ON STRONG DIGRAPHS WITH A UNIQUE MINIMALLY STRONG SUBDIGRAPH

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Abstract

In this paper we determine the maximum number of edges that a strong digraph can have if it has a unique minimally strong subdigraph. We show that this number equals $n(n+1)/2+1$, a surprisingly large number. Furthermore we show that there is, up to an isomorphism, a unique strong digraph which attains this maximum.

1. Introduction

A connected graph G with n vertices always has a minimally connected subgraph, namely a spanning tree. Moreover, the following three properties hold:

- Every minimally connected graph on n vertices has exactly $n-1$ edges.
- Every edge of G belongs to some minimally connected subgraph of G.
- G has a unique minimally connected subgraph if and only if G is itself a tree; or equivalently G has exactly $n-1$ edges.

We consider here the analogous properties for digraphs. A digraph D is *strong* (strongly connected) provided that for each ordered pair of distinct vertices x and y there is a path from x to y. A digraph

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D is *minimally strong* (minimally strongly connected) if D is strong but the removal of any edge results in a digraph that is not strong. A strong digraph always has a minimally strong subdigraph, but the analogy then begins to break down. A minimally strong digraph with n vertices can have as few as n edges - when it is a cycle on n vertices, and as may as $2(n-1)$ edges - when it is a symmetric digraph whose underlying graph is a tree [2,3]. The digraph D of Figure 1 shows that neither one of the other two properties indicated above for connected graphs holds in the directed case. First, not every edge of a strong digraph need belong to a minimally strong subdigraph of D. For example the digraph of Figure 1 has 3 edges in no minimally connected subdigraph. Moreover, a strong digraph D can have a unique minimally strong subdigraph different from D as does the digraph in Figure 1.

In this paper we determine the maximum number of edges that a strong digraph can have if it has a unique minimally strong subdigraph. We show that this number equals $n(n+1)/2+1$, a surprisingly large number. Furthermore we show that there is, up to an isomorphism, a unique strong digraph which attains this maximum.

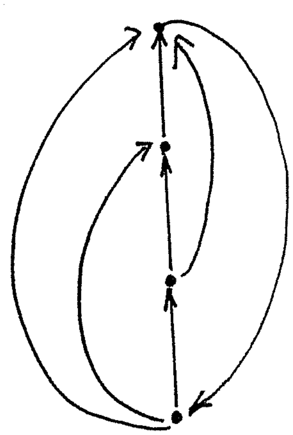

Figure 1

2. Main result

Let D be a strong digraph with n vertices. The digraphs in this paper are simple, unless otherwise stated. We first consider the case where the unique minimally strong subdigraph of D is a Hamiltonian cycle. Label the vertices of the digraph such that its unique Hamiltonian cycle is $C = (1,2,\ldots,n,1)$. All arithmetic operations on the labels are done modulo n. We denote the edge set of D by $E(D)$ and its vertex set by $V(D)$. The indegree and outdegree of a vertex v are denoted by $d^-(v)$ and $d^+(v)$ respectively.

Lemma 1: Let D be a strong digraph with n vertices whose unique minimally strong subdigraph is a Hamiltonian cycle C. Then $|E(D)| \leq n^2/2$.

Proof: We show that for every vertex $k = 1,2,\ldots,n$

$$d^+(k) + d^-(k+1) \leq n \tag{2.1}$$

With no loss of generality assume $k=1$. If there is no vertex $j > 2$ such that $(1,j) \in E(D)$, then $d^+(1) = 1$ and hence (2.1) holds. Otherwise let $i > 2$ be the smallest index such that $(1,i) \in E(D)$. Then $d^+(1) \leq n - i + 2$. Suppose there is an edge $(j,2)$ with $j \geq i$. Then the digraph obtained from D by deleting the edge $(1,2)$ is strong and hence D has a minimally strong subdigraph other than C. It follows that $d^-(2) \leq i-2$. Hence $d^+(1) + d^-(2) \leq n$. Thus, (2.1) holds for every $k=1,2,\ldots,n$ and hence

$$|E(D)| = ½ \sum_{k=1}^{n} (d^+(k) + d^-(k)) \leq ½n^2. \quad \square$$

Lemma 2: Let D be a strong digraph on n vertices and let $P=(1,\ldots,n)$ be a Hamiltonian path in D. If every minimally strong subdigraph of D contains P then $|E| \leq ½(n+2)(n-1)$. Equality holds if and only if D consists of the Hamiltonian path P together with all edges (i,j) such that $i > j$.

Proof: As in the proof of Lemma 1, we can conclude that

$$d^+(k) + d^-(k+1) \leq n \qquad k=1,\ldots,n-1. \tag{2.2}$$

Combining this with the fact that

$$d^+(n) \le n-1 \text{ and } d^-(1) \le n-1, \tag{2.3}$$

we get

$$|E| = \tfrac{1}{2}\left(\sum_{k=1}^{n} d^+(k) + \sum_{k=1}^{n} d^-(k)\right) \le \tfrac{1}{2}(n(n-1) + 2(n-1)) = \tfrac{1}{2}(n+2)(n-1).$$

It is clear that if D consists of P together with all edges (i,j) with $i > j$, then equality holds. Conversely, suppose that equality holds. Then we must have equalities in (2.2) and in (2.3). In particular $d^+(n) = n-1$ and hence $(n,i) \in E(D)$ for $i = 1, \ldots n-1$. It suffices to show that there is no edge (i,j) with $j > i+1$. Suppose that there is an edge (i,j) with $j > i+1$. Then the path $(i, j, j+1, \ldots n, i+1)$ joins i to $i+1$ in D and hence the digraph obtained from D by deleting the edge (1,2) is strong. It follows that D has a minimally strong subdigraph that does not contain P, a contradiction. □

Let S and R be two disjoint subsets of V(D). We denote by $(S:R)$ the set of edges with one endpoint in S and the other endpoint in R. We say that $(S:R)$ is the set of edges *between* S and R.

Lemma 3: Let D be a digraph on n vertices whose unique minimally strong subdigraph is the Hamiltonian cycle $C = (1, 2, \ldots, n, 1)$. Let p be an integer such that $2 \le p \le n$. Then

$$|(\{1\}:\{2, \ldots, p\}) - E(C)| \le p-1.$$

Equality holds only if $(p,1) \in E(D) - E(C)$.

Proof: The proof is by induction on p. The result is trivial if $p = 2$. Suppose it is true for $k \le p$ and let $k = p+1$. If there are no edges between 1 and $p+1$ except possibly edges of the cycle C, then the result follows from the inductive hypothesis and equality cannot hold. Similarly, if $|(\{1\}:\{2, \ldots, p\})| \le p-2$ then $|(\{1\}:\{2, \ldots, p+1\})| \le (p-2)+2 = p$ and equality implies that $(p+1, 1) \in E(D)$. It remains to consider the case where $|(\{1\}:\{2, \ldots, p\})| = p-1$. By the inductive hypothesis $(p,1) \in E(D)$. This implies that $(1, p+1) \notin E(D)$ because otherwise, there is a minimally strong subdigraph of D not containing the edge $(p, p+1)$. Hence $|(\{1\}:\{2, \ldots, p+1\})| \le p$ with equality only if $(p+1) \in E(D)$. □

Corollary 4: Let D be a digraph on n vertices whose unique minimally connected subdigraph is the Hamiltonian cycle $C = (1, 2, \ldots, n, 1)$. Then the number of edges between $\{1\}$ and $\{2, \ldots, n\}$ which are not edges of C is at most $n-2$.

Proof: By Lemma 3, if the number of edges between 1 and {2,...,n} is exactly $n-1$, then $(n,1) \in E(D)-E(C)$. This contradicts the fact that D is a simple digraph. □

Theorem 5: Let $D=(V,E)$ be a strong digraph on n vertices whose unique minimally strong subdigraph is a Hamiltonian cycle. Then

$$|E| \le \binom{n}{2} + 1$$

Proof: First suppose that for some i and j there holds $(i,j) \in E$ and $(j-1,i+1) \in E$. For simplicity assume that $i=1$ and $j=m+1$, $m \ge 1$. Let D_1 and D_2 be the vertex subgraphs induced by the sets $\{2,3,...,m+1\}$ and $\{m+2,...,n,1\}$ respectively. If there is an edge $(s,t) \ne (m+1,m+2)$ with $s \in D_1$ and $t \in D_2$ then the digraph resulting by deleting the edge $(j-1,j)$ is still strong. It follows that D has a minimally strong subdigraph other than the Hamiltonian cycle. Thus there are no edges, except $(m+1,m+2)$, directed from D_1 to D_2. Using a similar argument (deleting the edge $(i,i+1)$), we can conclude that there is no edge directed from D_2 to D_1 except for the cycle edge (1,2). Hence

$$|E| = |E(D_1)| + |E(D_2)| + 2. \tag{2.4}$$

The fact that D has a unique minimally strong subdigraph, does not imply that D_i $(i=1,2)$ has a unique minimally strong subdigraph. However, it does imply that every minimally strong subdigraph of D_1 (respectively D_2) contains the Hamiltonian path $(2,...,m+1)$ (respectively $(m+2,...n,1)$). By Lemma 2 and (2.4) we have

$$\begin{aligned} E(D) &\le \tfrac{1}{2}(m+2)(m-1) + \tfrac{1}{2}(n-m+2)(n-m-1) + 2 \\ &= [n^2 - m(n-m)] + n - m(n-m) \\ &\le \binom{n}{2} + n - m(n-m) \\ &\le \binom{n}{2} + 1. \end{aligned}$$

The last inequality is justified by the fact that the product xy with $x+y=n$ is minimized when the factors are 1 and $n-1$ or in otherwords $m(n-m) \ge n-1$.

We may now assume that there is no pair of vertices i and j for which both (i,j) and $(j-1,i+1)$ are edges of D. It is also clear that if $(i,j)\in E(D)$ then $(j,i+1)\notin E(D)$. Let $d^+(i)=k$. It then follows that $d^-(i+1)\leq n-(k+1)$. Hence $d^+(i)+d^-(i+1)\leq n-1$ for all i. Thus,

$$|E| = \tfrac{1}{2}\sum_{i=1}^{n}(d^+(i)+d^-(i+1))\leq \tfrac{1}{2}n(n-1) = \binom{n}{2}.$$

We now consider the case where the unique minimally strong subdigraph is not a Hamiltonian cycle.

Theorem 6: Let D be a strong digraph on $n>3$ vertices whose unique minimally strong subdigraph is not a Hamiltonian cycle. Then $|E(D)| \leq \binom{n}{2}$.

Proof: The proof is by induction on the number n of vertices of D. The case $n=4$ can be checked using the fact that unique minimally strong subdigraph D' can be one of the four digraphs of Figure 2. If $D' = D_1$ or D_2 then $D = D'$. In the other two cases it can be easily checked that $|E(D)|\leq 6$.

Now suppose the claim holds for $k<n$ and let D be a digraph with n vertices satisfying the conditions of the Theorem. Since D' is not a Hamiltonian cycle, D' contains a minimally strong subdigraph $D_0=(V_0,E_0)$ on m vertices and a simple path $(v_0,v_1,\ldots,v_{n-m+1})$, where v_0 and v_{n-m+1} are in V_0 while the vertices $v_2,\ldots,v_{n-m}$ are in $V-V_0$ [1]. Since D' is not a

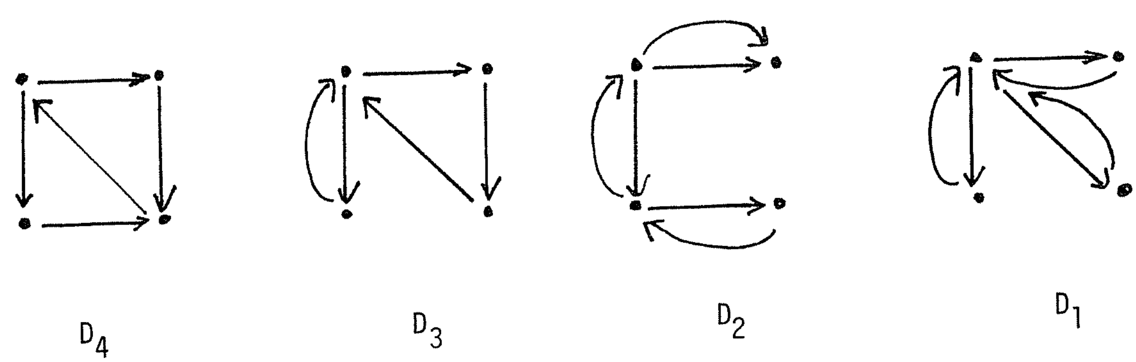

Figure 2.

Hamiltonian cycle, $m \geq 3$. It follows from the inductive hypothesis and from Theorem 4 that the vertex subgraph of D induced by V_0 has at most $\binom{n}{2} + 1$ edges. Let D^* be the simple digraph obtained from D by shrinking the set of vertices V_0 to a vertex V_0^* and identifying multiple edges to a single edge and eliminating self loops. The digraph D^* is clearly strong and since D has a unique minimally strong subdigraph, so does D^*. In fact the unique minimally strong subdigraph of D^* is a Hamiltonian cycle. Hence, By Theorem 5, D^* has at most $\binom{n-m+1}{2} + 1$ edges. Each edge in D^* between $V - V_0$ and $\{V_0^*\}$ corresponds to at most m edges in D. Thus the number of edges of D that are not in $E(D_0) \cup E(D^*)$ is at most $m-1$ times the number of edges between V_0^* and $\{v_1, \ldots, v_{n-m}\}$. By Corollary 4, the number of edges in D between V_0^* and $v_1, \ldots, v_{m-m}$ is at most $(n-m-1)$. Thus

$$|E(D)| \leq \binom{m}{2} + 1 + \binom{n-m+1}{2} + 1 + (n-m-1)(m-1) = \binom{n}{2} - (m-3) \leq \binom{n}{2}.$$

Let D be a strong digraph on n vertices with a unique minimally strong subdigraph D'. Our results show that the the number of edges of D is a number between n and $\frac{1}{2}n(n-1) + 1$. The two extremes are attained when D' is a Hamiltonian cycle, that is, when D' has the fewest possible number of edges in a minimally strong digraph on n vertices. On the other hand, suppose the unique minimally strong subdigraph D' of D has the largest possible number of edges a minimally strong digraph can have. Then, D' is a symmetric digraph whose underlying graph is a tree [2] and has exactly $2n-2$ edges [3]. In this case we must have $D = D'$.

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