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# Note

## Short Proofs on the Matching Polyhedron

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A short proof of Edmonds' matching polyhedron theorem and the total dual integrality of the associated system of linear inequalities, proved first by W. H. Cunningham and A. B. Marsh (*Math. Programming Stud.* 8 (1978), 50–72), is given.

### 1. THE MATCHING POLYHEDRON

Let $G=(V,E)$ be an undirected graph, with $|V|$ even, and let $P$ be the associated *perfect matching polytope*, i.e., $P$ is the convex hull of the incidence vectors (in $\{0,1\}^E$) of perfect matchings in $G$. In this paper we give a short proof of Edmonds' matching polyhedron theorem [3], which states that $P$ is equal to the set of vectors $x$ in $\mathbb{R}^E$ satisfying

$$\begin{array}{lll} \text{(i)} & x(e) \geqslant 0 & (e \in E), \\ \text{(ii)} & x(\delta(v)) = 1 & (v \in V), \\ \text{(iii)} & x(\delta(V')) \geqslant 1 & (V' \subseteq V, |V'| \text{ odd}). \end{array} \tag{1}$$

(Here $\delta(V')$ is the set of edges of $G$ intersecting $V'$ in exactly one point, $\delta(v) := \delta(\{v\})$, and $x(E') := \sum_{e \in E'} x(e)$ for $E' \subseteq E$.)

Let $P'$ be the set of vectors in $\mathbb{R}^E$ satisfying (1). As the incidence vector of any perfect matching satisfies (1), it follows that $P \subseteq P'$—the content of Edmonds' theorem is the converse inclusion; equivalently, that the polytope defined by (1) has integer vertices only. (For other proofs, see Lovász [5] and Seymour [8]. For applications, see Naddef and Pulleyblank [6].)

EDMONDS' MATCHING POLYHEDRON THEOREM. *The perfect matching polytope is determined by the inequalities* (1).



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*Proof.* Let $G$ be a smallest graph with $P' \not\subseteq P$ (that is, $|V| + |E|$ is minimal), and let $x$ be a vertex of $P'$ not contained in $P$. Then $0 < x(e) < 1$ for all $e$ in $E$—otherwise, we could delete $e$ from $G$ to obtain a smaller counterexample. Moreover, $|E| > |V|$—otherwise, either $G$ is disconnected (in which case one of the components of $G$ will be a smaller counterexample), or $G$ has a point $v$ of degree one (in which case the edge $e$ incident with $v$ has $x(e) = 1$), or $G$ is an even circuit (for which the theorem trivially holds).

Since $x$ is a vertex, there are $|E|$ independent constraints among (1) satisfied by $x$ with equality, and hence there is a $V' \subseteq V$ with $|V'|$ odd, $|V'| \geqslant 3$, $|V \backslash V'| \geqslant 3$, and $x(\delta(V')) = 1$. Let $G_1$ and $G_2$ arise from $G$ by contracting $V'$ and $V \backslash V'$, respectively, and let $x_1$ and $x_2$ be the corresponding projections of $x$ onto the edge sets of $G_1$ and $G_2$, respectively. Since $x_1$ and $x_2$ satisfy inequalities (1) for the smaller graphs $G_1$ and $G_2$, respectively, it follows that $x_1$ and $x_2$ can be decomposed as convex combinations of perfect matchings in $G_1$ and $G_2$, respectively. These decompositions can be easily glued together to form a decomposition of $x$ as a convex combination of perfect matchings, contradicting our assumption. (This glueing can be done, e.g., as follows: By the rationality of $x$ (as it is a vertex of $P'$), there exists a natural number $K$ such that, for $i = 1, 2$, $Kx_i$ is the sum of the incidence vectors of the perfect matchings $F_1^i, \ldots, F_K^i$ of $G_i$ (possibly with repetitions). Since, for each $e$ in $\delta(V')$, $e$ is contained in $Kx(e)$ of the $F_j^1$ as well as in $Kx(e)$ of the $F_j^2$, we may assume that $F_j^1 \cap F_j^2 \neq \varnothing$, for $j = 1, \ldots, K$. It follows that $Kx$ is the sum of the incidence vectors of the perfect matchings $F_1^1 \cup F_1^2, \ldots, F_K^1 \cup F_K^2$ of $G$, and hence that $x$ itself is a convex combination of perfect matchings in $G$.) ∎

By a standard construction we now derive Edmonds' characterization of the *matching polytope*, i.e., of the convex hull of (not-necessarily perfect) matchings. Again, $G = (V, E)$ is an undirected graph, but now $|V|$ may be odd. Edmonds showed that the matching polytope is determined by the inequalities

$$\begin{array}{llll} \text{(i)} & x(e) \geqslant 0 & (e \in E), & (2) \\ \text{(ii)} & x(\delta(v)) \leqslant 1 & (v \in V), & \\ \text{(iii)} & x(\langle V' \rangle) \leqslant \tfrac{1}{2}(|V'| - 1) & (V' \subseteq V, |V'| \text{ odd}). & \end{array}$$

(Here $\langle V' \rangle$ denotes the set of edges contained in $V'$.) Again it is clear that each vector in the matching polytope satisfies (2), as the incidence vector of each matching satisfies (2).

COROLLARY. *The matching polytope is determined by* (2).

*Proof.* Let $x \in \mathbb{R}^E$ satisfy (2). Let $G = (V^*, E^*)$ be a disjoint copy of $G$, where the copy of vertex $v$ will be denoted by $v^*$, and the copy of edge $e$

($= \{v, w\}$) will be denoted by $e^*$ ($= \{v^*, w^*\}$). Let $\tilde{G}$ be the graph with vertex set $V \cup V^*$ and with edge set $E \cup E^* \cup \{\{v, v^*\} \mid v \in V\}$. Define $\tilde{x}(e) = \tilde{x}(e^*) = x(e)$ for $e$ in $E$, and $\tilde{x}(\{v, v^*\}) := 1 - x(\delta(v))$, for $v$ in $V$. Now condition (1) is easily derived for $\tilde{x}$ with respect to $\tilde{G}$. (i) and (ii) are trivial. To prove (iii) we have to show, for $V_1, V_2 \subseteq V$ with $|V_1| + |V_2|$ odd, that $\tilde{x}(\delta(V_1 \cup V_2^*)) \geqslant 1$. Indeed, we may assume, without loss of generality, that $|V_1 \backslash V_2|$ is odd. Hence

$$\begin{aligned}\tilde{x}(\delta(V_1 \cup V_2^*)) &= \tilde{x}(\delta(V_1 \backslash V_2)) + \tilde{x}(\delta(V_2^* \backslash V_1^*)) \geqslant \tilde{x}(\delta(V_1 \backslash V_2)) \\ &= |V_1 \backslash V_2| - 2x(\langle V_1 \backslash V_2 \rangle) \geqslant 1,\end{aligned}$$

by (2)(iii).

Hence $\tilde{x}$ is a convex combination of perfect matchings of $\tilde{G}$. By restriction to $x$ and $G$ it follows that $x$ is a convex combination of matchings in $G$. ∎

## 2. Dual Integrality

The preceding corollary is equivalent to: the polytope defined by (2) has integer vertices only. In other words, for each "weight" function $w \in \mathbb{R}^E$, the linear program

$$\begin{aligned} &\max w^{\mathsf{T}} x, \\ &\text{subject to (2)} \end{aligned} \tag{3}$$

has an integer optimal solution. The dual program is

$$\begin{aligned} &\min \sum_{v \in V} y(v) + \sum_{V' \in \mathcal{O}} z(V') \tfrac{1}{2}(|V'| - 1) \\ &\text{subject to} \\ &\qquad y(v) \geqslant 0 \qquad (v \in V), \\ &\qquad z(V') \geqslant 0 \qquad (V' \in \mathcal{O}), \\ &\sum_{v \in e} y(v) + \sum_{\substack{V' \in \mathcal{O} \\ e \subset V'}} z(V') \geqslant w(e) \qquad (e \in E), \end{aligned} \tag{4}$$

where $\mathcal{O}$ denotes the collection of all subsets of $V$ of odd size. Cunningham and Marsh [2] (cf. Schrijver and Seymour [7]) showed that if $w$ is integral, this dual program also has an integer optimal solution, that is, the system of inequalities (2) is *totally dual integral* (cf. Edmonds and Giles [4]). Note that if we take $w \equiv 1$, this implies the Tutte–Berge theorem (Tutte [9], Berge

[1]): the maximum size of a matching in $G$ is equal to the minimum value of $|V| - \frac{1}{2}(|V'| + \sigma(V'))$, where $V'$ ranges over the subsets of $V$, and where $\sigma(V')$ denotes the number of odd-sized components of the subgraph induced by $V'$.

THEOREM. *If $w$ is integral, then problem* (4) *has an integer optimal solution.*

*Proof.* We may assume that $w$ is nonnegative. Suppose that $G$ and $w \in \mathbb{Z}_+^E$ form a smallest counterexample, i.e., that (4) has no integer optimal solution and that $|V| + |E| + \sum_{e \in E} w(e)$ is as small as possible. Then $w(e) \geqslant 1$ for all $e$ in $E$, otherwise, we could delete $e$. Let $\mathscr{F}$ be the collection of those matchings in $G$ whose incidence vector achieves the maximum (3). Then for each vertex $v$ there is a matching $F$ in $\mathscr{F}$ not covering $v$. Otherwise, we could decrease the weights of the edges incident with $v$ by one, thus decreasing the maximum (3), and therefore also the minimum (4), by one. For this smaller weight function there is an integer optimal solution $y, z$ for (4). By increasing $y(v)$ by one we obtain an integer optimal solution for $w$.

Now let $y, z$ be an optimal solution for (4) with $\sum_{V' \in \mathcal{O}} z(V') \, |V'| \cdot |V \backslash V'|$ as small as possible.

First, $y \equiv 0$, since if $y(v) > 0$, by complementary slackness each $F$ in $\mathscr{F}$ covers $v$.

Secondly, if $V', V'' \in \mathcal{O}$ with $z(V') > 0$, $z(V'') > 0$, and $V' \cap V'' \neq \varnothing$, then $V' \subseteq V''$ or $V'' \subseteq V'$. For let $v \in V' \cap V''$, and take $F$ in $\mathscr{F}$ not covering $v$. Then, by complementary slackness, $\frac{1}{2}(|V'| - 1)$ edges of $F$ are contained in $V'$, and $\frac{1}{2}(|V''| - 1)$ edges of $F$ are contained in $V''$. This directly implies that $|V' \cap V''|$ and $|V' \cup V''|$ are odd. Suppose now that $V' \backslash V'' \neq \varnothing \neq V'' \backslash V'$. Let $\varepsilon = \min\{z(V'), z(V'')\}$, and redefine $z$ by

$$\begin{aligned} z(V') &:= z(V') - \varepsilon, & z(V'') &:= z(V'') - \varepsilon, \\ z(V' \cap V'') &:= z(V' \cap V'') + \varepsilon, & z(V' \cup V'') &:= z(V' \cup V'') + \varepsilon, \end{aligned} \tag{5}$$

and let $z$ be unchanged in the other components. One easily checks that the new $y, z$ again is an optimal solution for (4), and moreover that $\sum_{V' \in \mathcal{O}} z(V') \, |V'| \cdot |V \backslash V'|$ is smaller than before, contradicting our assumption.

Finally, $z$ is integral, for suppose $z(V')$ is not an integer, with $V' \in \mathcal{O}$ and $|V'|$ as large as possible. Let $V_1, \ldots, V_k$ be the maximal elements (with respect to inclusion) of $\{V'' \in \mathcal{O} \mid z(V'') > 0, V'' \subset V'\}$. So $V_1, \ldots, V_k$ are pairwise disjoint. Now let $r$ be the fractional part of $z(V')$, and reset

$$z(V') := z(V') - r, \qquad z(V_i) := z(V_i) + r \qquad (\text{for } i = 1, \ldots, k). \tag{6}$$

One easily checks that $y, z$ again is a feasible solution for (4) (using that $w$ is integral), attaining a smaller criterion value, which contradicts that the original $y, z$ is optimal. ∎

We leave it to the reader to derive from this theorem that the dual of the linear program: max $w^{\mathrm{T}}x$ subject to (1), has a half-integer optimal solution for each $w$ in $\mathbb{Z}^E$ (the graph $K_4$ shows that there do not always exist integer optimal solutions).

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*Note added in proof.* Bill Cunningham (Bonn) informed the author that Jack Edmonds found a similar proof for the matching polyhedron theorem. An alternative short proof of both the matching polyhedron theorem and the dual integrality is given in: A. Schrijver, Min–max results in combinatorial optimization, *in* "Mathematical Programming Bonn 1982: The State of the Art," Springer-Verlag, Heidelberg, 1983.

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