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Combinatorial properties of generalized hypercube graphs

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Abstract

This paper investigates combinatorial properties of generalized hypercube graphs including best containers, wide diameter, and fault diameter. These properties have received much attention recently in the study of interconnection networks.

Keywords: Algorithms; Graph theory

1. Introduction

Generalized hypercube graphs are the underlying graphs of generalized hypercube networks [1] which were proposed for building massively parallel computer systems. Let $G(m_r, m_{r-1},\dots,m_1)$ denote a generalized hypercube graph of size $m_r \times m_{r-1} \times \cdots \times m_1$, where $m_i \geqslant 2$ for all $1 \leqslant i \leqslant r$. There are $N = m_r * m_{r-1} * \cdots * m_1$ nodes in $G(m_r, m_{r-1},\dots,m_1)$ which are assigned r-digit identifiers $x_r, x_{r-1} \dots x_1$, where $x_i \in [0, m_i - 1]$ for all $1 \leqslant i \leqslant r$. Two nodes in $G(m_r, m_{r-1},\dots, m_1)$ are adjacent if and only if their identifiers differ at exactly one digit position. In Fig. 1, the structure of $G(4, 3, 2)$ is depicted for illustration. Each node in $G(m_r, m_{r-1},\dots,m_1)$ has degree $\Sigma_{i=1\dots r}(m_i - 1)$.

In this paper, several combinatorial properties of $G(m_r, m_{r-1},\dots,m_1)$ are investigated. Specifically, best containers of width κ, the κ-wide diameter, and the fault diameter of $G(m_r, m_{r-1},\dots,m_1)$ are computed, where κ is the node connectivity of $G(m_r, m_{r-1},\dots,m_1)$. These properties have become more and more important recently in the study of reliability, fault tolerance, randomized routing, and transmission delay in interconnection networks [3,5,6]. It is of both theoretical interest and practical importance to determine a container of width κ between arbitrary two nodes because the existence of such a container means that messages can be transmitted in parallel using κ disjoint paths. Besides, the transmission will succeed even if $\kappa - 1$ node faults have occurred. The fault diameter, on the other hand, estimates the maximum transmission delay under the situation of at most $\kappa - 1$ node faults.

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2. Containers

Let X, Y be two arbitrary nodes in $G(m_r, m_{r-1}, \dots, m_1)$. An *$(X, Y)$-container* [5], denoted by $C(X, Y)$, is a set of node-disjoint paths between X and Y. The *width* of a $C(X, Y)$, denoted by $w(C(X, Y))$, is its cardinality. The *length* of a $C(X, Y)$, denoted by $l(C(X, Y))$, is defined as the maximum length of the paths in the $C(X, Y)$. A $C(X, Y)$ is the *best*, denoted by $C^*(X, Y)$, if its length is minimum among those with the same width.

Let $X = x_r x_{r-1} \dots x_1$ and $Y = y_r y_{r-1} \dots y_1$, be two arbitrary nodes in $G(m_r, m_{r-1}, \dots, m_1)$. A particular $C(X, Y)$ of width $\Sigma_{i=1\dots r}(m_i - 1)$ has been found in [1]. In this section, a general approach for constructing $C(X, Y)$s of width $\Sigma_{i=1\dots r}(m_i - 1)$ is first proposed. The constructed paths fall into three sets S_1, S_2 and S_3 according to their lengths.

Assume X and Y differ at d digit positions: $s(d), s(d-1), \dots, s(1)$. That is, $x_i = y_i$ if and only if $i \notin \{s(d), s(d-1), \dots, s(1)\}$. We let

$$
\begin{aligned}
S_1 = \{ & (X, X_{s(1)}, X_{s(1),s(2)}, \dots, X_{s(1),S(2),\dots,s(d)} = Y), \\
& (X, X_{s(2)}, X_{s(2),s(3)}, \dots, \\
& X_{s(2),s(3),\dots,s(d),s(1)} = Y), \\
& \vdots \\
& (X, X_{s(d)}, X_{s(d),s(1)}, \dots, \\
& X_{s(d),s(1),\dots,s(d-1)} = Y)\},
\end{aligned}
$$

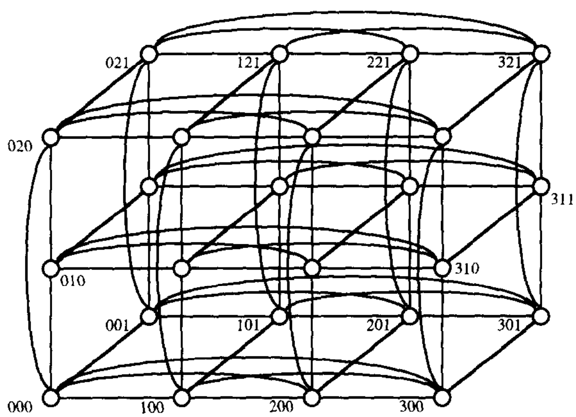

Fig. 1. The structure of $G(4, 3, 2)$.

where $X_{s(i), s(j), \dots, s(l)}$ represents the identifier resulting from X after replacing $x_{s(i)}, x_{s(j)}, \dots, x_{s(l)}$ with $y_{s(i)}, y_{s(j)}, \dots, y_{s(l)}$, respectively.

The set S_1 contains d paths whose length is d.

Further, let $X_{(k,a), s(i), s(j), \dots, s(l)}$ represent the identifier which is obtained by replacing the kth digit from the right of $X_{s(i), s(j), \dots, s(l)}$ with a. We let

$$S_2 = \{S_{2,1}, S_{2,2}, \dots, S_{2,d}\},$$

where for $i = 1, \dots, d$,

$$
\begin{aligned}
S_{2,i} = \{ & (X, X_{(s(i),j)}, X_{(s(i),j),s(1)}, X_{(s(i),j),s(1),s(2)}, \dots, \\
& X_{(s(i,)j),s(1),s(2),\dots,s(i-1)}, \\
& X_{(s(i),j),s(1),s(2),\dots,s(i-1),s(i+1)}, \dots, \\
& X_{(s(i),j),s(1),s(2),\dots,s(i-1),s(i+1),\dots,s(d)}, \\
& X_{s(1),s(2),\dots,s(d)} = Y) \mid \text{for } j = 0, \dots, m_i - 1 \\
& \text{and } j \neq x_{s(i)},\ j \neq y_{s(i)}\}.
\end{aligned}
$$

If $m_i = 2$, then no legal j can be found, in which the set $S_{2,i}$ is set to empty. The set S_2 contains $\Sigma_{i=1\dots d}(m_{s(i)} - 2)$ paths whose length is $d + 1$.

Assume $t(r-d), t(r-d-1), \dots, t(1)$ are the $r - d$ digit positions such that $x_i = y_i$ if and only if $i \in \{t(r-d), t(r-d-1), \dots, t(1)\}$. We let

$$S_3 = \{S_{3,1}, S_{3,2}, \dots, S_{3,r-d}\},$$

where for $i = 1, \dots, r - d$,

$$
\begin{aligned}
S_{3,i} = \{ & (X, X_{(t(i),j)}, X_{(t(i),j),s(1)}, X_{(t(i),j),s(1),s(2)}, \dots, \\
& X_{(t(i),j),s(1),s(2),\dots,s(d)}, \\
& X_{s(1),s(2),\dots,s(d)} = Y) \mid \text{for } j = 0, \dots, m_i - 1 \\
& \text{and } j \neq x_{s(i)}\}.
\end{aligned}
$$

When $r = d$, the set S_3 is empty. For nonempty S_3, it contains $\Sigma_{i=1\dots r-d}(m_{t(i)} - 1)$ paths whose length is $d + 2$.

For example, let us consider $X = 210$ and $Y = 201$ in $G(4, 3, 2)$. If $(s(2), s(1)) = (2, 1)$, then

$$S_1 = \{(210, 211, 201), (210, 200, 201)\},$$

$$S_2 = \{(210, 220, 221, 201)\},$$

$$
\begin{aligned}
s_3 = \{ & (210, 010, 011, 001, 201), \\
& (210, 110, 111, 101, 201), \\
& (210, 310, 311, 301, 201)\}.
\end{aligned}
$$

If (s(2), s(1)) = (1, 2), then

$S_1 = \{(210, 200, 201), (210, 211, 201)\}$,

$S_2 = \{(210, 220, 221, 201)\}$,

$S_3 = \{(210, 010, 000, 001, 201)$
$(210, 110, 100, 101, 201)$,
$(210, 310, 300, 301, 201)\}$.

There are totally

$$d + \sum_{i=1\ldots r}(m_{s(i)} - 2) + \sum_{i=1\ldots r-d}(m_{t(i)} - 1) = \sum_{i=1\ldots r}(m_i - 1)$$

paths in $S_1 \cup S_2 \cup S_3$. The node-disjoint property is very clear. So, for each instance of $s(d), s(d-1), \ldots, s(1)$, the paths contained in $S_1 \cup S_2 \cup S_3$ constitute a $C(X, Y)$ of width $\sum_{i=1\ldots r}(m_i - 1)$. The following lemma shows that the $C(X, Y)$ is the best.

Lemma 1. *Suppose X and Y are two nodes in $G(m_r, m_{r-1}, \ldots, m_1)$ and they differ at d digit positions denoted by $s(d), s(d-1), \ldots, s(1)$. Then, for each instance of $s(d), s(d-1), \ldots, s(1)$, the paths contained in $S_1 \cup S_2 \cup S_3$ constitute a $C^*(X, Y)$ of width $\sum_{i=1\ldots r}(m_i - 1)$.*

Proof. According to the structure of $G(m_r, m_{r-1}, \ldots, m_1)$, a shortest path between X and Y must contain d edges each of which equalizes a different digit. Hence, not more than d node-disjoint shortest paths between X and Y exist. A second shortest path between X and Y, whose length is $d+1$, has a form as shown in the set $S_{2,i}$. Each path in $S_{2,i}$ equalizes by two edges (the first and the last edges) the digit whose position is indicated by $s(i)$. There are at most $\sum_{i=1\ldots d}(m_{s(i)} - 2)$ node-disjoint second shortest paths between X and Y. Since $|S_1| = d$, $|S_2| = \sum_{i=1\ldots d}(m_{s(i)} - 2)$, and each path in S_3 has a length $d+2$, the correctness of the lemma follows. □

There are $d!$ ways to specify $s(d), s(d-1), \ldots, s(1)$. Carefully observing the construction of S_1, we find that two different instances of $s(d), s(d-1), \ldots, s(1)$ will construct the same S_1 if and only if one can be obtained from the other by continuous cyclic rotations. Therefore, the following lemma holds.

Lemma 2. *Suppose X and Y are two nodes in $G(m_r, m_{r-1}, \ldots, m_1)$ and they differ at d digit positions. Then, $(d-1)!$ different sets S_1 with respect to X, Y can be constructed.*

Suppose $a_d, a_{d-1}, \ldots, a_1$ and $b_d, b_{d-1}, \ldots, b_1$ are two different instances of $s(d), s(d-1), \ldots, s(1)$. Without loss of generality, assume $a_u = b_v$ and $u < v$. The two sets $S_{2,u}$ and $S_{2,v}$ that are derived from the two instances, respectively, are identical if and only if $a_k = b_k$ for $1 \leqslant k \leqslant u-1$ and $v+1 \leqslant k \leqslant d$, and $a_k = b_{k-1}$ for $u+1 \leqslant k \leqslant v$. But, under the latter condition, the two $S_{2,w}$, where $s(w) \neq a_u$ (and b_v), that are derived from the two instances, respectively, are different if $m_w > 2$, or empty if $m_w = 2$. This implies that under the assumption of $m_i > 2$ for all $1 \leqslant i \leqslant r$, different instances of $s(d), s(d-1), \ldots, s(1)$ will construct different S_2. Therefore, we have the following lemma.

Lemma 3. *Suppose X and Y are two nodes in $G(m_r, m_{r-1}, \ldots, m_1)$ and they differ at d digit positions. If $m_i > 2$ for all $1 \leqslant i \leqslant r$, then $d!$ different S_2 with respect to X, Y can be constructed.*

It is easy to see that when $r > d$, each instance of $s(d), s(d-1), \ldots, s(1)$ will uniquely construct a set S_3. Therefore, we have the following lemma.

Lemma 4. *Suppose X and Y are two nodes in $G(m_r, m_{r-1}, \ldots, m_1)$ and they differ at d digit positions. If $r > d$, then $d!$ different S_3 with respect to X, Y can be constructed. If $r = d$, no such set can be constructed.*

Combining Lemmas 1, 2, 3, and 4, we have the following theorem.

Theorem 5. *Suppose X and Y are two nodes in $G(m_r, m_{r-1}, \ldots, m_1)$, where $m_i > 2$ for all $1 \leqslant i \leqslant r$, and they differ at d digit positions. When $r > d$, $(d-1)! * (d!)^2 C^*(X, Y)$s of width $\sum_{i=1\ldots r}(m_i - 1)$ can be constructed. When $r = d$,*

$(d-1)! * d! C^*(X, Y)$s of width $\Sigma_{i=1\dots r}(m_i - 1)$ can be constructed.

The number of $C^*(X, Y)$s in Theorem 5 represents a lower bound. More $C^*(X, Y)$s still exist.

The *node connectivity* of a graph is defined as the minimum number of nodes whose removal will disconnect the graph. We note that no container of width larger than $\Sigma_{i=1\dots r}(m_i - 1)$ exists in $G(m_r, m_{r-1}, \dots, m_1)$ because each node in $G(m_r, m_{r-1}, \dots, m_1)$ has a degree $\Sigma_{i=1\dots r}(m_i - 1)$. According to Menger's theorem [2], which states that the node connectivity of a graph is κ if and only if there exist at least κ node-disjoint paths between any two distinct nodes of the graph, we know that the node connectivity of $G(m_r, m_{r-1}, \dots, m_1)$ is $\Sigma_{i=1\dots r}(m_i - 1)$.

Corollary 6. The node connectivity of $G(m_r, m_{r-1}, \dots, m_1)$ is $\Sigma_{i=1\dots r}(m_i - 1)$.

3. Wide diameter and fault diameter

Let $d_\kappa(X, Y)$ denote the length of $C^*(X, Y)$ of width κ. That is, $d_\kappa(X, Y) = \min\{l(C(X, Y)) \mid$ for all $C(X, Y)$s with $w(C(X, Y)) = \kappa\}$. The *κ-wide diameter* [5] of $G(m_r, m_{r-1}, \dots, m_1)$, which is denoted by $d_\kappa(G(m_r, m_{r-1}, \dots, m_1))$, is defined as the maximum of $d_\kappa(X, Y)$s for all pairs of nodes X, Y in $G(m_r, m_{r-1}, \dots, m_1)$. That is, $d_\kappa(G(m_r, m_{r-1}, \dots, m_1)) = \max\{d_\kappa(X, Y) \mid$ for all pairs of nodes X, Y in $G(m_r, m_{r-1}, \dots, m_1)\}$.

Let $\kappa = \Sigma_{i=1\dots r}(m_i - 1)$ denote the connectivity of $G(m_r, m_{r-1}, \dots, m_1)$. According to the discussion of Section 2, we know that for any two nodes X, Y in $G(m_r, m_{r-1}, \dots, m_1)$, $d_\kappa(X, Y) = d + 2$ if $d < r$, and $d + 1$ if $d = r$. Thus, $d_\kappa(X, Y)$ is maximized as $d = r$ or $d = r - 1$, and $d_\kappa(G(m_r, m_{r-1}, \dots, m_1))$ is computed as follows.

Theorem 7. $d_\kappa(G(m_r, m_{r-1}, m_1)) = r + 1$.

The fault diameter [6] of $G(m_r, m_{r-1}, \dots, m_1)$ is defined as its maximum diameter after at most $\kappa - 1$ nodes and their incident edges are removed. By definition, $d_\kappa(G(m_r, m_{r-1}, \dots, m_1))$ is an upper bound on the fault diameter of $G(m_r, m_{r-1}, \dots, m_1)$. Let $X = x_r x_{r-1} \dots x_1$ and $Y = y_r y_{r-1} \dots y_1$ be two nodes in $G(m_r, m_{r-1}, \dots, m_1)$, where $x_i \neq y_i$ for $1 \leqslant i \leqslant d$ and $x_j = y_j$ for $d + 1 \leqslant j \leqslant r$ such that $d_\kappa(X, Y)$ has maximum value. When $d = r$, if we remove the r nodes: $x_r x_{r-1} \dots x_2 y_1$, $x_r x_{r-1} \dots x_3 y_2 x_1, \dots$, $y_r x_{r-1} \dots x_1$, then the distance between X and Y will become $r + 1$. Similarly, when $d = r - 1$, the distance between X and Y will become $r + 1$ after removing the $r - 1$ nodes: $x_r x_{r-1} \dots x_2 y_1$, $x_r x_{r-1} \dots x_3 y_2 x_1, \dots$, $x_r y_{r-1} x_{r-2} \dots x_1$. This gives a lower bound of $r + 1$ on the fault diameter of $G(m_r, m_{r-1}, \dots, m_1)$. Hence, the following theorem holds.

Theorem 8. *The fault diameter of $G(m_r, m_{r-1}, \dots, m_1)$ is $r + 1$.*

In [8], Tien and Raghavendra showed that the diameter of a faulty n-dimensional hypercube with at most $2n - 3$ faults is bounded above by $n + 2$, provided the source node and the destination node are not isolated. This bound can be further reduced to $n + 1$ or n if extra conditions are satisfied.

4. Remarks

A method to construct ontainers with maximum width for the hypercube graph has been proposed in [7]. The generalized hypercube graph is a generalization of the hypercube graph. In this paper, besides containers with maximum width, we constructed the best containers and computed the wide diameter and the fault diameter for the generalized hypercube graph. It is of both theoretical interest and practical importance to solve the container problem. The existence of containers can increase transmission rate as well as transmission reliability. For a network having a container of width κ between every two nodes, a transmission path between the source node and the destination node can be guaranteed even if $\kappa - 1$ node faults have occurred. The fault diameter, on the other hand, estimates the maximum length of the transmission path between the source node and the destination node under the situation of at most $\kappa - 1$ node faults.

The k-ary n-cube graph [4] is another generalization of the hypercube graph. It differs from the generalized hypercube in two: (1) each dimension has the same width: (2) the nodes belonging to the same dimension are connected as a ring (they are connected as a clique for the generalized hypercube graph). The interested readers are encouraged to determine containers and the fault diameter of the k-ary n-cube graph.

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