realization to a minimal one. Ho and Kalman[3] use the Markov parameters Y_j for $j = 1, 2, \ldots$ of the proper rational transfer function matrix $G(s)$ to produce a minimal state-space realization. Other minimal realization algorithms exist which may be more computational efficient[4–7]. Note that in practice only the first $2n_{\max}$ parameters need be specified where $n_{\max}$ is an upper bound on the order of such a realization.

Let us consider, the linear time-invariant system with state-space equations

$$x(t) = Ax(t) + Bu(t),\ y(t) = C'x(t) + Du(t) \tag{1.1}$$

with u an r-vector input, y an m-vector output and x an n-vector state. Its transfer function is

$$D + C'(sI - A)^{-1}B = D + C'Bs^{-1} + C'ABs^{-2} + C'A^2Bs^{-3} + \ldots$$

Recalling that the Markov parameters Y_i of a transfer function $G(s)$ are defined from $G(s) = Y_0 + Y_1s^{-1} + Y_2s^{-2} + \ldots$ it is now clear that the system (1.1) is a realization of $G(s)$ if and only if

$$y_0 = D,\ Y_j = C'A^{j-1}B \quad \text{for} \quad j = 1, 2, \ldots \tag{1.2}$$

When only the first M Markov parameters are available and it is required to determine $[C, A, B]$ such that $Y_j = C'A^{j-1}B$ for $j = 1, 2, \ldots M$, then we have what is known as a partial realization problem. An algorithm to determine a minimal partial realization, in general not unique, is given in Tether[8] or Kalman[9] (note also [10]). Unfortunately this algorithm will not guarantee asymptotic stability of the realization even when it is known that the system giving rise to the Markov parameters is asymptotically stable. Efforts to include a stability constrant on this form of canonical realization are described in [16]. The algorithm presented can be implemented for relatively simple systems but would require further research effort to give a precise procedure to implement the various tests for higher order multivariable systems. In this paper decidability theory and polynomial factorization are applied to achieve minimal asymptotically stable partial realizations—admittedly at the expense of computational effort.

*Received 16 April 1975; revised 6 October 1975; revised 5 March 1976. The original version of this paper was presented at the 6*th IFAC Congress* which was held in Boston/Cambridge, MA. during August 1975. The published Proceedings of this IFAC Meeting may be ordered from: ISA—Instrument Society of America, 400 Stanwix Street, Pittsburgh, PA 15222, U.S.A. This paper was recommended for publication in revised form by associate editor J. Ackermann.

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1.2 *Decision methods*

The decision methods of interest to us here determine whether or not a vector υ exists such that $f(\upsilon)=0$ and $g(\upsilon)>0$ where $f(\cdot)$ and $g(\cdot)$ are vectors with each element a polynomial in any given element of υ. Such a decision can be made in a finite number of rational steps [11]. Unfortunately the number of steps increases exponentially with the number of unknowns, the elements of υ, and the number of inequalities. Reference [12] provides a method for determining a solution υ which satisfies the above equalities and inequalities—given of course knowledge that such a υ exists. This solution method involves polynomial factorization and thus in theory involves an infinite number of steps.

Perhaps the key contribution of this paper consists of the efficient use of results due to Ho and Kalman, to formulate the stable partial realization problem in a form suitable for solution by decision methods [11, 12]. Of the two solutions presented, the approach using the Ho–Kalman results, is most efficient for the case when the number of specified Markov parameters M satisfies $n\dagger \le M \le 2n\dagger$ where $n\dagger$ is the order of the minimal asymptotically stable partial realization. When $M>2n\dagger$ the Ho–Kalman algorithm may be applied directly. But when $M<n\dagger$ some of the ideas of a companion paper [13] for designing special classes of minimal order observers are employed. (Actually it was the observation by T. E. Fortmann that certain observer problems could be expressed as stable partial realization problems that initiated our investigations of this topic.) There is intentionally some duplication of [13] in this paper for the case $M<n\dagger$ since for the reader whose chief interest is the stable partial realization problem, much of [13] would be obscure.

Section 2 reviews some known realization results and extends these to derive a set of necessary and sufficient conditions for an asymptotically stable partial realization. A series of Hankel matrix properties are developed in Section 3 including a set of lower bounds for the order of a minimal realization. In Section 4, these properties are used to simplify the decision problem presented in Section 2 and a procedure is presented which may be used to evaluate the minimal order asymptotically stable partial realization. In Section 5 an alternative method to that described in Sections 2–4 is presented and is shown to be more efficient for the case $M<n\dagger$.

2. REALIZATION THEORY

In this section necessary and sufficient conditions for the existence of an asymptotically stable partial realization are presented. In developing these conditions, the relevant results of realization theory and partial realization theory are reviewed.

2.1 *Case* $Y_1, Y_2, \ldots$ *specified*

In the first instance we review realization results for the case where an infinite set of $m\times r$ Markov parameters $Y_1, Y_2, \ldots$ are specified—see Ho and Kalman [3].

First Ho–Kalman result. The transfer function $G(s)$ (possibly non minimal) with specified Markov parameters Y_j for $j=1,2,\ldots$ has a denominator polynomial $\beta(s)=s^q+\beta_{q-1}s^{q-1}+\ldots+\beta_1 s+\beta_0$ if and only if

$$Y_j\beta_0+Y_{j+1}\beta_1+\ldots+Y_{j+q-1}\beta_{q-1}+Y_{j+q}=0 \quad \text{for} \quad j=1,2,\ldots \tag{2.1}$$

Before summarizing a second result from [3] it is useful to define the Hankel matrices H_{qq} and $\tilde{H}_{qq}$ in terms of the specified Markov parameters Y_j for $j=1,2,\ldots$ as

$$H_{ij}=\begin{bmatrix} Y_1 & Y_2 & \cdot & \cdot & Y_j \\ Y_2 & Y_3 & \cdot & \cdot & Y_{j+1} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ Y_i & Y_{i+1} & \cdot & \cdot & Y_{i+j-1} \end{bmatrix};$$
$$\tilde{H}_{ij}=\begin{bmatrix} Y_2 & Y_3 & \cdot & \cdot & Y_{j\infty} \\ Y_3 & Y_4 & \cdot & \cdot & Y_{j+2} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ Y_{i+1} & Y_{i+2} & \cdot & \cdot & Y_{i+j} \end{bmatrix} \tag{2.2}$$

Second Ho–Kalman result. For a transfer function with Markov parameters Y_j for $j=1,2,\ldots$ and a denominator polynomial $\beta(s)$ of degree q, an irreducible realization is of order

$$n=\operatorname{rank} H_{qq} \tag{2.3}$$

Recall that the rank condition rank $H_{qq}\le p$ for some $p<\min\{mq, rq\}$ is equivalent to the determinant conditions

$$|H^{\ell}_{p+1}|=0 \quad \text{for all } \ell \tag{2.4}$$

where H^{ℓ}_{p+1} denotes the ℓth $(p+1)\times(p+1)$ submatrix of H_{qq}.

With knowledge of the order n of the realization, a system realization may be determined using the Ho–Kalman algorithm [3] or [4–7].

2.2 *Case* $Y_1, Y_2, \ldots Y_M$ *specified*

When only the first M Markov parameters

$Y_1, Y_2, \dots Y_M$ are specified, we have what is known as a partial realization problem. An algorithm described by Tether[8] and Kalman[9] achieves a realization for this case, but there is no guarantee that the realization is asymptotically stable. In this section we achieve asymptotically stable partial realizations using the Ho–Kalman results as a starting point. It now proves convenient to introduce the following point of notation.

Point of notation. An nth order realization having a transfer function with the first M Markov parameters $Y_1, Y_2, \dots Y_M$ and a qth order denominator polynomial $\beta(s)$ will be denoted as an $[n, q, M]$ realization. An asymptotically stable $[n, q, M]$ realization will be denoted an $[n, q, M, As]$ realization.

Now since an asymptotically stable partial realization is required, it is necessary that the characteristic polynomial $\beta(s)$ have left half plane zeros. Equivalently, there must exist parameters v_i, w_i and some positive constant ϵ such that

$$\beta(V_q, s) = \sum_{i=0}^{q} \beta_i(V_q)s^i \tag{2.5}$$

$$= \begin{cases} \prod_{i=1}^{q/2} [s^2 + (v_i^2 + \epsilon)s + (w_i^2 + \epsilon)] & \text{for } q \text{ even} \\ (s + w_0^2 + \epsilon) \prod_{i=1}^{(q-1)/2} [s^2 + (v_i^2 + \epsilon)s + (w_i^2 + \epsilon)] & \text{for } q \text{ odd} \end{cases}$$

where V_q (of dimension q) is used to denote the vector consisting of all v_i and w_i. Now we are in a position to make the following claim.

A necessary and sufficient condition for a $[\cdot, q, M, As]$ realization is that there exists a parameter vector V_q such that with $\beta_i(V_q)$ defined in (2.5)

$$\beta_0(V_q)Y_j + \beta_1(V_q)Y_{j+1} + \dots + \beta_{q-1}(V_q)Y_{j+q-1} + Y_{q+j} = 0$$

$$\text{for } j = 1, 2, \dots (M - q) \tag{2.6}$$

Necessity follows by application of the first Ho–Kalman result in the previous subsection. Sufficiency is seen by construction of such a realization. The parameters $Y_{M+1}, Y_{M+2}, \dots Y_{\bar{q}-1}$ (where $\bar{q} = \max\{q+1, M+1\}$) are chosen arbitrarily while $Y_{\bar{q}}, Y_{\bar{q}+1}, \dots$ may be calculated from

$$Y_i = -[\beta_0(V_q)Y_{i-q} + \beta_1(V_q)Y_{i-q+1} + \dots + \beta_{q-1}(V_q)Y_{i-1}] \tag{2.7}$$

for $i = \bar{q}, \bar{q}+1 \dots$. Having so defined an infinite series of Markov parameters, a realization may be constructed as in [3]. By the first Ho–Kalman result the transfer function denominator polynomial is $\beta(V_q, s)$ and since $\beta(V_q, s)$ is defined as in (2.5) the realization is asymptotically stable.

For $q = M$ equation (2.6) involves no restriction on $\beta(s)$. An arbitrary $\beta(s)$ may be chosen to have strictly negative zero and H_{MM} will be fully defined using (2.7), for any such choice. Using the second Ho–Kalman result above it is clear that the order of an asymptotically stable partial realization must in this case satisfy $n \le \min\{Mm, Mr\}$.

The key lemma of this section, now stated, is an immediate consequence of the above claims and the second Ho–Kalman result.

Lemma 2.1. A $[n, q, M, As]$ realization exists for some $n \le p$ if and only if the following condition (C1) is satisfied.

Condition (C1). There exists a $V_q, Y_{M+1}, Y_{M+2}, \dots Y_{\bar{q}-1}$ such that (2.6) is satisfied and, with $Y_{\bar{q}}, Y_{\bar{q}+1}, \dots Y_{2q-1}$ chosen satisfying (2.7), rank $H_{qq} \le p$ (or equivalently for $p < \min\{mq, rq\}$ the $(p+1) \times (p+1)$ submatrices H_{p+1}^{ℓ} of H_{qq} satisfy (2.4)).

Remarks. (1) The condition (C1) is in the form of multivariable polynomial equalities and thus the decision methods of [11, 12] may be applied. For this decision problem there are $q + rm \times (\max\{0, (q - M)\})$ unknowns. In the next sections we show how both the number and complexity of the equalities (2.4) can be reduced significantly so that the computational effort in testing condition (C1) can be reduced to manageable proportions.

(2) For the case of scalar input realizations or scalar output realizations, Y_i are either row or column vectors and thus $m = 1$ or $r = 1$, inevitably $n = \text{rank } H_{qq} \le q$. This means that in the statement of Lemma 2.1, $p = q$ and the inequality (rank $H_{qq} \le n$) in (C1) is automatically satisfied.

3. PROPERTIES OF H_{qq}

In this section we explore some properties of Hankel matrices defined in (2.2) where the Markov parameters $Y_1, Y_2, \dots$ satisfy (2.1) for some q. In particular, we explore the properties of H_{qq} since these will be useful for simplifying the condition (C1) of the previous section.

Property ($P1$)[5]. (i) Dependence of the ith row of H_{qq} on rows above the ith row of H_{qq} implies dependence of the $(i+km)$th row on rows above the $(i+km)$th row in H_{qq} for $k=0,1,\dots$ (ii) Dependence on the ith column of H_{qq} on columns to the left of the ith column of H_{qq} implies dependence of the $(i+kr)$th column on columns to the left of the $(i+kr)$th column in H_{qq} for $k=0,1,\dots$.

The next property of Hankel matrices is given in terms of a special class of submatrices of H_{qq} denoted $H(\mu,\lambda)$ obtained by an ordered deletion of rows and columns of H_{qq} defined from the integer indices $\lambda=[\lambda_1\lambda_2\dots\lambda_r]'$, $\mu=[\mu_1\mu_2\dots\mu_m]'$ as follows.

Definition: $H(\mu,\lambda)$ is obtained from H_{qq} by deleting the rows numbered $[i+(\mu_i+k)m]$ and the columns numbered $[j+(\lambda_j+k)r]$ for $i=1,2,\dots m$, $j=1,2,\dots r$ and $k=0,1,\dots$.

Property ($P2$). Given that there exists an n such that rank $H_{qq}\geq n$ then there exist vectors λ,μ satisfying

$$\sum_{i=1}^{r}\lambda_i=\sum_{i=1}^{m}\mu_i=n \tag{3.1}$$

such that rank $H(\mu,\lambda)=n$. Equivalently, given that rank $H(\mu,\lambda)\neq n$ for some n and all μ,λ satisfying (3.1), then rank $H_{qq}<n$.

Proof. The definition of rank tells us that the inequality rank $H_{qq}\geq n$ implies that there exists a full rank $n\times n$ matrix obtained from H_{qq} by deleting rows and columns according to the following rule. The ith row of H_{qq} is deleted if it is contained in the range space of the rows above it and the ith column of H_{qq} is deleted if it is contained in the range space of the column to the left of it. Application of property (P1) now ensures us that our full rank $n\times n$ matrix is in fact $H(\mu,\lambda)$ for some μ and λ satisfying (3.4) as desired.

We now move on to a consideration of an important property of the Hankel matrix H_{qq} defined from (2.2) with $Y_1, Y_2,\dots Y_M$ specified and $Y_{M+1}, Y_{M+2},\dots Y_{2M}$ unspecified. To determine a lower bound for the rank of such matrices irrespective of the selection of Y_{M+1}, $Y_{M+2},\dots Y_{2M}$, attention is focused on certain fully specified submatrices of H_{MM} namely the Hankel matrices $H_{r_j,M+1-r_j}$ and H_{M+1-c_j,c_j} where

r_j is the smallest integer $\geq j/m$
c_j is the smallest integer $\geq j/r$

Integer indices $\mu^*=[\mu_1^*\mu_2^*\dots\mu_m^*]'$ and $\lambda^*=[\lambda_1^*\lambda_2^*\dots\lambda_r^*]'$ are defined as follows

$$\mu_i^*=\sum_{k=0}^{q-1}R_{i+km},\ \lambda_i^*=\sum_{k=0}^{q-1}C_{i+kr} \tag{3.2}$$

where

$$R_j=\begin{cases}0 \text{ if the } j\text{th row of } H_{r_j,M+1-r_j} \text{ is}\\ \text{dependent on rows above it.}\\ 1 \text{ otherwise}\end{cases} \tag{3.3}$$

$$C_j=\begin{cases}0 \text{ if the } j\text{th columns of } H_{M+1-c_j,c_j} \text{ is}\\ \text{dependent on columns to the left of it.}\\ 1 \text{ otherwise}\end{cases} \tag{3.4}$$

The above choice of rows (or columns) is the same as the choice of independent rows (or columns) in the algorithm of [8]. The indices $\mu^*(\lambda^*)$ are the observability (controllability) Kronecker indices[14] of the minimal partial realization, and as a consequence we have the following property adapted from [8].

Property ($P3$). With H_{qq} defined in (2.2) where $Y_1, Y_2,\dots Y_M$ is specified and $Y_{M+1}, Y_{M+2},\dots Y_{2M}$ unspecified, and μ^* and λ^* defined above, a lower bound on the rank H_{qq} is given from

$$\text{rank}\quad H_{qq}\geq n^* \tag{3.5}$$

where

$$n^*=\sum_{i=1}^{r}\lambda_i^*=\sum_{i=1}^{m}\mu_i^*=\text{rank}\quad H(\mu^*,\lambda^*) \tag{3.6}$$

Some useful bounds. (1) The order of the minimal partial realization n^* is clearly a lower bound for the order of the minimal stable partial realization $n\dagger$. (2) Let the Hankel matrix H_{ab} be the smallest submatrix of H_{MM} which contains all rows for which $R_j=1$ (i.e. "independent" rows) and all columns for which $C_j=1$ (i.e. "independent" columns), also let $q^*=\max(a,b)$. With the first Ho–Kalman result in mind, upon reflection we see that q^* is a lower bound for q in any $[\cdot,q,M]$ realization and is also a lower bound for the $q\dagger$ of the minimal stable partial realization denoted $[n\dagger,q\dagger,M,As]$.

4. MINIMAL ORDER ASYMPTOTICALLY STABLE PARTIAL REALIZATION

To determine a minimal order asymptotically stable partial realization denoted an $[n\dagger,q\dagger,M,As]$ realization, the value of $n\dagger$, the minimal order itself, is first calculated being the following efficient procedure.

4.1 *Steps to calculate* n†

Step 1. Test for the existence of a $[\cdot,q,M,As]$ realization, for $q=q^*,q^*+1,\dots$ by applying decision methods to determine whether or not for $q=q^*,q^*+1,\dots$ a parameter vector V_q exists such that with $\beta_i(V_q)$ defined in (2.5) polynomial equality condition (2.6) is satisfied. The lowest integer q for which such a realization exists is denoted $q_{\min}$. Note that q^* is readily calculated as described in the previous sections.

Step 2. Apply decision methods to the condition (C1) for $\beta(V_n, s)$ defined as in (2.5) to test for the existence of an $[n, n, M, As]$ realization for $n = \bar{n}, \bar{n}+1, \ldots$ where $\bar{n} = \max(q_{\min}, n^*)$, until the conditions are satisfied. The lowest integer n is denoted $n\dagger$. Note that n^* is readily calculated as described in the previous section.

There is a considerable saving in the above procedure for calculating $n\dagger$ in testing for an $[n, n, M, As]$ realization for $n = \bar{n}, \bar{n}+1, \ldots$ rather than in testing for an $[n, q, M, As]$ realization for $n = \bar{n}, \bar{n}+1, \ldots$ and $q = q_{\min}, q_{\min+1}, \ldots$. The justification stems from the following lemma.

Lemma 4.1. The non-existence of an $[n, n, M, As]$ realization for some n implies the non-existence of an $[n, q, M, As]$ realization for all q.

The above lemma is established by application of the following two properties of realization:

(a) The existence of an $[n, q, M, As]$ realization of $G(s)$ implies the existence of an $[n, q+\ell, M, As]$ realization of $G(s)L(s)/L(s) = G(s)$ for any polynomial $L(s)$ of degree ℓ, where $\ell = 1, 2, \ldots$.

(b) The existence of an $[n, q, M, As]$ realization of $G(s)$ for $q > n$ implies the existence of an $[n, q, M, As]$ realization for $q = n$, since there must be at least $(q-n)$ pole–zero cancellations possible in $G(s)$.

Remarks. (1) The algorithm must terminate since $n\dagger \leq M \times \min(r, m)$. (2) For the case of scalar input or scalar output realizations, $n\dagger = q_{\min}$. To see this note that with step 1 implemented, lemma 2.1 tells us that there exists an $[n, q_{\min}, M, As]$ realization for some $n \leq q_{\min}$. But the property (b) above is violated unless $n = q_{\min}$. Thus $n\dagger = q_{\min}$ and step 2 above is not required. (3) For the multiple-input, multiple-output realization cases $(m > 1, r > 1)$, by setting $q_{\min} = q^*$ in step 2, step 1 can be eliminated at a possibly small increase in computational effort. It is shown in [17] that there exist cases where $q_{\min} > n^*$ so either term may be a more useful lower bound. (4) Having determined $n, n\dagger$ and $q\dagger$ may be evaluated by testing the conditions (C1) for the existence of an $[n\dagger, q, M, As]$ realization for $q = q_{\min}, q_{\min+1}, \ldots, q\dagger$ which is the lowest values of q for which a realization exists. Alternatively one may test for $q = n\dagger, n\dagger - 1, \ldots$ until a realization fails to exist, depending on whether q is expected closer to $n\dagger$ than $q_{\min}$. (5) To test the condition (C1) there are of the order of $(n!)^2$ determinant equalities. Fortunately (C1) may be considerably simplified, both in the number and complexity of the determinant tests as we now go to show. The simplifications we describe are crucial to achieve a practical algorithm but unfortunately the manipulations are intricate.

4.2 *Simiplification of condition (C1)*

(This subsection may be omitted as a first reading of the paper). The properties (P1)–(P3) of Hankel matrices are exploited to achieve a simplification of the condition (C1) for the cases $m > 1, r > 1$. In particular the conditions (2.4) are simplified. For convenience we adapt the following convention.

Convention. The columns and rows of a submatrix of H_{qq}, possibly with its rows and columns interchanged, are assigned the numbers of the corresponding rows and columns of H_{qq}.

Four steps are now described to achieve a simplification of (2.4) given that H_{qq} is defined from (2.2) where the Markov parameters $Y_1, Y_2, \ldots$ satisfy (2.1) for some q, and $Y_1, Y_2, \ldots Y_M$ are specified. Using the Convention above, the simplified equalities are readily stated although one has to be careful with bookkeeping details in practice. The four steps are now described along with a brief justification of each step.

Step 1, *description.* Examine in turn the first $m(M-q+1)$ rows of H_{qq} for linear dependence on the rows above the row in question. If the ith row is dependent on the rows $1, 2, \ldots i-1$, then delete this row. Likewise if the ith column is dependent on the columns $1, 2, \ldots i-1$, then delete this column. Note that property (P1) is used to accomplish this step. Our depleted matrix or submatrix is denoted $\hat{H}$.

Assertions (i) Rank $\hat{H} = \text{rank } H_{qq}$. (ii) Property (P1) holds with H_{qq} replaced by $\hat{H}$ and the rows [columns] of $\hat{H}$ denoted according to the convention above.

Justification. The above assertions are justified since only dependent rows [columns] of H_{qq} are deleted, and so whether or not a row [column] of $\hat{H}$ is independent or not can be determined without knowledge of these dependent rows [columns].

Step 2, *description.* Calculate μ^* and λ^* via (3.5)–3.7) and thereby obtain $H(\mu^*, \lambda^*)$ a submatrix of H_{MM}. ($H(\mu^*, \lambda^*)$ is also submatrix of H_{qq} and $\hat{H}$) . Rearrange rows and columns of $\hat{H}$ to form

$$\bar{H} = \begin{bmatrix} H(\mu^*, \lambda^*) & \bar{H}_2 \\ \bar{H}_3 & \bar{H}_4 \end{bmatrix}$$

where both $\bar{H}_4$ and $H(\mu^*, \lambda^*)$ are submatrices of $\hat{H}$.

Remark. This operation merely groups the independent rows and columns of H_{MM}.

Assertions. (i) Rank $H(\mu^*, \lambda^*) = n^*$. (ii) Rank $\bar{H} = \text{rank } \hat{H} = \text{rank } H_{qq}$. (iii) Property (P1) part (i) holds with H_{qq} replaced by $[\bar{H}_3, \bar{H}_4]$ and the rows of $[\bar{H}_3, \bar{H}_4]$ denoted according to the convention above. Also, property (P1) (ii) holds with H_{qq} replaced by $\begin{bmatrix}\bar{H}_2\\ \bar{H}_4\end{bmatrix}$ and the columns of $\begin{bmatrix}\bar{H}_2\\ \bar{H}_4\end{bmatrix}$ denoted according to the convention above.

Justification. The first two assertions above are immediate. The third assertion follows from property (P1) holding for $\hat{H}$ and the properties of $H(\mu^*, \lambda^*)$. Thus in constructing $[\bar{H}_3, \bar{H}_4]$ from $\hat{H}$, if the ith row is deleted, then the $(i-km)$th row is also deleted for $k = 1, 2, \ldots$. As a consequence, property (P1) part (i) holds with H_{qq} replaced by $[\bar{H}_3, \bar{H}_4]$ as desired. In a similar manner the remainder of assertion (iii) can be extablished.

Step 3: *Description.* Form the matrix

$$\bar{H}_1 = \bar{H}_4 - \bar{H}_3 H^{-1}(\mu^*, \lambda^*)\bar{H}_2$$

Assertions. (i) Rank $H_{qq} = n^* + \text{rank } \bar{H}_1$. (ii) Property (P1) holds with H_{qq} replaced by $\bar{H}_1$ and the rows and columns of $\bar{H}_1$ designated by the numbers given to the corresponding rows of $\bar{H}_4$ when these are assigned according to the convention above.

Justification. We have the following

$$\begin{tikzcd}[column sep=huge, row sep=large]
\begin{bmatrix} H(\mu^*, \lambda^*) & \bar{H}_2 \\ \bar{H}_3 & \bar{H}_4 \end{bmatrix} \arrow[r, "\text{column operations}"] \arrow[d, "\text{row operations}"] & \begin{bmatrix} H(\mu^*, \lambda^*) & 0 \\ \bar{H}_3 & \bar{H}_1 \end{bmatrix} \arrow[d, "\text{row operations}"] \\
\begin{bmatrix} H(\mu^*, \lambda^*) & \bar{H}_2 \\ 0 & \bar{H}_1 \end{bmatrix} \arrow[r, "\text{column operations}"] & \begin{bmatrix} I_{n^*} & 0 \\ 0 & \bar{H}_1 \end{bmatrix}
\end{tikzcd}$$

Assertion (i) above is now immediate. From the above diagram note that $[\bar{H}_3\, \bar{H}_1]$ is obtained from column operations on $[\bar{H}_3\, \bar{H}_4]$. As noted earlier, property (P1) part (i) holds for $[\bar{H}_3\, \bar{H}_4]$ and since the property is not affected by column operations, including deletions, we have that it also holds for $[\bar{H}_3\, \bar{H}_1]$ and $\bar{H}_1$. Likewise it may be shown that (P1) part (ii) holds for $\begin{bmatrix}\bar{H}_2\\ \bar{H}_1\end{bmatrix}$ and thus for $\bar{H}_1$. We have thus shown that (P1) parts (i) and (ii) hold for $\bar{H}_1$.

Remark. $H(\mu^*, \lambda^*)$ may contain elements which are polynomial functions of the unknown parameters making the expression of the determinant and adjoint of $H(\mu^*, \lambda^*)$ more difficult. The difficulty may be avoided by separating in step 2 a full rank submatrix of $H(\mu^*, \lambda^*)$ containing only specified elements. The simplification is achieved at the expense of not producing a least order $\bar{H}_1$ in step 3.

Step 4, *description.* Form the submatrices $\bar{H}_1(\bar{\mu}_n, \bar{\lambda}_n)$ of $\bar{H}_1$ where $\bar{\mu}'_n = [(\mu_1 - \mu_1^*)(\mu_2 - \mu_2^*) \ldots (\mu_m - \mu_m^*)]$ and $\bar{\lambda}_n = [(\lambda_1 - \lambda_1^*)(\lambda_2 - \lambda_2^*) \ldots (\lambda_r - \lambda_r^*)]$ with (3.1) holding. (Here $n = \sum_{i=1}^{m} u_i = \sum_{i=1}^{r} \lambda_i$.)

Assertions. (i) Given that there exists an n such that rank $\bar{H}_1 = n - n^*$, then there exist vectors $\bar{\mu}_n$ and $\bar{\lambda}_n$ as defined above with (3.1) holding such that rank $\bar{H}_1(\bar{\mu}_n, \bar{\lambda}_n) = (n - n^*)$. (ii) The conditions (2.4) are equivalent to the conditions

$$|\bar{H}_1(\bar{\mu}_{n+1}, \bar{\lambda}_{n+1})| = 0 \text{ for all possible } \bar{\mu}_{n+1}, \bar{\lambda}_{n+1} \tag{4.1}$$

Justification. The arguments used above to demonstrate that property (P2) follows from (P1) for H_{qq} also demonstrate that the above assertion (i) follows from assertions (i) and (ii) of step 3. Assertion (ii) follows from assertion (i).

As a consequence of assertion (ii) above, we can restate condition (C1) as

Condition ($C2$). There exists a V_q, Y_{M+1}, $Y_{M+2}, \ldots Y_{q-1}$ such that (2.6) is satisfied and with Y_q, $Y_{q+1}, \ldots Y_{2q-1}$ chosen to satisfy (2.7), the equations (4.1) are satisfied.

Remarks (1). Observe that the submatrices $H^{\prime}_{p+1}$ of (2.4) are of dimension $(n+1) \times (n+1)$ whereas the submatrices $\bar{H}_1(\bar{\mu}_{n+1}, \bar{\lambda}_{n+1})$ of (4.1) are of dimension $(n+1-n^*) \times (n+1-n^*)$. This means that a determinant equality $|\bar{H}_1(\bar{\mu}_{n+1}, \bar{\lambda}_{n+1})| = 0$ is considerably simpler than the equality $|H^{\prime}_{n+1}| = 0$ for even moderate n^*. Moreover, the number of equalities in (4.1) is considerably less.

The number of equalities in (2.4) is

$$N_{2.4} = \binom{qm}{n+1}\binom{qr}{n+1}$$

where

$$\binom{A}{B} = \frac{A!}{B!(A-B)!}.$$

The number of equalities in (4.1) is no greater than

$$N_{4.1} = \binom{n - n^* + \hat{m} - 1}{n - n^*}\binom{n - n^* + \hat{r} - 1}{n - n^*}$$

where $\hat{m}$ is the number of $\mu_i^\dagger \geq M - q + 1$ and $\hat{r}$ is the number of $\lambda_i^\dagger \geq M - q + 1$. For the case $n = 6$, $n^* = 4$, $\hat{m} = \hat{r} = 3$, $q = 3$, $N_{2.4} = 7392$ and $N_{4.1} = 36$—a spectacular reduction. (2) If $H(\mu^*, \lambda^*)$ contains some elements not fully specified, then, the determinant test (4.1) will not be *polynominal* equalities in the unknown parameters. This problem is avoided by looking instead at determinants of submatrices of $\det[H(\mu^*, \lambda^*)]\bar{H}_1$. (3) For any λ_n and μ_n for which a realization exists, this realization may be constructed having $\lambda_n(\mu_n)$ as the controllability (observability) indices of that realization. Of course at least one such set of indices for which a realization exists are Kronecker indices so these can be identified from the set of λ_n or μ_n for which a realization exists and a canonical realization as in [15] constructed using these.

Example. $M = 2, r = m = 2$

$$Y_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad Y_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Assuming for the moment that ϵ in (2.5) is zero, Y_3 is defined in terms of Y_1 and Y_2 for $q = 2$ as

$$Y_3 = \begin{bmatrix} -w^2 - v^2 & -v^2 \\ 0 & w^2 - v^2 \end{bmatrix}$$

giving

$$H_{qq} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & -w^2 - v^2 & -v^2 \\ 0 & 1 & 0 & w^2 - v^2 \end{bmatrix}$$

$$H(\mu^*, \lambda^*) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, n^* = 2$$

Now step 1 of the simplification of (C1) is not relevant, step 2 is trivial as $\bar{H} = H_{qq}$, and step 3 gives

$$\bar{H}_1 = \begin{bmatrix} -w^2 - v^2 & -v^2 \\ 0 & w^2 - v^2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= -\begin{bmatrix} w^2 + v^2 + 1 & v^2 \\ 0 & -w^2 + v^2 - 1 \end{bmatrix}.$$

4.3 *A minimal order asymptotically stable partial realization*

With knowledge that an $[n\dagger, q, M, As]$ realization exists for a known $n\dagger$ and some $q\dagger < q < n\dagger$ obtained using the techniques of the previous subsection, such realization may be calculated as follows.

Step 1. The polynomial equalities of (C2) are solved by polynomial factorization for the elements of $V_q, Y_{M+1}, Y_{M+2}, \ldots Y_q$.

Step 2. The parameters $Y_{q+1}, Y_{q+2}, \ldots Y_{2q}$ are calculated via (2.7). Thus $Y_1, Y_2, \ldots Y_{2q}$ and thereby H_{qq} and $\tilde{H}_{qq}$ are determined.

Step 3. The Ho–Kalman algorithm [3] or one of the simpler algorithms [4–7] is applied to H_{qq} and $\tilde{H}_{qq}$ to yield a realization of order $n\dagger$ with the desired properties that it is minimal and asymptotically stable. Of course, in the event that scalar input or scalar output realizations are involved the full power of the multivariable algorithms are not required. For this case realizations in the observable or controllable canonical form are readily achieved once the β_i are calculated from V_q.

The steps of the algorithm and to simplification are now demonstrated on a simple example.

In this example it is clear that there are no v and w such that rank $\bar{H}_1 = 0$ so clearly $n\dagger > n^* = 2$. Checking for rank $\bar{H}_1 = 1$ we examine $|\bar{H}_1|$ which is zero for $v^2 = 1 + w^2$, since (2.6) is trivially satisfied we may choose the solution $w^2 = 1$, $v^2 = 2$. This gives an extension

$$Y_3 = \begin{bmatrix} -3 & -2 \\ 0 & -1 \end{bmatrix}$$

and

$$Y_4 = \begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix},$$

thus

$$H_{MM} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 3 & -2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

Using the Ackerman–Bucy procedure [4] a realization of third order may be calculated which turns out to be asymptotically stable. In particular,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For this example the minimal realization is of order 2 while the minimal stable realization is of order 3.

5. ALTERNATIVE APPROACH

In this section we formulate our realization problem as a decision problem which may be solved by decision methods using a specialization of the results in [13]. The alternative approach includes fewer unknowns, in some cases when $n\dagger > M$, than the methods of earlier sections.

For the case $Y_1, Y_2, \dots Y_M$ specified and $Y_{M+1}, Y_{M+2}, \dots$ unspecified, equation (1.2) reduces to

$$[Y_1 Y_2 \dots Y_M] = C'R, \qquad R = [B\,AB \dots A^{M-1}B]$$

or

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_M \end{bmatrix} = QB, \qquad Q = \begin{bmatrix} C' \\ C'A \\ \vdots \\ C'A^{M-1} \end{bmatrix} \tag{5.1}$$

For a fully specified sequence of Markov parameters, Kalman showed that any state realization is irreducible if and only if the system is completely controllable and observable. We now state a corresponding result for asymptotically stable partial realizations.

Lemma 5.1. A necessary condition for the system (1.1) with system matrix $[A, B, C]$, an asymptotically stable partial realization of the Markov parameters $Y_1 Y_2 \dots Y_M$, to be of minimum order is that R and Q defined above be full rank matrices.

Proof. Necessary conditions for minamality are that $[A, B]$ is completely controllable and $[A, C]$ completely observable. For the case $M \le n$ the rank conditions on R and Q are equivalent to the controllability and observability conditions and the lemma is immediate. For the case $M \ge n$, we proceed by showing that if the lemma is not true we are led to an contradiction. In particular we show that if $[A, B, C]$ is an asymptotically stable partial realization with R not of full rank then a lower order asymptotically stable partial realization can be constructed.

For R not of full rank, we have for some singular D, that $R = DR$. Now this relationship implies that the system $[A, B, C]$ and $[AD, B, D'C]$ have identical Markov parameters $Y_1 Y_2 \dots Y_M$ since $C'B = C'DB$, $C'AB = C'D(AD)B, \dots C'A^{M-1}B = C'D(AD)^{M-1}B$.

Now the pair $[AD, D'C]$ is not completely observable since its observability matrix can be expressed as the product of two matrices one of which is the singular matrix D. Denoting the observable component of the system $[AD, B, D'C]$ as $[\bar{A}_D, \bar{B}_D, \bar{C}_D]$, it is clear that $[\bar{A}_D, \bar{B}_D, \bar{C}_D]$ has Markov parameters $Y_1, Y_2, \dots Y_M$ and is lower order than the realization $[A, B, C]$. It remains to be shown that $[\bar{A}_D, \bar{B}_D, \bar{C}_D]$, for some choice of D, is an asymptotically stable realization, to complete the proof. In fact we show that $\{\lambda_i[\bar{A}_D]\}$ for some D is a subset of $\{\lambda_i[A]\}$.

Without loss of generality let A be upper triangular and let the $n-i$th row of R be dependent on the rows below it. For this case

$$D = \begin{bmatrix} I_{n-i-1} & 0 & 0 \\ 0 & 0 & x \\ 0 & 0 & I_i \end{bmatrix}, \qquad A = \begin{bmatrix} A_1 & x & x \\ 0 & x & x \\ 0 & 0 & A_2 \end{bmatrix} \tag{5.2}$$

where A_1 and A_2 are upper triangular matrix of order $(n-i-1)$ and i, respectively and x denotes a submatrix of appropriate dimension. Manipulations yield the following forms for (AD) and $\bar{A}_D$ since it is the $n-i$th state of the system which is clearly unobserable.

$$AD = \begin{bmatrix} A_1 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & A_2 \end{bmatrix}, \quad \bar{A}_D = \begin{bmatrix} A_1 & x \\ 0 & A_2 \end{bmatrix}$$

since A_1 and A_2 are upper diagonal and submatrices of A this shows that $\{\lambda_i(\bar{A}_D)\}$ is a subset of $\{\lambda_i(A)\}$.

The above rank condition allows us to reduce the number of unknown parameters in (5.1) by considering the equivalent conditions.

$$C' = [Y_1 Y_2 \dots Y_M]R'(RR')^{-1} \tag{5.3}$$

$$[Y_1 Y_2 \dots Y_M][\det(RR')I - R'(RR')^A R] = 0 \tag{5.4}$$

$$\det(RR') > 0 \tag{5.5}$$

or

$$B' = [Y_1' \dots Y_M']Q[Q'Q]^{-1} \tag{5.6}$$

$$[Y_1' \dots Y_M'][\det(Q'Q)I - Q(Q'Q)^A Q'] = 0 \tag{5.7}$$

$$\det(Q'Q) > 0 \quad (5.8)$$

where X^A denotes the adjoins of the matrix X.

To determine the existence of an nth order asymptotically stable partial realization decision methods can be applied to the polynomial expressions (5.4) and (5.5) or (5.7) and (5.8) with the stability constraint on A. This test only involves the unknowns of A and B (or A and C) rather than all parameters of A, B and C as in (5.1).

This decision problem is a specialisation of one studied in [13]. In [13] the above decision problem is greatly simplified by use of $[A, C]$ in a special form with a set of structural indices n_i where $\sum_{i=1}^{r} n_i = n$. In [15] several structural forms are available with a reduced parameter set but the lower block triangular form for A described in [4] appears most desirable here. For this form the diagonal blocks of A are in the single output observable canonical form, thus the stability, constraint on A is satisfied by choosing the parameters describing the characteristic polynomial in (2.5). The number of unknowns in the decision problem is now $\sum_{i=1}^{r} (r+1-i)n_i$. An upper limit on the number of such unknowns is rn. Similarly if $[A, B]$ is chosen in the special form the number of parameters would be less than mr. Thus an upper bound for the number of unknowns which are involved using these structural forms is $\min(rn, mn)$.

Since the complexity of the decision problem depends largely on the numbers of unknowns, and the number of unknowns for the methods of the previous section totals $(n-M)rm$ for $n > M$, the methods of the section should be used for the case $(n-M)rm \geq \min(rn, mn)$.

Since for multivariate systems it is possible for an nth order realization to exist for one set of structural indices n_i and not for another set, it should be noted that the decision problem must be repeated for all possible choices of n_i. This implication of the property for realizations is treated in [18]. We will have $\binom{n+r-1}{n}$ decision problems for the $[A, C]$ form and $\binom{n+M-1}{n}$ for the $[A, B]$ form: where

$$\binom{X}{y} = \frac{X!}{y!(X-y)!}$$

Unfortunately the complexity of the computations for both procedures given in the paper for the design of minimal order stable partial realizations increases exponentially with the order of the realization. For the special case $m = 1$ or $r = 1$, single output or input systems, it is only necessary to test one simple set of equalities involving n unknowns. If $r > 1$ and $m > 1$ the decision problem is inevitably more tedious to implement.

6. CONCLUSIONS

In the paper we have pointed out that the decision methods of [12] can be applied to yield a solution to the problem of minimal order asymptotically stable partial realizations. In exploring the application of decision methods to this problem, the objective is to minimize by analytic means, the number of unknowns in the underlying polynomial equalities and so minimize the complexity of the decision problem and thereby achieve more efficient solutions than could otherwise be obtained as in [16]. Proceeding towards the objective has been fruitful, both in defining a relevant simplified decision problem and in giving insight into the nature of the minimal order asymptotically stable partial realization problem itself. Unfortunately the complexity of the completions for both procedures given in the paper for the design of such realization increases exponentially with the order of the realization. A "curse of dimensionality" overshadows the work as in Dynamics Programming for control, Viterbi Algorithms for communications and shortest route methods for operations research. For the special case $r = 1$, $m = 1$, single input or output systems, there is no difficulty as it is only necessary to test one simple set of equation involving n unknowns. For $r > 1$ and $m > 1$ the decision problem is inevitably more tedious to implement.

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