# Circuit Preserving Edge Maps 

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It is proved than any one-to-one edge map $f$ from a 3-connected graph $G$ onto a graph $G^{\prime}, G$ anf $G^{\prime}$ possibly infinite, satisfying $f(C)$ is a circuit in $G^{\prime}$ whenever $C$ is a circuit in $G$ is induced by a vertex isomorphism. This generalizes a result of Whitney which hypothesizes $f(C)$ is a circuit in $G^{\prime}$ if and only if $C$ is a circuit in $G$.

## 1. INTRODUCTION

In 1932, Whitney proved [3] that every circuit isomorphism (one-to-one onto edge map $f$ such that $C$ is a circuit if and only if $f(C)$ is a circuit) between two 3 -connected graphs is induced by a vertex isomorphism. The following year Whitney observed [4] that this result could be strengthened by hypothesizing the 3 -connectivity of only one of the graphs. It is necessary to also assume the other graph has no isolated vertices. In 1966, Jung pointed out[1] that Whitney's result also holds for infinite graphs.

In this paper, we further generalize Whitney's result by proving that any circuit injection $f$ (a one-to-one edge map such that if $C$ is a circuit then $f(C)$ is a circuit) from a 3-connected graph $G$ onto a graph $G^{\prime}$ is induced by a vertex isomorphism. Throughout we will understand the terminology that $f$ is a circuit injection from $G$ onto $G^{\prime}$ to preclude the possibility of $G^{\prime}$ having isolated vertices. The term graph refers to undirected graphs, finite or infinite, without loops or multiple edges.

We note that a circuit injection $f: G \rightarrow G^{\prime}$ where $G$ is 2 -connected is not necessarily a vertex or circuit isomorphism no matter what connectivity $n$ is assumed for $G^{\prime}$ as illustrated by the following example. For any prime $p>2$ let $G$ be the graph consisting of $p$ paths $P_{i}, i \in Z_{p}$ (where $Z_{p}$ is the integers modulo $p$ ), each path having the same two endpoints but otherwise mutually disjoint, and each $P_{i}$ consisting of $p$ edges $e_{i \cdot j}, j \in Z_{p}$. Let $G^{\prime}$ be the complete bipartite graph on the vertex sets $\left\{b_{i}: i \in Z_{p}\right\}$ and $\left\{c_{i}: i \in Z_{p}\right\}$; and define the edge map $f: G \rightarrow G^{\prime}$ by $f\left(e_{i \cdot j}\right)=\left(b_{j}, c_{i+j}\right)$ where $i \in Z_{p}, j \in Z_{p}$. Then $G$ is 2 -connected, $G^{\prime}$ is $p$-connected and it can be checked that $f(C)$ is a circuit whenever $C$ is a circuit.

## 2. THEOREMS AND PROOFS

Our principal result is Theorem 6 whose Proof consists of the application of Theorems 1 through 5 .

THEOREM 1 Let $G$ and $G^{\prime}$ be graphs without isolated vertices, $G$ without isolated edges, and $g: G \rightarrow G^{\prime}$ is a one-to-one map of the edges of $G$ onto the edges of $G^{\prime}$ such that for each vertex $v$ of $G$ the star subgraph $S(v)$ is mapped by $g$ onto the star subgrapgh $S\left(v^{\prime}\right)$ for some vertex $v^{\prime}$ of $G^{\prime}$. Then $g$ is induced by a vertex isomorphism $\lambda$.

Proof. For each vertex $v$ of $G$ let $\lambda(v)=v^{\prime}$ be a vertex such that $g(S(v))=$ $S\left(v^{\prime}\right)$. It can be verified that $v^{\prime}$ is then uniquely determined, but this is not necessary. To see that $\lambda$ is one-to-one note that if $\lambda(u)=\lambda(v)$ then $S(\lambda(u))=S(\lambda(v))$, thus $g(S(u))=g(S(v))$, which implies $S(u)=S(v)$, which implies $u=v$, edge $(u, v)$ is isolated, or $u$ and $v$ are isolated vertices. To see that $\lambda$ is onto, given any vertex $w$ of $G^{\prime}$ let $e$ be an edge incident to $w$ and then using the definition of $\lambda$ and that $\lambda$ is one-to-one it is seen that $\lambda$ must map one of the vertices of $g^{-1}(e)$ into $w$. To see that $\lambda$ induces $g$, we observe that there exists an edge $(\lambda(u), \lambda(v))$ in $G^{\prime}$ if and only if $S(\lambda(u)) \cap S(\lambda(v)) \neq \phi$ if and only if $g^{-1}(S(\lambda(u)) \cap S(\lambda(v))) \neq \phi$ if and only if $g^{-1}(S(\lambda(u))) \cap g^{-1}(S(\lambda(v))) \neq \phi$ if and only if $S(u) \cap S(v) \neq \phi$ if and only if there exists an edge $(u, v)$ in $G$.

LEMMA 1 Let $a, b, c$ be three distinct vertices of a 2-connected graph $G$. Then there exists a circuit $C$ containing $a$ and $b$ and a path $P(c, t)$ where $t$ is $a$ vertex on $C$ different from $a$ and $b$ and no other vertex of $P(c, t)$ is on $C$. We allow the possibility $c=t$ and $P(c, t)=\phi$.

Proof. Take any circuit containing $a$ and $b$. if $c$ is on $C$ then we have the case with $P(c, t)=\phi$. If $c$ is not on $C$ choose any vertex $v$ of $C, v \neq a, v \neq b$ and let $C_{1}=P_{1}(c, v) \cup P_{2}(c, v)$ be a circuit through $c$ and $v$. Let $t_{1}$ and $t_{2}$ be the first vertices of $P_{1}(c, v)$ respectively $P_{2}(c, v)$ which lie on $C$. If $\left\{t_{1}, t_{2}\right\}=\{a, b\}$ then $C_{1}$ is a circuit containing $a, b, c$ and again we have the case with $P(c, t)=\phi$. Otherwise at least one of the $t_{i}$ is different from $a$ and $b$ and the corresponding $P_{i}\left(c, t_{i}\right)$ with $C$ are desired path and circuit.

LEMMA 2 Let $f$ be a circuit injection from $G$ onto $G^{\prime}$, G 3-connected, and $S(v)$ a star subgraph of $G$. Then $f(S(v))$ is either a star subgraph of $G^{\prime}$ or an independent (i.e., pairwise nonadjacent) set of edges.

Proof. If $f(S(v))$ is not an independent set of edges then there are two edges $e_{1}=\left(a_{1}, v\right)$ and $a_{2}, v$ of $S(v)$ with $f\left(e_{1}\right)$ and $f\left(e_{2}\right)$ adjacent in $G^{\prime}$ at some vertex $w$. Suppose some other edge $e_{3}=\left(a_{3}, v\right)$ of $S(v)$ does not have its image $f\left(e_{3}\right)$ incident to $w$. Since $G-v$ is 2-connected, by Lemma 1 there is a circuit $C=P_{1}\left(a_{1}, a_{3}\right) \cup P_{2}\left(a_{1}, a_{3}\right)$ and a path $P\left(a_{2}, t\right)$ with no vertex on $C$ except t. $C_{1}=P_{1} \cup\left\{e_{1}, e_{3}\right\}$ is a circuit in $G$ so $f\left(C_{1}\right)=f\left(P_{1}\right) \cup\left\{f\left(e_{1}\right), f\left(e_{3}\right)\right\}$ is a circuit in $G^{\prime}$. By hypothesis $f\left(C_{1}\right)$ passes through $w$ and $f\left(e_{2}\right)$ does not. So some edge $f\left(p_{1}\right)$ of $f\left(P_{1}\right)$ must be incident to $w$. Similarly some edge $f\left(P_{1}\right)$ of $f\left(p_{2}\right)$ must be incident to $w$. We derive a contadiction to $f\left(p_{1}\right), f\left(p_{2}\right), f\left(e_{2}\right)$ each incident to $w$ by finding a circuit in $G$ containing $p_{1}, p_{2}$, and $e_{2}$. Since $t$ lies on $C$, we have $t$ on $P_{1}$ or $P_{2}$. Suppose without loss of generality $t$ lies on $P_{1}$ so that we may write $P_{1}\left(a_{1}, a_{3}\right)=P_{1}\left(a_{1}, t\right) \cup P_{1}\left(t, a_{3}\right)$. If $p_{1}$ is on $P_{2}\left(a_{1}, t\right)$ then the circuit $P_{1}\left(a_{1}, t\right) \cup P\left(a_{2}, t\right) \cup\left\{e_{2}, e_{3}\right\} \cup P_{2}\left(a_{1}, a_{3}\right)$ contains $p_{1}, p_{2}$ and $e_{2}$. If $p_{1}$ is on $P_{1}\left(t, a_{3}\right)$, then the desired circuit is $P_{1}\left(t, a_{3}\right) \cup P\left(a_{2}, t\right) \cup\left\{e_{1}, e_{2}\right\} \cup$ $P_{2}\left(a_{1}, a_{3}\right)$.

Thus we have shown that if $f(S(v))$ is not an independent set of edges $f(S(v))$ is a subset of a star subgraph $S(w)$ of $G^{\prime}$. To finish the proof suppose there were some edge $f\left(e_{4}\right)$ at $w$ with $e_{4} \notin S(v)$. Pick any edge $e$ of $S(v)$ and a circuit $C^{\prime}$ in $G$ contaning $e$ and $e_{4}$. $C^{\prime}$ must contain another edge $e^{\prime}$ of $S(v)$ but then we have the contradiction that $f\left(C^{\prime}\right)$ cannot be a circuit because $f(e) \cdot f\left(e^{\prime}\right)$, and $f\left(e_{4}\right)$ are each incident at $w$.

THEOREM 2 Let $f$ be a circuit injection from $G^{\prime}$ onto $G$ 3-connected, and $S(w)$ a star subgraph of $G$. Then $f^{-1}(S(w))$ is either a star subgraph of $G$ or an independent set of edges.

Proof. If $f^{-1}(S(w))$ is not an independent set of edges, then there exist $e_{1}$ and $e_{2} \in f^{-1}(S(w))$ such that $e_{1}$ and $e_{2}$ have common vertex $v$. By lemma $2, f(S(v))$ is either an independent set or a star subgraph of $G^{\prime}$.The former case is ruled out since $f\left(e_{1}\right)$ and $f\left(e_{2}\right)$ are adjacent at $w$. Thus $f(S(v))=$ $S\left(w^{\prime}\right)$ for some vertex $w^{\prime}$ of $G^{\prime}$. But since $\left\{f\left(e_{1}\right), f\left(e_{2}\right)\right\} \subset S(w) \cap S\left(w^{\prime}\right)$ we have $w=w^{\prime}$. Thus $f(S(v))=S(w)$, hence $f^{-1}(S(w))=S(v)$.

THEOREM 3 Let $f$ be a circuit injection from $G$ onto $G^{\prime}, G$ 2-connected, and $S=S(v)$ a star subgraph of $G^{\prime}$. Then $G=G_{1} \cup G_{2} \cup f^{-1}(S)$, where $G_{1}$
and $G_{2}$ are connected components of $G-f^{-1}(S)$. (with $G-f^{-1}(S)$ ) denoting the subgraph of $G$ containing the same vertices as $G$ but only those edges of $G$ not in $f^{-1}(S)$ and where each edge of $f^{-1}(S)$ has one vertex in $G_{1}$ and one vertex in $G_{2}$.

Proof. Let $G_{\alpha}, \alpha \in I$ be the connected components of $G-f^{-1}(S)$. Each edge $e=(a, b) \in f^{-1}(S)$ cannot have both vertices $a, b$ in the same connected component $G_{\alpha}$, for otherwise there would exist a path $P(a, b) \subset G_{\alpha}$, a circuit $C=\{e\} \cup P(a, b)$ and therefore a circuit $f(C)$ containing only one edge $f(e)$ of $S(v)$, an impossibility. It remains only to show $|I|=2$. From the preceding, $|I|>1$, so assume $|I| \geq 3$. Take any three connected components $G_{1}, G_{2}, G_{3}$ of $G-f^{-1}(S)$. If there were edges $e_{12}=\left(a_{1}, a_{2}\right) \cdot e_{23}=\left(b_{2}, b_{3}\right), e_{31}=\left(c_{3}, c_{1}\right)$ of $f^{-1}(S)$ joining $G_{1}$ to $G_{2}, G_{2}$ to $G_{3}, G_{3}$ to $G_{1}$, respectively, there would be a circuit $C_{1}$ in $G$ consisting of $\left\{e_{12}, e_{23}, e_{31}\right\}$ and paths $P\left(c_{1}, a_{1}\right)$ in $G_{1}, P\left(a_{2}, b_{2}\right)$ in $G_{2}$, and $P\left(b_{3}, c_{3}\right)$ in $G_{3}$. Then we have the contradiction that there is a circuit $f\left(C_{1}\right)$ in $G^{\prime}$ containing three edges $f\left(e_{12}\right), f\left(e_{23}\right)$, and $f\left(e_{31}\right)$ of $S(v)$. So at least two of the components, say $G_{1}$ and $G_{2}$, are not joined by any edge of $f^{-1}(S)$. Choose a vertex $v_{1}$ in $G_{1}$ and a vertex $v_{2}$ in $G_{2}$. Since $G$ is 2connected there is a circuit $C_{2}$ in $G$ containing $v_{1}$ and $v_{2}, C_{2}$ consisting of two paths $P_{1}\left(v_{1}, v_{2}\right)$ and $P_{2}\left(v_{1}, v_{2}\right)$ having only $v_{1}$ and $v_{2}$ in common. Because no edge of $f^{-1}(S)$ joins $G_{1}$ and $G_{2}, P_{1}$ and $P_{2}$ each contain two edges of $f^{-1}(S)$. But then we have the contradiction that $f\left(C_{2}\right)$ contains four or more edges of $S(v)$. This $|I|=2$ and the Proof is complete.

DEFINITION 1 Let $G$ be a graph consisting of two vertex disjoint circuits $A$ and $B$, two edges $e_{1}=\left(a_{1}, b_{1}\right), e_{2}=\left(a_{2}, b_{2}\right)$ and a path $P\left(a_{3}, b_{3}\right)$ vertex disjoint except for $a_{3}$ and $b_{3}$ from $A$ and $B$, where $a_{1}, a_{2}, a_{3}$ are distinct vertices of $A$ and $b_{1}, b_{2}, b_{3}$ are distinct vertices of $B$. Let $e_{3}$ be an arbitrary edge of $P\left(a_{3}, b_{3}\right)$. We say $G$ is a graph of type $X$ with connectors $e_{1}, e_{2}$, and $e_{3}$.

THEOREM 4 Let $G$ be 3-connected and let $A=\left\{e_{1}, e_{2}, \cdots, e_{n}\right\}$ be a set of independent edges of $G$ such that $G-A$ has two connected components $G_{1}$ and $G_{2}$ and each edge of $A$ has one vertex in $G_{1}$ and one vertex in $G_{2}$. Then either $G$ has a subgraph of type $X$ with three connectors from $A$ or there exists a circuit containing at least four distinct edge in $A$.

Proof. We consider two cases.

Case 1. $\quad G_{1}$ and $G_{2}$ are both 2-connected. By the 3-connectivity of $G$ there must be at least three edges in $A, e_{1}=\left(a_{1}, b_{1}\right), e_{2}=\left(a_{2}, b_{2}\right)$, and $e_{3}=\left(a_{3}, b_{3}\right)$ with the $a^{\prime} s$ distinct and in $G_{1}$, the $b^{\prime} s$ distinct and in $G_{2}$. By Lemma 1 there exist a circuit $C_{1}$ containing $a_{1}$ and $a_{2}$ and a path $P_{1}\left(a_{3}, t\right)$ having no vertex in common with $C_{1}$ except $t$ which is different from $a_{1}, a_{2}$. Similarly, there is a circuit $C_{2}$ containing $b_{1}$ and $b_{2}$ and a path $P_{2}\left(b_{3}, t^{\prime}\right)$ vertex disjoint from $C_{2}$ except for $t^{\prime} \neq b_{1}, b_{2}$. Then $C_{1}, C_{2},\left\{e_{1}, e_{2}\right\}$, and $P_{1}\left(a_{3}, t\right) \cup\left\{e_{3}\right\} \cup P_{2}\left(b_{3}, t^{\prime}\right)$ constitute a subgraph of type $X$ with connectors $e_{1}, e_{2}$ and $e_{3}$.

Case 2. $\quad G_{1}$ and $G_{2}$ are not both 2-connected. Then at least one of $G_{1}$ and $G_{2}$, say $G_{1}$ has a cutpoint $v$. Choose vertices $a$ and $b$ in different components of $G_{1}-v$. By the 3-connectivity of $G$ there are two paths $P_{1}(a, b)$ and $P_{2}(a, b)$ in $G-v$ having only $a$ and $b$ in common, and each of these paths must have at least two edges of $A$. This gives a circuit containing at least four distinct edges of $A$.

THEOREM 5 If $G$ is a graph of type $X$ with connectors $e_{1}, e_{2}, e_{3} \in P\left(a_{3}, b_{3}\right)$ and $f$ is a circuit injection from $G$ onto $G^{\prime}$, then $f\left(e_{1}\right)$ and $f\left(e_{2}\right)$ do not have a common vertex.

Proof. For any edge, path, or circuit $P$ of $G$ let $P^{\prime}=f(P)$. Suppose $f\left(e_{1}\right)$ and $f\left(e_{2}\right)$ have a common vertex so we may write $e_{1}^{\prime}=\left(v_{1}, v_{0}\right)$ and $e_{2}^{\prime}=\left(v_{2}, v_{0}\right)$. In the notation of Definition 1 we may also write $A=$ $P\left(a_{1}, a_{2}\right) \cup P\left(a_{2}, a_{3}\right) \cup P\left(a_{3}, a_{1}\right)$ and $B=P\left(b_{1}, b_{2}\right) \cup P\left(b_{2}, b_{3}\right) \cup P\left(b_{3}, b_{1}\right)$. Since $\left\{e_{1}, e_{2}\right\} \cup P\left(a_{1}, a_{2}\right) \cup P\left(b_{1}, b_{2}\right)$ is a circuit in $G,\left\{e_{1}^{\prime}, e_{2}^{\prime}\right\} \cup P^{\prime}\left(a_{1}, a_{2}\right) \cup P^{\prime}\left(b_{1}, b_{2}\right)$ is a circuit in $G^{\prime}$. Thus the edges of $P^{\prime}\left(a_{1}, a_{2}\right) \cup P^{\prime}\left(b_{1}, b_{2}\right)$ form a path $P\left(v_{1}, v_{2}\right)$. Let $v \neq v_{1}, v_{2}$ be a vertex in $G^{\prime}$ where an edge $e_{0}^{\prime}$ of $P^{\prime}\left(a_{1}, a_{2}\right)$ and an edge of $P^{\prime}\left(b_{1}, b_{2}\right)$ meet. $A^{\prime}$ is a circuit containing $P^{\prime}\left(a_{1}, a_{2}\right)$ and disjoint from $P^{\prime}\left(b_{1}, b_{2}\right)$. Let $e^{\prime}$ be an edge of $A^{\prime}$ at $v, e^{\prime} \neq e_{0}^{\prime}$. We have $e^{\prime} \notin P^{\prime}\left(a_{1}, a_{2}\right)$ since otherwise there would be two edges of $P^{\prime}\left(a_{1}, a_{2}\right)$ and an edge of $P^{\prime}\left(b_{1}, b_{2}\right)$ incident at $v$ contradicting $P^{\prime}\left(a_{1}, a_{2}\right) \cup P^{\prime}\left(b_{1}, b_{2}\right)$ is a path. Also, $e^{\prime} \notin P^{\prime}\left(a_{2}, a_{3}\right)$ since otherwise $v$ is a vertex of degree at least 3 in the subgraph $P^{\prime}\left(a_{1}, a_{2}\right) \cup P^{\prime}\left(a_{2}, a_{3}\right) \cup P^{\prime}\left(b_{1}, b_{2}\right)$ which is contained in the circuit $P^{\prime}\left(a_{1}, a_{2}\right) \cup P^{\prime}\left(a_{2}, a_{3}\right) \cup P^{\prime}\left(a_{3}, b_{3}\right) \cup P^{\prime}\left(b_{2}, b_{3}\right) \cup P^{\prime}\left(b_{1}, b_{2}\right) \cup\left\{e_{1}\right\}$. Similarly, $e^{1} \notin P^{\prime}\left(a_{3}, a_{1}\right)$ since otherwise $v$ has degree at least 3 in the subgraph $P^{\prime}\left(a_{1}, a_{2}\right) \cup P^{\prime}\left(b_{1}, b_{2}\right) \cup P^{\prime}\left(a_{3}, a_{1}\right)$ which is contained in the circuit $P^{\prime}\left(a_{1}, a_{2}\right) \cup P^{\prime}\left(a_{3}, a_{1}\right) \cup P^{\prime}\left(a_{3}, b_{3}\right) \cup P^{\prime}\left(b_{3}, b_{1}\right) \cup P^{\prime}\left(b_{1}, b_{2}\right) \cup\left\{e_{1}\right\}$. Thus we have a condition to $e^{\prime} \in A^{\prime}=P^{\prime}\left(a_{1}, a_{2}\right) \cup P^{\prime}\left(a_{2}, a_{3}\right) \cup P^{\prime}\left(a_{3}, a_{1}\right)$ and the Proof is complete.

THEOREM 6 If $f$ is a circuit injection from $G$ onto $G^{\prime}$ where $G$ is 3connected, then $f$ is induced by a vertex isomorphism.

Proof. We prove $f$ is induced by a vertex isomorphism by applying Theorem 1 to $f^{-1}$ to show it is induced by a vertex isomorphism. Note Theorem 1 can apply to $f^{-1}$ since $G^{\prime}$ has no isolated veertices by the assumption that $f$ is onto, and no isolated edges by the fact that any two edges $e_{1}$ and $e_{2}$ of $G^{\prime}$ must lie on some circuit $f(C)$ where $C$ is a circuit containing $f^{-1}\left(e_{1}\right)$ and $f^{-1}\left(e_{2}\right)$. To complete the Proof we must show for any star subgraph $S(v)$ of $G^{\prime}$ that $f^{-1}(S(v))$ is also a star subgraph. Theorems 2 and 3 tell us the only other possibility for $f^{-1}(S(v))$ is that it is a set of independent edges of $G$ such that $G-f^{-1}(S(v))$ consists of two connected components $G_{1}$ and $G_{2}$ with each edge of $f^{-1}(S(v))$ having one vertex in $G_{1}$ and one vertex in $G_{2}$. But in this event Theorem 4 asserts that either three edges of $f^{-1}(S(v))$ are connectors in a subgraph of $G$ of type $X$ or atleast four edges of $f^{-1}(S(v))$ lie on some circuit $C^{\prime}$ in $G$. The first situation is ruled out by Theorem 5 . The second case is also impossible since it implies $\left|f\left(C^{\prime}\right) \cap S(v)\right| \geq 4$ and the theorem is proved.

## 3. GENERALIZATIONS

Possible generalization of Theorem 6 could be attempted by dropping the hypothesis that $f$ is one-to-one. An interesting result of dropping this hypothesis is that the theorem remains true for finite 3-connected graphs, but not for infinite graphs of arbitrarily large connectivity.

Further generalization could follow the route of assuming $G^{\prime}$ is not necessarily a graph but a (binary) matroid. Using Tutte's definition of 3-connected for matroids [2], $G$ could also be assumed to be a matriod. The existence of these generalizations will be explored in a following paper.

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