Robotics Research Technical Report

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DETERMINING THE SHAPE OF A
CONVEX n-SIDED POLYGON BY USING
2n+k TACTILE PROBES

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Courant Institute of Mathematical Sciences

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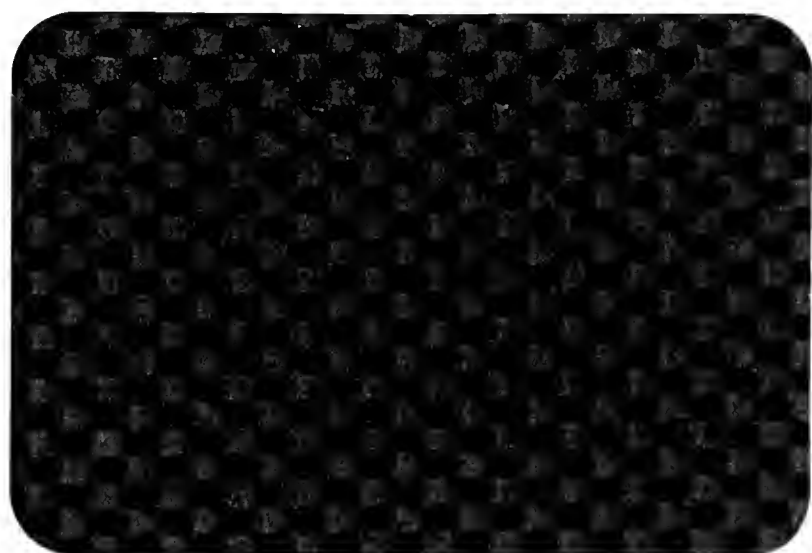
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Determining the Shape of a Convex n-sided Polygon by Using $2n+k$ Tactile Probes

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1. Introduction.

A robot may see into a workspace in many ways. It may have cameras which present entire projections of a scene at once. Often, however, the robot will have to rely on simple probes into the workspace. For each probe the robot moves along some line until it encounters some sort of object boundary. By recording the locations of such contacts, the robot can infer object locations.

Cole and Yap [1] considered the question of determining the shape of a convex polygon by such tactile probes. They did not restrict the polygons to a finite set. They showed that the shape of a polygon known to contain the origin can be determined with no more than $3n$ probes, where n is the number of sides of the polygon. They also showed that $3n-1$ probes are necessary. Under a mild assumption, they showed that $3n$ probes are necessary. Schwartz and Sharir [3] considered the question of selecting one polygon from a finite set by such probing and asserted that "normally, very few probes will be required." In this note we show that $2n+k$ probes suffice to determine the shape of a convex polygon of n sides selected from a finite set of polygons. The number of polygons in the set enters only indirectly and some infinite sets can be handled by the same technique. We show that $k = 3$ under the assumptions of Cole and Yap. For slightly stronger assumptions we show that $k = 2$. Under the assumptions of Schwartz and Sharir we obtain $k = -1$. If we add the assumption that the number of sides n is known, then k drops by one in each case.

Necessity is shown in a case closely related to the Schwartz and Sharir assumptions.

The algorithm presented is not intended to be most efficient in all cases, or even feasible in practice, but to present an upper bound on the number of probes required for efficient algorithms.

Earlier references on the subject of tactile probing can be found in [2].

2. Preliminaries.

Let Γ be a set of convex polygons. Let $P \in \Gamma$ be a polygon with edges $e_1,\ldots,$ e_n. Define $d(P)$ as the minimum over i of the distance from the line containing the line segment e_i to points on edges not adjacent to e_i. Define d_{min} as the infimum of $d(P)$ over all $P \in \Gamma$. Since $d(P)$ must be strictly positive for convex P, d_{min} must be non-zero for finite Γ. It is also possible for d_{min} to be non-zero for an infinite set of convex polygons. We restrict our attention to Γ for which $d_{min} > 0$.

A probe p of a polygon is a directed line. If a probe intersects both the interior and the exterior of a polygon, the first point of intersection with the boundary of the polygon will be called its contact point. A probe may well intersect the boundary of the polygon, but not its interior. We make the assumption that such a probe will not be considered to have a contact point. Cole and Yap make a slightly weaker "mild assumption" that a probe which just touches a vertex will not make contact. The difference between these two assumptions is that in our case, a probe just along the line through an edge will be assumed to miss, while in the Cole and Yap case it will be assumed to have a contact at a vertex.

When we address the necessary number of probes, we will need to account for some of the probes which fail to contact the polygon. For convenience in this, we say that a probe p is *involved* with an edge e_i of convex polygon P if:

1. The probe p has a contact point on the edge e_i; or

2. The probe p does not have any contact point on the P, but is directed along the line through the edge e_i; or

3. The probe p does not contact P and is not directed along the line through any edge of P, but passes through a vertex incident on e_i.

Note that case 1, above, includes contacts at vertices. Such probes will be involved with two edges, as will those in case 3.

Cole and Yap start their probing process with the origin known to be within the (open) interior of the polygon. This allows one to isolate the problem of recognizing the shape of the polygon from the problem of locating it within the workspace. With this in mind we state:

Theorem 1. Let Γ be a set of convex polgons for which $d_{\min} > 0$. Let $P \in \Gamma$ have n sides and contain the origin O within its interior. The shape of P can be determined by $2n+3$ tactile probes. If n is known *a priori* then the shape of P can be determined by $2n+2$ tactile probes.

If there is no lower bound on the ratio of the size of the polygon to the size of the workspace, there is no bound on the number of probes required to locate the polygon. Except in the case of triangles, having $d_{\min} > 0$ forces a minimum size for the polygons under consideration. If we have a finite workspace and polygons bounded below in size, we could use additional probes to allow us to assume not only that the origin is in the interior of the polygon but that we also know the size of some disk centered on the origin contained entirely within the polygon. In this case we assert the following theorem:

Theorem 2. Let Γ be a set of convex polygons for which $d_{\min} > 0$. Let $P \in \Gamma$ have n sides and contain a disk of known radius ρ around the origin O. The shape of P can be determined by $2n+2$ tactile probes. If n is known *a priori* then the shape of P can be determined by $2n+1$ tactile probes.

Schwartz and Sharir remove the requirement for convexity but assume that some first edge is known. In order to find that first edge, however, they reintroduce convexity. We retain the convexity assumption and will show:

Theorem 3. Let Γ be a set of convex polygons for which $d_{\min} > 0$. Let $P \in \Gamma$ have n sides and contain the origin O within its interior. Let the line through some edge of P be given. The shape of P can be determined by $2n-1$ tactile probes. If n is known *a priori* then the shape of P can be determined by $2n-2$ tactile probes.

It would be nice to be able to prove necessity. Starting with the assumption of a known finite set makes this difficult, since we cannot introduce new polygons to challenge the probe algorithm partway through its execution. We can probably construct cases to force $\log(card(\Gamma))$ behavior for some properly chosen Γ, but

there will be many cases where enumeration of the features of the polygons in Γ will allow identification by 2 probes. Rather than confront this difficult aspect of the finite case, let us consider a more manageable infinite case in the following theorem:

Theorem 4. Suppose a probe algorithm can determine the shape of any convex polygon P for which $d(P) \geq d_{min}$ for some given d_{min}, and for which a line through some edge of P is given. Then for each $n \geq 3$ there is a such a polygon P for which $2n-2$ tactile probes will be required to determine its shape.

3. Proofs.

Let us start by proving Theorem 3. Let Γ, d_{min}, and $P \in \Gamma$ be given. Let the origin O be in the interior of P and let l_1 be the known line through some edge e_1 of P. Clearly, l_1 cannot pass through O, so we can make probes parallel to l_1 which pass between l_1 and O. By construction of d_{min}, if such a probe is made at a distance less than d_{min} from l_1 it must encounter an edge adjacent to e_1. Make two such probes from the same end of l_1, as in Figure 1. Both probes must be on the same edge e_2 adjacent to e_1. If not, we would have P with O outside. Thus the two probe points determine a line l_2 through e_2. We may now continue inductively with $2(n-1)$ probes to determine the lines through $n-1$ edges. If we know n, we are done. If not, we need one more probe to discover that we have closed the polygon.
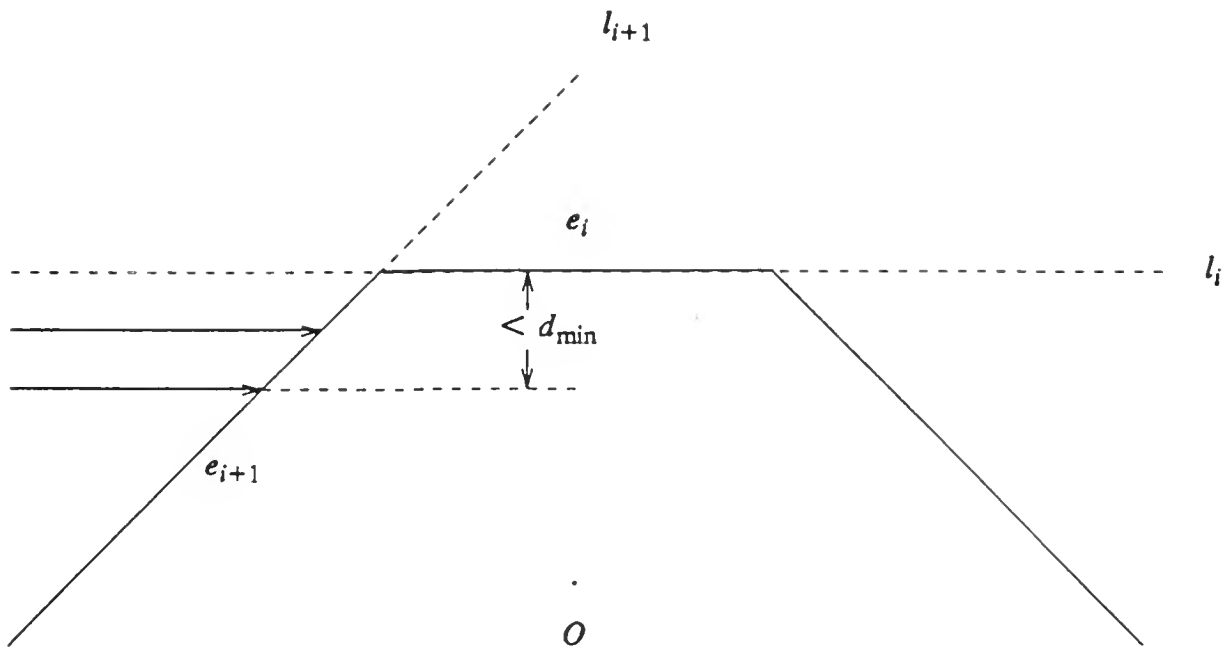
Figure 1. Probes to Find Next Edge.

To prove Theorem 4 we will take a starting polygon P for which $d(P) > d_{min}$ and make arbitrarily small changes of edge positions to force the total number of probes involved with each edge to be at least 2 in the form of two contact points in the interior of an edge, or one contact point in the interior of an edge and one probe along the direction of the edge. We do this by making the inductive assumption:

The first probe involved with an edge contacts that edge at an interior point. The second probe involved with an edge either contacts that edge at an interior point or along the line through the edge.

Start the algorithm and run until there is a probe which would be involved with an edge, but which would not contact the interior of that edge. There are two cases to consider: a probe along the direction of the edge, or a probe intersecting a vertex incident on the edge.

Consider a probe along the direction of the edge. If it is the first probe involved with the edge, we may simply move the edge slightly in towards the interior of P to avoid the probe entirely. If it is the second probe involved with the edge, by our induction hypotheses, the first probe must have contacted an interior

point of the edge, and we need not make any change. If it is the third or later probe, we certainly need not make any change.

Consider a probe intersecting a vertex incident on the edge. It is also involved with a second edge. The probe may be the first or second probe involved with either one. If it is the third or later probe involved with both, we need not make any change.

Suppose the probe is the first probe involved with one of the edges. It is not a probe along the direction of the other edge, so we may move the first edge slightly to make the probe miss the vertex, making the probe either miss P or contact P at an interior point of an edge.

Suppose the probe is the second probe involved with one of the edges, but not the first involved with either. Since the first probe must have contacted P at an interior point of an edge, and the new probe cannot be along the direction of the other edge, we are still free to rock the first edge around the interior contact point and make the new probe miss the vertex, as in Figure 2.

Clearly we can continue in this manner until we have used two probes of the type allowed to determine the direction of each of the $n-1$ unknown edges or until we have used $2n-2$ probes. Either case proves the theorem.
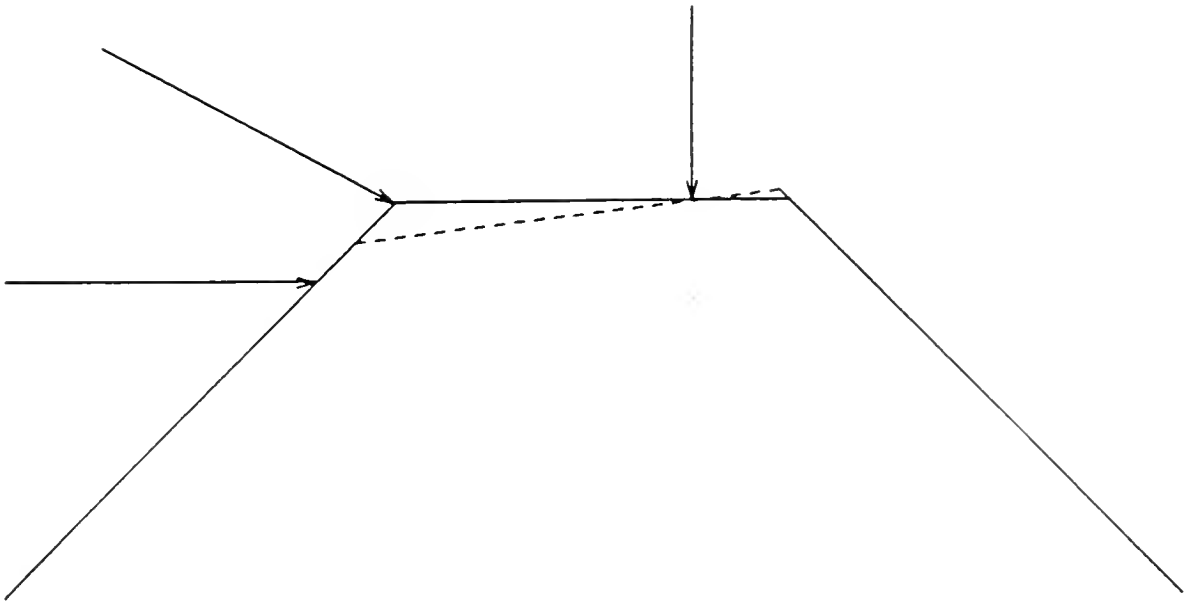
Figure 2. Rocking an Edge With Only One Contact Point

We can now prove Theorems 1 and 2 by showing that a first two adjacent edges can be identified by no more than 6 probes under the hypotheses of Theorem 1 and by 5 probes under the hypotheses of Theorem 2. The proofs can then be completed as for Theorem 3. We will need the following lemma:

Lemma. Let $P \in \Gamma$. Let x_1, x_2 be points on the boundary of P such that the distance from x_1 to x_2 is less than d_{min}. Then x_1 and x_2 are on the same edge of P or are on two adjacent edges of P.

Proof of Lemma. Suppose not. Then by definition of d_{min}, x_2 is at a distance at least d_{min} from the line through the edge containing x_1, *a fortiori* at a distance at least d_{min} from x_1 itself, a contradiction.
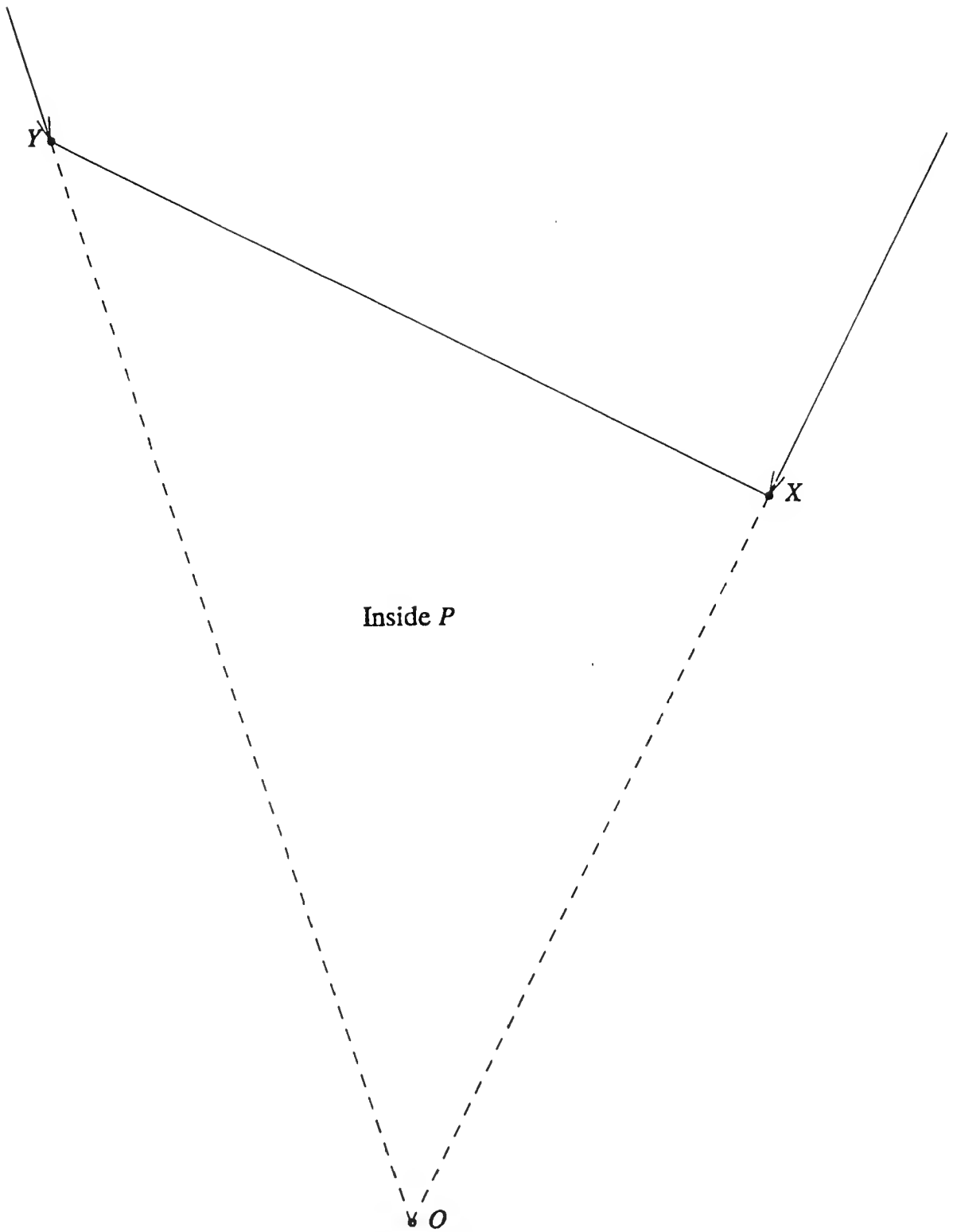
Inside *P*

Figure 3. Initial Probes to *X* and *Y*
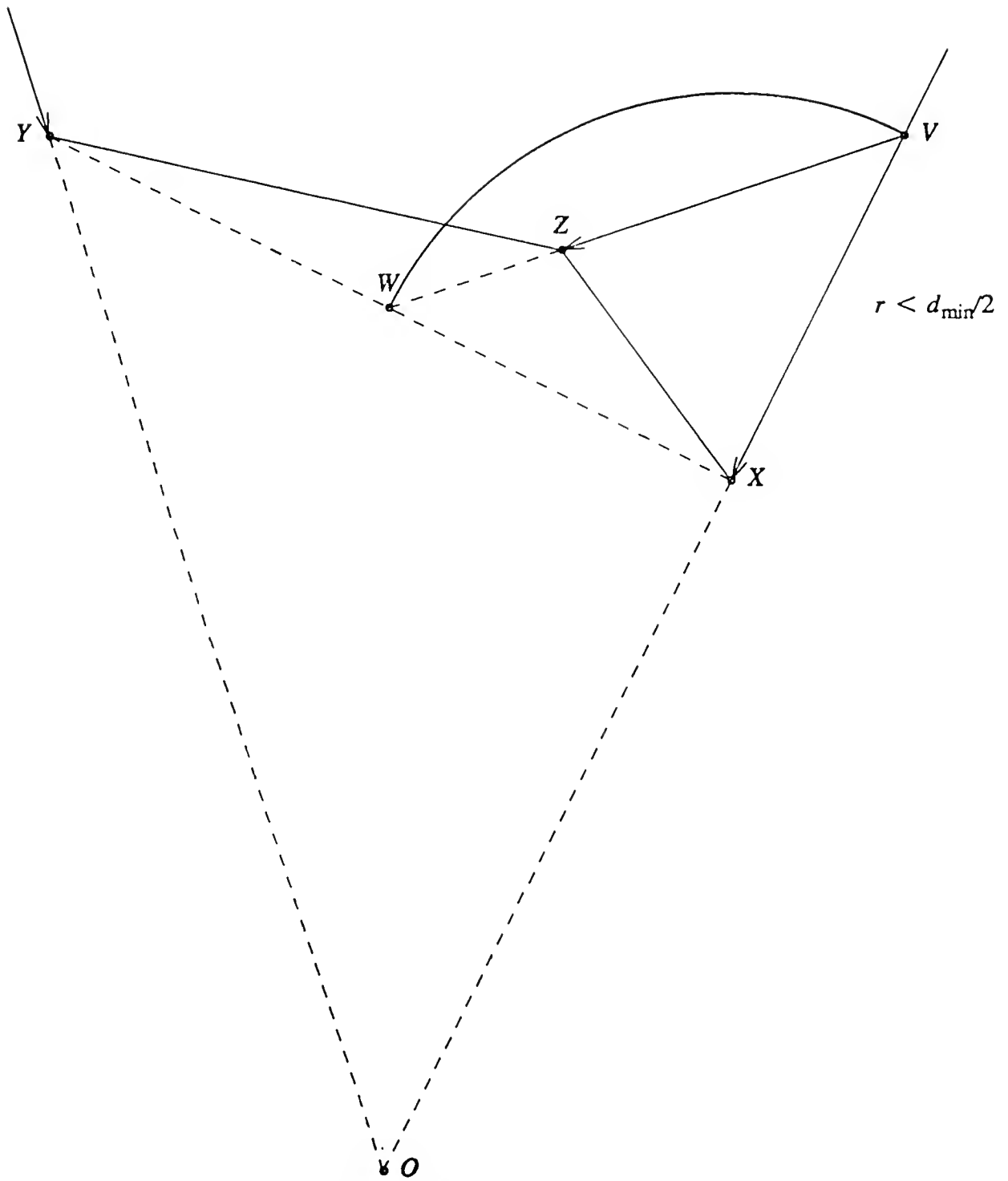
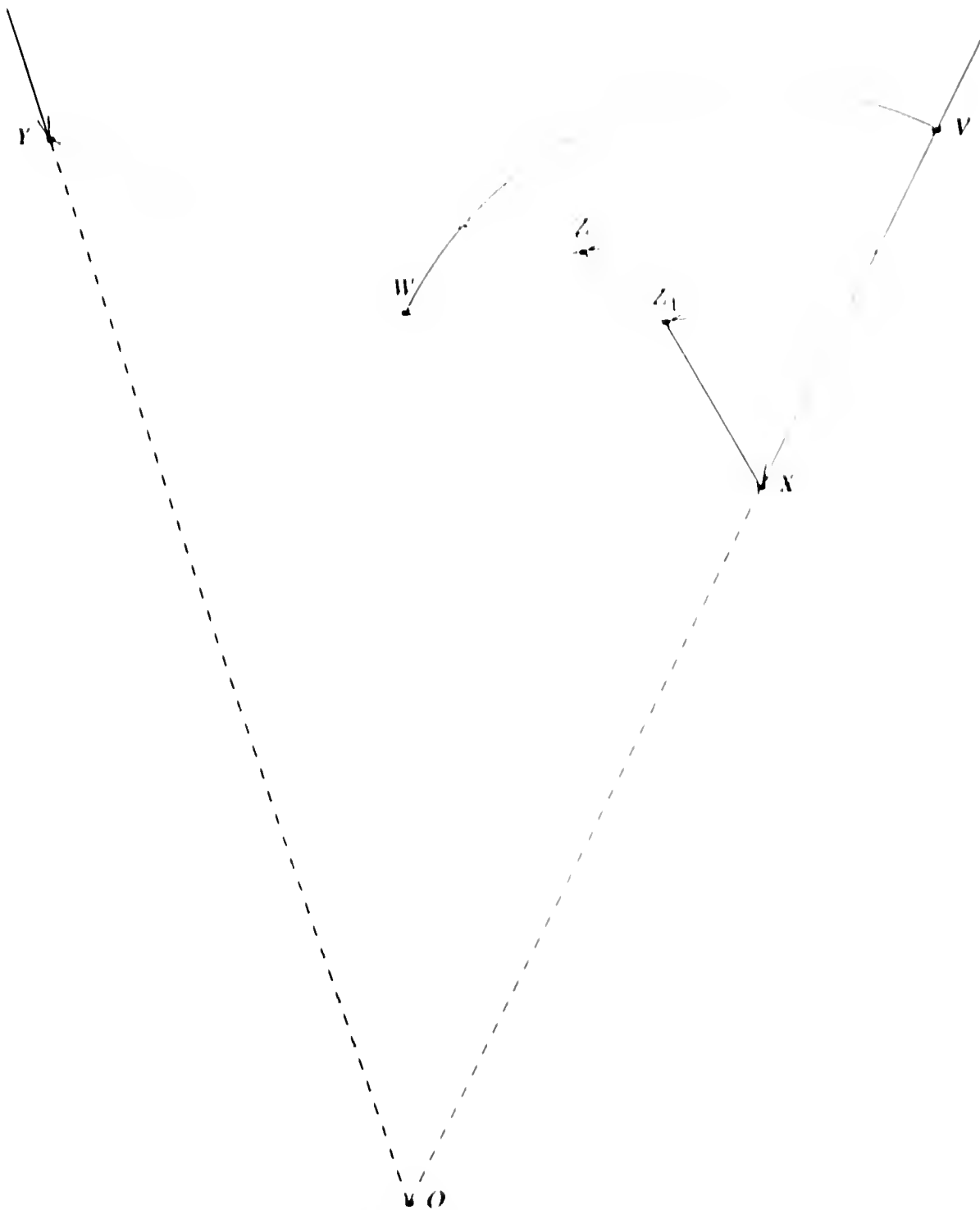$r < d_{\min}/2$

Figure 4. Probe from V towards W
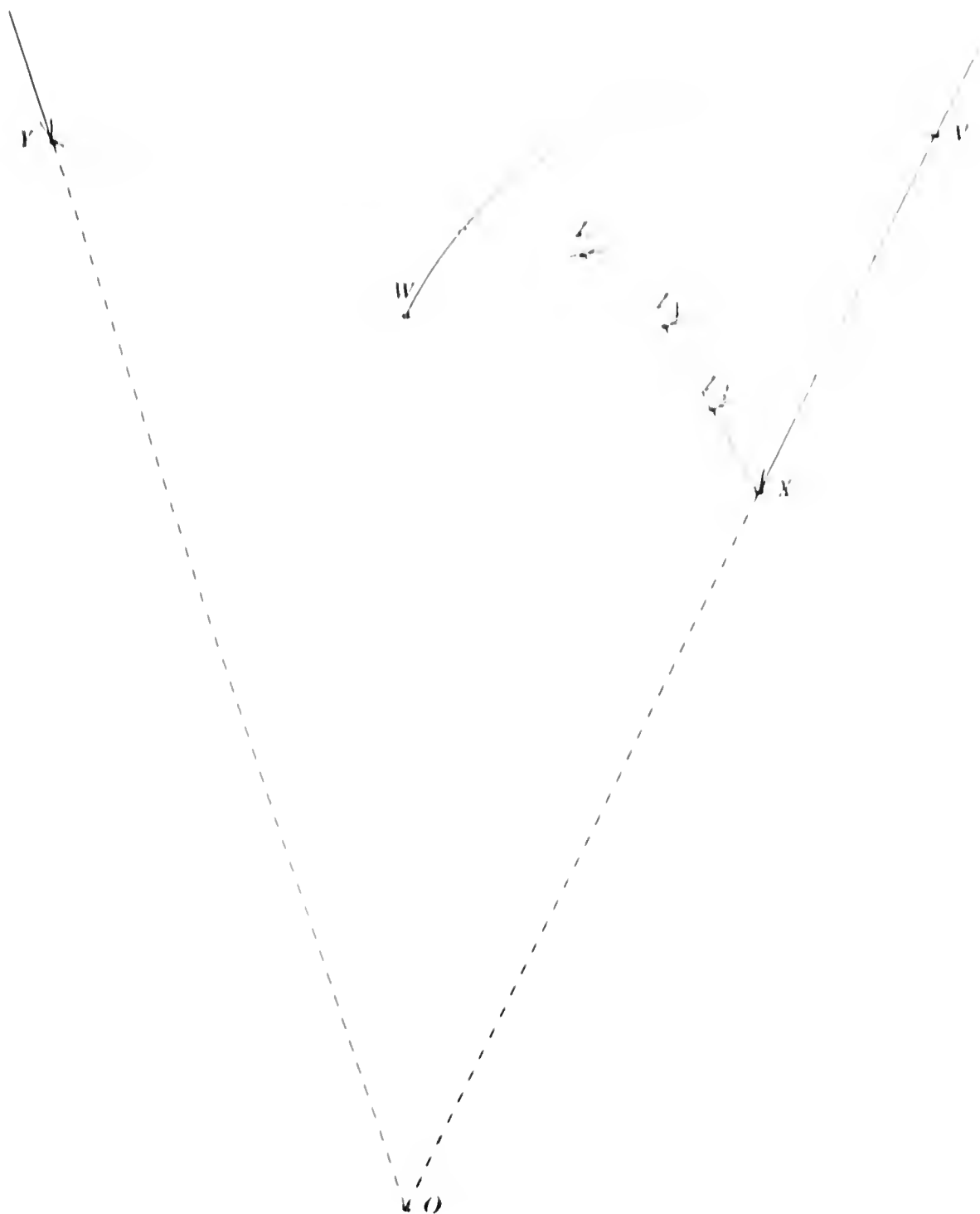
Figure 5. Probe to Z_1 parallel to VW

Figure 6. Probe to Z_2 parallel to VW

Let P be a polygon known to contain the origin O. Let X and Y be two probe points obtained by probes directed at O such that XOY is a small acute

angle, as in Figure 3. (We do not use the size of angle XOY in this proof, but the smaller this angle, the fewer the number of probes likely to be required to start the process, since fewer edges are likely to be subtended.) Without loss of generality, we assume that angle OXY is no smaller than angle OYX. Take a point W at distance r, $0 < r < d_{min}/2$, along the line segment XY and a point V at distance r from the point X on the XO probe exterior to the polygon. The point W must be on an edge or in the interior of the polygon. Therefore, a probe from V to W must contact a polygon edge. Since this probe is along a chord of a circle of radius r, the point Z of contact must be at a distance no more than r from X. (See Figure 4.) Note that by the lemma, the probe contact points within this circle can only be involved with at most three edges, since all must be within $2r < d_{min}$ of one another. (We could have taken a tighter bound for r knowing the angle of YOX, but that would have no qualitative effect on the proof.) By the lemma, Z is on an edge adjacent to the edge containing X or on the same edge as X. If Z is colinear with X and Y then we have found the first edge with 3 probes. If Z is not colinear with X and Y, make at most 2 more probes with contact points Z_1 and Z_2 between Z and X along probe directions parallel to the VW probe, as in Figures 5 and 6. If Z_1 is colinear with Z and X then we have found the first edge with 4 probes and need not make the probe to Z_2. If Z_1 is not colinear with Z and X then, since at most three edges are involved and the three edge case would leave the origin outside P, either Z_2 is colinear with Z and Z_1 or Z_2 is colinear with Z_1 and X. In either case we have found a first edge with 5 probes, one of which (Z or X) is certain to be on an adjacent edge. Now make a probe parallel to the known edge from the side containing the point on the adjacent edge at less than the known point's distance from the line through the known edge. This sixth probe must determine a second edge since, by the lemma, it must be on an adjacent edge to the known edge. This completes the proof of Theorem 1.

The proof of Theorem 2 is essentially the same once we recognize that inclusion of a circle of known radius around the origin allows us to make the YOX probe angle sufficiently small to have the contact point Y play the role of contact point Z, saving one probe.

4. Additional Remarks

It appears that this approach can be extended to $3n+k$ probing of convex polyhedra drawn from a finite set. One would have to make narrow bundles of three probes near to and parallel to previously determined faces.

In any attempt to implement such algorithms, one would be faced with the serious problem that small errors in contact position would be greatly magnified away from the contacts. This is particularly true with respect to the starting method chosen. Extra probes would certainly be required in practice.

Acknowledgement.

Our thanks to Chee Yap and Frances Bernstein for the time they took reading drafts of this note and making helpful suggestions.

References.

[1] R. Cole, C. Yap, "Shape from Probing", Technical Report No. 104, Robotics Report No. 15, New York University, Courant Institute of Mathematical Sciences, Computer Science Division, December 1983, 14 pp.

[2] P. C. Gaston, T. Lozano-Perés, "Tactile Recognition and Localization Using Object Models: The Case of Polyhedra on a Plane," M.I.T. Artificial Intelligence Lab Memo 705, March, 1983.

[3] J. T. Schwartz, M. Sharir, "Some Remarks on Robot Vision", Technical Report No. 119, Robotics Report No. 25, New York University, Courant Institute of Mathematical Sciences, Computer Science Division, April 1984, 36 pp.

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