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Circuit Preserving Edge Maps

## JON HENRY SANDERS

jon\_sanders@partech.com

AND

# DAVID SANDERS

davidhs and ers@earthlink.net

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It is proved than any one-to-one edge map f from a 3-connected graph G onto a graph G',G and G' possibly infinite, satisfying f(C) is a circuit in G' whenever C is a circuit in G is induced by a vertex isomorphism. This generalizes a result of Whitney which hypothesizes f(C) is a circuit in G' if and only if C is a circuit in G.

#### 1. INTRODUCTION

In 1932, Whitney proved [3] that every circuit isomorphism (one-to-one onto edge map f such that C is a circuit if and only if f(C) is a circuit) between two 3-connected graphs is induced by a vertex isomorphism. The following year Whitney observed [4] that this result could be strengthened by hypothesizing the 3-connectivity of only one of the graphs. It is necessary to also assume the other graph has no isolated vertices. In 1966, Jung pointed out[1] that Whitney's result also holds for infinite graphs.

In this paper, we further generalize Whitney's result by proving that any circuit injection f (a one-to-one edge map such that if C is a circuit then f(C) is a circuit) from a 3-connected graph G onto a graph G' is induced by a vertex isomorphism. Throughout we will understand the terminology that f is a circuit injection from G onto G' to preclude the possibility of G' having isolated vertices. The term graph refers to undirected graphs, finite or infinite, without loops or multiple edges.

We note that a circuit injection  $f: G \to G'$  where G is 2-connected is not necessarily a vertex or circuit isomorphism no matter what connectivity n is assumed for G' as illustrated by the following example. For any prime p > 2let G be the graph consisting of p paths  $P_i, i \in Z_p$  (where  $Z_p$  is the integers modulo p), each path having the same two endpoints but otherwise mutually disjoint, and each  $P_i$  consisting of p edges  $e_{i\cdot j}, j \in Z_p$ . Let G' be the complete bipartite graph on the vertex sets  $\{b_i : i \in Z_p\}$  and  $\{c_i : i \in Z_p\}$ ; and define the edge map  $f: G \to G'$  by  $f(e_{i\cdot j}) = (b_j, c_{i+j})$  where  $i \in Z_p, j \in Z_p$ . Then G is 2-connected, G' is p-connected and it can be checked that f(C) is a circuit whenever C is a circuit.

#### 2. THEOREMS AND PROOFS

Our principal result is Theorem 6 whose Proof consists of the application of Theorems 1 through 5.

**THEOREM 1** Let G and G' be graphs without isolated vertices, G without isolated edges, and  $g: G \to G'$  is a one-to-one map of the edges of G onto the edges of G' such that for each vertex v of G the star subgraph S(v) is mapped by g onto the star subgraph S(v') for some vertex v' of G'. Then g is induced by a vertex isomorphism  $\lambda$ .

Proof. For each vertex v of G let  $\lambda(v) = v'$  be a vertex such that g(S(v)) = S(v'). It can be verified that v' is then uniquely determined, but this is not necessary. To see that  $\lambda$  is one-to-one note that if  $\lambda(u) = \lambda(v)$  then  $S(\lambda(u)) = S(\lambda(v))$ , thus g(S(u)) = g(S(v)), which implies S(u) = S(v), which implies u = v, edge(u, v) is isolated, or u and v are isolated vertices. To see that  $\lambda$  is onto, given any vertex w of G' let e be an edge incident to w and then using the definition of  $\lambda$  and that  $\lambda$  is one-to-one it is seen that  $\lambda$  must map one of the vertices of  $g^{-1}(e)$  into w. To see that  $\lambda$  induces g, we observe that there exists an edge  $(\lambda(u), \lambda(v))$  in G' if and only if  $S(\lambda(u)) \cap S(\lambda(v)) \neq \phi$  if and only if  $g^{-1}(S(\lambda(u))) \cap g^{-1}(S(\lambda(v))) \neq \phi$  if and only if  $S(u) \cap S(v) \neq \phi$  if and only if there exists an edge (u, v) in G.

**LEMMA 1** Let a, b, c be three distinct vertices of a 2-connected graph G. Then there exists a circuit C containing a and b and a path P(c,t) where t is a vertex on C different from a and b and no other vertex of P(c,t) is on C. We allow the possibility c = t and  $P(c,t) = \phi$ .

*Proof.* Take any circuit containing a and b. if c is on C then we have the case with  $P(c,t) = \phi$ . If c is not on C choose any vertex v of  $C, v \neq a, v \neq b$  and let  $C_1 = P_1(c,v) \cup P_2(c,v)$  be a circuit through c and v. Let  $t_1$  and  $t_2$  be the first vertices of  $P_1(c,v)$  respectively  $P_2(c,v)$  which lie on C. If  $\{t_1, t_2\} = \{a, b\}$  then  $C_1$  is a circuit containing a, b, c and again we have the case with  $P(c,t) = \phi$ . Otherwise at least one of the  $t_i$  is different from a and b and the corresponding  $P_i(c, t_i)$  with C are desired path and circuit.

**LEMMA 2** Let f be a circuit injection from G onto G', G 3-connected, and S(v) a star subgraph of G. Then f(S(v)) is either a star subgraph of G' or an independent (i.e., pairwise nonadjacent) set of edges.

Proof. If f(S(v)) is not an independent set of edges then there are two edges  $e_1 = (a_1, v)$  and  $a_2, v$  of S(v) with  $f(e_1)$  and  $f(e_2)$  adjacent in G' at some vertex w. Suppose some other edge  $e_3 = (a_3, v)$  of S(v) does not have its image  $f(e_3)$  incident to w. Since G - v is 2-connected, by Lemma 1 there is a circuit  $C = P_1(a_1, a_3) \cup P_2(a_1, a_3)$  and a path  $P(a_2, t)$  with no vertex on Cexcept t.  $C_1 = P_1 \cup \{e_1, e_3\}$  is a circuit in G so  $f(C_1) = f(P_1) \cup \{f(e_1), f(e_3)\}$ is a circuit in G'. By hypothesis  $f(C_1)$  passes through w and  $f(e_2)$  does not. So some edge  $f(p_1)$  of  $f(P_1)$  must be incident to w. Similarly some edge  $f(P_1)$ of  $f(p_2)$  must be incident to w. We derive a contadiction to  $f(p_1), f(p_2), f(e_2)$ each incident to w by finding a circuit in G containing  $p_1, p_2$ , and  $e_2$ . Since tlies on C, we have t on  $P_1$  or  $P_2$ . Suppose without loss of generality t lies on  $P_1$  so that we may write  $P_1(a_1, a_3) = P_1(a_1, t) \cup P_1(t, a_3)$ . If  $p_1$  is on  $P_2(a_1, t)$ then the circuit  $P_1(a_1, t) \cup P(a_2, t) \cup \{e_2, e_3\} \cup P_2(a_1, a_3)$  contains  $p_1, p_2$  and  $e_2$ . If  $p_1$  is on  $P_1(t, a_3)$ , then the desired circuit is  $P_1(t, a_3) \cup P(a_2, t) \cup \{e_1, e_2\} \cup P_2(a_1, a_3)$ .

Thus we have shown that if f(S(v)) is not an independent set of edges f(S(v)) is a subset of a star subgraph S(w) of G'. To finish the proof suppose there were some edge  $f(e_4)$  at w with  $e_4 \notin S(v)$ . Pick any edge e of S(v) and a circuit C' in G containing e and  $e_4$ . C' must contain another edge e' of S(v) but then we have the contradiction that f(C') cannot be a circuit because  $f(e) \cdot f(e')$ , and  $f(e_4)$  are each incident at w.

**THEOREM 2** Let f be a circuit injection from G' onto G 3-connected, and S(w) a star subgraph of  $G_i$ . Then  $f^{-1}(S(w))$  is either a star subgraph of G or an independent set of edges.

*Proof.* If  $f^{-1}(S(w))$  is not an independent set of edges, then there exist  $e_1$  and  $e_2 \in f^{-1}(S(w))$  such that  $e_1$  and  $e_2$  have common vertex v. By lemma 2, f(S(v)) is either an independent set or a star subgraph of G'. The former case is ruled out since  $f(e_1)$  and  $f(e_2)$  are adjacent at w. Thus f(S(v)) = S(w') for some vertex w' of G'. But since  $\{f(e_1), f(e_2)\} \subset S(w) \cap S(w')$  we have w = w'. Thus f(S(v)) = S(w), hence  $f^{-1}(S(w)) = S(v)$ .

**THEOREM 3** Let f be a circuit injection from G onto G', G 2-connected, and S = S(v) a star subgraph of G'. Then  $G = G_1 \cup G_2 \cup f^{-1}(S)$ , where  $G_1$  and  $G_2$  are connected components of  $G - f^{-1}(S)$ . (with  $G - f^{-1}(S)$ ) denoting the subgraph of G containing the same vertices as G but only those edges of G not in  $f^{-1}(S)$  and where each edge of  $f^{-1}(S)$  has one vertex in  $G_1$  and one vertex in  $G_2$ .

Let  $G_{\alpha}, \alpha \in I$  be the connected components of  $G - f^{-1}(S)$ . Each Proof. edge  $e = (a, b) \in f^{-1}(S)$  cannot have both vertices a, b in the same connected component  $G_{\alpha}$ , for otherwise there would exist a path  $P(a, b) \subset G_{\alpha}$ , a circuit  $C = \{e\} \cup P(a, b)$  and therefore a circuit f(C) containing only one edge f(e)of S(v), an impossibility. It remains only to show |I| = 2. From the preceding, |I| > 1, so assume  $|I| \ge 3$ . Take any three connected components  $G_1, G_2, G_3$ of  $G - f^{-1}(S)$ . If there were edges  $e_{12} = (a_1, a_2) \cdot e_{23} = (b_2, b_3), e_{31} = (c_3, c_1)$  of  $f^{-1}(S)$  joining  $G_1$  to  $G_2$ ,  $G_2$  to  $G_3$ ,  $G_3$  to  $G_1$ , respectively, there would be a circuit  $C_1$  in G consisting of  $\{e_{12}, e_{23}, e_{31}\}$  and paths  $P(c_1, a_1)$  in  $G_1, P(a_2, b_2)$ in  $G_2$ , and  $P(b_3, c_3)$  in  $G_3$ . Then we have the contradiction that there is a circuit  $f(C_1)$  in G' containing three edges  $f(e_{12}), f(e_{23})$ , and  $f(e_{31})$  of S(v). So at least two of the components, say  $G_1$  and  $G_2$ , are not joined by any edge of  $f^{-1}(S)$ . Choose a vertex  $v_1$  in  $G_1$  and a vertex  $v_2$  in  $G_2$ . Since G is 2connected there is a circuit  $C_2$  in G containing  $v_1$  and  $v_2$ ,  $C_2$  consisting of two paths  $P_1(v_1, v_2)$  and  $P_2(v_1, v_2)$  having only  $v_1$  and  $v_2$  in common. Because no edge of  $f^{-1}(S)$  joins  $G_1$  and  $G_2$ ,  $P_1$  and  $P_2$  each contain two edges of  $f^{-1}(S)$ . But then we have the contradiction that  $f(C_2)$  contains four or more edges of S(v). This |I| = 2 and the Proof is complete.

**DEFINITION 1** Let G be a graph consisting of two vertex disjoint circuits A and B, two edges  $e_1 = (a_1, b_1), e_2 = (a_2, b_2)$  and a path  $P(a_3, b_3)$  vertex disjoint except for  $a_3$  and  $b_3$  from A and B, where  $a_1, a_2, a_3$  are distinct vertices of A and  $b_1, b_2, b_3$  are distinct vertices of B. Let  $e_3$  be an arbitrary edge of  $P(a_3, b_3)$ . We say G is a graph of type X with connectors  $e_1, e_2$ , and  $e_3$ .

**THEOREM 4** Let G be 3-connected and let  $A = \{e_1, e_2, \dots, e_n\}$  be a set of independent edges of G such that G - A has two connected components  $G_1$  and  $G_2$  and each edge of A has one vertex in  $G_1$  and one vertex in  $G_2$ . Then either G has a subgraph of type X with three connectors from A or there exists a circuit containing at least four distinct edge in A.

*Proof.* We consider two cases.

Case 1.  $G_1$  and  $G_2$  are both 2-connected. By the 3-connectivity of G there must be at least three edges in A,  $e_1 = (a_1, b_1)$ ,  $e_2 = (a_2, b_2)$ , and  $e_3 = (a_3, b_3)$ with the a's distinct and in  $G_1$ , the b's distinct and in  $G_2$ . By Lemma 1 there exist a circuit  $C_1$  containing  $a_1$  and  $a_2$  and a path  $P_1(a_3, t)$  having no vertex in common with  $C_1$  except t which is different from  $a_1, a_2$ . Similarly, there is a circuit  $C_2$  containing  $b_1$  and  $b_2$  and a path  $P_2(b_3, t')$  vertex disjoint from  $C_2$  except for  $t' \neq b_1, b_2$ . Then  $C_1, C_2, \{e_1, e_2\}$ , and  $P_1(a_3, t) \cup \{e_3\} \cup P_2(b_3, t')$ constitute a subgraph of type X with connectors  $e_1, e_2$  and  $e_3$ .

Case 2.  $G_1$  and  $G_2$  are not both 2-connected. Then at least one of  $G_1$  and  $G_2$ , say  $G_1$  has a cutpoint v. Choose vertices a and b in different components of  $G_1 - v$ . By the 3-connectivity of G there are two paths  $P_1(a, b)$  and  $P_2(a, b)$  in G- v having only a and b in common, and each of these paths must have at least two edges of A. This gives a circuit containing at least four distinct edges of A.

**THEOREM 5** If G is a graph of type X with connectors  $e_1, e_2, e_3 \in P(a_3, b_3)$ and f is a circuit injection from G onto G', then  $f(e_1)$  and  $f(e_2)$  do not have a common vertex.

*Proof.* For any edge, path, or circuit P of G let P' = f(P). Suppose  $f(e_1)$  and  $f(e_2)$  have a common vertex so we may write  $e'_1 = (v_1, v_0)$  and  $e'_2 = (v_2, v_0)$ . In the notation of Definition 1 we may also write A = $P(a_1, a_2) \cup P(a_2, a_3) \cup P(a_3, a_1)$  and  $B = P(b_1, b_2) \cup P(b_2, b_3) \cup P(b_3, b_1)$ . Since  $\{e_1, e_2\} \cup P(a_1, a_2) \cup P(b_1, b_2)$  is a circuit in  $G, \{e'_1, e'_2\} \cup P'(a_1, a_2) \cup P'(b_1, b_2)$ is a circuit in G'. Thus the edges of  $P'(a_1, a_2) \cup P'(b_1, b_2)$  form a path  $P(v_1, v_2)$ . Let  $v \neq v_1, v_2$  be a vertex in G' where an edge  $e'_0$  of  $P'(a_1, a_2)$ and an edge of  $P'(b_1, b_2)$  meet. A' is a circuit containing  $P'(a_1, a_2)$  and disjoint from  $P'(b_1, b_2)$ . Let e' be an edge of A' at  $v, e' \neq e'_0$ . We have  $e' \notin P'(a_1, a_2)$  since otherwise there would be two edges of  $P'(a_1, a_2)$  and an edge of  $P'(b_1, b_2)$  incident at v contradicting  $P'(a_1, a_2) \cup P'(b_1, b_2)$  is a path. Also,  $e' \notin P'(a_2, a_3)$  since otherwise v is a vertex of degree at least 3 in the subgraph  $P'(a_1, a_2) \cup P'(a_2, a_3) \cup P'(b_1, b_2)$  which is contained in the circuit  $P'(a_1, a_2) \cup P'(a_2, a_3) \cup P'(a_3, b_3) \cup P'(b_2, b_3) \cup P'(b_1, b_2) \cup \{e_1\}.$ Similarly,  $e^1 \notin P'(a_3, a_1)$  since otherwise v has degree at least 3 in the subgraph  $P'(a_1, a_2) \cup P'(b_1, b_2) \cup P'(a_3, a_1)$  which is contained in the circuit  $P'(a_1, a_2) \cup P'(a_3, a_1) \cup P'(a_3, b_3) \cup P'(b_3, b_1) \cup P'(b_1, b_2) \cup \{e_1\}$ . Thus we have a condition to  $e' \in A' = P'(a_1, a_2) \cup P'(a_2, a_3) \cup P'(a_3, a_1)$  and the Proof is complete.

**THEOREM 6** If f is a circuit injection from G onto G' where G is 3-connected, then f is induced by a vertex isomorphism.

We prove f is induced by a vertex isomorphism by applying Theo-Proof. rem 1 to  $f^{-1}$  to show it is induced by a vertex isomorphism. Note Theorem 1 can apply to  $f^{-1}$  since G' has no isolated veertices by the assumption that f is onto, and no isolated edges by the fact that any two edges  $e_1$  and  $e_2$  of G' must lie on some circuit f(C) where C is a circuit containing  $f^{-1}(e_1)$  and  $f^{-1}(e_2)$ . To complete the Proof we must show for any star subgraph S(v) of G' that  $f^{-1}(S(v))$  is also a star subgraph. Theorems 2 and 3 tell us the only other possibility for  $f^{-1}(S(v))$  is that it is a set of independent edges of G such that  $G - f^{-1}(S(v))$  consists of two connected components  $G_1$  and  $G_2$ with each edge of  $f^{-1}(S(v))$  having one vertex in  $G_1$  and one vertex in  $G_2$ . But in this event Theorem 4 asserts that either three edges of  $f^{-1}(S(v))$  are connectors in a subgraph of G of type X or at least four edges of  $f^{-1}(S(v))$ lie on some circuit C' in G. The first situation is ruled out by Theorem 5. The second case is also impossible since it implies  $|f(C') \cap S(v)| \ge 4$  and the theorem is proved.

#### **3. GENERALIZATIONS**

Possible generalization of Theorem 6 could be attempted by dropping the hypothesis that f is one-to-one. An interesting result of dropping this hypothesis is that the theorem remains true for finite 3-connected graphs, but not for infinite graphs of arbitrarily large connectivity.

Further generalization could follow the route of assuming G' is not necessarily a graph but a (binary) matroid. Using Tutte's definition of 3-connected for matroids [2], G could also be assumed to be a matriod. The existence of these generalizations will be explored in a following paper.

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