

**A FORMULA FOR COMPUTATION OF THE  
REAL STABILITY RADIUS**

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# A Formula for Computation of the Real Stability Radius\*

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## Abstract

This paper presents a readily computable formula for the real structured stability radius with respect to an arbitrary stability region in the complex plane.

**Keywords:** Stability; robustness; perturbation techniques; matrix analysis; linear systems.

## 1 Introduction

In many engineering applications it is required that a square matrix has all of its eigenvalues in a prescribed area in the complex plane. We will use the word stability to describe such an eigenvalue clustering property. Furthermore, it is often desired that the matrix should maintain this stability property when its elements are subject to certain perturbations. The real structured stability radius measures the ability of a matrix to preserve its stability under a certain class of real perturbations.

Let us partition the complex plane  $\mathbb{C}$  into two disjoint subset  $\mathbb{C}_g$  and  $\mathbb{C}_b$ , i.e.,  $\mathbb{C} = \mathbb{C}_g \dot{\cup} \mathbb{C}_b$ , such that  $\mathbb{C}_g$  is open. A matrix is said to be stable if its eigenvalues are contained in  $\mathbb{C}_g$ . Denote the singular values of  $M \in \mathbb{C}^{p \times m}$ , ordered nonincreasingly, by  $\sigma_i(M)$ ,  $i = 1, 2, \dots, \min\{p, m\}$ . Also denote  $\sigma_1(M)$  by  $\bar{\sigma}(M)$  and  $\sigma_{\min\{p, m\}}(M)$  by  $\underline{\sigma}(M)$ . Let  $\mathbb{F}$  be either the real field  $\mathbb{R}$  or the complex field  $\mathbb{C}$ . Following (Hinrichsen and Pritchard, 1986b), we define the (*structured*) *stability radius* of a matrix triple  $(A, B, C) \in \mathbb{F}^{n \times n} \times \mathbb{F}^{n \times m} \times \mathbb{F}^{p \times n}$  as

$$r_{\mathbb{F}}(A, B, C) := \inf \{ \bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{m \times p} \text{ and } A + B\Delta C \text{ is unstable} \}.$$

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We abbreviate  $r_{\mathbb{F}}(A, I, I)$  by  $r_{\mathbb{F}}(A)$  and call it the *(unstructured) stability radius* of  $A$ . For real  $(A, B, C)$ ,  $r_{\mathbb{R}}(A, B, C)$  is called the *real* stability radius and for complex  $(A, B, C)$ ,  $r_{\mathbb{C}}(A, B, C)$  is called the *complex* stability radius. The stability radius problem concerns the computation of  $r_{\mathbb{F}}(A, B, C)$  when  $(A, B, C)$  is given.

Let  $\partial\mathbb{C}_g$  denote the boundary of  $\mathbb{C}_g$ . By continuity, we can easily show that for stable  $A$ ,

$$\begin{aligned} r_{\mathbb{F}}(A, B, C) &= \inf\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{m \times p} \text{ and } A + B\Delta C \text{ has an eigenvalue on } \partial\mathbb{C}_g\} \\ &= \inf_{s \in \partial\mathbb{C}_g} \inf\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{m \times p} \text{ and } \det(sI - A - B\Delta C) = 0\} \\ &= \inf_{s \in \partial\mathbb{C}_g} \inf\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{m \times p} \text{ and } \det[I - \Delta C(sI - A)^{-1}B] = 0\}. \end{aligned}$$

Hence the key issue in the computation of the stability radius is to solve the following linear algebra problem: given  $M \in \mathbb{C}^{p \times m}$ , compute

$$\mu_{\mathbb{F}}(M) := \{\inf\{\bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{m \times p} \text{ and } \det(I - \Delta M) = 0\}\}^{-1}.$$

Simple singular value arguments show that  $\mu_{\mathbb{C}}(M) = \bar{\sigma}(M)$ . Hence

$$r_{\mathbb{C}}(A, B, C) = \left\{ \sup_{s \in \partial\mathbb{C}_g} \bar{\sigma}[C(sI - A)^{-1}B] \right\}^{-1}, \quad (1)$$

which was essentially obtained in (Doyle and Stein, 1981; Chen and Desoer, 1982; Hinrichsen and Pritchard, 1986b). Equation (1) relates the complex stability radius to the concept of the  $\mathcal{H}_{\infty}$  norm.

This paper concerns the computation of  $r_{\mathbb{R}}(A, B, C)$ . As we have seen,

$$r_{\mathbb{R}}(A, B, C) = \left\{ \sup_{s \in \partial\mathbb{C}_g} \mu_{\mathbb{R}}[C(sI - A)^{-1}B] \right\}^{-1}. \quad (2)$$

Our main result is a simple formula for  $\mu_{\mathbb{R}}$  which allows computation of the real stability radius using (2).

**Main Result:**

$$\mu_{\mathbb{R}}(M) = \inf_{\gamma \in (0,1]} \sigma_2 \left( \begin{bmatrix} \operatorname{Re} M & -\gamma \operatorname{Im} M \\ \gamma^{-1} \operatorname{Im} M & \operatorname{Re} M \end{bmatrix} \right). \quad (3)$$

*The function to be minimized is a unimodal function on  $(0, 1]$ .*

Since the function to be minimized in (3) is unimodal, any local minimum is a global minimum. Many standard search algorithms, such as golden section search, can be used with guaranteed convergence to a global minimum.

In a sense the stability radius problem, although not having been called so, has been studied since decades ago. It is difficult to trace the exact history, partly because it has been treated by several authors in different fields independently. A theorem in (Rudin, 1973, p. 239) and its proof immediately leads to  $r_{\mathbb{F}}(A) \geq \left\{ \sup_{s \in \partial \mathbb{C}_g} \bar{\sigma}[(sI - A)^{-1}] \right\}^{-1}$ . Various versions of this inequality have appeared in many text books. The fact that this inequality is actually an equality when  $\mathbb{F} = \mathbb{C}$  follows from Schmidt/Mirsky's classical theorem (often also attributed to Eckart and Young) of approximating a matrix by a lower rank one, see e.g., (Stewart and Sun, 1990, Theorem 4.18). For contributions to various aspects of the complex unstructured stability radius  $r_{\mathbb{C}}(A)$ , see also (Van Loan, 1985; Hinrichsen and Pritchard, 1986a; Martin, 1987; Byers, 1988).

The stability radius  $r_{\mathbb{F}}(A, B, C)$  has been motivated from several different viewpoints. It arises in the stability robustness analysis of a feedback loop consisting of a fixed linear time-invariant system and a norm bounded uncertain gain representing uncertain parameters. It can also be posed from a pure matrix perturbation point of view, in which the matrices  $B$  and  $C$  reflect the structural information of the perturbation matrix  $B\Delta C$ , as in (Hinrichsen and Pritchard, 1986b). The solution to the structured complex stability radius problem, again, is a simple application of Schmidt/Mirsky's lower rank matrix approximation theorem.

When the stability radius is used to analyze the stability of a linear time-invariant system under parametric perturbation, the real stability radius is more natural than its complex counterpart. It however turns out to be a much more difficult problem. Obviously,  $r_{\mathbb{R}}(A, B, C) \geq r_{\mathbb{C}}(A, B, C)$ . The ratio  $r_{\mathbb{R}}(A, B, C)/r_{\mathbb{C}}(A, B, C)$  can actually be arbitrarily large. Hinrichsen, Pritchard, and associates studied various properties of the real stability radius and surveyed their results in (Hinrichsen and Pritchard, 1990). Several lower bounds on  $r_{\mathbb{R}}(A)$  were obtained in (Qiu and Davison, 1991) by using tensor product techniques. Conditions under which  $r_{\mathbb{R}}(A)$  equals  $r_{\mathbb{C}}(A)$  were investigated in (Lewkowicz, 1992).

The specialization of the right hand sides of (2) and (3) to the case when  $B = C = I$  was shown to be a lower bound on  $r_{\mathbb{R}}(A)$  in (Qiu and Davison, 1992) and was also conjectured to be actually equal to  $r_{\mathbb{R}}(A)$ . Our main result stated above completely solves the general real structured stability radius problem. In particular, it shows that the conjecture by Qiu and Davison is indeed true.

The paper is organized in the following way. Section 2 gives a proof of the main result. The idea is to rewrite the mixed problem involving a complex matrix and a realness constraint into a purely real problem. It is then easy to prove that the left hand side of (3) is less than or equal to the right hand side. To prove the opposite inequality, we construct a specific real  $\Delta$  such that  $I - \Delta M$  is singular and  $[\bar{\sigma}(\Delta)]^{-1}$  is equal to the right hand side of (3). Section 3

addresses the sensitivity of  $\mu_{\mathbb{R}}(M)$  to the changes in  $M$ . In Section 4 we specialize the results to the unstructured real stability radius and also generalize the definition of the structured stability radius so that it covers linear fractional perturbations. Section 5 presents several examples which illustrate different possible behaviors of the function on the right hand side of (3) at its minimum and also illustrate the extra sweep over  $\partial\mathbb{C}_g$  needed for the real stability radius computation. Section 6 presents some concluding remarks.

## 2 Proof of the Main Result

Let  $M \in \mathbb{C}^{p \times m}$  be given. Introduce  $X := \operatorname{Re} M$  and  $Y := \operatorname{Im} M$ . The case when  $Y = 0$  is trivial; we then have  $\mu_{\mathbb{R}}(M) = \mu_{\mathbb{C}}(M) = \bar{\sigma}(X)$ . Hence we assume  $Y \neq 0$  in the following proof. For  $\Delta \in \mathbb{R}^{m \times p}$  the matrix  $I - \Delta M$  is singular if and only if there are  $v_1, v_2 \in \mathbb{R}^m$  with  $(v_1, v_2) \neq (0, 0)$  such that

$$[I - \Delta(X + jY)](v_1 + jv_2) = 0. \quad (4)$$

An equivalent form of (4) is

$$\left( I - \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \right) \begin{bmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{bmatrix} = 0. \quad (5)$$

The advantage of (5) is that only real numbers are involved. Since  $(v_1, v_2) \neq (0, 0)$ , the columns of  $\begin{bmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{bmatrix}$  are linearly independent, therefore

$$\operatorname{rank} \left( I - \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \right) \leq 2m - 2. \quad (6)$$

To proceed, we need a version of the Schmidt/Mirsky theorem, tailored according to our need.

**Lemma 1** *Let  $E \in \mathbb{F}^{m \times p}$  and  $F \in \mathbb{F}^{p \times m}$ . Then for  $i = 1, \dots, \min\{p, m\}$ ,*

$$\inf\{\bar{\sigma}(E) : \operatorname{rank}(I_m - EF) \leq m - i\} = [\sigma_i(F)]^{-1}.$$

To avoid the conservatism caused by applying Lemma 1 directly to (6), we resort to the widely used technique of scaling. Let  $\gamma \in \mathbb{R} \setminus \{0\}$ . From (6) we get

$$\begin{aligned} & \operatorname{rank} \left( I - \begin{bmatrix} \gamma I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & I \end{bmatrix} \right) \\ &= \operatorname{rank} \left( I - \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} X & -\gamma Y \\ \gamma^{-1} Y & X \end{bmatrix} \right) \\ &\leq 2m - 2. \end{aligned} \quad (7)$$

Let us introduce a notation:

$$P(\gamma) = \begin{bmatrix} X & -\gamma Y \\ \gamma^{-1}Y & X \end{bmatrix}.$$

Lemma 1 and inequality (7) imply that

$$\bar{\sigma}(\Delta) = \bar{\sigma} \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \geq \sigma_2^{-1}[P(\gamma)], \quad \forall \gamma \neq 0.$$

Consequently,

$$\mu_{\mathbb{R}}(M) \leq \inf_{\gamma \neq 0} \sigma_2[P(\gamma)] = \inf_{\gamma \in (0,1]} \sigma_2[P(\gamma)].$$

Here, the search over  $\gamma$  has been restricted to  $(0, 1]$  due to the fact that  $P(\gamma)$ ,  $P(-\gamma)$  and  $P(\gamma^{-1})$  all have the same singular values.

The rest of this section is devoted to the proof of the reverse inequality:

$$\mu_{\mathbb{R}}(M) \geq \inf_{\gamma \in (0,1]} \sigma_2[P(\gamma)] =: \sigma^*, \quad (8)$$

which is significantly more difficult. We only need to prove this for the case when  $\sigma^* > 0$ . The proof is done by an explicit construction of a real  $\Delta$  such that  $I - \Delta M$  is singular and  $\bar{\sigma}(\Delta) = \sigma^{*-1}$ . Let us use  $(\cdot)^\dagger$  to denote the Moore-Penrose generalized inverse. The following lemma, whose proof is left to the reader, is needed in the construction.

**Lemma 2** *Let  $U \in \mathbb{R}^{p \times k}$  and  $V \in \mathbb{R}^{m \times k}$ . If  $U^T U = V^T V \neq 0$ , then  $\bar{\sigma}(VU^\dagger) = 1$  and  $VU^\dagger U = V$ .*

First we will treat the case when  $\inf_{\gamma \in (0,1]} \sigma_2[P(\gamma)]$  is attained for some  $\gamma^* \in (0, 1]$ . Let  $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  and  $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  be a pair of left and right singular vectors of  $P(\gamma^*)$  corresponding to  $\sigma^*$ , with  $u_1, u_2 \in \mathbb{R}^p$  and  $v_1, v_2 \in \mathbb{R}^m$ , and set

$$\Delta = \sigma^{*-1} \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \end{bmatrix}^\dagger. \quad (9)$$

If  $u$  and  $v$  can be chosen so that

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix}^T \begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T \begin{bmatrix} v_1 & v_2 \end{bmatrix}, \quad (10)$$

then it follows from Lemma 2 that  $\bar{\sigma}(\Delta) = \sigma^{*-1}$  and that

$$[I - \Delta(X + jY)](v_1 + j\gamma^* v_2) = v_1 + j\gamma^* v_2 - \Delta \sigma^* u_1 - \Delta j\gamma^* \sigma^* u_2 = 0$$

which means that  $I - \Delta M$  is singular. Hence  $\Delta$  given by (9) is the desired construction. What follows is a long elaboration which shows that the singular vectors  $u$  and  $v$  can always be chosen so that (10) is satisfied when  $\gamma^* \in (0, 1]$ .

The proof for the case when  $\inf_{\gamma \in (0, 1]} \sigma_2[P(\gamma)]$  is attained only as  $\gamma \rightarrow 0$ , which occurs if and only if  $\text{rank}(Y) = 1$ , is carried out in a different way, in which an explicit formula for  $\mu_{\mathbb{R}}(M)$ , involving no minimization, and a more direct construction of  $\Delta$  are available.

We start with several claims on the singular vectors of  $P(\gamma)$ . The first one is of a purely algebraic nature.

**Claim 1** *Let  $\gamma \in \mathbb{R} \setminus \{-1, 0, 1\}$  and let  $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  and  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  be a pair of left and right singular vectors of  $P(\gamma)$  corresponding to some nonzero singular value  $\sigma$ . Then  $u_1^T u_2 = v_1^T v_2$ .*

**PROOF** The singular vectors satisfy

$$\begin{bmatrix} X & -\gamma Y \\ \gamma^{-1} Y & X \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \sigma \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (11)$$

$$\begin{bmatrix} X^T & \gamma^{-1} Y^T \\ -\gamma Y^T & X^T \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \sigma \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \quad (12)$$

The difference between  $\begin{bmatrix} u_2^T & u_1^T \end{bmatrix}$  times (11) and  $\begin{bmatrix} v_2^T & v_1^T \end{bmatrix}$  times (12) gives

$$(\gamma + \gamma^{-1})(u_1^T Y v_1 - u_2^T Y v_2) = 2\sigma(u_1^T u_2 - v_1^T v_2). \quad (13)$$

Similarly, the sum of  $\begin{bmatrix} u_2^T & -u_1^T \end{bmatrix}$  times (11) and  $\begin{bmatrix} v_2^T & -v_1^T \end{bmatrix}$  times (12) gives

$$(\gamma - \gamma^{-1})(u_1^T Y v_1 - u_2^T Y v_2) = 0. \quad (14)$$

Since  $\sigma \neq 0$  and  $\gamma \neq 0$  or  $\pm 1$ , the claim follows from (13) and (14).  $\square$

The second claim concerns the singular vectors of  $P(\gamma)$  corresponding to singular values at extrema. We need several lemmas.

**Lemma 3** *Let  $F(\gamma) \in \mathbb{R}^{p \times m}$  be a (real) analytic matrix function on an open set  $\Gamma \subset \mathbb{R}$ . Then there exist an analytic diagonal matrix function  $\tilde{\Sigma}(\gamma) = \text{diag}(\tilde{\sigma}_1(\gamma), \dots, \tilde{\sigma}_{\min\{p, m\}}(\gamma)) \in \mathbb{R}^{p \times m}$  and analytic orthogonal matrix functions  $\tilde{U}(\gamma) = [\tilde{u}_1(\gamma) \cdots \tilde{u}_p(\gamma)] \in \mathbb{R}^{p \times p}$  and  $\tilde{V}(\gamma) = [\tilde{v}_1(\gamma) \cdots \tilde{v}_m(\gamma)] \in \mathbb{R}^{m \times m}$ , all of which are defined on  $\Gamma$ , such that*

$$\tilde{\Sigma}(\gamma) = \tilde{U}^T(\gamma) F(\gamma) \tilde{V}(\gamma).$$

Furthermore,

$$\frac{d\tilde{\sigma}_i(\gamma)}{d\gamma} = \tilde{u}_i^T(\gamma) \frac{dF(\gamma)}{d\gamma} \tilde{v}_i(\gamma), \quad (15)$$

for  $i = 1, \dots, \min\{p, m\}$ .

**PROOF** The first statement above follows from a similar result for Hermitian matrices in (Baumgärtel, 1985, p. 149), see also (Kato, 1966, Section II.6.2). To prove (15), differentiate  $F(\gamma)\tilde{v}_i(\gamma) = \tilde{\sigma}_i(\gamma)\tilde{u}_i(\gamma)$ . This gives

$$\frac{dF}{d\gamma}(\gamma)\tilde{v}_i(\gamma) + F(\gamma)\frac{d\tilde{v}_i}{d\gamma}(\gamma) = \frac{d\tilde{\sigma}_i}{d\gamma}(\gamma)\tilde{u}_i(\gamma) + \tilde{\sigma}_i(\gamma)\frac{d\tilde{u}_i}{d\gamma}(\gamma).$$

Multiplying both sides by  $\tilde{u}_i^T(\gamma)$  from the left and noticing that  $\tilde{u}_i^T(\gamma)F(\gamma) = \tilde{\sigma}_i(\gamma)\tilde{v}_i^T(\gamma)$ , we obtain

$$\tilde{u}_i^T(\gamma)\frac{dF}{d\gamma}(\gamma)\tilde{v}_i(\gamma) + \sigma_i(\gamma)\tilde{v}_i^T(\gamma)\frac{d\tilde{v}_i}{d\gamma}(\gamma) = \frac{d\tilde{\sigma}_i}{d\gamma}(\gamma) + \tilde{\sigma}_i(\gamma)\tilde{u}_i^T(\gamma)\frac{d\tilde{u}_i}{d\gamma}(\gamma).$$

From  $\tilde{u}_i^T(\gamma)\tilde{u}_i(\gamma) = 1$  and  $\tilde{v}_i^T(\gamma)\tilde{v}_i(\gamma) = 1$ , it follows that  $\tilde{u}_i^T(\gamma)\frac{d\tilde{u}_i}{d\gamma}(\gamma) = 0$  and  $\tilde{v}_i^T(\gamma)\frac{d\tilde{v}_i}{d\gamma}(\gamma) = 0$ . Hence (15) follows from (2).  $\square$

Apparently,  $|\tilde{\sigma}_i(\gamma)|$ ,  $i = 1, \dots, \min\{p, m\}$ , are singular values of  $F(\gamma)$ . However, they are not in any particular order. In the following, we will also use the ordered singular values  $\sigma_1(\gamma) \geq \dots \geq \sigma_{\min\{p, m\}}(\gamma) \geq 0$  of  $F(\gamma)$ . The difference between  $\tilde{\sigma}_i(\gamma)$  and  $\sigma_i(\gamma)$  is that the former are analytic whereas the latter are generally not and the latter are nonnegative and ordered nonincreasingly whereas the former are generally not. Despite its lack of analyticity on the whole  $\Gamma$ ,  $\sigma_i(\gamma)$  is continuous and piecewise analytic.

**Lemma 4** *Let  $F(\gamma) \in \mathbb{R}^{p \times m}$  be an analytic matrix function on an open set  $\Gamma \subset \mathbb{R}$ . Let  $\sigma_1(\gamma) \geq \dots \geq \sigma_{\min\{p, m\}}(\gamma) \geq 0$  be its ordered singular values. If  $\sigma_i(\gamma)$  has a nonzero local extremum  $\gamma^* \in \Gamma$ , then there exists a pair of left and right singular vectors  $u \in \mathbb{R}^p$  and  $v \in \mathbb{R}^m$  of  $F(\gamma^*)$  corresponding to  $\sigma_i(\gamma^*)$  such that  $u^T \frac{dF}{d\gamma}(\gamma^*)v = 0$ .*

**PROOF** If  $\sigma_i(\gamma)$  is analytic at  $\gamma^*$ , then we can assume, without loss of generality, that  $\sigma_i(\gamma)$  is equal to  $\tilde{\sigma}_1(\gamma)$  in an open neighborhood of  $\gamma^*$ . Thus  $\gamma^*$  is also a stationary point of  $\tilde{\sigma}_1(\gamma)$ . Let  $\tilde{u}_1(\gamma)$  and  $\tilde{v}_1(\gamma)$  be a pair of left and right analytic singular vectors corresponding to  $\tilde{\sigma}_1(\gamma)$ . The lemma then follows since (15) gives

$$\tilde{u}_1^T(\gamma^*)\frac{dF}{d\gamma}(\gamma^*)\tilde{v}_1(\gamma^*) = 0.$$

If instead  $\sigma_i(\gamma)$  is not analytic at  $\gamma^*$ , then we can assume, without loss of generality, that in an open neighborhood of  $\gamma^*$ ,  $\sigma_i(\gamma) = \tilde{\sigma}_1(\gamma)$  for  $\gamma \leq \gamma^*$  and  $\sigma_i(\gamma) = \tilde{\sigma}_2(\gamma)$  for  $\gamma \geq \gamma^*$ . Let  $\tilde{u}_k(\gamma)$  and  $\tilde{v}_k(\gamma)$ ,  $k = 1, 2$ , be the a pair of left and right analytic singular vectors corresponding to  $\tilde{\sigma}_k(\gamma)$ . Then (15) gives

$$\begin{aligned} \frac{d\tilde{\sigma}_1}{d\gamma}(\gamma^*) &= \tilde{u}_1^T(\gamma^*)\frac{dF}{d\gamma}(\gamma^*)\tilde{v}_1(\gamma^*) \\ \frac{d\tilde{\sigma}_2}{d\gamma}(\gamma^*) &= \tilde{u}_2^T(\gamma^*)\frac{dF}{d\gamma}(\gamma^*)\tilde{v}_2(\gamma^*). \end{aligned}$$



Put  $u_\alpha = \alpha \tilde{u}_1 + (1 - \alpha^2)^{1/2} \tilde{u}_2$  and  $v_\alpha = \alpha \tilde{v}_1 + (1 - \alpha^2)^{1/2} \tilde{v}_2$  for  $\alpha \in [0, 1]$ . Then  $u_\alpha(\gamma^*)$  and  $v_\alpha(\gamma^*)$  also form a pair of singular vectors of  $F(\gamma^*)$  corresponding to the singular value  $\sigma_i(\gamma^*)$ . Define

$$f(\alpha) = u_\alpha^T(\gamma^*) \frac{dF}{d\gamma}(\gamma^*) v_\alpha(\gamma^*).$$

Since  $\gamma^*$  is a local extremum of  $\sigma_i(\gamma)$ , we must have  $f(0)f(1) = \frac{d\tilde{\sigma}_1}{d\gamma}(\gamma^*) \frac{d\tilde{\sigma}_2}{d\gamma}(\gamma^*) \leq 0$ . By continuity,  $f(\alpha) = 0$  has a solution in  $[0, 1]$ . This proves the lemma.  $\square$

For the matrix  $P(\gamma)$ , the singular vectors described in Lemma 4 satisfy some pleasant alignment conditions:

**Claim 2** *Let  $\gamma \in \mathbb{R} \setminus \{0\}$  and let  $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  and  $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  be a pair of left and right singular vectors of  $P(\gamma)$  corresponding to a nonzero singular value  $\sigma$ . If the extra condition  $u^T \frac{dP}{d\gamma}(\gamma) v = 0$  is satisfied, then  $u_1^T u_1 = v_1^T v_1$  and  $u_2^T u_2 = v_2^T v_2$ .*

**PROOF** The singular vectors satisfy (11), (12), and

$$\begin{bmatrix} u_1^T & u_2^T \end{bmatrix} \begin{bmatrix} 0 & -Y \\ -\gamma^{-2}Y & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0. \quad (16)$$

Equation (16) gives

$$u_1^T Y v_2 + \gamma^{-2} u_2^T Y v_1 = 0.$$

Multiplying (11) by  $[u_1^T \ -u_2^T]$  from the left and (12) by  $[v_1^T \ -v_2^T]$  from the left, we obtain

$$u_1^T X v_1 - \gamma u_1^T Y v_2 - \gamma^{-1} u_2^T Y v_1 - u_2^T X v_2 = u_1^T X v_1 - u_2^T X v_2 = \sigma(u_1^T u_1 - u_2^T u_2)$$

and

$$v_1^T X^T u_1 + \gamma^{-1} v_1^T Y^T u_2 + \gamma v_2^T Y^T u_1 - v_2^T X^T u_2 = v_1^T X^T u_1 - v_2^T X^T u_2 = \sigma(v_1^T v_1 - v_2^T v_2).$$

Since  $\sigma > 0$ , we get

$$u_1^T u_1 - u_2^T u_2 = v_1^T v_1 - v_2^T v_2.$$

Claim 2 now follows from  $u_1^T u_1 + u_2^T u_2 = v_1^T v_1 + v_2^T v_2 = 1$ .  $\square$

We are now ready to show inequality (8). We need to treat three different cases separately.

**Case 1:**  $\sigma^* = \sigma_2[P(\gamma^*)]$  for some  $\gamma^* \in (0, 1)$ .

Lemma 4, together with Claims 1 and 2, tells us that a pair of singular vectors  $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  and  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  of  $P(\gamma^*)$  corresponding to  $\sigma^*$  can be chosen to satisfy (10). Then (8) follows as discussed previously.

**Case 2:**  $\sigma^* = \sigma_2[P(1)]$  but  $\sigma^* < \sigma_2[P(\gamma)]$  for all  $\gamma \in (0, 1)$ .

We have to treat this case separately since Claim 1 is not valid for  $\gamma = 1$ . We however know that the singular values of  $P(1)$  are paired so that  $\sigma_{2i-1}[P(1)] = \sigma_{2i}[P(1)] = \sigma_i(M)$  for all  $i$ . In particular, the largest and the second largest singular values of  $P(1)$  are equal to  $\sigma^*$ . We need to consider two possibilities. The first possibility is that the multiplicity of the largest singular value of  $P(1)$  is two. Without loss of generality, assume  $\sigma_1[P(1)] = \sigma_2[P(1)] = \tilde{\sigma}_1(1) = \tilde{\sigma}_2(1)$ , where  $\tilde{\sigma}_1(\gamma)$  and  $\tilde{\sigma}_2(\gamma)$  are analytic singular values of  $P(\gamma)$ . Note also that if  $\sigma(\gamma)$  is a singular value then so is  $\sigma(\gamma^{-1})$ . Since  $\gamma = 1$  is a minimum of  $\sigma_2(\gamma)$  which is equal to  $\min\{\tilde{\sigma}_1(\gamma), \tilde{\sigma}_2(\gamma)\}$  locally around  $\gamma = 1$ , it follows that  $\gamma = 1$  must be a local minimum of  $\tilde{\sigma}_1(\gamma)$  and  $\tilde{\sigma}_2(\gamma)$ . Let  $\begin{bmatrix} u_1(\gamma) \\ u_2(\gamma) \end{bmatrix}$  and  $\begin{bmatrix} v_1(\gamma) \\ v_2(\gamma) \end{bmatrix}$  be a pair of analytic singular vectors of  $P(\gamma)$  corresponding to  $\tilde{\sigma}_2(\gamma)$ . By Claim 1 we know that

$$u_1^T(\gamma)u_2(\gamma) = v_1^T(\gamma)v_2(\gamma) \quad \gamma \neq 0, \pm 1.$$

By continuity, we must therefore have

$$u_1^T(1)u_2(1) = v_1^T(1)v_2(1).$$

Using the fact that  $\frac{d\tilde{\sigma}_2}{d\gamma}(1) = 0$ , we conclude from derivative relation (15) and Claim 2 that

$$\begin{aligned} u_1^T(1)u_1(1) &= v_1^T(1)v_1(1) \\ u_2^T(1)u_2(1) &= v_2^T(1)v_2(1). \end{aligned}$$

Putting  $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1(1) \\ u_2(1) \end{bmatrix}$  and  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1(1) \\ v_2(1) \end{bmatrix}$ , it follows that (10) holds.

We have completed the proof for the first possibility of Case 2. However, the construction above is not quite readily implementable numerically. Here we pause for an interesting observation which renders surprising numerical advantages.

**Remark** If (10) holds for a pair of left and right singular vectors  $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  and  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  of  $P(1)$  corresponding to a nonzero singular value  $\sigma$  of multiplicity 2, then it holds for every such pair.

**PROOF** It is easy to check that  $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  and  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  form a pair of singular vectors of  $P(1)$  corresponding to a nonzero singular value  $\sigma$  of multiplicity 2 if and only if  $u_1 + ju_2$  and  $v_1 + jv_2$  form a pair of singular vectors of  $M$  corresponding to the same singular value of multiplicity 1. Now suppose  $\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}$  and  $\begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix}$  are another such pair. Then since  $\sigma$  is a distinct nonzero singular value of  $M$ , we have

$$\tilde{u}_1 + j\tilde{u}_2 = (u_1 + ju_2)e^{j\theta} \quad \text{and} \quad \tilde{v}_1 + j\tilde{v}_2 = (v_1 + jv_2)e^{j\theta}$$

for some  $\theta \in [0, 2\pi)$ . These can be rewritten as

$$\begin{bmatrix} \tilde{u}_1 & \tilde{u}_2 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

and

$$\begin{bmatrix} \tilde{v}_1 & \tilde{v}_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.$$

We immediately see that if  $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  and  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  satisfy (10), then so do  $\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}$  and  $\begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix}$ .  $\square$

The second possibility of Case 2 is that the multiplicity of the largest singular value is greater than two. This means that the largest four singular values of  $P(1)$  are equal to  $\sigma^*$ , i.e. the two (or more) largest singular values of  $M = X + jY$  are equal to  $\sigma^*$ . This possibility is related to a problem considered in (Lewkowicz, 1992), which inspired our solution. Bring in a singular value decomposition

$$M = \sigma^*(\mu_1 \nu_1^H + \mu_2 \nu_2^H) + \sum_{i=3}^{\min\{p,m\}} \sigma_i(M) \mu_i \nu_i^H.$$

where  $(\cdot)^H$  means conjugate transpose. Introduce

$$\begin{bmatrix} \mu \\ \nu \end{bmatrix} := \begin{bmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{bmatrix} \xi \quad (17)$$

where  $\xi \in \mathbb{C}^2$  is a unit length vector. Then  $\mu$  and  $\nu$  also form a pair of singular vectors of  $M$  corresponding to  $\sigma^*$ . If  $\xi$  can be found so that  $\mu^T \mu - \nu^T \nu = 0$ , then  $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} := \begin{bmatrix} \operatorname{Re} \mu \\ \operatorname{Im} \mu \end{bmatrix}$  and  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} := \begin{bmatrix} \operatorname{Re} \nu \\ \operatorname{Im} \nu \end{bmatrix}$  form a pair of left and right singular vectors of  $P(1)$  corresponding to  $\sigma^*$  and condition (10), which is equivalent to  $\mu^T \mu - \nu^T \nu = 0$ , is satisfied. Inequality (8) now follows as discussed previously. To show the existence of such a desired  $\xi$  one needs to study how  $\mu^T \mu - \nu^T \nu$  varies with  $\xi$ . This can be done straightforwardly with elementary calculus as in (Qiu et al., 1993), but an alternative is to introduce the Takagi factorization (Horn and Johnson, 1985, Corollary 4.4.4):

$$\begin{bmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{bmatrix} = \Xi^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Xi$$

where  $\Xi$  is a unitary matrix and  $\lambda_1 \geq \lambda_2 \geq 0$ . Clearly, a desired  $\xi$  can be chosen as follows:

$$\xi = \begin{cases} \text{any unit length vector in } \mathbb{C}^2 & \text{if } \lambda_1 = 0 \\ \Xi^H \begin{bmatrix} j\sqrt{\lambda_2} \\ \sqrt{\lambda_1} \end{bmatrix} \frac{1}{\sqrt{\lambda_1 + \lambda_2}} & \text{if } \lambda_1 \neq 0. \end{cases}$$

**Case 3:**  $\sigma^* = \lim_{\gamma \rightarrow 0} \sigma_2[P(\gamma)]$  but  $\sigma^* < \sigma_2[P(\gamma)]$  for all  $\gamma \in (0, 1]$ .

It follows from e.g. (Stewart and Sun, 1990, Theorem 1.4.4)) that  $\sigma_2[P(\gamma)] \geq \sigma_2(\gamma^{-1}Y)$ , so Case 3 is relevant only if  $\text{rank } Y = 1$ . We need a lemma to proceed.

**Lemma 5** *Let  $F(\gamma) = G(\gamma) + \gamma^{-1}H \in \mathbb{R}^{p \times m}$ , where  $G(\gamma)$  is analytic on an open interval  $\Gamma$  around 0 and  $H$  is a constant matrix with  $\text{rank}(H) =: r < \min\{p, m\}$ . Let  $\sigma_1(\gamma) \geq \dots \geq \sigma_{\min\{p, m\}}(\gamma) \geq 0$  be the ordered singular values of  $F(\gamma)$  defined on  $\Gamma \setminus \{0\}$ . Assume a singular value decomposition of  $H$  is given by*

$$H = [U_1 \ U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} [V_1 \ V_2]^T$$

where  $\Sigma_1 \in \mathbb{R}^{r \times r}$ . Then

$$\lim_{\gamma \rightarrow 0} \sigma_{r+i}(\gamma) = \sigma_i[U_2^T G(0) V_2]$$

for  $i = 1, \dots, \min\{p, m\} - r$ .

**PROOF** Without loss of generality, assume an analytic singular value decomposition of  $\gamma F(\gamma)$  is

$$\gamma F(\gamma) = \begin{bmatrix} \tilde{U}_1(\gamma) & \tilde{U}_2(\gamma) \end{bmatrix} \begin{bmatrix} \tilde{\Sigma}_1(\gamma) & 0 \\ 0 & \tilde{\Sigma}_2(\gamma) \end{bmatrix} \begin{bmatrix} \tilde{V}_1(\gamma) & \tilde{V}_2(\gamma) \end{bmatrix}^T$$

where  $\tilde{\Sigma}_1(0) \in \mathbb{R}^{r \times r}$  and  $\tilde{\Sigma}_2(0) = 0$ . Then

$$\gamma^{-1} \tilde{\Sigma}_2(\gamma) = \tilde{U}_2^T(\gamma) F(\gamma) \tilde{V}_2(\gamma) = \tilde{U}_2^T(\gamma) G(\gamma) \tilde{V}_2(\gamma) + \gamma^{-1} \tilde{U}_2^T(\gamma) H \tilde{V}_2(\gamma),$$

and

$$H = \begin{bmatrix} \tilde{U}_1(0) & \tilde{U}_2(0) \end{bmatrix} \begin{bmatrix} \tilde{\Sigma}_1(0) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{V}_1(0) & \tilde{V}_2(0) \end{bmatrix}^T.$$

Since both  $\tilde{U}_2^T(\gamma) \tilde{U}_1(0)$  and  $\tilde{V}_1^T(0) \tilde{V}_2(\gamma)$  are analytic and vanishing at  $\gamma = 0$ , it follows that

$$\lim_{\gamma \rightarrow 0} \gamma^{-1} \tilde{U}_2^T(\gamma) H \tilde{V}_2(\gamma) = \lim_{\gamma \rightarrow 0} \gamma^{-1} \tilde{U}_2^T(\gamma) \tilde{U}_1(0) \tilde{\Sigma}_1(0) \tilde{V}_1^T(0) \tilde{V}_2(\gamma) = 0.$$

Therefore

$$\lim_{\gamma \rightarrow 0} \gamma^{-1} \tilde{\Sigma}_2(\gamma) = \tilde{U}_2^T(0) G(0) \tilde{V}_2(0).$$

Since all singular values of  $\gamma^{-1} \tilde{\Sigma}_1(\gamma)$  go to infinity as  $\gamma \rightarrow 0$ ,

$$\lim_{\gamma \rightarrow 0} \sigma_{r+i}(\gamma) = \lim_{\gamma \rightarrow 0} \sigma_i[\gamma^{-1} \tilde{\Sigma}_2(\gamma)] = \sigma_i[\tilde{U}_2^T(0) G(0) \tilde{V}_2(0)] = \sigma_i[U_2^T G(0) V_2].$$

Notice that  $\tilde{U}_2(0)$  and  $\tilde{V}_2(0)$  can be replaced by  $U_2$  and  $V_2$  since they have the same ranges respectively.  $\square$

Following the notation in the lemma put

$$G(\gamma) = \begin{bmatrix} X & -\gamma Y \\ 0 & X \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ Y & 0 \end{bmatrix},$$

and let a real singular value decomposition of  $Y$  be

$$U^Y \Sigma^Y (V^Y)^T = \begin{bmatrix} U_1^Y & U_2^Y \end{bmatrix} \begin{bmatrix} \sigma_1(Y) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^Y & V_2^Y \end{bmatrix}^T.$$

Then a singular value decomposition of  $H$  is

$$H = \begin{bmatrix} 0 & I \\ U^Y & 0 \end{bmatrix} \begin{bmatrix} \Sigma^Y & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V^Y & 0 \\ 0 & I \end{bmatrix}^T.$$

Applying Lemma 5, we obtain

$$\begin{aligned} \lim_{\gamma \rightarrow 0} \sigma_2[P(\gamma)] &= \bar{\sigma} \left( \begin{bmatrix} 0 & I \\ U_2^Y & 0 \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} V_2^Y & 0 \\ 0 & I \end{bmatrix} \right) \\ &= \max\{\bar{\sigma}[(U_2^Y)^T X], \bar{\sigma}(X V_2^Y)\}. \end{aligned}$$

Now we want to show that  $\lim_{\gamma \rightarrow 0} \sigma_2[P(\gamma)] = \inf_{\gamma \in (0,1]} \sigma_2[P(\gamma)]$ . If  $u$  and  $v$  are a pair of left and right real singular vectors of  $(U_2^Y)^T X$  corresponding to  $\bar{\sigma}[(U_2^Y)^T X]$ , then the choice

$$\Delta = -vu^T(U_2^Y)^T / \bar{\sigma}[(U_2^Y)^T X] \tag{18}$$

satisfies  $[I + \Delta(X + jY)]v = 0$  and  $\bar{\sigma}(\Delta)^{-1} = \bar{\sigma}[(U_2^Y)^T X]$ . Similarly, if  $u$  and  $v$  are a pair of left and right real singular vectors of  $X V_2^Y$  corresponding to  $\bar{\sigma}(X V_2^Y)$ , then the choice

$$\Delta = -V_2^Y v u^T / \bar{\sigma}(X V_2^Y) \tag{19}$$

satisfies  $u^T[I + (X + jY)\Delta] = 0$  and  $\bar{\sigma}(\Delta)^{-1} = \bar{\sigma}(X V_2^Y)$ . Together this shows that

$$\max\{\bar{\sigma}[(U_2^Y)^T X], \bar{\sigma}(X V_2^Y)\} \leq \mu_{\mathbb{R}}(M),$$

so

$$\inf_{\gamma \in (0,1]} \sigma_2[P(\gamma)] \leq \lim_{\gamma \rightarrow 0} \sigma_2[P(\gamma)] = \max\{\bar{\sigma}[(U_2^Y)^T X], \bar{\sigma}(X V_2^Y)\} \leq \mu_{\mathbb{R}}(M) \leq \inf_{\gamma \in (0,1]} \sigma_2[P(\gamma)]$$

and therefore the inequalities above can be replaced by equalities.

Note that if  $\min\{p, m\} = 1$  then  $U_2^Y$  or  $V_2^Y$  will be empty. We define the largest singular value of an empty matrix to be zero.

We have completed the proof of the equality (3). Now suppose that  $\sigma_2[P(\gamma)]$  has a local extremum (either minimum or maximum)  $\gamma^{**} \in (0, 1)$  such that  $\sigma_2[P(\gamma^{**})] > \sigma^*$ . Then using exactly the same arguments as in Case 1, one can construct a real  $\Delta$  such that  $I - \Delta M$  is singular and  $\bar{\sigma}(\Delta) = \{\sigma_2[P(\gamma^{**})]\}^{-1} < \sigma^{*-1}$ . This contradicts (3) and, therefore, can not happen. This shows that  $\sigma_2[P(\gamma)]$  is a unimodal function on  $(0, 1]$ .

Note also that the proof shows that  $\mu_{\mathbb{R}}(M) = \mu_{\mathbb{C}}(M)$  if and only if the minimum value of  $\sigma_2[P(\gamma)]$  is attained at  $\gamma = 1$ .

To recap, we summarize what we have proved in this section in the following theorem:

**Theorem** *If  $X, Y \in \mathbb{R}^{p \times m}$  and  $M = X + jY$ , then*

$$\mu_{\mathbb{R}}(M) = \inf_{\gamma \in (0, 1]} \sigma_2 \left( \begin{bmatrix} X & -\gamma Y \\ \gamma^{-1} Y & X \end{bmatrix} \right). \quad (20)$$

*The function to be minimized is a unimodal function on  $(0, 1]$ . If  $\text{rank } Y = 1$ , then furthermore*

$$\mu_{\mathbb{R}}(M) = \lim_{\gamma \rightarrow 0} \sigma_2 \left( \begin{bmatrix} X & -\gamma Y \\ \gamma^{-1} Y & X \end{bmatrix} \right) = \max\{\bar{\sigma}[(U_2^Y)^T X], \bar{\sigma}(X V_2^Y)\}$$

*where  $U_2^Y$  and  $V_2^Y$  come from any singular value decomposition of  $Y$*

$$Y = \begin{bmatrix} U_1^Y & U_2^Y \end{bmatrix} \begin{bmatrix} \bar{\sigma}(Y) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^Y & V_2^Y \end{bmatrix}^T.$$

We also summarize a procedure to construct a worst  $\Delta$ .

#### Construction of a worst $\Delta$

1. *If  $Y = 0$ , find a pair of left and right real singular vectors  $u$  and  $v$  of  $X$  corresponding to  $\bar{\sigma}(X)$  and set  $\Delta = vu^T / \bar{\sigma}(X)$ .*
2. *If  $\text{rank}(Y) = 1$ , compute a real singular value decomposition*

$$Y = \begin{bmatrix} U_1^Y & U_2^Y \end{bmatrix} \begin{bmatrix} \bar{\sigma}(Y) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^Y & V_2^Y \end{bmatrix}^T.$$

- *If  $\bar{\sigma}[(U_2^Y)^T X] \geq \bar{\sigma}(X V_2^Y)$ , find a pair of left and right real singular vectors  $u$  and  $v$  of  $(U_2^Y)^T X$  corresponding to  $\mu_{\mathbb{R}}(M)$  and set*

$$\Delta = -vu^T (U_2^Y)^T / \bar{\sigma}[(U_2^Y)^T X].$$

- *If  $\bar{\sigma}(X V_2^Y) > \bar{\sigma}[(U_2^Y)^T X]$ , find a pair of left and right real singular vectors  $u$  and  $v$  of  $X V_2^Y$  corresponding to  $\mu_{\mathbb{R}}(M)$  and set*

$$\Delta = -V_2^Y vu^T / \bar{\sigma}(X V_2^Y).$$

3. If  $\text{rank}(Y) > 1$ , find a minimum  $\gamma^* \in (0, 1]$  of  $\sigma_2[P(\gamma)]$ .

- If  $\gamma^* \in (0, 1)$  and  $\sigma_2[P(\gamma^*)]$  has multiplicity 1 or if  $\gamma^* = 1$  and  $\sigma_2[P(1)]$  has multiplicity 2, find a pair of left and right real singular vectors  $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$  and  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  of  $P(\gamma^*)$  corresponding to  $\mu_{\mathbb{R}}(M)$ .
- If  $\gamma^* \in (0, 1)$  and  $\sigma_2[P(\gamma^*)]$  has multiplicity  $r > 1$ , find matrices  $U \in \mathbb{R}^{2m \times r}$  and  $V \in \mathbb{R}^{2p \times r}$  with orthonormal columns such that

$$P(\gamma^*)V = \sigma_2[P(\gamma^*)]U.$$

Carry out a real Schur decomposition

$$U^T \frac{dP}{d\gamma}(\gamma^*)V + V^T \frac{dP^T}{d\gamma}U = W \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_r) W^T$$

where  $W$  is orthogonal and  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$ . Lemma 4 implies that  $\lambda_1 \lambda_r \leq 0$ . Take

$$w = \begin{cases} \text{any unit length vector in } \mathbb{R}^r & \text{if } \lambda_1 = \lambda_r = 0 \\ W \begin{bmatrix} \sqrt{-\lambda_r} & 0 & \dots & 0 & \sqrt{\lambda_1} \end{bmatrix}^T / \sqrt{\lambda_1 - \lambda_r} & \text{if } \lambda_1 \neq \lambda_r, \end{cases}$$

$$\text{and } \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = Uw \text{ and } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = Vw.$$

- If  $\gamma^* = 1$  and  $\sigma_2[P(1)]$  has multiplicity  $2r > 2$ , find matrices  $\begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix} \in \mathbb{C}^{m \times 2}$  and  $\begin{bmatrix} \nu_1 & \nu_2 \end{bmatrix} \in \mathbb{C}^{p \times 2}$  with orthonormal columns such that

$$M \begin{bmatrix} \nu_1 & \nu_2 \end{bmatrix} = \sigma_2[P(1)] \begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix}.$$

Carry out the Takagi factorization

$$\begin{bmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{bmatrix} = \Xi^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Xi$$

where  $\Xi$  is a unitary matrix and  $\lambda_1 \geq \lambda_2 \geq 0$ . Take

$$\xi = \begin{cases} \text{any unit length vector in } \mathbb{C}^2 & \text{if } \lambda_1 = 0 \\ \Xi^H \begin{bmatrix} j\sqrt{\lambda_2} \\ \sqrt{\lambda_1} \end{bmatrix} \frac{1}{\sqrt{\lambda_1 + \lambda_2}} & \text{if } \lambda_1 \neq 0, \end{cases}$$

and

$$\begin{bmatrix} \mu \\ \nu \end{bmatrix} = \begin{bmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{bmatrix} \xi, \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \text{Re } \mu \\ \text{Im } \mu \end{bmatrix}, \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \text{Re } \nu \\ \text{Im } \nu \end{bmatrix}.$$

Finally set

$$\Delta = \sigma_2[P(\gamma^*)] \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \end{bmatrix}^\dagger.$$

### 3 Continuity Properties

In computation of the real stability radius it is of interest to know how sensitive  $\mu_{\mathbb{R}}(M)$  is to changes in  $M$ . The example

$$\mu_{\mathbb{R}}(1 + j\varepsilon) = \begin{cases} 1 & \text{if } \varepsilon = 0 \\ 0 & \text{if } \varepsilon \neq 0 \end{cases}$$

shows that  $\mu_{\mathbb{R}}(M)$  can be discontinuous at certain  $M$ . Upper semicontinuity of the map  $M \mapsto \mu_{\mathbb{R}}(M)$  however follows from the following general argument: If  $I - \Delta M$  is invertible for all  $\Delta \in \mathcal{K}$ , where  $\mathcal{K}$  is compact, then  $I - \Delta \tilde{M}$  is invertible for all  $\Delta \in \mathcal{K}$  and all  $\tilde{M}$  in an open neighborhood of  $M$ .

In addition,  $M \mapsto \mu_{\mathbb{R}}(M)$  is continuous at any  $M$  with  $\text{rank}(\text{Im } M) > 1$  as the following relative error bound shows:

**Proposition** *If  $\text{rank}(\text{Im } M) > 1$ , then for all  $E \in \mathbb{C}^{p \times m}$*

$$\frac{|\mu_{\mathbb{R}}(M + E) - \mu_{\mathbb{R}}(M)|}{\mu_{\mathbb{R}}(M)} \leq \frac{\bar{\sigma}(E)}{\sigma_2(\text{Im } M)}.$$

**PROOF** Let  $\gamma^* \in (0, 1]$  be a minimum of  $\sigma_2[P(\gamma)]$ . Then

$$\begin{aligned} \mu_{\mathbb{R}}(M + E) &\leq \mu_{\mathbb{R}}(M) + \bar{\sigma} \left( \begin{bmatrix} \text{Re } E & -\gamma^* \text{Im } E \\ \gamma^{*-1} \text{Im } E & \text{Re } E \end{bmatrix} \right) \\ &\leq \mu_{\mathbb{R}}(M) + \gamma^{*-1} \bar{\sigma} \left( \begin{bmatrix} \text{Re } E & -\text{Im } E \\ \text{Im } E & \text{Re } E \end{bmatrix} \right) \\ &= \mu_{\mathbb{R}}(M) + \gamma^{*-1} \bar{\sigma}(E). \end{aligned}$$

Noting that  $\gamma^{*-1} \sigma_2(\text{Im } M) \leq \mu_{\mathbb{R}}(M)$  gives

$$\frac{\mu_{\mathbb{R}}(M + E)}{\mu_{\mathbb{R}}(M)} \leq 1 + \frac{\bar{\sigma}(E)}{\sigma_2(\text{Im } M)}$$

To obtain the other half of the inequality we exchange the role of  $M$  and  $M + E$  and invert:

$$\frac{\mu_{\mathbb{R}}(M + E)}{\mu_{\mathbb{R}}(M)} \geq \left( 1 + \frac{\bar{\sigma}(E)}{\sigma_2[\text{Im}(M + E)]} \right)^{-1} = 1 - \frac{\bar{\sigma}(E)}{\sigma_2[\text{Im}(M + E)] + \bar{\sigma}(E)} \geq 1 - \frac{\bar{\sigma}(E)}{\sigma_2(\text{Im } M)}$$

□

The only possible discontinuity points are therefore at  $M$  with  $\text{rank}(\text{Im } M) \leq 1$ . For  $\text{Im } M = 0$ , we have shown the existence above, but we have not been able to find any example of a discontinuity at any  $M$  with  $\text{rank}(\text{Im } M) = 1$ .



## 4 A Specialization and a Generalization

It is well known that a more convenient formula for the complex unstructured stability radius is given by

$$r_{\mathbb{C}}(A) = \min_{s \in \partial \mathbb{C}_g} \underline{\sigma}(A - sI).$$

Analogously, an alternative formula for the real unstructured stability radius is available, which might be sometimes simpler to apply.

**Corollary** *Assume  $A \in \mathbb{R}^{n \times n}$  ( $n > 1$ ) is stable. Then*

$$r_{\mathbb{R}}(A) = \min_{s \in \partial \mathbb{C}_g} \max_{\gamma \in (0,1]} \sigma_{2n-1} \left( \begin{bmatrix} A - \operatorname{Re} sI & -\gamma \operatorname{Im} sI \\ \gamma^{-1} \operatorname{Im} sI & A - \operatorname{Re} sI \end{bmatrix} \right).$$

*For each fixed  $s \in \partial \mathbb{C}_g$ , the function to be maximized is a quasiconcave function.*

We leave it to the reader to derive this from (2) and (3) and to justify the use of “max” and “min” instead of “sup” and “inf”. Note also that due to the proposition the only possible discontinuity of the function to be minimized occurs at the intersection of  $\partial \mathbb{C}_g$  with the real axis.

In the definition of  $r_{\mathbb{F}}(A, B, C)$ , the perturbed matrix  $A + B\Delta C$  depends on the perturbation matrix  $\Delta$  in an affine way. In applications, however, a perturbed matrix may depend on the perturbation in a linear fractional way. This motivates a more general definition of the structured stability radius. For  $(A, B, C, D) \in \mathbb{F}^{n \times n} \times \mathbb{F}^{n \times m} \times \mathbb{F}^{p \times n} \times \mathbb{F}^{p \times m}$ , introduce

$$\begin{aligned} r_{\mathbb{F}}(A, B, C, D) \\ := \inf \{ \bar{\sigma}(\Delta) : \Delta \in \mathbb{F}^{m \times p}, \det(I - \Delta D) = 0 \text{ or } A + B(I - \Delta D)^{-1} \Delta C \text{ is unstable} \}. \end{aligned}$$

Again we leave it to the reader to show that if  $\mathbb{C}_b$  is unbounded and  $A$  is stable, then

$$r_{\mathbb{F}}(A, B, C, D) = \left\{ \sup_{s \in \partial \mathbb{C}_g} \mu_{\mathbb{F}}[D + C(sI - A)^{-1}B] \right\}^{-1}.$$

In the case when  $\mathbb{C}_g = \{s \in \mathbb{C} : \operatorname{Re}(s) < 0\}$  or  $\mathbb{C}_g = \{s \in \mathbb{C} : |s| < 1\}$ ,  $r_{\mathbb{F}}(A, B, C, D)$  gives the smallest norm of a complex ( $\mathbb{F} = \mathbb{C}$ ) or real ( $\mathbb{F} = \mathbb{R}$ ) perturbation  $\Delta$  which destabilizes the feedback system shown in Fig. 1.

## 5 Examples

**Example 1:** Recall from Section 2 the notation

$$P(\gamma) = \begin{bmatrix} X & -\gamma Y \\ \gamma^{-1} Y & X \end{bmatrix}.$$

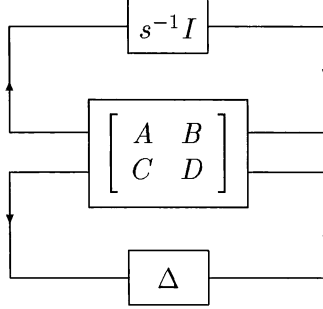


Figure 1: Feedback interpretation of the generalized stability radius

In this example, we illustrate various behavior of  $\sigma_2[P(\gamma)]$  at its minimum  $\gamma^*$ . The data and computed results are listed in Table 1. There are essentially five possibilities:

1.  $\gamma^* \in (0, 1)$  and  $\sigma_2[P(\gamma)]$  is smooth at  $\gamma^*$ .
2.  $\gamma^* \in (0, 1)$  and  $\sigma_2[P(\gamma)]$  is nonsmooth at  $\gamma^*$ .
3.  $\gamma^* = 1$  and  $\sigma_2[P(1)]$  has multiplicity 2.
4.  $\gamma^* = 1$  and  $\sigma_2[P(1)]$  has multiplicity greater than 2.
5.  $\inf_{\gamma \in (0, 1]} \sigma_2[P(\gamma)]$  is attained as  $\gamma \rightarrow 0$ .

The construction of a smallest  $\Delta$  such that  $I - \Delta M$  is singular has to be carried out in different ways for these different possibilities as done in Section 2.

**Example 2:** Assume  $\mathbb{C}_g = \{s \in \mathbb{C} : \text{Re } s < 0\}$ . Find  $r_{\mathbb{R}}(A, B, C)$  for

$$A = \begin{bmatrix} 79 & 20 & -30 & -20 \\ -41 & -12 & 17 & 13 \\ 167 & 40 & -60 & -38 \\ 33.5 & 9 & -14.5 & -11 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2190 & 0.9347 \\ 0.0470 & 0.3835 \\ 0.6789 & 0.5194 \\ 0.6793 & 0.8310 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.0346 & 0.5297 & 0.0077 & 0.0668 \\ 0.0535 & 0.6711 & 0.3834 & 0.4175 \end{bmatrix}.$$

We plot  $\mu_{\mathbb{R}}[C(j\omega I - A)^{-1}B]$ , computed by using golden section search, and  $\mu_{\mathbb{C}}[C(j\omega I - A)^{-1}B]$  in Fig. 2. Their maximal values are 1.9450 and 2.5546 respectively. These maxima occur at  $\omega = 1.38$  and  $\omega = 9.9$  respectively. We get  $r_{\mathbb{R}}(A, B, C) = 0.5141$  and  $r_{\mathbb{C}}(A, B, C) =$

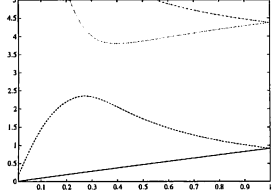
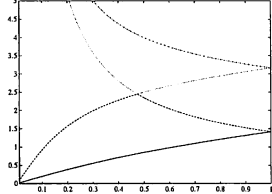
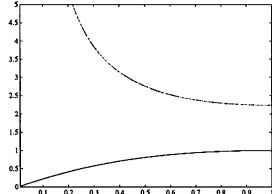
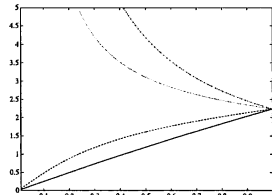
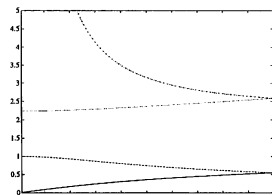
$M$	singular values of $P(\gamma)$	$\mu_{\mathbb{R}}(M)$	worst $\Delta$	remark
$\begin{bmatrix} 4+j & 1 \\ -1 & j \end{bmatrix}$		3.8042	$\begin{bmatrix} 0.1382 & -0.2236 \\ 0.2236 & 0.1382 \end{bmatrix}$	$\sigma_2[P(\gamma)]$ is smooth at $\gamma^*$
$\begin{bmatrix} 2+j & 1 \\ 1 & 2+j \end{bmatrix}$		2.4495	$\begin{bmatrix} 0.3333 & -0.2357 \\ 0.2357 & 0.3333 \end{bmatrix}$	$\sigma_2[P(\gamma)]$ is nonsmooth at $\gamma^*$
$\begin{bmatrix} 1+j & -1 \\ 1 & 1+j \end{bmatrix}$		2.2361	$\begin{bmatrix} 0.2000 & 0.4000 \\ -0.4000 & 0.2000 \end{bmatrix}$	$\gamma^* = 1$ and $\sigma_2[P(1)]$ has multiplicity 2
$\begin{bmatrix} 2+j & 0 \\ 0 & 1+j2 \end{bmatrix}$		2.2361	$\begin{bmatrix} 0.3333 & -0.2981 \\ 0.2981 & 0.3333 \end{bmatrix}$	$\gamma^* = 1$ and $\sigma_2[P(1)]$ has multiplicity greater than 2
$\begin{bmatrix} 1+j & 2 \\ 0 & 1 \end{bmatrix}$		2.2361	$\begin{bmatrix} 0.0000 & 0.0000 \\ 0.4000 & 0.2000 \end{bmatrix}$	$\inf_{\gamma \in (0,1]} \sigma_2[P(\gamma)]$ is attained as $\gamma \rightarrow 0$ since $\text{rank}(\text{Im } M) = 1$

Table 1: For Example 1: behavior of  $\sigma_2[P(\gamma)]$ .

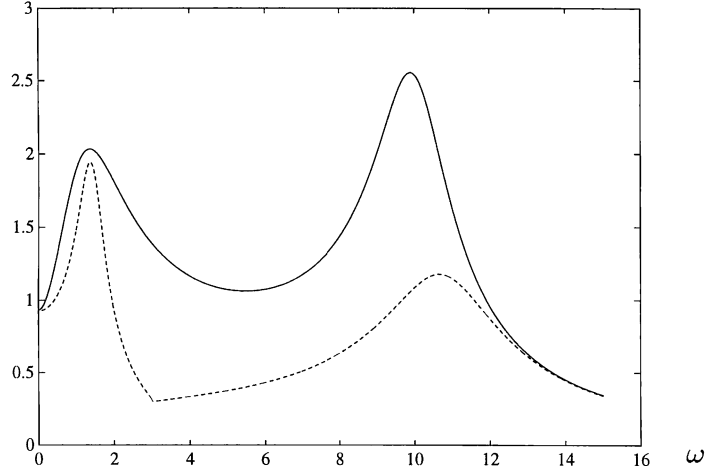


Figure 2: For Example 2, solid line is  $\mu_{\mathbb{C}}[C(j\omega I - A)^{-1}B]$  and dashed line is  $\mu_{\mathbb{R}}[C(j\omega I - A)^{-1}B]$ .

0.3914. Note that the critical frequencies for  $\mu_{\mathbb{R}}[C(j\omega I - A)^{-1}B]$  and  $\mu_{\mathbb{C}}[C(j\omega I - A)^{-1}B]$  are dramatically different.

To obtain a smallest real perturbation  $\Delta$  such that  $A + B\Delta C$  is unstable, we need to find a smallest  $\Delta$  such that  $I - \Delta C(j\omega I - A)^{-1}B$  is singular at  $\omega = 1.38$ . At this frequency, the minimum of the second singular value of

$$\begin{bmatrix} \operatorname{Re} C(j\omega I - A)^{-1}B & -\gamma \operatorname{Im} C(j\omega I - A)^{-1}B \\ \gamma^{-1} \operatorname{Im} C(j\omega I - A)^{-1}B & \operatorname{Re} C(j\omega I - A)^{-1}B \end{bmatrix}$$

occurs at  $\gamma = 0.2267$ . Its corresponding left and right singular vectors are

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0.3892 \\ 0.9178 \\ -0.0553 \\ 0.0558 \end{bmatrix}, \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -0.1615 \\ 0.9837 \\ 0.0669 \\ 0.0411 \end{bmatrix}$$

A smallest real  $\Delta$  is given by

$$\Delta = r_{\mathbb{R}}(A, B, C) \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \end{bmatrix}^{\dagger} = \begin{bmatrix} -0.4996 & 0.1214 \\ 0.1214 & 0.4996 \end{bmatrix}.$$

## 6 Concluding Discussion

This paper presents a formula for computation of the real stability radius. The basic problem is a pure linear algebra problem: Given a complex matrix  $M$ , find the smallest *real* matrix  $\Delta$  such that  $I - \Delta M$  is singular. Our main result reduces this problem to the minimization of a unimodal function in the interval  $(0, 1]$ . Our proof also gives a way to construct a worst  $\Delta$  such that  $I - \Delta M$  is singular. This then gives a computationally efficient way to compute the real structured stability radius and to construct a smallest destabilizing  $\Delta$ .

It is of interest to notice that the linear algebra problem that we have considered in this paper has rather deep and rich connections to many other problems in linear algebra, in particular the theory of complex symmetric matrices (Horn and Johnson, 1985, Chapter 4). The first three authors have recently shown the following extension of Lemma 1 (the Schmidt/Mirsky approximation theorem): For  $M \in \mathbb{C}^{p \times m}$ , the smallest spectral norm of real  $\Delta$  such that  $\text{rank}(I - \Delta M) \leq m - k$  is given by (3) with  $\sigma_2$  replaced by  $\sigma_{2k}$ . This result will be published elsewhere.

## References

- Baumgärtel, H. (1985). *Analytic perturbation theory for matrix and operators*. Birkhäuser, Basel.
- Byers, R. (1988). A bisection method for measuring the distance of a stable matrix to the unstable matrices. *SIAM J. Sci. Stat. Comput.*, 9:875–881.
- Chen, M. J. and Desoer, C. A. (1982). Necessary and sufficient condition for robust stability of linear distributed systems. *Int. J. Control*, 35:255–267.
- Doyle, J. C. and Stein, G. (1981). Multivariable feedback design: Concepts for a classical/modern synthesis. *IEEE Trans. Automat. Contr.*, AC-26:4–16.
- Hinrichsen, D. and Pritchard, A. J. (1986a). Stability radii of linear systems. *Systems & Control Letters*, 7:1–10.
- Hinrichsen, D. and Pritchard, A. J. (1986b). Stability radius for structured perturbations and the algebraic Riccati equation. *Systems & Control Letters*, 8:105–113.
- Hinrichsen, D. and Pritchard, A. J. (1990). Real and complex stability radii: a survey. In Hinrichsen, D. and Mårtensson, B., editors, *Control of Uncertain Systems*. Birkhäuser, Boston.

- Horn, R. A. and Johnson, C. R. (1985). *Matrix Analysis*. Cambridge University Press, Cambridge, England.
- Kato, T. (1966). *Perturbation Theory for Linear Operator*. Springer-Verlag, New York.
- Lewkowicz, I. (1992). When are the complex and the real stability radius equal? *IEEE Trans. Automat. Contr.*, 37:880–883.
- Martin, J. M. (1987). State-space measure for stability robustness. *IEEE Trans. Automat. Contr.*, AC-32:509–512.
- Qiu, L., Bernhardsson, B., Rantzer, A., Davison, E. J., Young, P. M., and Doyle, J. C. (1993). On the real structured stability radius. In *Proc. 12th IFAC World Congress*.
- Qiu, L. and Davison, E. J. (1991). The stability robustness determination of state space models with real unstructured perturbations. *Math. Control Signals Systems*, 4:247–267.
- Qiu, L. and Davison, E. J. (1992). Bounds on the real stability radius. In M. Mansour and S. Balemi and W. Truöl, editor, *Robustness of dynamic system with parameter uncertainties*, pages 139–145. Birkhäuser Verlag, Basel, Switzerland.
- Rudin, W. (1973). *Functional Analysis*. McGraw-Hill Book Company, New York.
- Stewart, G. W. and Sun, J. G. (1990). *Matrix Perturbation Theory*. Academic Press, San Diego.
- Van Loan, C. (1985). How near is a stable matrix to an unstable matrix? *Contemporary Math.*, 47:465–477.

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