8311772

Mahoney, Carolyn Ray Boone

ON THE UNIMODALITY OF THE INDEPENDENT SET NUMBERS OF A CLASS OF MATROIDS

The Ohio State University PH.D. 1983

University
Microfilms
International 300 N. Zeeb Road, Ann Arbor, MI 48106

PLEASE NOTE:

In all cases this material has been filmed in the best possible way from the available copy. Problems encountered with this document have been identified here with a check mark __√__.

1. Glossy photographs or pages ______
2. Colored illustrations, paper or print ______
3. Photographs with dark background ______
4. Illustrations are poor copy ______
5. Pages with black marks, not original copy ______
6. Print shows through as there is text on both sides of page ______
7. Indistinct, broken or small print on several pages __√__
8. Print exceeds margin requirements ______
9. Tightly bound copy with print lost in spine ______
10. Computer printout pages with indistinct print __√__
11. Page(s) ____________ lacking when material received, and not available from school or author.
12. Page(s) ____________ seem to be missing in numbering only as text follows.
13. Two pages numbered ____________. Text follows.
14. Curling and wrinkled pages ______
15. Other ____

ON THE UNIMODALITY OF THE INDEPENDENT SET NUMBERS OF A CLASS OF MATROIDS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in the Graduate School of The Ohio State University

BY

Carolyn Ray Boone Mahoney, B.S., M.A.

* * * * *

The Ohio State University

1982

Reading Committee:

Dr. Thomas A. Dowling
Dr. Dijen K. Ray-Chaudhuri
Dr. Paul D. Seymour
Dr. Neil Robertson

Approved By

Thomas A. Dowling
Advisor
Department of Mathematics

ACKNOWLEDGEMENTS

I wish to thank Dr. A. E. Ross who made it possible for me to come to The Ohio State University.

I am grateful to Professors D. K. Ray-Chaudhuri, T. A. Dowling, R. M. Wilson, E. Bannai, N. Robertson, and P. D. Seymour who have taught me combinatorics. I especially wish to thank Dr. Ray-Chaudhuri for his inspiration and encouragement and Dr. Seymour for his many ideas and suggestions.

I am most grateful for the time, energy and guidance given me by my advisor, Professor Thomas Dowling.

I wish to thank Charles Mahoney, Joe Fiedler, Alayne Parson and Cindy Mahoney for their faith and active support during my graduate school years.

Thanks to JoAnn Stokes for typing this thesis.

VITA

December 22, 1946	Born - Memphis, Tennessee
1970.	B.S., Sienna College, Memphis, Tennessee
1970-1971	National Science Foundation Fellowship, The Ohio State University, Columbus, Ohio
1972.	M.S., The Ohio State University, Columbus, Ohio
1972-1975, 1979-1982. . .	Graduate Teaching Associate, Department of Mathematics, The Ohio State University, Columbus, Ohio

FIELD OF STUDY

Major Field: Mathematics

Studies in Combinatorial Theory.

Dr. Dijen K. Ray-Chaudhuri
Dr. Richard M. Wilson
Dr. Thomas A. Dowling
Dr. Neil Robertson
Dr. Eiichi Bannai
Dr. Paul D. Seymour

TABLE OF CONTENTS

LIST OF FIGURES

INTRODUCTION

Let M be a finite matroid of rank r on a set E. We denote by I_k the number of independent sets of size k. An <u>independent i-partition</u> of M is an ordered partition (A, B) of E such that A and B are independent in M and $|A| = i$. We denote by $\Pi_i(M)$ the set of independent i-partitions of M and by $\pi_i(M)$ the cardinality of $\Pi_i(M)$. Over the past thirteen years, several interesting conjectures have been made about the sequences (I_k) and (π_i). Welsh's <u>unimodal conjecture</u> [71] was made in 1969.

Conjecture 1. $I_k \geq \min\{I_{k-1}, I_{k+1}\}$ $1 \leq k \leq r-1$.

In 1972, Mason [72] made the following conjectures:

Conjecture 2. $I_k^2 \geq I_{k-1}I_{k+1}$,

Conjecture 3. $I_k^2 \geq \frac{k+1}{k} I_{k-1}I_{k+1}$,

Conjecture 4. $I_k^2 \geq \frac{k+1}{k} \frac{m-k+1}{m-k} I_{k-1}I_{k+1}$,
for $2 \leq k \leq r-1$, where $m = |E|$.

Stronger than the <u>logarithmic concavity conjecture</u> (Conjecture 2) is Dowling's [80] <u>independent partition</u> conjecture appearing in 1980, Conjecture 5. $\pi_{k-1}(M) \leq \pi_k(M)$ for $|E| = 2k$

or more generally,

Conjecture 6. $\pi_{i-1}(M) \leq \pi_i(M)$ for $i \leq \frac{|E|+1}{2}$

There are many results which lend support to the truth of these conjectures. Mason [72] proved that $I_k \leq I_{r-k}$ for $k \leq \frac{r}{2}$. Seymour [75], in his doctoral dissertation, proved that if k is such that any circuit C of M has $|C| = 1$, $|C| = 2$ or $|C| \geq k$ then,

$$I_k^2 \geq \frac{k+1}{k} I_{k-1} I_{k+1} .$$

Further, if all circuits of M have size at least k, then

$$I_k^2 \geq \frac{k+1}{k} \frac{m-k+1}{m-k} I_{k-1} I_{k+1} .$$

Dowling proved Conjecture 5 for $k \leq 7$. In Chapter 4, we prove Conjecture 6 for polygon matroids of outerplanar graphs.

Our initial interest in independent partitions grew out of a desire to prove the log-concavity conjecture. We hoped to use the "method of proof" in Dowling [80]. That is, to each matroid M of size $2k$ a certain bipartite graph $H = H(M)$ with bipartition $\pi_{k-1}(M) \cup \pi_k(M)$ can be associated. Dowling proves that this graph $H(M)$ has a matching of $\pi_{k-1}(M)$ into $\pi_k(M)$ (and hence $\pi_{k-1}(M) \leq \pi_k(M)$) if $k \leq 7$). We believed initially that the graph

H(M) had such a matching for every value of k, but we found this to be untrue for $k \geq 16$. The details of this discovery are given in Chapter 3.

In Chapter 2 we list some well-known unimodality conjectures and indicate any progress toward their resolution.

Dowling has written a computer program which computes the numbers π_i for outerplanar graphs. We thank him for allowing us access to this program. We make some conjectures about the behavior of the sequence (π_i) in Chapter 5.

Chapter 1 gives the necessary definitions and basic propositions.

CHAPTER I

Background

1. Set Theory Notation

We will use $\cup, \cap, \subseteq, \subset$ to denote set-union, set intersection, set-inclusion and proper inclusion, respectively. The empty set is denoted by $\emptyset$. For sets $A, B, A \setminus B$ is the set of elements in A but not in B . When it is clear from the context that we are referring to a set rather than an element we abbreviate $\{x\}$ to x . If A and B are sets such that $A \cap B = \emptyset$, we may use $A + B$ instead of $A \cup B$. Thus, if x is an element not in a set A , we may write $A + x$ to mean $A \cup \{x\}$. $|A|$ denotes the cardinality of the set A . 2^A denotes the set of all subsets of A .

The set of integers is denoted by $\mathbb{Z}$, the set of nonnegative integers by $\mathbb{Z}^+$, and the set of positive integers by $\mathbb{N}$.

2. Graph Theory Preliminaries

A graph G is a triple (V, E, I) where V and E are disjoint sets and I is a mapping $I: V \times E \mapsto \{0, 1, 2\}$ such that for $e \in E$,

$$\sum_{v \in V} I(v, e) = 2,$$

and for $v \in V$ there is an $e \in E$ such that $I(v, e) > 0$. The elements of V and E are called vertices and edges, respectively. An edge e is said to be incident with a vertex v if and only if $I(v, e) > 0$; if e is incident with v we say v is an end of e.

If the edge e is incident with distinct vertices u and v, e is called a link. We say e joins u and v and we call u, v adjacent vertices. A loop of a graph is an edge e such that $I(v, e) = 2$ for some $v \in V$. Two links e_1 and e_2 are parallel if they join the same pair of vertices. A graph is simple if it has no loops or parallel edges.

The degree of a vertex v in a graph G is $\sum_{e \in E} I(v, e)$, and is denoted $d_G(v)$. Thus in a simple graph, $d_G(v)$ is the number of vertices adjacent to v.

Two graphs $G_1 = (V_1, E_1, I_1)$, $G_2 = (V_2, E_2, I_2)$ are isomorphic if there are bijections $\varphi: V_1 \mapsto V_2$ and $\psi: E_1 \to E_2$ such that $I_1(v, e) = I_2(\varphi(v), \psi(e))$.

A bipartition of a graph is an ordered partition (V_1, V_2) of V such that for all $e \in E$, there exist vertices $v_i \in V_i$, $i = 1, 2$, incident with e. A graph which admits a bipartition is called bipartite.

Let $G = (V, E, I)$ and $G_1 = (V_1, E_1, I_1)$ be graphs. G_1 is said to be a <u>subgraph</u> of G if $V_1 \subseteq V$, $E_1 \subseteq E$ and I_1 is the restriction of I to $V_1 \times E_1$, i.e., $I_1(v, e) = I(v, e)$ for all $v \in V_1, e \in E_1$. When it is clear from the context, we omit reference to the incidence map I_1. For $A \subseteq E$, let $V(A)$ be the set of vertices incident with some $e \in A$. Then $G|A = (V(A), A)$ is the <u>subgraph generated by A</u>. Often we abbreviate $G|A$ to A when the meaning is clear from the context. If $X \subseteq V$, we let $G[V \setminus X]$, or simply $G \setminus X$, denote the subgraph obtained by deleting X and all edges incident with some vertex in X from G.

For $G = (V, E, I)$ a <u>path</u> in G is a finite sequence $v_0, e_1, v_1, \ldots, v_{m-1}, e_m, v_m$, $m \geq 0$, such that $v_i \in V$, $e_j \in E$ for $0 \leq i \leq m$ and $1 \leq j \leq m$, and e_i has ends v_{i-1} and v_i, $1 \leq i \leq m$. If the ver- vertices v_i are all distinct P is called <u>simple</u>. The <u>length</u> of P is the number of edges m; P is said to connect v_0 and v_m. The vertices v_0 and v_m are called the <u>initial</u> and <u>terminal</u> vertices of P, respectively, or <u>end vertices</u> when order is not important. The vertices v_i, $1 \leq i \leq m-1$ are <u>interior vertices</u> of P. We may identify a simple path with its edge set $\{e_1, e_2, \ldots, e_m\}$. A circuit is a path in which $v_i \neq v_j$ for $i \neq j$, except $v_0 = v_m$.

For $G = (V, E, I)$, the relation $\sim$ on V defined by $x \sim y$ if $x = y$ or there is a path in G joining x and y is an equivalence relation on V. If $V_1, \ldots, V_n$ are the distinct equivalence classes, then the subgraphs $G_k = G[V_k]$ are called the <u>connected components</u> of G. If $n = 1$, G is <u>connected</u>.

Let $G_k = (V_k, E_k, I_k)$, $1 \leq k \leq n$ be graphs such that the sets V_k, E_k, $1 \leq k \leq n$, are pairwise disjoint. Then

$$G_1 \oplus \cdots \oplus G_n = (V_1 + \cdots + V_n, E_1 + \cdots + E_n, I)$$

where

$$I(v, e) = \begin{cases} 0 \text{ if } v \in V_k, e \in E_\ell \text{ and } k \neq \ell , \\ I_k(v, e) \text{ if } v \in V_k, e \in E_k , \end{cases}$$

is called the direct sum of $G_1, \ldots, G_n$.

Let $G = (V, E, I)$. If $A \subseteq E$, we write $\eta(A)$ for the number of common vertices of the subgraphs generated by A and $E \setminus A$. Then for k a positive integer, we say G is <u>k-separated</u> if and only if G is connected and there exists $A \subseteq E$ with $|A| \geq k$, $|E \setminus A| \geq k$ and $\eta(A) = k$. We say G is <u>0-separated</u> if it is not connected. If there exists a least nonnegative integer k such that G is k-separated, we call it the <u>connectivity</u> of G and denote it by $\lambda(G)$. If there is no such integer we write $\lambda(G) = \infty$. We say G is <u>n-connected,</u> where n is a positive integer, if $n \leq \lambda(G)$. If $\lambda(G) = 1$ there is a vertex $v \in V$ such that v is the only common vertex of a set A of edges and $E \setminus A$, both non-empty. We call such a vertex v a <u>cut-vertex</u>. If G is connected with no cut- vertex we say G is <u>nonseparable</u>.

Let $G=(V,E,I)$ be a graph and let $e \in E$. $G\setminus e$, pronounced "G delete e", is defined to be the subgraph $(V_1, E\setminus e)$ where $V_1 = \{v \in V : I(V,a) \neq 0$ for some $a \in E\setminus e\}$. G/e, pronounced "G contract e", is defined as follows: let e be incident with u_1 and u_2,

(i) if $u_1 = u_2$, then G/e is $G\setminus e$.

(ii) if $u_1 \neq u_2$, then $G/e = (V_1, E\setminus e, I_1)$ where $V_1 = V - \{u_1, u_2\} + \{u\}$ $(u \notin V)$ and $I_1(v,a) = I(v,a)$ for $v \neq u$ and $I_1(u,a) = I(u_1,a) + I(u_2,a)$.

If $A \subseteq E$, then $G\setminus A$ is defined to be

$((\ldots((G\setminus e_1)\setminus e_2)\ldots)\setminus e_n)$ and G/A is $((\ldots((G/e_1)/e_2\ldots)/e_n)$

where $e_1, \ldots, e_n$ are the elements of A (in any order). It can be shown that these terms are well-defined. A minor of G is a graph H which is obtained from G by a series of deletions and contractions. It is easy to see that if H is a minor of G, then there are subsets $A, B \subseteq E$ such that $H = (G\setminus A)/B = (G/B)\setminus A$. Also, if K is a minor of H and H is a minor of G, then K is a minor of G.

3. Matroid Theory Fundamentals

A matroid M is a finite set E and a collection $\mathcal{I}$ of subsets of E (called independent sets) such that

(i) $\emptyset \in \mathcal{I}$

(ii) if $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$

(iii) if X, Y are members of $\mathcal{I}$ with $|X| = |Y| + 1$ then there exists an element $x \in X \setminus Y$ such that $Y \cup x \in \mathcal{I}$.

$|E|$ is called the size of M.

A basis of M is a maximal independent subset of E. A subset of E not belonging to $\mathcal{I}$ is called dependent. A circuit of M is a minimal dependent subset of E.

The function $r : 2^E \to \mathbb{Z}$ defined by

$$r(A) = \max(|X| : X \subseteq A, X \in \mathcal{I})$$

is called the rank function of M. The rank of a subset A of E is $r(A)$; the rank of the matroid M is $r(E)$ and will be written $r(M)$. It is not difficult to prove that every basis of M has cardinality $r(M)$.

For $e \in E$ and $A \subseteq E$, we say e is in the closure of A if and only if $r(A \cup e) = r(A)$. A set A is said to be a flat if its closure is itself.

Let e_1, e_2 be elements of E. If $r(\{e_1\}) = 0$, then e_1 is called a loop. If $r(\{e_1, e_2\}) = r(\{e_1\}) = r(\{e_2\}) = 1$, then

e_1 and e_2 are called <u>parallel</u> elements. A matroid with no loops or parallel elements is called <u>simple</u>.

An element $e \in E$ such that for $A \subseteq E$, $e \notin A$ implies $r(A \cup e) = r(A) + 1$ is called a <u>coloop</u>. A coloop of M belongs to every basis of M.

Based on the work of Whitney [35] and Tutte [66], we now show that a matroid M on E induces two matroids on $E \setminus e$. <u>$M \setminus e$</u>, pronounced "M delete e", is defined to be the matroid on $E \setminus e$ in which $X \subseteq E \setminus e$ is independent in $M \setminus e$ if and only if X is independent in M. <u>M/e</u>, pronounced "M contract e", is defined to be the matroid on $E \setminus e$ given by one of the following:

(i) if e is a loop, then M/e is $M \setminus e$.

(ii) if e is not a loop, then $X \subseteq E \setminus e$ is independent in M/e if and only if $X \cup e$ is independent in M.

It is easily shown that these definitions do indeed give matroids.

We say that a matroid N is a <u>minor</u> of M if N can be obtained from M by a series of deletions and contractions. If $e_1, e_2 \in E$, then it is easy to see that $((M \setminus e_1) \setminus e_2) = ((M \setminus e_2) \setminus e_1)$, $((M/e_1)/e_2) = ((M/e_2)/e_1)$ and $((M \setminus e_1)/e_2) = ((M/e_2) \setminus e_1)$. Hence a minor N of M can be realized as $(M \setminus X)/Y$ for subsets X, Y of E where $M \setminus X = (M \setminus x_1) \setminus x_2) \setminus \ldots \setminus x_k)$ and $M/X = ((M/x_1)/x_2) \ldots)/x_k$ for $X = \{x_1, \ldots, x_k\}$. Sometimes we use the notation $M | X$ for $M \setminus (E \setminus X)$.

Tutte [66] also has developed a theory of n-connection for matroids which we shall need in the sequel. If M is a matroid on

E with rank function r, define a function $\xi : 2^E \to \mathbb{Z}$ by

$$\xi(A) = r(A) + r(E \setminus A) - r(E) + 1 \qquad (A \subseteq E)$$

Call the matroid M <u>k-separated</u> for k a positive integer if there exists $A \subseteq E$, $|A| \geq k$, $|E \setminus A| \geq k$, with $\xi(A) = k$. If there is a least positive integer k such that M is k-separated we call k the <u>connectivity</u> of M and denote it by $\lambda(M)$. If there is no such integer, then $\lambda(M) = \infty$. In either case, M is <u>n-connected</u> for any $n \leq \lambda(M)$. If $\lambda(M) > 1$, that is, if

$$r(A) + r(E \setminus A) > r(E)$$

for any proper subset A of E, then M is <u>connected</u>. If $\lambda(M) = 1$, that is, if there exists a nonempty subset $A \subset E$ such that

$$r(A) + r(E \setminus A) = r(E) ,$$

we can form two deletion matroids $M_1 = M \setminus (E \setminus A)$, $M_2 = M \setminus A$. M_1 and M_2 are matroids on disjoint sets E_1 and E_2, respectively and a subset $X \subseteq E = E_1 \cup E_2$ is independent in M if and only if $X_i = X \cap E_i$ is independent in M_i, $i = 1, 2$.

Let M_i, $i = 1, \dots, k$ be matroids on disjoint sets E_i, $i = 1, \dots, k$. The matroid M on $E = E_1 + E_2 + \dots + E_k$ given by $X \subseteq E$ is independent in M if and only if $X_i = X \cap E_i$, is independent in M_i, $i = 1, \dots, k$, is called the <u>direct sum</u> of $M_1, M_2, \dots, M_k$. Thus, M is a direct sum of smaller matroids if and only if $\lambda(M) = 1$.

4. Graphic Matroids

In this section we relate the two previous sections by showing that every graph $G = (V, E)$ gives rise to a matroid $\mathcal{M}(G)$ on E. Actually, there are two well-known classes of matroids arising from graphs, but we consider only one of them. For a detailed treatment of both graphic and cographic matroids, and for proofs of statements appearing below, see Welsh [76].

1.1 Proposition: Let G be a graph with edge set E. The collection $\mathcal{J}$ of subsets of E given by $X \in \mathcal{J}$ if and only if X contains no circuit is the collection of independent subsets of a matroid on E.

We denote this matroid by $\mathcal{M}(G)$, and call it the polygon matroid of G.

1.2 Proposition:

(i) An edge e is a loop of the graph G if and only if it is a loop in the matroid $\mathcal{M}(G)$.

(ii) Two edges are parallel in the graph G if and only if they are parallel in the matroid $\mathcal{M}(G)$.

(iii) For any subset $A \subseteq E$, the rank of A in $\mathcal{M}(G)$ is given by

$$r(A) = |V(A)| - k(A)$$

where $k(A)$ denotes the number of connected components in the subgraph generated by A.

(iv) If $A \subseteq E$ and $e \in E \setminus A$, then e belongs to the closure of A in $\mathcal{M}(G)$ if and only if there is a circuit C of G with $e \in C \subseteq A \cup e$.

(v) For $A \subseteq E$, $\mathcal{M}(G \setminus A) = \mathcal{M}(G) \setminus A$.

(vi) For $A \subseteq E$, $\mathcal{M}(G / A) = \mathcal{M}(G) / A$.

(vii) A connected graph G is k-separated if and only if its matroid is k-separated. That is, $\lambda(\mathcal{M}(G)) = \lambda(G)$ for any connected graph G.

5. Independent Partitions

Let M be a finite matroid of size m and rank r on a set E. An independent i-partition of M is an ordered partition (A, B) of E (i.e. $A \cup B = E$, $A \cap B = \emptyset$) such that

(i) $|A| = i$, $|B| = m - i$.

(ii) A and B are independent sets in M.

We denote by $\Pi_i(M)$ the set of all independent i-partitions of M, and by $\pi_i(M)$ the cardinality of $\Pi_i(M)$.

Example. Let $M = \mathcal{M}(G)$, where G is the graph in Fig. 1.

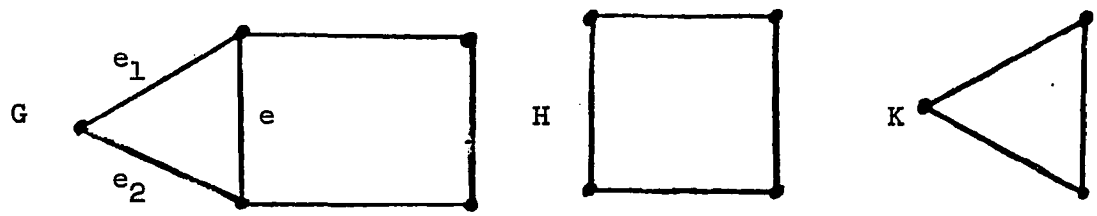

Figure 1

Since $r(M) = 4$, $\pi_0(M) = \pi_1(M) = \pi_5(M) = \pi_6(M) = 0$. For $i = 2, 3$ let $(A, B) \in \Pi_i(M)$. If $e_1, e_2 \in B$, then $e \in A$ and $(A - e, B - \{e_1 e_2\}) \in \Pi_{i-1}(\mathcal{M}(K))$. Conversely, if $(C, D) \in \Pi_{i-1}(M / \{e, e_1, e_2\})$ then $(C + e, D + \{e_1, e_2\}) \in \Pi_i(M)$. And, $e_1 \in A$, $e_2 \in B$ iff $(A - e_1, B - e_2) \in \Pi_{i-1}(\mathcal{M}(H))$. So,
$\pi_2(M) = \pi_1(\mathcal{M}(K)) + \pi_1(\mathcal{M}(H)) + \pi_0(\mathcal{M}(K)) = 3 + 2 \cdot 4 = 11$, and
$\pi_3(M) = \pi_2(\mathcal{M}(K)) + \pi_2(\mathcal{M}(H)) + \pi_1(\mathcal{M}(K)) = 3 + 2 \cdot 6 + 3 = 18$. (We will use "this method" of counting independent partitions in Chapter 3).

Since (A, B) is an independent i-partition of M if and only if (B, A) is an independent $(m-i)$-partition of M, we have $\pi_i(M) = \pi_{m-i}(M)$ so that the sequence $\pi_i(M)$, $i = 0, \ldots, m$ is symmetric.

If M has a loop or if $r(M) < \frac{m}{2}$, it is clear that $\pi_i(M) = 0$. In fact, Edmonds and Fulkerson [65] have shown that M has an independent i-partition for some i if and only if $r(A) \geq \frac{|A|}{2}$ for all $A \subseteq E$.

We began studying independent partitions as a tool for solving some long-standing conjectures about the number I_k of independent sets of size k in a matroid M of rank r. Welsh [71] has conjectured

$$I_k \geq \min\{I_j, I_\ell\} \quad \text{for} \quad 0 \leq j < k < \ell \leq r \quad .$$

Stronger than Welsh's unimodal conjecture is the logarithmic concavity conjecture of Mason [72]

$$I_k^2 \geq I_{k-1} I_{k+1} \text{ for } 0 < k < r \; .$$

Dowling [80] showed that stronger still is the independent partition conjecture

$$\pi_{k-1}(M) \leq \pi_k(M) \quad \text{for } M \text{ of size } 2k \quad ,$$

or more generally, that if M has size m,

$$\pi_{i-1}(M) \leq \pi_i(M) \quad \text{for} \quad i \leq \frac{m+1}{2} \quad ,$$

that is, the symmetric sequence $\pi_i(M)$, $i = 0, \ldots, m$ is unimodal. In Chapter 3, we prove the latter conjecture for polygon matroids of outerplanar graphs.

Let M be a finite matroid of rank r on a set $E = \{e_1, \dots, e_m\}$. We consider the polynomial ring $R = \mathbb{Z}[e_1, \dots, e_m]$ freely generated over the integers by the elements of E, and define a partial order on R as follows: $f \leq g$ if the coefficient of each term of f does not exceed the coefficient of the corresponding term of g.

Define a homomorphism $\sigma : R \to \mathbb{Z}$ under which the image of a polynomial is the sum of its coefficients. Clearly, $f \leq g$ implies $\sigma(f) \leq \sigma(g)$. Now, for $0 \leq \ell \leq r$, we let

$$f_\ell = \sum_A \Big(\prod_{e_i \in A} e_i \Big) ,$$

where the sum is extended over all independent sets A of size ℓ in M. Then f_ℓ is homogeneous of degree ℓ and $\sigma(f_\ell) = I_\ell$. Then since

$$I_\ell^2 = (\sigma(f_\ell))^2 = \sigma(f_\ell^2) \text{ and } \sigma(f_{\ell-1} f_{\ell+1}) = \sigma(f_{\ell-1})\sigma(f_{\ell+1}) = I_{\ell-1} I_{\ell+1}$$

we have that

(1) $$f_\ell^2 \geq f_{\ell-1} f_{\ell+1} \quad \text{for } 0 < \ell < r ,$$

implies

$$I_\ell^2 \geq I_{\ell-1} I_{\ell+1} \quad \text{for } 0 < \ell < r .$$

To obtain an equivalent form of (1) we need the following notations and definitions. Given a matroid M on a set E, and disjoint subsets X and Y of E, the <u>size</u> of the minor $M|(X \cup Y)/Y$ is the cardinality of X, and its <u>depth</u> in M is the rank of Y in M.

<u>1.3 Proposition</u>. Let M be a finite matroid and ℓ a positive integer. Then

$$f_{\ell}^{2}(M) \geq f_{\ell-1}(M) f_{\ell+1}(M) \tag{1}$$

if and only if, for every $k \leq \ell$ and every minor N of M of size $2k$ and depth $\ell - k$,

$$\pi_{k}(N) \geq \pi_{k-1}(N) \quad . \tag{2}$$

<u>Proof</u>. We interpret the coefficients in the homogeneous polynomials f_{ℓ}^{2} and $f_{\ell-1} f_{\ell+1}$. Each term of either is an integer multiple of a monomial of the form

$$g = \prod_{e_i \in Y} e_i \cdot \prod_{e_j \in Z} e_j^2 \quad ,$$

where Y and Z are disjoint subsets of X satisfying $|Y| + 2|Z| = 2\ell$. Let $|Y| = 2k$, so $|Z| = \ell - k$. Then the coefficient of g in f_{ℓ}^{2} is the number of ordered pairs $(A \cup Z, B \cup Z)$ of independent sets in the restriction $M|(Y \cup Z)$ such that (A, B)

is an ordered partition of Y and $|A| = |B| = k$. These correspond to the independent k-partitions of the minor $N = M \mid (Y \cup Z)/Z$ of size $2k$ and depth $\ell - k$ in M. Hence the coefficient is $\pi_k(N)$. In like manner, the coefficient of g in $f_{\ell-1}f_{\ell+1}$ is $\pi_{k-1}(N)$. So (2) implies (1).

Conversely, suppose (1) holds and let $N = M \mid (Y \cup Z)/Z$ be a minor of size $2k$ and depth $\ell - k$ in M. We may assume Z is independent in M, so that $|Z| = \ell - k$. Define a monomial g as above, and observe that the coefficients of g in f_ℓ^2 and $f_{\ell-1}f_{\ell+1}$ are given by the left and right sides of (2), respectively, so that (2) follows from (1).

A class $\mathcal{M}$ of matroids in <u>closed under minors</u> if $N \in \mathcal{M}$ whenever $M \in \mathcal{M}$ and N is a minor of M.

<u>1.4 Corollary</u>. Let $\mathcal{M}$ be a class of matroids closed under minors and suppose that (2) holds for every $k \leq \ell$ and every N in $\mathcal{M}$. Then (1) holds for every M in $\mathcal{M}$.

CHAPTER II

Unimodality and Combinatorial Sequences: a Survey

A sequence a_k, $k = 0, \ldots, r$ of non-negative real numbers is said to be <u>unimodal</u> if

$$a_j \geq \{a_i , a_k\} \quad \text{for} \quad i < j < k ,$$

that is, the sequence has no local minimum. A sequence a_k, $k = 0, \ldots, r$ of non-negative real numbers is said to be <u>logarithmically concave</u> if

$$a_k^2 \geq a_{k-1}a_{k+1} \quad \text{for} \quad k = 1, \ldots, r-1 .$$

In the introduction we looked at some sequences which have been conjectured to be unimodal or logarithmically concave (log concave). In this chapter we continue this study by collecting some of the well-known unimodality conjectures in combinatorics and indicating the related results.

One form of the Newton inequality, which is extensively discussed in Hardy, Polya, Littlewood [52] is the following:

<u>2.1 Theorem</u>. If the generating polynomial

$$P(x) = \sum_{0 \leq k \leq r} a_k x^k , \; a_r \neq 0$$

of a finite sequence a_k , $0 \le k \le r$, has only real roots (≤ 0) , then

$$a_k^2 \ge a_{k-1}a_{k+1} \cdot \frac{k}{k-1} \cdot \frac{r-k+1}{r-k} \quad , \; 2 \le k \le r-1 \; .$$

This theorem provides a powerful tool for proving unimodality of certain combinatorial sequences, for example, the sequence of Stirling numbers. The Stirling number of the first kind, $s(n,k)$ is defined by $(-1)^{n-k}s(n,k)$ is the number of permutations of n symbols which have exactly k cycles. It is well-known that the generating function for $s(n,k)$ is

$$\sum_{k=1}^{n} s(n,k)\,x^k = x(x-1)\ldots(x-n+1) \; ,$$

a polynomial with only real roots. Hence, Erdos [53] uses Theorem 2.1 to prove:

2.2 Corollary. The sequence of the absolute values of the Stirling numbers of the first kind, $s(n,k)$, n fixed (≥ 3) , k variable $(\le n)$ is log-concave.

The number $S(n,k)$ of partitions of a set of n elements into k non-empty subsets is called the Stirling number of the second kind. Harper [67] has shown that the generating function

$$\sum_{k=1}^{n} S(n,k)x^k$$

has only non-positive real roots. He uses this result to show

2.3 Corollary. The sequence $S(n, k)$ of the Stirling numbers of the second kind, n fixed (≥ 3), k variable $(\leq n)$ is log-concave.

Lieb [68] gets this same result using a slightly different generating function.

There are other unimodality conjectures which have not been settled. Before we present them we need to give some notation and definitions. We will assume the reader is familiar with elementary lattice theory as given in Birkhoff [67]. If M is a matroid on E, we can associate with M a partially ordered set $\mathcal{L}(M)$ whose elements are the flats of M ordered by inclusion. Under this inclusion ordering $\mathcal{L}(M)$ is a geometric lattice with atoms the flats of rank one in M and with hyperplanes the flats of rank equal to one less than the rank of M. Conversely, a finite geometric lattice is isomorphic to the lattice of flats of a matroid. We also assume the reader has knowledge of the Mobius function μ, of a partially ordered set as given in Rota [64].

Let G be a graph. A k-coloring of a graph G is the assignment to each vertex of G one of a specified set of k colors. A proper k-coloring of G is a coloring in which no two adjacent vertices are the same color. The chromatic polynomial $P(G; \lambda)$ is the function, defined on the positive integers, which when λ is a positive integer equals the number of different ways of properly coloring G in λ colors. This definition can be generalized to matroids. That is, for any matroid M on a set E the chromatic polynomial of M is given by

$$P(M\,;\lambda) = \sum_{A \subseteq E} (-1)^{|A|} \lambda^{r(E)-r(A)}$$

where again $r(A)$ is the rank of A. Thus for a graph G with no loops or parallel edges

$$\lambda^{k(G)} P(\mathcal{M}(G)\,;\lambda) = P(G\,;\lambda) \quad ,$$

where $k(G)$ denotes the number of connected components of G. Birkhoff [12] and Whitney [32] were the first to study the chromatic polynomial.

2.4 Theorem. If G is a graph without loops or parallel edges and e is an edge of G then

$$P(G\,;\lambda) = P(G \setminus e\,;\lambda) - P(G/e\,;\lambda) \quad .$$

From this theorem many properties of the chromatic polynomial can be derived. For example, the maximum degree of $P(G\,;\lambda)$ is $|V(G)|$, the coefficients of the chromatic polynomial alternate in sign, and the chromatic polynomial has a representation

$$P(G\,;\lambda) = \sum_{i=1}^{\lambda} \binom{\lambda}{i}\, i\,!\, b_i \;,$$

where b_i is the number of ways of properly coloring G in exactly

i colors with color indifference. Read has conjectured that if G is a graph with chromatic polynomial

$$P(G\,;\lambda) = \sum_{k=1}^{|V(G)|} (-1)^k a_k \lambda^k ,$$

then the sequence a_k , $1 \le k \le |V(G)|$ is unimodal. Welsh [75] made the stronger conjecture that if M is a matroid with chromatic polynomial

$$P(M\,;\lambda) = \sum_{k=0}^{r(M)} (-1)^k a_k \lambda^k ,$$

then the sequence a_k , $0 \le k \le r(M)$ is log-concave. These conjectures are true in the instances where the chromatic polynomial is known.

For example,

1) For the complete graph K_n

$$P(K_n\,;\lambda) = \lambda(\lambda-1)\ldots(\lambda-n+1)$$

2) For G a tree with n vertices

$$P(G\,;\lambda) = \lambda(\lambda-1)^{n-1}$$

(In fact, a graph G is a tree with n vertices if and only if it has chromatic polynomial $\lambda(\lambda-1)^{n-1}$

3) For G a cycle with n vertices

$$P(G\,;\lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$$

4) For G the Fano matroid

$$P(G\,;\lambda) = \lambda^3 - 7\lambda^2 + 14\lambda - 8 \quad .$$

Further Heron [72], has proved:

2.5 Theorem. If G is a connected graph with n vertices and chromatic polynomial

$$P(G\,;\lambda) = \lambda^n - a_1\lambda^{n-1} + \ldots + (-1)^{n-1}a_{n-1}\lambda$$

then for all $i \leq (n-1)/2$,

$$a_{i-1} \leq a_i \quad .$$

Rota [64] has shown that if M is a simple matroid on E and $\mathcal{L}$ is its lattice of flats

$$P(M\,;\lambda) = \sum_{F \in \mathcal{L}} \mu(\emptyset, F)\lambda^{r(E) - r(F)}$$

where μ is the Mobius function of $\mathcal{L}$. The coefficient w_k of $\lambda^{r(E) - k}$ in $P(M\,;\lambda)$ is known as the kth Whitney number of the first kind. Evidently,

$$w_k = \sum_{r(F)=k} \mu(\emptyset, F) \quad , \quad 0 \leq k \leq r(E) \quad .$$

Using this characterization of the coefficients of the chromatic polynomial, Dowling and Wilson [74] have proved:

<u>2.6 Theorem</u>. Let $\mathcal{L}$ be a finite geometric lattice of rank r with m atoms. Then

$$|w_k| \geq \binom{r-1}{k-1}(m-r) + \binom{r}{k}, \quad 0 \leq k \leq r ,$$

and equality holds for some k , $2 \leq k \leq r$ if and only if $\mathcal{L}$ is isomorphic to the direct product of a line and a free matroid.

On the other hand, Heron [72] has shown that

$$|w_k| \leq \binom{m}{k} \quad .$$

Also of interest is the sequence W_k of <u>Whitney numbers of the second kind</u>, where W_k denotes the number of flats of rank k in a finite matroid. Rota has conjectured that the sequence W_k , $0 \leq k \leq r(M)$, is unimodal. This is known to be true, for example, for the partition lattices (see Harper [67] and Lieb [68]), paving matroids, and projective or affine geometries. In fact, the stronger conjecture that the sequence W_k is log-concave seems to hold. Mason [72] makes the plausible conjecture that the ratio $W_k^2/W_{k+1}W_{k-1}$ is minimized over all matroids on m points by the free matroid, that is,

$$W_k^2 \geq \frac{k+1}{k}\,\frac{m-k+1}{m-k}\,W_{k-1}W_{k+1} ,$$

where $m = W_1$.

The most general results concerning these conjectures were proved by Dowling and Wilson [75].

2.7 Theorem. For any finite geometric lattice $\mathcal{L}$, the Whitney numbers satisfy

$$W_1 + W_2 + \cdots + W_k \le W_{r-k} + \cdots + W_{r-2} + W_{r-1}$$

where r is the rank of $\mathcal{L}$, and $1 \le k \le r-1$. If equality holds for some k then $\mathcal{L}$ is a modular lattice.

2.8 Theorem. Let $\mathcal{L}$ be a finite geometric lattice of rank r with m atoms. Then

$$W_k \ge \binom{r-2}{k-1}(m-r) + \binom{r}{k}, \quad 0 \le k \le r .$$

When $r \ge 4$, equality holds for some k, $2 \le k \le r-2$ if and only if $\mathcal{L}$ is isomorphic to the direct product of a modular plane and a free geometry. (A free geometry with j atoms is the geometric lattice of all subsets of a j-set. A modular plane is a modular geometric lattice of rank 3.)

A special case of the previous conjecture for the sequence W_k has been studied recently. The Points-Lines-Planes conjecture is:

$$W_2^2 \ge \frac{3}{2} W_1 W_3 .$$

Stonesifer [75] proved this conjecture for graphic matroids; Seymour [82] proved this conjecture for matroids in which no line has five or more points. (A point is a flat of rank 1, a line is a flat of rank 2, a plane is a flat of rank 3.)

Analogous conjectures have been made for the sequence I_k of independent set numbers of a matroid M on E, where I_k denotes the number of independent sets of size k. Welsh [71] made the following conjecture in 1969

$$I_k \geq \min\{I_{k-1}, I_{k+1}\} \quad , \quad 2 \leq k \leq r(M) - 1 .$$

Mason's [72] log-concavity conjectures for the independent set numbers are: If $2 \leq k \leq r(M) - 1$ and $m = |E|$, then

$$I_k^2 \geq I_{k-1} I_{k+1} \quad ,$$

$$I_k^2 \geq \frac{k+1}{k} I_{k-1} I_{k+1} \quad ,$$

$$I_k^2 \geq \frac{k+1}{k} \frac{m-k+1}{m-k} I_{k-1} I_{k+1} .$$

These conjectures are known to hold for M a projective or affine geometry. Further, M is a paving matroid if and only if

$$I_k = \binom{m}{k} \quad , \qquad 0 \leq k \leq r - 1 .$$

More general results which lend support to the truth of these conjectures can be found in Mason [72] where he proves

2.9 Theorem. If M is a matroid of rank r, then

$$I_k \leq I_{r-k} \quad \text{for} \quad k \leq r/2 \quad .$$

Seymour [75] proves the following three theorems.

2.10 Theorem. If M is a matroid of rank r on a set E, and $0 \leq k \leq r-1$, then

$$I_{k+1} \geq \left(\frac{r+d}{k+1} - 1\right) I_k \quad \text{and} \quad I_{k+1} \leq \frac{m-k}{k+1} I_k \ ,$$

where d is the size of the smallest cocircuit of M, and $m = |E|$. (A subset A of E is a cocircuit of M if and only if A has non-null intersection with every base of M, and is minimal with respect to this property.)

2.11 Theorem. If M is a matroid and k is a positive integer such that for any circuit C of M, either $|C| = 1$ or $|C| = 2$ or $|C| \geq k$, then

$$I_k^2 \geq \frac{k+1}{k} I_{k-1} I_{k+1} \quad .$$

<u>2.12 Theorem</u>. If all circuits of M have size at least k, then

$$I_k^2 \geq \frac{k+1}{k}\,\frac{m-k+1}{m-k}\,I_{k-1}I_{k+1}\ , \qquad m = |E|\ .$$

Dowling [80] proved

$$I_k^2 \geq I_{k-1}I_{k+1} \quad \text{for} \quad 0 < k < r\ ,\ k \leq 7\ .$$

Two other interesting log-concave sequences are given by Stanley [81]. We first give a definition. A matroid M of rank r on a set E of size m is called <u>unimodular</u> (or <u>regular</u>) if there exists a mapping $\vartheta : E \to \mathbb{R}^r$ such that

a) a subset A of E is independent if and only if the $|A|$ column vectors $\vartheta(E)$, $e \in A$, are linearly independent, and

b) the $r \times m$ matrix $[\vartheta(e_1)\ldots\vartheta(e_m)]$ is totally unimodular, i.e. every minor has determinant 0 or ± 1.

The most familiar class of unimodular matroids are the graphic matroids.

Stanley's main results are the following:

<u>2.13 Theorem</u>. Let M be a unimodular matroid of rank r on a finite set E, and let $A \subseteq S$. Let f_i be the number of bases B of M satisfying $|B \cap A| = i$, and set $g_i = f_i \big/ \binom{r}{i}$. Then the sequence $g_0, g_1, \ldots, g_r$ is log-concave.

<u>2.14 Theorem</u>. Let P be a finite partially ordered set with n elements, and let $x \in P$. Let N_i be the number of order-preserving bijections $\sigma : P \to \{1, 2, \ldots n\}$ satisfying $\sigma(x) = i$. Then the sequence $N_1, N_2, \ldots, N_n$ is log-concave.

We discuss the independent partition conjectures in the ensuing chapters.

CHAPTER III

The Independent Partition Conjecture

1. Dowling's Theorem

We turn, then, to the study of independent partitions of a matroid of size $2k$. In [80], two bipartite graphs are associated with such a matroid M. Each has bipartition $\pi_{k-1}(M) \cup \pi_k(M)$. The graphs $G = g(M)$ and $H = H(M)$ are given as follows: let $a \in A$ be such that $(A, B) \in \pi_k(M)$ and $(A \setminus a, B + a) \in \pi_{k-1}(M)$, then in G $(A \setminus a, B + a)$ is adjacent to (A, B)., and in H $(A \setminus a, B + a)$ is adjacent to both (A, B) and (B, A). An alternate description is helpful. An independent partition may be regarded as a 2-coloring of the elements of M so that each color represents an independent set in M. Adjacency in G describes the ability to switch colors on one element. In H, $(A, B) \in \pi_{k-1}(M)$ is adjacent to $(C, D) \in \pi_k(M)$ if (C, D) is arrived at in one of two ways:

(1) the color of one element of B is switched,

(2) the color is switched on one element of B and then all colors are switched.

In a bipartite graph with bipartition $V_1 \cup V_2$, a <u>matching of V_1 into V_2</u> is a set A of pairwise non-adjacent edges such that every vertex in V_1 is incident with some edge in A. If we could show that there is a matching of $\pi_{k-1}(M)$ into $\pi_k(M)$ in either graph, G or H, then we would have the inequality $\pi_{k-1}(M) \leq \pi_k(M)$.

When the graph H has such a matching we say M admits an H-matching. In 1980, Dowling [80] proved that any matroid M of size 2k admits an H-matching when $k \leq 7$. We include his proof here. In the next section we exhibit a matroid which has no H-matching.

We first need a result on matchings in bipartite graphs. For vertices x, y of a graph H , let $x \sim y$ mean x is adjacent to y .

<u>3.1 Proposition</u>. Let H be a finite simple bipartite graph with bipartition $X \cup Y$. Suppose that $d_H(x) \geq 1$ for all $x \in X$ and that

(3) $$\sum_{x \sim y} \frac{1}{d_H(x)} \leq 1 \quad \text{for each } y \in Y .$$

Then G admits a matching of X into Y .

<u>Proof</u>. Let A be a subset of X and let N(A) be the set of vertices in Y adjacent to some vertex in A . By Hall's Theorem, it suffices to show $|A| \leq |N(A)|$. Using (3), we have

$$|A| = \sum_{x \in A} \frac{d_H(x)}{d_H(x)} = \sum_{x \in A} \sum_{x \sim y} \frac{1}{d_H(x)} \leq \sum_{y \in N(A)} \sum_{x \sim y} \frac{1}{d_H(x)} \leq |N(A)| .$$

Denote by $\kappa(A)$ the closure of a set $A \subseteq E$. For $(C, D) \in \Pi_{k-1}(M)$, we have $(C+d, D \setminus d) \in \Pi_k(M)$ if and only if $d \in D - \kappa(C)$,

$$d_G((C, D)) = |D - \kappa(C)| \quad \text{and} \quad d_H((C, D)) = 2d_G((C, D)) = 2|D - \kappa(C)| .$$

And for $(A, B) \in \pi_k(M)$

$$d_H(A, B) = d_G(A, B) + d_G(B, A) = |A - \kappa(B)| + |B - \kappa(A)| .$$

Let $X = \pi_{k-1}(M)$ and $Y = \pi_k(M)$

<u>3.2 Lemma</u>. If $r(M) \leq k$ or $r(M) = k+1$ and M has a coloop, then the graph H admits a matching of X into Y. Otherwise, for $(C, D) \in X$, $(A, B) \in Y$

$$(4) \qquad |D - \kappa(c)| \geq 3 \quad \text{and}$$

$$(5) \qquad |B - \kappa(A)| \geq 2$$

<u>Proof</u>. If $r(M) \leq k$ then $\pi_{k-1}(M) = 0$ and so the result is true vacuously. If $r(M) = k+1$ and e is a coloop, then for $(C, D) \in X$ we have $|D| = k+1$ and so D is a basis and hence $e \in D$. We match (C, D) to $(C+e, D \setminus e)$. If $r(M) \geq k+2$, or if $r(M) = k+1$ and M has no coloop, then there are at least 3 elements not in the closure of an independent $(k-1)$-set, and at least 2 elements not in the closure of an independent k-set, so (4) and (5) hold.

For $(A, B) \in \pi_k(M)$, we call the pair (B, A) and (A, B) mates.

3.3 Theorem. Let M be a matroid on $2k$ elements, $k \leq 7$. Then H admits a matching of X into Y.

Proof. We may assume $r(M) \geq k+1$ and if $r(M) = k+1$ then M has no coloop, so (4) and (5) hold. Since the theorem is true for $k = 1$, we may assume $k \geq 2$. We shall show that under these assumptions, H satisfies (3).

Let us fix $(A, B) \in Y$ and define sets

$$B_0 = B - \kappa(A) = \{b_1, b_2, \ldots, b_s\}$$

$$A_0 = A - \kappa(B) = \{a_1, a_2, \ldots, a_t\}$$

where $|B_0| = x$ and $|A_0| = t$. Then (A, B) is adjacent to the $s + t$ vertices

(6) $$(B \setminus b_j, A + b_j), \; b_j \in B_0$$

(7) $$(A \setminus a_i, B + a_i), \; a_i \in A_0$$

To determine the degree in H of these vertices, we define

$$B_i = B - \kappa(A \setminus a_i), \quad 1 \leq i \leq t$$

$$A_j = A - \kappa(B - b_j), \quad 1 \le j \le s$$

and let $t_j = |A_j|$, $s_i = |B_i|$. Since $A_j \supseteq A_0$ for $1 \le j \le s$ and $B_i \supseteq B_0$ for $1 \le i \le t$, we have using (5)

$$(8) \qquad 2 \le t \le t_j \le k, \quad 2 \le s \le s_j \le k \quad .$$

The vertices adjacent in H to (7) are (A, B), the s_i vertices $(A - a_i + b_j, B + a_i - b_j)$ for $b_j \in B_i$, and the mates of these. Thus $d_H(A - a_i, B + a_i) = 2(s_i + 1)$. The vertices adjacent in H to (6) are (B, A), the t_j vertices $(B - b_j + a_i, A + b_j - a_i)$ for $a_i \in A_j$ and the mates of these. Thus $d_H(B - b_j, A + b_j) = 2(t_j + 1)$ and thus

$$\sum_{(C,D)\sim(A,B)} \frac{1}{d_H(C,D)} = \sum_{i=1}^{t} \frac{1}{2(s_i+1)} + \sum_{j=1}^{s} \frac{1}{2(t_j+1)} .$$

To establish (3) we must show

$$(9) \qquad \sum_{i=1}^{t} \frac{1}{s_i + 1} + \sum_{j=1}^{s} \frac{1}{t_j + 1} \le 2 \quad .$$

Let $\int = \int(s_1, s_2, \ldots, s_t, t_1, t_2, \ldots, t_s)$ denote the function given by the left side of (9) whose domain is the set of admissible (s_i) and (t_j). $\int$ is a decreasing function of each s_i and t_j, and so by (8)

$$(10) \qquad \int \le \frac{t}{s+1} + \frac{s}{t+1} \quad .$$

The right hand side of (10) is ≤ 2 when

$$t(t+1)+s(s+1)\leq 2(t+1)(s+1) ,$$

that is when

$$(t-s)^2 \leq t+s+2 \quad . \tag{11}$$

We may, without loss of generality, assume that $t \geq s$. Then (11) holds and hence (9) for all (t, s) satisfying (8) with $k \leq 8$ except for $(t, s) = (6, 2), (7, 3), (7, 2), (8, 4), (8, 3), (8, 2)$. To deal with these cases for $k \leq 7$ we need the following Lemmas.

3.4 Lemma. For each subset $I \subseteq \{1, 2, \ldots, t\}$

$$|\bigcup_{i\in I} B_i| \geq |I| \quad . \tag{12}$$

If I is a proper subset of $\{1, 2, \ldots, n\}$

$$|\bigcup_{i\in I} B_i| > |I| \quad . \tag{13}$$

Proof. Clearly (12) holds if B is empty, so assume $B \neq \emptyset$. Let F be the flat of M defined by

$$F = \bigcap_{i\in I} \kappa(A - a_i) ,$$

where $\kappa(A)$ is the closure of A. Since the sets $A - a_i$ are subsets of an independent set A, we have

$$F = \kappa\Big(\bigcap_{i\in I}(A - a_i)\Big) = \kappa(A - \bigcup_{i\in I} a_i) ,$$

a flat of rank $k - |I|$. Since B is independent, $|F \cap B| \leq k - |I|$. But

$$F \cap B = \bigcap_{i\in I}(\kappa(A - a_i) \cap B) = \bigcap_{i\in I}(B - B_i) = B - \bigcup_{i\in I} B_i ,$$

so

$$|B - \bigcup_{i\in I} B_i| \leq k - |I| ,$$

which gives (12). Suppose now that equality holds in (12). Then $B - \bigcup_{i\in I} B_i$ is a basis of F, and therefore spans $A - \bigcup_{i\in I} a_i$. Thus

$$\kappa(B) \supseteq \kappa(B - \bigcup_{i\in I} B_i) \supseteq A - \bigcup_{i\in I} a_i .$$

But $\kappa(B) \cap A = A - A_0$, $A_0 \subseteq \bigcup_{i\in I} a_i$ and we conclude that $I = \{1, 2, \ldots, t\}$.

We assume $t > s$, and note that (13) implies that at most $s - 1$ of the B_i equal B_0. Hence the upper bound of (10) can be improved to

$$(14) \qquad \int \leq \frac{s-1}{s+1} + \frac{t-s+1}{s+2} + \frac{s}{t+1} ,$$

and the right hand side of (14) is ≤ 2 for $(t, s) = (6, 2), (7, 3)$ and $(8, 4)$.

<u>3.5 Lemma</u>. Let $C(b_j)$, $j = s+1, \ldots, k$ be the unique circuit $b_j \in C(b_j) \subseteq A \cup b_j$. Then

$$\sum_{i=1}^{t} (s_i - s) = \sum_{j=s+1}^{k} |C(b_j) \cap A_0| \quad . \tag{15}$$

<u>Proof</u>. For $1 \leq i \leq t$, $s_i - s = |B_i - B_0|$. $b_j \in B_i - B_0$ means $j \in \{s+1, \ldots, k\}$, $b_j \notin \kappa(A - a_i)$ but $b_j \in \kappa(A)$, which means $a_i \in C(b_j)$, $j \in \{s+1, \ldots, k\}$. To prove (15) we count ordered pairs $(b_j , B_i - B_0)$ where $b_j \in B_i - B_0$.

There remains the case $(t, s) = (7, 2)$. Since $s = 2$, at most one $s_i = 2$. If $s_i \geq 3$ for $1 \leq i \leq 7$, then $\int \leq \frac{7}{4} + \frac{2}{8} = 2$. If $s_1 = 2$ and at least two $s_i \geq 4$, then $\int \leq \frac{1}{3} + \frac{4}{4} + \frac{2}{5} + \frac{2}{8} < 2$. The remaining possibility is $s_1 = 2$ and (at least) $s_i = 3$, then, by (15),

$$\sum_{i=1}^{t} (s_i - 2) = \sum_{j=3}^{k} |C(b_j) \cap A_0| \geq \sum_{j=3}^{k} 2 = 2(k-2)$$

(i.e., $|C(b_j)| \geq 3$ as otherwise $a_i \in \kappa(B)$ for some $i \in \{1, \ldots, t\}$) . So, $2(k-2) + 2 \cdot 7 \leq \sum_{i=1}^{7} s_i = 2 + 5 \cdot 3 + s_7$. And $s_7 \geq 2k - 7 \geq 7$, for $k \geq 7$. Hence, $\int \leq \frac{1}{3} + \frac{5}{4} + \frac{1}{8} + \frac{2}{8} < 2$. This completes the proof of the theorem.

Actually, we have shown that (9) holds for all (t, s) satisfying (8) and (15) with $k \leq 8$ except for $(t, s) = (8, 3), (8, 2)$.

The case $(t, s) = (8, 3)$ is similar to the case $(t, s) = (7, 2)$. We see that since $s = 3$, $s_i = 3$ for at most two i's. If $s_i \geq 4$ for $i = 2, \ldots, 8$ then $\int \leq \frac{1}{4} + \frac{7}{5} + \frac{3}{9} < 2$. So we may assume $s_1 = s_2 = 3$ and $s_i \geq 4$ for $i \geq 3$. Then (15) gives

$$\sum_{i=1}^{8} (s_i - 3) \geq \sum_{j=4}^{8} 2 = 10 , \qquad \text{or} \qquad \sum_{i=1}^{8} s_i \geq 34 .$$

All 5-tuples $(s_3, \ldots, s_8)$ such that $4 \leq s_i \leq 8$ and $\sum_{i=3}^{8} s_i \geq 28$ are such that $\sum_{i=3}^{8} \frac{1}{s_i + 1} \leq 2 - \frac{3}{9} - \frac{2}{4}$. Hence, $\int \leq 2$ in the case $(t, s) = (8, 3)$.

The case $(t, s) = (8, 2)$ is a special case of the following theorem, where we show $\pi_7(M) \leq \pi_8(M)$ without showing the existence of an H-matching.

<u>3.6 Theorem</u>. Let M be a matroid of size $2k$ on E, and let $e_1, e_2 \in E$ be such that for $N = M \setminus \{e_1, e_2\}$, $r(N) = k$, $r(\{e_1, e_2\}) = 2$ and N has an $H(N)$-matching. Then $\pi_{k-1}(M) \leq \pi_k(M)$.

<u>Proof</u>. Let $\varphi : \Pi_{k-2}(N) \to \Pi_{k-1}(N)$ be the bijection defined by $\varphi(x) = y$ if x and y are the ends of some edge of the matching in $H(N)$. That is, for $(C, D) \in \Pi_{k-2}(N)$, $\varphi((C, D)) = (C + d, D - d)$ or $(D - d, C + d)$ for some $d \in D$. Using φ, we will define a bijection $\psi : \Pi_{k-1}(M) \to \Pi_k(M)$. ($\psi$ may not describe an $H(M)$-matching.) We will use the notation $\varphi(C, D)$ for $\varphi((C, D))$, $\psi(C, D)$ for $\psi((C, D))$.

We see that $r(M) \leq r(N) + r(\{e_1, e_2\}) = k+2$, and we may assume $r(M) > k$.

Case 1. $r(M) = k+2$. Then M is not connected and we can map as follows: Let $(C, D) \in \Pi_{k-1}(M)$;

(i) if $e_1, e_2 \in D$ let $\psi(C, D) = (C+e_1, D-e)$

(ii) if $e_1 \in C$, $e_2 \in D$ let $\psi(C, D) = (C+e_2, D-e_2)$

(iii) if $e_2 \in C$, $e_1 \in D$ let $\psi(C, D) = (D-e_1, C+e_1)$.

It is easily checked that this map is well-defined.

Case 2. $r(M) = k+1$. If $e_1 \in \kappa(N)$ then e_2 is a coloop and Lemma 3.2 applies. Hence we may assume $e_i \notin \kappa(N)$ for $i = 1, 2$. Define ψ as follows: Let $(C, D) \in \Pi_{k-1}(M)$,

(i) if $e_1 \in C$, $e_2 \in D$ and $\varphi(C-e_1, D-e_2) = (A_1, B_1)$ then let $\psi(C, D) = (A_1+e_1, B_1+e_2)$

(ii) if $e_2 \in C$, $e_1 \in D$, and $\varphi(C-e_2, D-e_1) = (A_1, B_1)$ then let $\psi(C, D) = (C+e_1, D-e_1)$ if $C+e_1$ is independent and let $\psi(C, D) = (A_1+e_2, B_1+e_1)$ if $C+e_1$ is dependent.

(iii) if $e_1e_2 \in D$, let $D_1 = D - e_1e_2$.

if there is no $\varphi^{-1}(D_1, C)$, then let $\psi(C, D) =$ $(D-e_2, C+e_2)$, if $\varphi^{-1}(D_1, C) = (D_1 - d, C + d)$, then let $\psi(C, D) = (C + d, D - d)$,

if $\varphi^{-1}(D_1, C) = (C - c, D_1 + c)$, then let

$$\psi(C, D) = \begin{cases} (D_1 + c, C - c + e_1e_2) & \text{if } C - c + e_1e_2 \text{ is independent} \\ (C + e_2, D - e_2) & \text{if } C - c + e_1e_2 \text{ is dependent.} \end{cases}$$

We must show that ψ is well-defined. Let $(A, B) \in \Pi_k(M)$. There are several cases for (A, B).

(1) If $e_1 \in A$, $e_2 \in B$, let $(A_1, B_1) = (A - e_1, B - e_2)$. Then (A, B) has pre-image in (i) iff $\varphi^{-1}(A_1, B_1)$ exists and (A, B) has pre-image in (iii) iff $\varphi^{-1}(A_1, B_1)$ does not exist.

(2) If $e_2 \in A$, $e_1 \in B$, let $(A_1, B_1) = (A - e_2, B - e)$. Then (A, B) has pre-image in (ii) iff $\varphi^{-1}(A_1, B_1) = (C_1, D_1)$ and $C_1 + e_1e_2$ is dependent. And (A, B) has pre-image in (iii) if $\varphi^{-1}(B_1, A_1) = (A_1 - a, B_1 + a)$ and $A_1 - a + e_1e_2$ is dependent, and $B + e_2$ is independent. If possible, suppose that $(C_1 + e_2, D_1 + e_1) \to (A, B)$ from (ii) and $(A_1, B_1 + e_1e_2) \to (A, B)$ from (iii). Then either $C_1 \subset A_1$ or $C_1 \subset B_1$. If $C_1 \subset A_1$, then $C_1 + e_1e_2$ is dependent and $A_1 - a + e_1e_2$ is dependent, but this is impossible as $A_1 + e_1$ is independent. If $C_1 \subset B_1$ then (ii) gives $C_1 + e_1e_2$ dependent and (iii) gives $B_1 + e_1e_2$ independent, another contradiction.

(3) The cases $e_1, e_2 \in A$ and $e_1, e_2 \in B$ are easily checked. We omit the details. We conclude that $\pi_{k-1}(M) \leq \pi_k(M)$.

As a corollary we get

<u>3.7 Corollary</u>. Let M be a matroid of size 2k , $k \leq 8$. Then $\pi_{k-1}(M) \leq \pi_k(M)$.

2. A counterexample

K :

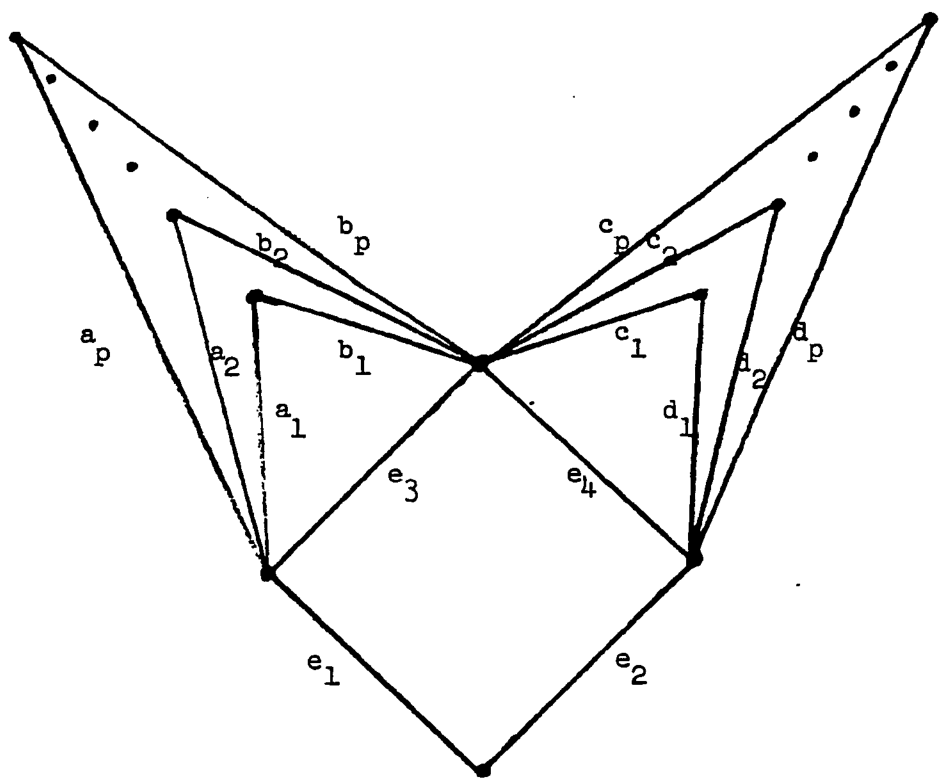

Figure 2

Consider the graph K of Fig. 2 with edge set $E = A \cup B \cup C \cup \{e_1, e_2, e_3, e_4\}$ where $A = \{a_1, \ldots, a_p\}$, $B = \{b_1, \ldots, b_p\}$, $C = \{c_1, \ldots, c_p\}$, $D = \{d_1, \ldots, d_p\}$. $|E| = 4p + 4 = 2(2p + 2)$. Let $k = 2(p + 1)$. Let $M = \mathcal{M}(K)$.

We see that for $y = (A + C + e_3 + e_4 , B + D + e_1 + e_2) \in \pi_k(M)$, $(t , s) = (k , 2)$ and

$$\Sigma = \sum_{x \sim y} \frac{1}{d_H(x)} = \frac{2p}{8} + \frac{2}{2(p+3)} + \frac{2}{2(2p+3)} = \frac{p}{4} + \frac{1}{p+3} + \frac{1}{2p+3}$$

That is, $(A + c + e_3 + e_4 , B + D + e_1 + e_2) = (X , Y)$ is adjacent to the $2p$ vertices $(X - a_i , Y + a_i)$, $(X - c_i , Y + c_i)$ of degree $2 \cdot 4$ and the 2 vertices $(X - e_3 , Y + e_3)$, $(X - e_4 , Y + e_4)$ of degree $2(p+3)$ and the 2 vertices $(Y - e_1 , X + e_1)$, $(Y - e_2 , X + e_2)$ of degree $2(2p+3)$.

If $p \geq 4$ then clearly $\Sigma > 1$ while if $p = 3$ then $\Sigma = \frac{37}{36} > 1$. Hence condition (3), sufficient but not necessary for a matching, is violated where $p \geq 3$. We next show that for p large enough, the graph $H(M)$ indeed does not have a matching, although the inequality $\pi_{k-1}(M) \leq \pi_k(M)$ is satisfied.

Sets of the form $\{a_i , b_i\}$ and $\{c_i , d_i\}$ we shall call wings. Let $W = A \cup B \cup C \cup D$ and

$$\mathcal{A} = \{(X , Y) \in \pi_{k-1}(M) : e_3 , e_4 \in X\} .$$

We will show that $|\mathcal{A}| > |N(\mathcal{A})|$.

For $(X , Y) \in \mathcal{A}$, $|Y \cap \{e_1 , e_2\}| \geq 1$, since otherwise the circuit $\{e_1 , e_2 , e_3 , e_4\} \subseteq X$. Let $\mathcal{A}_1 = \{(X , Y) \in \mathcal{A} : e_1 , e_2 \in Y\}$ and $\mathcal{A}_2 = \{(X , Y) \in \mathcal{A} : |Y \cap \{e_1 , e_2\}| = 1\}$, then $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$.

$(X, Y) \in \mathcal{A}_1$ implies Y cannot contain two wings. On the other hand, $|Y \cap W| = |Y| - |\{e_3, e_4\}| = 2p+1$ means Y must contain a wing. Since $e_3, e_4 \in X$, X cannot contain a wing. Hence $|\mathcal{A}_1| = (2p)2^{2p-1}$.

$(X, Y) \in \mathcal{A}_2$ implies $|X \cap W| = |X| - 3 = 2p-2$ and $|Y \cap W| = |Y| - 1 = 2p+2$. Hence Y must contain two wings and must meet every wing. So $|\mathcal{A}_2| = 2(p^2 2^{2p-1})$. So $|\mathcal{A}| = p(p+2)2^{2p-1}$.

$N(\mathcal{A}) = N \cup N'$ where N is the set of neighbors in $G(M)$, and N' is the set of mates of elements of N. Hence $|N(\mathcal{A})| = 2|N|$. Again, $N = N_1 \cup N_2$ where $N_1 = \{X, Y) \in N : e_1, e_2 \in Y\}$ and $N_2 = \{(X, Y) \in N : |Y \cap \{e_1, e_2\}| = 1\}$

$(X, Y) \in N_1$ implies $|X \cap W| = |X| - 2 = 2p$, and $|Y \cap W| = |Y| - 2 = 2p$. Now, X cannot contain a wing, as $\{e_3, e_4\} \subseteq X$, and hence neither can Y. So $|N_1| = 2^{2p}$.

$(X, Y) \in N_2$ implies $|X \cap W| = |X| - 3 = 2p-1$ and $|Y \cap W| = |Y| - 1 = 2p+1$. Then Y must meet every wing and contain one wing. So $|N_2| = 2(2p(2^{2p-1}))$.

Thus $|N(\mathcal{A})| = 2|N| = 2(2^{2p} + 2p\,2^{2p}) = 4(2p+1)2^{2p-1}$, $|\mathcal{A}| = p(p+2)2^{2p-1}$. For $p \geq 7$, $p(p+2) > 4(2p+1)$. Hence for every $k \geq 16$ we see that there is a matroid of size $2k$ which has no H-matching.

It is, however, easily verified that

$$\pi_{k-1}(M) = (p^2 + 6p + 8)2^{2p-1} \quad ,$$

$$\pi_k(M) = (p^2 + 6p + 6)2^{2p} \quad ,$$

that is, $\pi_{k-1}(M) \leq \pi_k(M)$.

3. The Independent Partition Conjecture

Since we cannot expect to produce a matching in the graph H to prove the inequality $\pi_{k-1}(M) \leq \pi_k(M)$, for M of size $2k$ we now turn to some results about the stronger

Conjecture 6: If M is a matroid of size m and

$$i \leq \frac{m+1}{2}, \text{ then } \pi_{i-1}(M) \leq \pi_i(M) .$$

Our first result shows that a smallest counterexample to Conjecture 6, should one exist, must be connected. Theorem 3.9 is based on a result of Andrews [75] which states that the product of two polynomials with symmetric unimodal coefficients is a polynomial with symmetric unimodal coefficients.

<u>3.8 Lemma</u>. Let $M = M_1 \oplus M_2$ where M_j has size m_j, $j = 1, 2$. Then

$$\pi_i(M) - \pi_{i-1}(M) = \sum_{0 \leq t \leq \frac{1}{2}m_1} [\pi_t(M_1) - \pi_{t-1}(M_1)][\pi_{i-t}(M_2) - \pi_{i-m_1+t-1}(M_2)] .$$

<u>Proof</u>. $(A, B) \in \Pi_i(M)$ if and only if $(A, B) = (A_1 + A_2, B_1 + B_2)$ where $(A_1, B_1) \in \Pi_t(M_1)$ and $(A_2, B_2) \in \Pi_{i-t}(M_2)$, $t = |A_1|$. Then

$$\pi_i(M) = \sum_{0 \leq t \leq m_1} \pi_t(M_1)\, \pi_{i-t}(M_2) .$$

And so

$$\pi_i(M) - \pi_{i-1}(M) = \sum_{t=-\infty}^{\infty} \pi_t(M_1)\pi_{i-t}(M_2) - \sum_{t=-\infty}^{\infty} \pi_t(M_1)\pi_{i-1-t}(M_2)$$

$$= \sum_{t=-\infty}^{\infty} \pi_t(M_1)\pi_{i-t}(M_2) - \sum_{t=-\infty}^{\infty} \pi_{t-1}(M_1)\pi_{i-t}(M_2)$$

$$= \sum_{t\le\frac{1}{2}(m_1+1)} [\pi_t(M_1) - \pi_{t-1}(M_1)][\pi_{i-t}(M_2)] + \sum_{\ell>\frac{1}{2}(m_1+1)} [\pi_\ell(M_1) - \pi_{\ell-1}(M_1)][\pi_{i-\ell}(M_2)]$$

$$= \sum_{t\le\frac{1}{2}(m_1+1)} [\pi_t(M_1) - \pi_{t-1}(M_1)][\pi_{i-t}(M_2)] + \sum_{t<\frac{1}{2}(m_1+1)} [\pi_{t-1}(M_1) - \pi_t(M_1)][\pi_{i-m_1+t-1}(M_2)] .$$

To get the last line we let $t = m_1 - \ell + 1$, then $\pi_\ell(M_1) = \pi_{m_1-\ell}(M_1) = \pi_{t-1}(M_1)$ and $\pi_{\ell-1}(M_1) = \pi_{m_1-\ell+1}(M_1) = \pi_t(M_1)$; further, if $\frac{1}{2}(m_1+1) < t \le m_1+1$ then $0 \le \ell < \frac{1}{2}(m_1+1)$. Also, $\pi_{i-\ell}(M_2) = \pi_{i-m_1+t-1}(M_2)$. Now, if we use the fact that $\pi_t(M_1) = \pi_{t-1}(M_1)$ if $t = \frac{1}{2}(m_1+1)$, then we get

$$\pi_i(M) - \pi_{i-1}(M) = \sum_{t\le\frac{1}{2}m_1} [\pi_t(M_1) - \pi_{t-1}(M_1)][\pi_{i-t}(M_2) - \pi_{i-m_1+t-1}(M_2)] .$$

<u>3.9 Theorem</u>. Let $M = M_1 \oplus M_2$ where M_j has size m_j, $j = 1, 2$. If

$$\pi_{i-1}(M_j) \le \pi_i(M_j) \quad \text{for} \quad i \le \tfrac{1}{2}(m_j+1) \quad \text{and} \quad j = 1, 2 ,$$

then

$$\pi_{i-1}(M) \le \pi_i(M) \quad \text{for} \quad i \le \tfrac{1}{2}(m+1) \ .$$

Proof. From Lemma 3.8 we have

$$\pi_i(M) - \pi_{i-1}(M) = \sum_{t \le \frac{1}{2}m_1} [\pi_t(M_1) - \pi_{t-1}(M_1)][\pi_{i-t}(M_2) - \pi_{i-m_1+t-1}(M_2)] \ .$$

We shall show that if $i \le \frac{1}{2}(m_1 + 1)$ then every term in this sum is nonnegative. By assumption, $\pi_t(M_1) \ge \pi_{t-1}(M_1)$ for $t \le \frac{1}{2}(m_1)$. Also, $t \le \frac{1}{2}(m_1)$, $i \le \frac{1}{2}(m_1 + m_2 + 1)$ implies $i - m_1 + t - 1 < \frac{1}{2}(m_2)$, and both $i - t \ge i - m_1 + t - 1$ and $m_2 - i + t \ge i - m_1 + t - 1$. Since one of $i - t$ or $m_2 - i + t$ is less than $\frac{1}{2}(m_2 + 1)$, we see that

$$\pi_{i-t}(M_2) \ge \pi_{i-m_1+t-1}(M_2) \ .$$

Our next result shows that a smallest counterexample to Conjecture 6, should one exist, has a flat whose rank is larger than half its size.

3.10 Theorem. If M has a flat F such that $r(F) \le \frac{|F|}{2}$ and Conjecture 6 holds for M/F and $M|F$, then Conjecture 6 is true for M.

Proof. If M has a flat F such that $r(F) < \frac{|F|}{2}$ then $\pi_i(M) = 0$ for all i. That is, if $(A, B) \in \Pi_i(M)$ then one of $|A \cap F| \ge \frac{|F|}{2}$ or $|B \cap F| \ge \frac{|F|}{2}$ is true. So we may assume $r(F) = \frac{|F|}{2}$. Then for $(A, B) \in \Pi_i(M)$ we must have $|A \cap F| = |B \cap F| = \frac{|F|}{2} = f$. Then $\pi_i(M) = \pi_f(M|F) \pi_{i-f}(M/F)$ and $i \le \frac{1}{2}(m+1)$ implies $i - f \le \frac{m - 2f + 1}{2}$.

CHAPTER IV

The Main Theorem

1. Series-parallel Networks

There is a simple type of electrical network termed a <u>series-parallel connection</u> which occurs frequently in both theoretical and applied electrical engineering. We have taken the following description from Duffin [65].

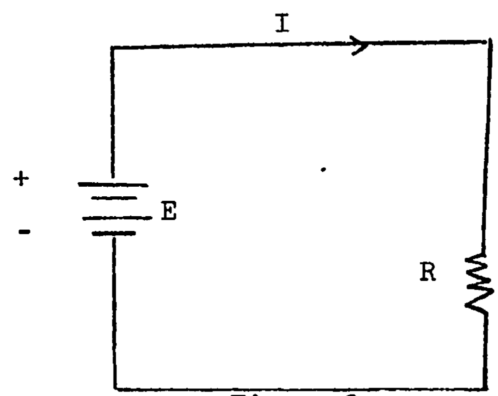

Figure 3

A simple circuit

Shown in Fig. 3 is a simple circuit containing a battery of voltage E and a resistor of resistance R ohms. Then the current I flowing in the circuit is determined by the relation

$$E/I = R > 0$$

This is Ohm's law.

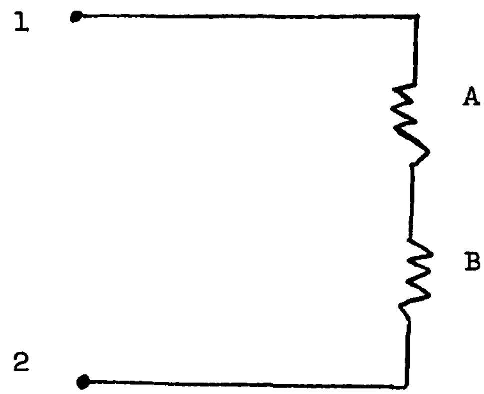

Figure 4

Resistors in series

Shown in Fig. 4 are two resistors connected in series. One resistor has resistance A ohms and the other has resistance B ohms. Then the joint resistance R between terminals 1 and 2 is given by the formula

$$R = A + B \quad .$$

On the other hand, the two resistors could be connected in parallel as shown in Fig. 5.

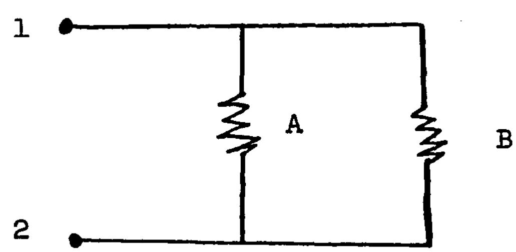

Figure 5

Resistors in parallel

Conductance is the reciprocal of resistance and conductances add in the parallel connection, so

$$R^{-1} = A^{-1} + B^{-1} \quad .$$

Solving for R gives

$$R = \frac{AB}{A+B} \quad ,$$

and this is the formula for the joint resistance R of two resistors in parallel. Let

$$A : B = \frac{AB}{A+B}$$

Then $A : B$ may be regarded as a new operation termed <u>parallel addition</u>. Parallel addition is defined for any nonnegative numbers not both zero. The network model shows that parallel addition is commutative and associative. Moreover, multiplication is distributive over this operation.

Consider now an algebraic expression in the operations $(+)$ and $(:)$ operating on positive numbers A, B, C, etc. An example is

$$R = A + B : (C + D : E) \quad . \tag{16}$$

To give a network interpretation of such a polynomial, read $A + B$ as "A series B" and $A : B$ as "A parallel B"; then it is clear that the expression (16) is the joint resistance of the network shown in <u>Fig. 6</u>.

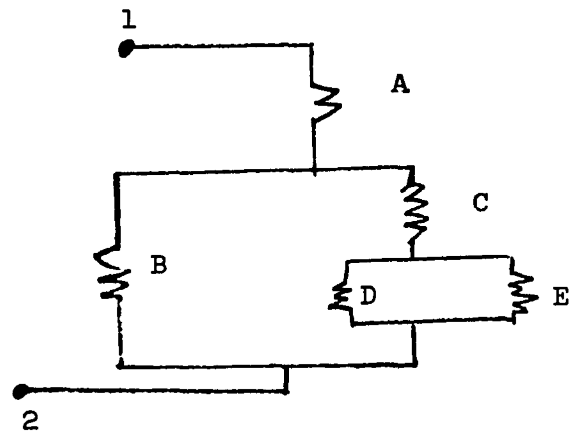

Figure 6

A series-parallel connection

Networks obtained from such polynomials are termed series-parallel connections.

Not every network is a series-parallel connection. In particular, it can be checked that the Wheatstone bridge connection of Fig. 7 is not a series-parallel connection.

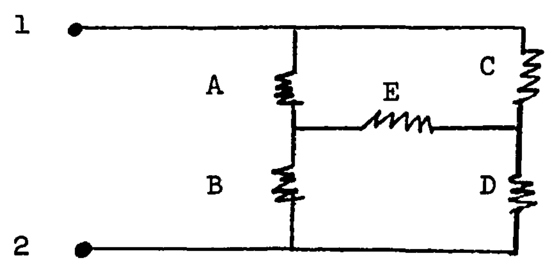

Figure 7

The Wheatstone Bridge connection

We can generalize these concepts to matroids, as suggested by Minty [66] and given in Welsh [76].

Let M be a matroid on E and let $e \in E$ and suppose $x \notin E$. The series extension of M at e by x is the matroid

$sM(e, x)$ on $E+x$ whose independent sets are of the form 1) or 2)

1) $A+x$, where A is independent in M

2) $A+e$, where A is independent in M and $e \notin A$.

Similarly the <u>parallel extension</u> of M is the matroid $pM(e, x)$ on $E+x$ whose independent sets are of the form 3) or 4)

3) A, where A is independent in M

4) $(A \setminus e)+x$ where $e \in A$ and A is independent in M.

A matroid N is a <u>series-parallel extension</u> of a matroid M if N is a series or parallel extension of M or if $N = M \oplus M_1$ where M_1 is a matroid of size 1. A <u>series-parallel matroid</u> is one which can be obtained from a matroid on one element by successive series-parallel extensions.

It is clear that if we replace a given edge e of a graph G by a pair of edges e_1, e_2 in series or by a pair of parallel edges e_1, e_2, and if G_1, G_2 denote respectively these series and parallel extensions of G, then

$$\mathfrak{m}(G_1) \cong s\mathfrak{m}(G)(e, e_1)$$

$$\mathfrak{m}(G_2) \cong p\mathfrak{m}(G)(e, e_1) .$$

A graph is a <u>series-parallel network</u> if each connected component can be obtained from a link or loop graph by successive series and parallel extensions. Trivially, if G is a series-parallel network then $\mathcal{M}(G)$ is a series-parallel matroid. Minty points out that the converse is also true. That is, a series-parallel matroid is nothing more than the polygon matroid of a series-parallel network.

Using the following Lemma due to Dirac [52], we can give an alternate characterization of series-parallel networks. A subdivision of a graph G is a graph that can be obtained from G by a sequence of edge subdivisions.

4.1 Lemma. A nonseparable simple graph in which the degree of every vertex is at least three has a subgraph which is a subdivision of K_4.

Proof. Let G be such a graph. Then G has no cut-vertex and so has a circuit. We show first that G must have a circuit of length at least four. Since G is simple, it has no circuit of length less than three. If possible, let C be a circuit of length three with vertices v_1, v_2, v_3. Since the degree of v_1 is greater than or equal to 3, there is a vertex $v \neq v_2, v_3$ adjacent to v_1. Then v is joined to v_2 by a path avoiding v_1 and hence to one of v_2, v_3 by a path whose interior avoids v_1, v_2, v_3. But this gives a circuit of size at least four.

Let C be a circuit of maximum length, and let the vertices of C in cyclic order be $v_1, v_2, \ldots, v_n, v_1$ $(n \geq 4)$. A path between two non-adjacent vertices of C having only its initial and terminal vertices in common with C will be called a chord of C.

We claim: every vertex of C is connected by a chord to another vertex of C. Without loss of generality, we may consider v_1. It is joined to v_2 and v_n and to at least one other vertex u of G. If $u \in V(C)$ then we have the claim. Assume $u \notin V(C)$. The vertex v_1 is not a cut-vertex so there is a path P from u to v_2 which avoids v_1. One of its interior vertices is on C as C is a longest circuit. Let v_i be the first vertex of C in the interior of P. Then $i \neq n$ and $i \neq 2$, as then C would not be a longest circuit. Hence there is a chord from v_1 to v_i.

Now, if there are two chords of C which join different pairs of vertices of C and which have a common vertex we obtain a subgraph homeomorphic to K_4 using three vertices on C and the vertex common to the two chords. So assume that every pair of chords from distinct pairs of vertices of C do not intersect. Every chord divides C into two arcs. Let P be a chord which has one of the two arcs of shortest length. Say P' joins non-adjacent vertices v_i, v_k where $i < k$, and the arc with vertices $v_i, v_{i+1}, \ldots, v_k$ is shortest. Then there is v_j, $i < j < k$, on this arc. v_j is joined by a chord to some non-adjacent vertex v_ℓ of C. Then, because of our choice of shortest arc, $\ell \notin \{i, i+1, \ldots, k\}$, and we obtain a subdivision of K_4 using vertices v_i, v_k, v_j, v_ℓ, the two chords, and the circuit C.

We are now ready to state the excluded minor characterization of series-parallel matroids as suggested in Duffin [65].

4.2 Theorem. Let M be a matroid. The following are equivalent:

(a) M is a series-parallel matroid.

(b) M is graphic and does not contain $\mathcal{M}(K_4)$ as a minor.

(c) M is the polygon matroid of a graph G which does not contain a subgraph which is a subdivision of K_4.

The characterization we most often use is given in (c). For other characterizations of series-parallel matroids see Welsh [76].

2. Outerplanar Graphs

A graph is said to be *embedded* in a surface S when it is drawn on S so that no two edges intersect at an interior point. A graph is *planar* if it can be embedded in the plane; a *plane graph* has already been embedded in the plane. We will refer to the regions defined by a plane graph as its *faces*, the unbounded region being called the *exterior face*. A *plane map* is a connected plane graph together with all its faces. A planar graph is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the same face; we usually choose this face to be the exterior. It is easy to see that if G is outerplanar and e is an edge of G, then $G - e$ and G/e are both outerplanar. Hence any minor of an outerplanar graph is outerplanar.

All plane embeddings of K_4 and $K_{2,3}$ are of the forms shown in *Fig. 8* in which each has a vertex inside the exterior circuit.

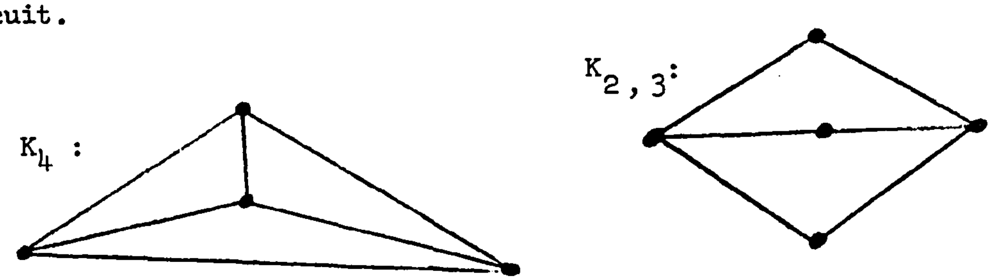

Figure 8

Chartrand and Harary [67] observed that these are the two basic non-outerplanar graphs.

4.3 Theorem. A graph is outerplanar if and only if it has no subgraph which is a subdivision of K_4 or $K_{2,3}$.

Proof. The "only if" statement is clear. We prove the "if" statement. If possible, assume that there is a graph which is not outerplanar and which has no subgraph which is a subdivision of K_4 or $K_{2,3}$. Among all such graphs let G be such that $|E(G)|$ is minimum. Then G is nonseparable and simple. Hence, by Lemma 4.1, G has a vertex v of degree 2. Let v be incident with edges e_1 and e_2 which join v to v_1 and v_2, respectively. It must be that G/e_1 has no subgraph which is a subdivision of K_4 or $K_{2,3}$; hence G/e_1 is outerplanar. Consider a planar embedding of G/e_1 in which all the vertices lie on the exterior face. If there is such an embedding in which e_2 is on the exterior face, then G is outerplanar, as G is obtained from G/e_1 by subdividing the edge e_2. So assume e_2 is not on the exterior face in any outerplanar embedding of G/e_1. The vertices v_1 and v_2 divide the boundary of the exterior face into two arcs. Since e_2 is not on this boundary, there is a vertex on each arc, say v_3 and v_4. But then G has a subgraph which is a subdivision of $K_{2,3}$ using the vertices v_1, v_2 and v_3, v_4, v. This contradiction proves the theorem.

4.4 Corollary. An outerplanar graph is a series-parallel network.

3. The Main Theorem

Let G be a graph. A handle is a simple path in G whose interior vertices have degree 2. A graph G which has a handle P and an edge $e \notin P$ such that $P+e$ is a circuit is said to have a handle on an edge.

4.4 Lemma. Let G be a nonseparable graph of size ≥ 2.

1) If G is a series-parallel network, then G has handles P, Q with common end vertices.

2) If G is outerplanar, then G has a handle on an edge.

Proof. We proceed by induction on the number of edges of G.

1) Suppose G is a series-parallel network. If G is not simple, then there are parallel edges u, v and hence handles $P . \{u\}$, $Q = \{v\}$. So we may assume G is simple of size ≥ 3 and nonseparable and therefore, by Lemma 4.1, G has a vertex v of degree 2 on edges e, e'. Then G/e is series-parallel, nonseparable of size ≥ 2 and hence has handles P', Q' with common end vertices. Since G is obtained from G/e by subdividing an edge e, G has handles P, Q with common end vertices.

2) Suppose G is outerplanar. Then G is a series-parallel network and therefore has handles P, Q with common end vertices. Among all such "double handles" choose a pair such that

the length of P is longest. If Q has length one we are done. So assume Q has length greater than one. Let the common end vertices be u, v. We must have $d_G(u) \geq 3$, as otherwise we would contradict our choice of P. Since G is nonseparable, there is another path R from u to v. If R has length one we are done. But R cannot have length greater than one, as then G would have a subgraph which is a subdivision of $K_{2,3}$.

If G is series-parallel with double handles P and Q of sizes p, q respectively, let

$$\Pi_i(G) = \Pi_i(\mathcal{M}(G)) \ , \ \pi_i(G) = \pi_i(\mathcal{M}(G)) \ ,$$

$$H = G \setminus (P+Q) \qquad\qquad K = G/(P+Q)$$

$$V = \{(A, B) \in \Pi_i(G) : P \subseteq A \ , \ Q \subseteq B\} \ , \ W = \{(A, B) \in \Pi_i(G) : Q \subseteq A \ , \ P \subseteq B\}$$

$$X = \{(A, B) \in \Pi_i(G) : P \cap A \neq \emptyset \ , \ P \cap B \neq \emptyset\} \ , \ Y = \{(A, B) \in \Pi_i(G) : Q \cap A \neq \emptyset \ , \ Q \cap B \neq \emptyset\}$$

$$Z = \{(A, B) \in \Pi_i(G) : P \cap A \neq \emptyset \ , \ P \cap B \neq \emptyset \ , \ Q \cap A \neq \emptyset \ , \ Q \cap B \neq \emptyset\} \quad .$$

Then,

$$\pi_i(G) = |V| + |W| + |X| + |Y| - |Z| \ , \tag{17}$$

We now derive formulas for the cardinalities of the sets appearing in (17). First we note that if C_p is a circuit of size p , then

$$\pi_i(C_p) = \begin{cases} \binom{p}{i} & \text{if } o < i < p \\ 0 & \text{otherwise.} \end{cases}$$

Now, $(A, B) \in V$ if and only if $(A \setminus P, B \setminus Q) \in \Pi_{i-p}(K)$ and $(A, B) \in W$ if and only if $(A \setminus Q, B \setminus P) \in \Pi_{i-q}(K)$. Further, $(A, B) \in X$ if and only if $(A - P, B - P) \in \Pi_{i-t}(G \setminus P)$ where $t = |A \cap P|$; and $(A, B) \in Y$ if and only if $(A \setminus Q, B \setminus Q) \in \Pi_{i-t}(G \setminus Q)$ where $t = |A \cap Q|$. Finally, $(A, B) \in Z$ if and only if $(A \setminus (P+Q), B \setminus (P+Q) \in \Pi_{i-t-s}(H)$ where $t = |A \cap P|$, $s = |A \cap Q|$. Hence,

$$|V| + |W| + |X| + |Y| - |Z| = \pi_{i-p}(K) + \pi_{i-q}(K) +$$

$$\sum_{t=1}^{p-1} \binom{p}{t} \pi_{i-t}(G \setminus P) + \sum_{t=1}^{q-1} \binom{q}{t} \pi_{i-t}(G \setminus Q) - \sum_{t=1}^{p-1} \binom{p}{t} \sum_{s=1}^{q-1} \binom{q}{s} \pi_{i-t-s}(H)$$

$$= \pi_{i-p}(K) + \pi_{i-q}(K) + \sum_{t=0}^{p} \pi_t(C_p)\, \pi_{i-t}(G \setminus P) + \sum_{t=0}^{q} \pi_t(C_q)\, \pi_{i-t}(G \setminus Q)$$

$$- \sum_{t=0}^{p} \sum_{s=0}^{q} \pi_t(C_p)\, \pi_s(C_q)\, \pi_{i-t-s}(H) .$$

Hence the following formula for a series-parallel network G ,

$$\pi_i(G) = \pi_i((G\setminus P)\oplus C_p) + \pi_i((G\setminus Q)\oplus C_q) - \pi_i(C_p\oplus C_q\oplus H) + \pi_{i-p}(K) + \pi_{i-q}(K) .$$

When G is outerplanar, we may choose Q so that Q has length one, so C_q is a loop. Then we have

$$\pi_i(G) = \pi_i((G\setminus P)\oplus C_p) + \pi_{i-p}(K) + \pi_{i-1}(K) .$$

Lemma 4.5. Let G be a nonseparable outerplanar graph. Either G is circuit or G has handles P_1, P_2 on edges e_1, e_2, respectively, sucht that $|(P_1+e_1)\cap(P_2+e_2)| \leq 1$.

Proof. Let m be the number of edges of G . We proceed by induction on m . The theorem is true if G is a circuit, so we may assume G is not a circuit and $m \geq 3$.

Let P be a handle on an edge e^* . If $G\setminus P$ is a circuit C , then P and $C\setminus e^*$ are disjoint handles on e^* . Hence we may assume $G\setminus P$ is not a circuit. Let $G\setminus P$ have handles Q_1, Q_2 on edges e_1, e_2, respectively, such that $|(Q_1+e_1)\cap(Q_2+e_2)| \leq 1$.

If $e^* \notin Q_1+e_1$, then G has disjoint handles P, Q_1 on distinct edges e^*, e_1, respectively. Similarly, the theorem is true if $e^* \notin Q_2+e_2$.

If $e^* = e_1 = e_2$, then G has pairwise disjoint handles P, Q_1, Q_2 on the edge e^* .

If $e^* \in Q_1$ and $e^* = e_2$, then $Q_1 = \{e_2\} = \{e^*\}$ and G has disjoint handles $P, \{e_1\}$ on the edge e^*. Similarly, the theorem is true if $e^* = e_1 \in Q_2$.

If $e^* \in Q_1 \cap Q_2$, then $e_1 \neq e_2$ and $Q_1 = Q_2 = \{e^*\}$. In this case G has pairwise disjoint handles P, $\{e_1\}$, $\{e_2\}$ on the edge e^*.

Since we have exhausted all the cases, the theorem is proved.

<u>4.6 Main Theorem</u>. Let G be an outerplanar graph with m edges and no loops, and let e be an edge of G. Then for $i \leq \frac{1}{2}(m+1)$,

$$1) \quad \pi_i(G) - \pi_{i-1}(G) \geq \pi_i(G/e) - \pi_{i-1}(G/e)$$

and

$$2) \quad \pi_i(G) \geq \pi_{i-1}(G) .$$

<u>Proof</u>. For $k \in \mathbb{Z}$, let

$$\Delta_i^k(G, e) = \pi_i(G) - \pi_{i-k}(G) - \pi_i(G/e) + \pi_{i-k}(G/e) .$$

To prove 1) we must show $\Delta_i^1(G, e) \geq 0$ for $i \leq \frac{1}{2}(m+1)$. Since, for $k \geq 1$,

$$\Delta_i^k(G, e) = \sum_{0 \leq \ell < k} \Delta_{i-\ell}^1(G, e) ,$$

we have $\Delta_i^k(G, e) \geq 0$ for $k \geq 1$, whenever $\Delta_{i-\ell}^1(G, e) \geq 0$ for $0 \leq \ell < k$.

We proceed by induction on m. The theorem is true for $m = 0$ as then $\pi_0(G) = 1$. The theorem is true for $m = 1$ as then $\pi_1(G) = \pi_0(G) = \pi_0(G/e) = 1$, and $\pi_1(G/e) = 0$. Hence we may assume $m \geq 2$.

If $i = \frac{1}{2}(m+1)$ then $\pi_i(G) = \pi_{i-1}(G)$ and $\pi_{i-1}(G/e) \geq \pi_{i-2}(G/e) = \pi_i(G/e)$. Hence we may assume $i \leq \frac{1}{2}(m)$.

If G is not simple, then G has parallel edges e_1, e_2. Let $G' = G/\{e_1, e_2\}$, then for $i \in \mathbb{Z}$,

$$\pi_i(G) = 2\pi_{i-1}(G') \quad .$$

If G' has a loop, then $\pi_i(G) = \pi_i(G/e) = 0$ for $i \in \mathbb{Z}$ and any edge e of G. Hence we may assume that G' has no loops so that the theorem is true for G'. Let e be an edge of G. If $e = e_1$, then $\pi_i(G/e) = 0$ for $i \in \mathbb{Z}$; hence, for $i \leq \frac{1}{2}(m)$,

$$\Delta_i^1(G) = \pi_i(G) - \pi_{i-1}(G) = 2[\pi_{i-1}(G') - \pi_{i-2}(G')] \geq 0 \ .$$

If $e \notin \{e_1, e_2\}$, then e_1 and e_2 are parallel edges in G/e and

$$\pi_i(G/e) = 2\pi_{i-1}(G'/e)$$

for $i \in \mathbb{Z}$; hence for $i \leq \frac{1}{2}(m)$,

$$- \Delta_i^1(G) = 2\Delta^1{}_{i-1}(G') \geq 0 .$$

Hence we may assume G is simple.

If G is separable, then G has subgraphs G_1, G_2 such that

$$\pi_i(G) = \pi_i(G_1 \oplus G_2) .$$

Let G_j have $m_j \geq 1$ edges, $j = 1, 2$. Since G does not have a loop, G_1 and G_2 do not have loops. Using Lemma 3.8 we have

$$\pi_i(G) - \pi_{i-1}(G) = \sum_{0 \leq t \leq \frac{1}{2}m} [\pi_t(G_1) - \pi_{t-1}(G_1)][\pi_{i-t}(G_2) - \pi_{i-m_1+t-1}(G_2)] .$$

Suppose e is an edge of G_2. Without loss of generality we may assume G_2 is nonseparable. If $m_2 = 1$, then $G/e = G_1$ and

$$\pi_i(G) = \pi_i(G/e) + \pi_{i-1}(G/e) = \pi_i(G_1) + \pi_{i-1}(G_1) .$$

Therefore, for $i \leq \frac{1}{2}(m)$,

$$\Delta_i^1(G, e) = [\pi_i(G_1) + \pi_{i-1}(G_1)] - [\pi_{i-1}(G_1) + \pi_{i-2}(G_1)] - \pi_i(G_1) + \pi_{i-1}(G_1)$$

$$= \pi_{i-1}(G_1) - \pi_{i-2}(G_1) \geq 0 .$$

If $m_2 > 1$, then G simple, G_2 nonseparable implies $m_2 \geq 3$. Then

$$\Delta_i^1(G, e) = \sum_{0 \leq t \leq \frac{1}{2}m_1} [\pi_t(G_1) - \pi_{t-1}(G_1)][\pi_{i-t}(G_2) - \pi_{i-m_1+t-1}(G_2)]$$

$$- \sum_{0 \leq t \leq \frac{1}{2}m_1} [\pi_t(G_1) - \pi_{t-1}(G_1)][\pi_{i-t}(G_2/e) - \pi_{i-m_1+t-1}(G_2/e)]$$

$$= \sum_{0 \leq t \leq \frac{1}{2}m_1} [\pi_t(G_1) - \pi_{t-1}(G_1)][\Delta_{i-t}^{m_1-2t+1}(G_2, e)] .$$

Now, $0 \leq t \leq \frac{1}{2}m_1$ implies $m_1 - 2t + 1 \geq 1$, hence if $i - t \leq \frac{1}{2}(m_2 + 1)$ we have

$$\Delta_{i-t}^{m_1-2t+1}(G_2, e) \geq 0 .$$

If $i - t > \frac{1}{2}(m_2 + 1)$, then $i - m_1 + t - 1 \leq m_2 - i + t < \frac{1}{2}(m_2 - 1)$, and

$$\Delta_{i-t}^{m_1-2t+1}(G_2, e) = \pi_{i-t}(G_2) - \pi_{i-m_1+t-1}(G_2) - \pi_{i-t}(G_2/e) + \pi_{i-m_1+t-1}(G_2/e)$$

$$= \pi_{m_2-i+t}(G_2) - \pi_{i-m_1+t-1}(G_2) - \pi_{m_2-1-i+t}(G_2/e) + \pi_{i-m_1+t-1}(G_2/e)$$

$$= \pi_{m_2-i+t}(G_2) - \pi_{m_2-i+t-1}(G_2) + \pi_{m_2-i+t-1}(G_2) - \pi_{i-m_1+t-1}(G_2)$$

$$- \pi_{m_2-i+t-1}(G_2/e) + \pi_{i-m_1+t-1}(G_2/e)$$

$$= \pi_{m_2-i+t}(G_2) - \pi_{m_2-i+t-1}(G_2) + \Delta^{m-2i}_{m_2-i+t-1}(G_2, e) .$$

Since $m_2 - i + t \leq \frac{1}{2}(m_2+1)$, if $m - 2i \geq 1$ then we can use induction to prove 1) . But if $m = 2i$, then

$$\Delta^{m-2i}_{m_2-1+t-1}(G_2, e) = 0 .$$

We use Theorem 3.9 to prove 2) .

Hence we may assume G is simple, nonseparable and $m \geq 3$. Let P be a handle of length $p \geq 2$ on an edge e^* . Let e be an edge of G . There are three cases to consider.

<u>Case 1</u>. $e \notin P + e^*$.

Then G/e is outerplanar with no loops and P is a handle on the edge e^* . Let $H = G \setminus P$, $K = G/(P+e^*)$, $C = C_p$, $\overline{H} = H/e$, $\overline{K} = K/e$, and $h = |E(H)|$. Since G is simple, h is greater than 2 , and K has no loops.

$$\Delta^1_i(G) = [\pi_i(C \oplus H) + \pi_{i-1}(K) + \pi_{i-p}(K)] - [\pi_{i-1}(C \oplus H) + \pi_{i-2}(K) + \pi_{i-1-p}(K)]$$

$$- [\pi_i(C \oplus \overline{H}) + \pi_{i-1}(\overline{K}) + \pi_{i-p}(\overline{K})] + [\pi_{i-1}(C \oplus \overline{H}) + \pi_{i-2}(\overline{K}) + \pi_{i-1-p}(\overline{K})]$$

$$= \pi_i(C \oplus H) - \pi_{i-1}(C \oplus H) - \pi_i(C \oplus \overline{H}) + \pi_{i-1}(C \oplus \overline{H})$$

$$+ \pi_{i-1}(K) - \pi_{i-2}(K) - \pi_{i-1}(\overline{K}) + \pi_{i-1}(\overline{K})$$

$$+ \pi_{i-p}(K) - \pi_{i-p-1}(K) - \pi_{i-p}(\overline{K}) + \pi_{i-p-1}(\overline{K})$$

$$= \sum_{0 \le t \le \frac{p}{2}} [\pi_t(C) - \pi_{t-1}(C)][\pi_{i-t}(H) - \pi_{i-p+t-1}(H) - \pi_{i-t}(\overline{H}) + \pi_{i-p+t}(\overline{H})]$$

$$+ \Delta^1_{i-1}(K , e) + \Delta^1_{i-p}(K , e)$$

$$= \sum_{0 \le t \le \frac{p}{2}} [\pi_t(C) - \pi_{t-1}(C)[\Delta^{p-2t+1}_{i-t}(H , e)] + \Delta^1_{i-1}(K , e) + \Delta^1_{i-p}(K , e)$$

$$= \sum_{2 \le t \le \frac{p}{2}} [\pi_t(C) - \pi_{t-1}(C)][\Delta^{p-2t+1}_{i-t}(H , e)] + [\Delta^{p-1}_{i-1}(H , e) + \Delta^1_{i-1}(K , e)]$$

$$+ [\Delta^1_{i-p}(K , e)] .$$

To complete the proof of 1), we show that each bracketed term is nonnegative. Since C is not a loop, $\pi_t(C) \ge \pi_{t-1}(C)$ for $1 \le t \le \frac{p}{2}$. If $1 \le t \le \frac{p}{2}$, then $p - 2t + 1 \ge 1$. Hence if $i - t \le \frac{1}{2}(h + 1)$ we have

$$\Delta^{p-2t+1}_{i-t}(H , e) \ge 0 .$$

If $i - t > \frac{1}{2}(h + 1)$, then $h - i + t < \frac{1}{2}(h - 1)$ and

$$\Delta^{p-2t+1}_{i-t}(H , e) = \pi_{i-t}(H) - \pi_{i-p+t-1}(H) - \pi_{i-t}(H / e) + \pi_{i-p+t-1}(H / e)$$

$$= \pi_{h-i+t}(H) - \pi_{h-i+t-1}(H) + \Delta^{m-2i}_{h-i+t-1}(H , e) \ge 0 .$$

If $i-1 \leq \frac{1}{2}(|E(K)|+1)$, then

$$\Delta^{p-1}_{i-1}(H, e) + \Delta^{1}_{i-1}(K, e) \geq 0 .$$

If $i-1 > \frac{1}{2}(h)$, then $h-i+1 < \frac{1}{2}(h)$ and

$$\Delta^{p-1}_{i-1}(H, e) + \Delta^{1}_{i-1}(K, e) = \pi_{i-1}(H) - \pi_{i-p}(H) - \pi_{i-1}(\overline{H}) + \pi_{i-p}(\overline{H})$$

$$+ \pi_{i-1}(K) - \pi_{i-2}(K) - \pi_{i-1}(\overline{K}) + \pi_{i-2}(\overline{K})$$

$$= \pi_{h-i+1}(H) - \pi_{i-p}(H) - \pi_{h-i}(\overline{H}) + \pi_{i-p}(\overline{H}) + \pi_{h-i}(K) - \pi_{h-i+1}(K) - \pi_{h-i-1}(\overline{K})$$

$$+ \pi_{h-i}(\overline{K})$$

$$= \pi_{h-i-1}(H) - \pi_{h-i}(H) - \pi_{h-i+1}(K) + \pi_{h-i}(K) + \Delta^{m-2i}_{h-i}(H, e) + \pi_{h-i}(\overline{K})$$

$$- \pi_{h-i-1}(\overline{K})$$

$$= \Delta^{1}_{h-i+1}(H, e^*) + \Delta^{m-2i}_{h-i}(H, e) + \pi_{h-i}(\overline{K}) - \pi_{h-i-1}(\overline{K}) \geq 0 .$$

(If $\overline{K}$ has a loop, then $\pi_{h-i}(\overline{K}) = \pi_{h-i-1}(\overline{K}) = 0$; if $\overline{K}$ has no loops then, by induction, $\pi_{h-i}(\overline{K}) \geq \pi_{h-i-1}(\overline{K})$.)

Since $i \leq \frac{1}{2}(m) = \frac{1}{2}(h+p)$ inplies $i-p \leq \frac{1}{2}(h-p) < \frac{1}{2}(|E(K)|+1)$, $i-p \leq \frac{1}{2}(h-p) < \frac{1}{2}(|E(K)|+1)$, we have

$$\Delta^{1}_{i-p}(K, e) \geq 0 .$$

This proves 1). By induction, since G/e has no loops and $i \leq \frac{1}{2}(m-1)+1)$, we have

$$\pi_i(G/e) - \pi_{i-1}(G/e) \geq 0 .$$

Hence 1) and 2) are true in this case.

Case 2. $e \in P$.

If G is the circuit $P+e^*$, then 1) and 2) are easily verified. Assume G is not a circuit. By Lemma 4.5, G has a handle P_2 on an edge e_2 such that $|(P+e^*) \cap (P_2+e_2)| \leq 1$. Since P has length greater than one, $e \notin P_2+e_2$. Hence, we can use Case 1 to prove the theorem.

Case 3. $e = e^*$.

If G has a handle P_2 on an edge e_2 such that $e \notin P_2+e_2$, we use Case 1 to prove the theorem; if G is a circuit, we use Case 2. Hence we may assume that G is simple, is not a circuit, has a handle $Q \neq P$ on an edge and, further, that every handle P_2 on an edge e_2 is such that $e \in P+e_2$. Since G has no subgraph which is a subdivision of $K_{2,3}$, G has exactly two handles P, Q on the edge e . Let Q have length $q \geq 2$. Then, for $i \leq \frac{1}{2}(m)$,

$$\pi_i(G) = \pi_i(C_p \oplus C_{q+1}) + \pi_{i-1}(C_q) + \pi_{i-p}(C_q) ,$$

and

$$\pi_i(G/e) = \pi_i(C_p \oplus C_q) .$$

Hence, for $i \le \frac{1}{2}(m) = \frac{1}{2}(p+q+1)$ we have $i-p \le \frac{1}{2}(q+1-p) < \frac{1}{2}(q)$, and

$$\Delta_i^1(G) = \pi_i(C_p \oplus C_{q+1}) - \pi_{i-1}(C_p \oplus C_{q+1}) - \pi_i(C_p \oplus C_q) + \pi_{i-1}(C_p \oplus C_q)$$

$$+ \pi_{i-1}(C_q) - \pi_{i-2}(C_q) + \pi_{i-p}(C_q) - \pi_{i-p-1}(C_q)$$

$$= \sum_{1 \le t \le \frac{p}{2}} [\pi_t(C_p) - \pi_{t-1}(C_p)][\Delta_{i-t}^{p-2t+1}(C_{q+1}, e)] + \pi_{i-1}(C_q) - \pi_{i-2}(C_q)$$

$$+ \pi_{i-p}(C_q) - \pi_{i-p-1}(C_q) \ge \Delta_{i-1}^{p-1}(C_{q+1}, e) + \pi_{i-1}(C_q) - \pi_{i-2}(C_q) .$$

As before, if $i-1 \le \frac{1}{2}(q+1)$, then $\pi_{i-1}(C_q) \ge \pi_{i-2}(C_q)$; and if $i-1 > \frac{1}{2}(q+1)$, then

$$\Delta_{i-1}^{p-1}(C_{q+1}, e) + \pi_{i-1}(C_q) - \pi_{i-2}(C_q)$$

$$= \pi_{i-1}(C_{q+1}) - \pi_{i-p}(C_{q+1}) - \pi_{i-1}(C_q) + \pi_{i-p}(C_q) + \pi_{i-1}(C_q) - \pi_{i-2}(C_q)$$

$$= \pi_{q-i+2}(C_{q+1}) - \pi_{i-p}(C_{q+1}) - \pi_{q-i+2}(C_q) + \pi_{i-p}(C_q)$$

$$= \Delta_{q-i+2}^{m+1-2i}(C_{q+1}, e) \ge 0 ,$$

since $q-i+2 \le \frac{1}{2}(q+1)$, $m+1-2i \ge 1$, and C_{q+1} is not a loop. As in Case 1, we use 1) to prove 2).

<u>4.11 Corollary</u>. Let G be an outerplanar graph with m edges. Then, for $i \leq \frac{1}{2}(m+1)$,

$$\pi_i(G) \geq \pi_{i-1}(G) .$$

<u>Proof</u>. If G has a loop, then $\pi_i(G) = 0$ for all i . If G does not have a loop, use Theorem 4.10.

<u>4.12 Corollary</u>. If G is an outerplanar graph with independent set numbers I_i , $0 \leq i \leq r(G)$, then

$$I_i^2 \geq I_{i-1}I_{i+1} \quad , \qquad 1 \leq i \leq r(G) - 1 .$$

CHAPTER V

Independent Partition Numbers for some Outerplanar Graphs

In this chapter all outerplanar graphs are embedded in the plane so that all vertices lie on the exterior face.

1. Maximal Outerplanar graphs.

A simple nonseparable outerplanar graph G is maximal outerplanar if no edge can be added without losing outerplanarity. Clearly, every maximal outerplanar graph is a triangulation of a polygon. Harary [71] proved the following simple but important theorem.

5.1 Theorem. Let G be a maximal outerplanr graph with $p \geq 3$ vertices all lying on the exterior face. Then G has $p-2$ interior faces.

Proof. Obviously the results holds for $p = 3$. Suppose it is true for $p = n$ and let G have $p = n+1$ vertices and m interior faces. G has a handle P of size 2 on an edge; thus $G \setminus P$ has $p-1 = n$ vertices and $m-1$ interior faces. By induction $m-1 = n-2$, or $m = p-2$.

5.2 Corollary. Every maximal outerplanar graph with p vertices has $2p-3$ edges

Proof. Use the Euler formula, $p - q + r = 2$, where p, q and r are the number of vertices, edges and faces, respectively.

Let G be maximal outerplanar with $n + 1 \geq 3$ vertices and $2n - 1$ edges. Then

$$\pi_i(G) = 0 \quad \text{for} \quad i < n - 1$$

and, since G has a handle P on an edge e ,

$$\pi_{n-1}(G) = \pi_n(G) = \pi_{n-1}(K) + 2\pi_{n-1}(H) + \pi_{n-2}(K) ,$$

where $K = G/(P + e)$, $H = G \setminus P$. K has $2n - 4$ edges and $n - 1$ vertices, hence

$$\pi_{n-1}(K) = \pi_{n-3}(K) = 0 \quad .$$

H is maximal outerplanar with n vertices and $2n - 3$ edges.

If $n = 2$ we see that

$$\pi_n(G) = 2 \cdot 1 + 1 = n2^{n-2} + 2^{n-2} = (n + 1)2^{n-2} \quad .$$

If $n > 2$, then K has parallel edges e_1, e_2 and so

$$\pi_{n-2}(K) = 2\pi_{n-3}(K / \{e_1, e_2\}) \quad .$$

We see, therefore, that

$$\pi_{n-2}(K) = 2^{n-2} \quad .$$

We have shown:

5.3 Theorem. If G is a maximal outerplanar graph with $n+1 \geq 3$ vertices, then

$$\pi_{n-1}(G) = \pi_n(G) = (n+1)2^{n-2} \quad ,$$

$$\pi_i(G) = 0 \quad \text{for} \quad i < n-1 \quad \text{and} \quad i > n \quad .$$

2. Some Conjectures.

When G is outerplanar but not maximal we do not have such an explicit formula for $\pi_i(G)$; however, since G has a handle P of size p on an edge e , if we let $H = G \setminus P$, $K = G/(P+e)$, then we can use the reduction formula,

$$\pi_i(G) = \pi_i(H \oplus C_p) + \pi_{i-1}(K) + \pi_{i-p}(K)$$

to compute $\pi_i(G)$. A computer program which does this computation was written in BASIC for the Apple II$^+$. After briefly explaining how the program works, we use it to compare the numbers $\pi_i(G)$ to the binomial coefficients $\binom{m}{i}$, where $m = |E(G)|$.

Let G be a simple nonseparable outerplanar graph with $m \geq 3$ edges and n interior faces. We define two sequences. If G has exactly one interior face F_1 , then let $p_1 = m-1$ and $a_1 = 0$. Assume G has more than one interior face, let P be a handle of size p on an edge e and let F be the face bounded by $P+e$. Assume that the $n-1$ faces of $G \setminus P$ are ordered $F_1, \ldots, F_{n-1}$ and that $p_1, \ldots, p_{n-1}$ and $a_1, \ldots, a_{n-1}$ have been defined. Let $p_n = p$ and $a_n = \ell$ where $\ell \in \{1, \ldots, n-1\}$ is such that e lies on the boundary of F_ℓ. In this manner, the graph G determines sequences p_t , $1 \leq t \leq n$, called a <u>handle sequence</u>, and a_t , $1 \leq t \leq n$, called a <u>parent</u> sequence. Since G may have several handles on edges, these sequences are not unique. However, it is easy to see that these sequences determine a tree $T(G)$ with vertex set $V = \{(t, p_t) ; 1 \le t \le n\}$ where (t, p_t) is adjacent to (ℓ, p_ℓ), $t > \ell$, if $\ell = a_t$.

Conversely, given a tree T and a function $w : V(T) \to \mathbb{N}$ we can define an outerplanar graph $G = G(T)$ such that $T(G) \cong T$; $\mathcal{M}(G)$ is uniquely determined. For an example, see Figure 9.

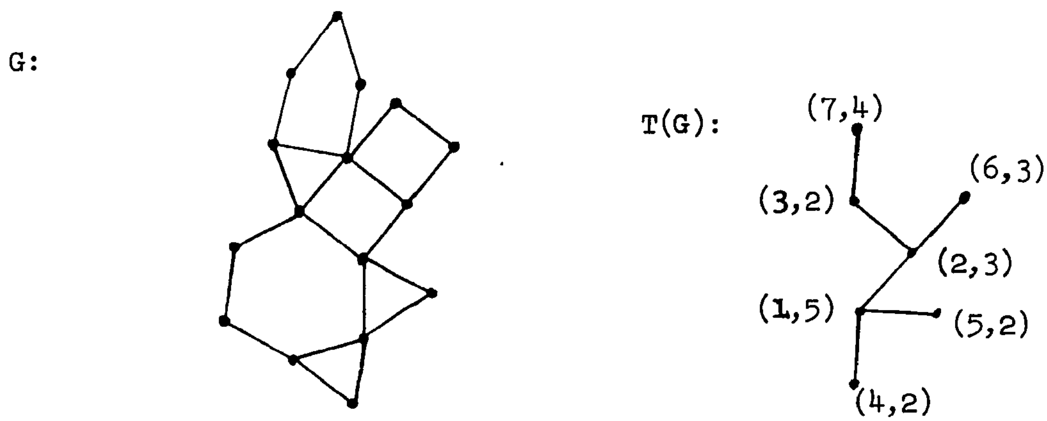

Figure 9

An outerplanar graph G and tree T(G)

Let σ be a function from $[n] = \{1, 2, \ldots, n\}$ to $\{0, 1, \ldots, n-1\}$ such that $\sigma(t) < t$, and let $\mathbb{N}^t = \mathbb{N} \times \ldots \times \mathbb{N}$ (t factors). We now recursively define functions $f_i^t : \mathbb{N} \to \mathbb{Z}^+$. Let

$$f_i^1(x_1) = \begin{cases} \binom{x_1 + 1}{i} & \text{if } 1 \le i \le x_1 \\ 0 & \text{otherwise} \end{cases} .$$

Assume f_j^ℓ, $\ell < t$, $j < i$ have been defined. Let

$$f_i^t(x_1, \dots, x_t) = \sum_{1 \le \ell \le x_t} \binom{x_t + 1}{\ell} f_{i-\ell}(x_1, \dots, x_{t-1})$$

$$+ f_{i-1}(x_1, \dots, x_{\sigma(t)} - 1, \dots, x_{t-1}) + f_{i-x_t}(x_1, \dots, x_{\sigma(t)} - 1, \dots, x_{t-1}) .$$

It is clear that if $\sigma(t) = b_t$, $1 \le t \le n$, then $f_i^n(x_1, \dots, x_n) = \pi_i(G)$; the numbers f_i^t correspond to independent partition numbers of minors of G .

The program computes the numbers $f_i^n(x_1, \dots, x_n)$. To minimize storage requirements, the program only computes the f_i^t needed in the computation of π_i .

In the tables below and in the appendix, the following notation is used. Let G be the graph.

M = the number of edges,

V = the number of vertices,

$R = V - 1$ (The rank of the graph),

N = the number of faces,

$P(G)$ = the parent sequence,

$H(G)$ = the handle sequence,

$$PI(I) = \pi_I(G) ,$$

$$BC(I) = \binom{M}{I} ,$$

$$PI / BC = \pi_I(G) / \binom{M}{I} ,$$

$$PR(I) = \pi_I(G) / \pi_{I-1}(G) ,$$

$$PR / BCR = \frac{\pi_I(G) / \pi_{I-1}(G)}{\binom{M}{I} / \binom{M}{I-1}} = \frac{M - I + 1}{I} PR(I) .$$

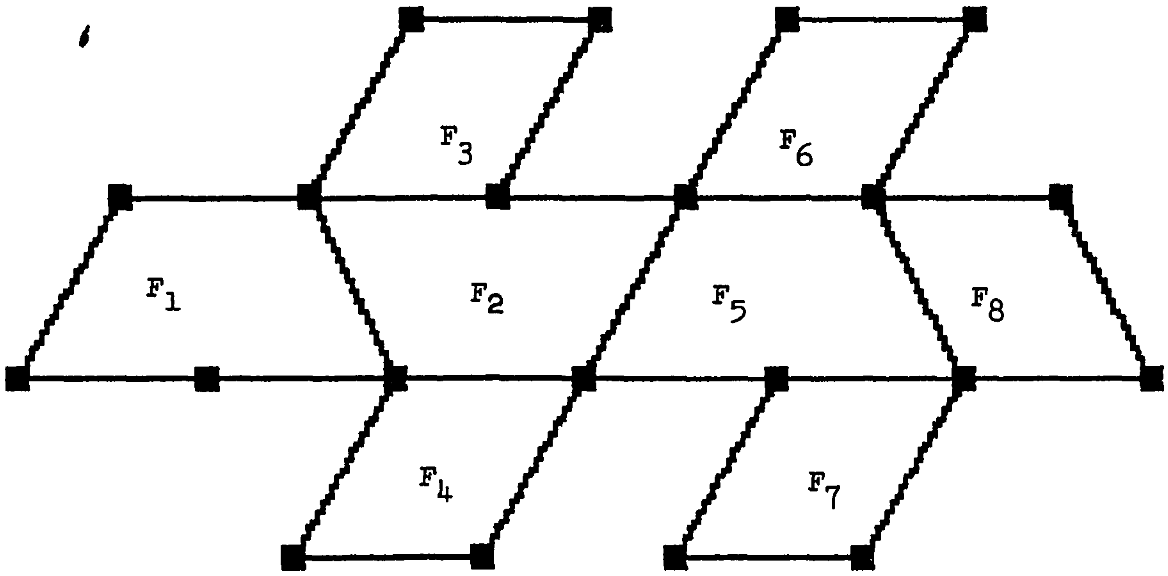

Figure 10. An Outerplanar Graph

M=28 V=21 P(G)=X1222555 21 MINORS
R=20 N=8 H(G)=44334333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
8	88303	3108105	0.0284	**	**
9	691374	6906900	0.1	7.8295	3.5233
10	2708787	13123110	0.2064	3.9179	2.062
11	6973458	21474180	0.3247	2.5743	1.5732
12	13080837	30421755	0.4299	1.8758	1.324
13	18784128	37442160	0.5016	1.436	1.1667
>14	21142322	40116600	0.527	1.1255	1.0505
15	18784128	37442160	0.5016	0.8884	0.9519
16	13080837	30421755	0.4299	0.6963	0.857
17	6973458	21474180	0.3247	0.5331	0.7552
18	2708787	13123110	0.2064	0.3884	0.6356
19	691374	6906900	0.1	0.2552	0.4849
20	88303	3108105	0.0284	0.1277	0.2838

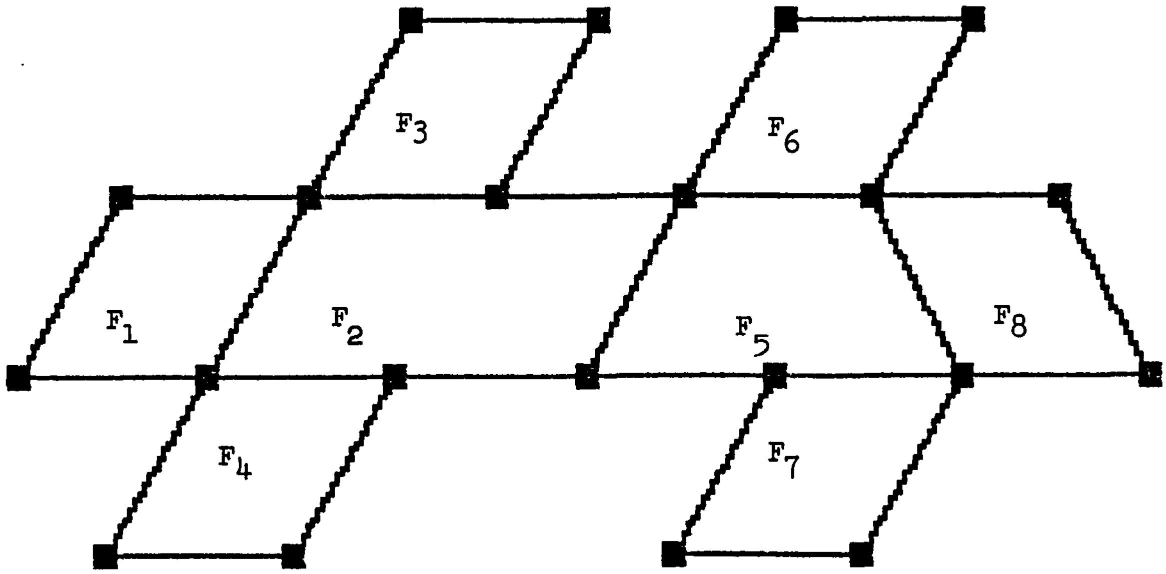

Figure 11. An Outerplanar Graph

M=28 V=21 P(G)=X1222555 21 MINORS
R=20 N=8 H(G)=35334333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
8	87232	3108105	0.028	**	**
9	680031	6906900	0.0984	7.7956	3.508
10	2657391	13123110	0.2024	3.9077	2.0567
11	6832179	21474180	0.3181	2.571	1.5711
12	12809076	30421755	0.421	1.8748	1.3233
13	18391086	37442160	0.4911	1.4357	1.1665
>14	20699450	40116600	0.5159	1.1255	1.0504
15	18391086	37442160	0.4911	0.8884	0.9519
16	12809076	30421755	0.421	0.6964	0.8572
17	6832179	21474180	0.3181	0.5333	0.7556
18	2657391	13123110	0.2024	0.3889	0.6364
19	680031	6906900	0.0984	0.2559	0.4862
20	87232	3108105	0.028	0.1282	0.285

M=28 V=21 P(G)=X1222555 19 MINORS
R=20 N=8 H(G)=53433333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
8	74403	3108105	0.0239	XX	XX
9	593730	6906900	0.0859	7.9799	3.5909
10	2358891	13123110	0.1797	3.973	2.091
11	6137154	21474180	0.2857	2.6017	1.5899
12	11597913	30421755	0.3812	1.8897	1.3339
13	16728444	37442160	0.4467	1.4423	1.1719
>14	18856242	40116600	0.47	1.1271	1.052
15	16728444	37442160	0.4467	0.8871	0.9505
16	11597913	30421755	0.3812	0.6933	0.8532
17	6137154	21474180	0.2857	0.5291	0.7496
18	2358891	13123110	0.1797	0.3843	0.6289
19	593730	6906900	0.0859	0.2516	0.4782
20	74403	3108105	0.0239	0.1253	0.2784

M=28 V=21 P(G)=X1222555 20 MINORS
R=20 N=8 H(G)=44433333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
8	82773	3108105	0.0266	XX	XX
9	653850	6906900	0.0946	7.8993	3.5546
10	2576331	13123110	0.1963	3.9402	2.0738
11	6658614	21474180	0.31	2.5845	1.5794
12	12522591	30421755	0.4116	1.8806	1.3275
13	18009072	37442160	0.4809	1.4381	1.1684
>14	20279682	40116600	0.5055	1.126	1.051
15	18009072	37442160	0.4809	0.888	0.9514
16	12522591	30421755	0.4116	0.6953	0.8558
17	6658614	21474180	0.31	0.5317	0.7532
18	2576331	13123110	0.1963	0.3869	0.6331
19	653850	6906900	0.0946	0.2537	0.4822
20	82773	3108105	0.0266	0.1265	0.2813

M=28 V=21 P(G)=*1222555 20 MINORS
R=20 N=8 H(G)=45333333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
8	82008	3108105	0.0263	**	**
9	644985	6906900	0.0933	7.8649	3.5392
10	2533905	13123110	0.193	3.9286	2.0676
11	6538041	21474180	0.3044	2.5802	1.5768
12	12286332	30421755	0.4038	1.8792	1.3264
13	17664462	37442160	0.4717	1.4377	1.1681
>14	19890486	40116600	0.4958	1.126	1.0509
15	17664462	37442160	0.4717	0.888	0.9515
16	12286332	30421755	0.4038	0.6955	0.856
17	6538041	21474180	0.3044	0.5321	0.7538
18	2533905	13123110	0.193	0.3875	0.6341
19	644985	6906900	0.0933	0.2545	0.4836
20	82008	3108105	0.0263	0.1271	0.2825

M=28 V=21 P(G)=*1222555 20 MINORS
R=20 N=8 H(G)=36333333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
8	78192	3108105	0.0251	**	**
9	613548	6906900	0.0888	7.8466	3.531
10	2412099	13123110	0.1838	3.9313	2.0691
11	6239376	21474180	0.2905	2.5866	1.5807
12	11759580	30421755	0.3865	1.8847	1.3304
13	16945092	37442160	0.4525	1.4409	1.1707
>14	19096290	40116600	0.476	1.1269	1.0518
15	16945092	37442160	0.4525	0.8873	0.9507
16	11759580	30421755	0.3865	0.6939	0.8541
17	6239376	21474180	0.2905	0.5305	0.7516
18	2412099	13123110	0.1838	0.3865	0.6326
19	613548	6906900	0.0888	0.2543	0.4832
20	78192	3108105	0.0251	0.1274	0.2832

M=28 V=21 P(G)=*1222555 19 MINORS
R=20 N=8 H(G)=63333333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
8	68769	3108105	0.0221	**	**
9	549423	6906900	0.0795	7.9893	3.5952
10	2189511	13123110	0.1668	3.9851	2.0974
11	5719491	21474180	0.2663	2.6122	1.5963
12	10849707	30421755	0.3566	1.8969	1.339
13	15690510	37442160	0.419	1.4461	1.175
>14	17702874	40116600	0.4412	1.1282	1.053
15	15690510	37442160	0.419	0.8863	0.9496
16	10849707	30421755	0.3566	0.6914	0.851
17	5719491	21474180	0.2663	0.5271	0.7468
18	2189511	13123110	0.1668	0.3828	0.6264
19	549423	6906900	0.0795	0.2509	0.4767
20	68769	3108105	0.0221	0.1251	0.2781

M=28 V=21 P(G)=*1222555 20 MINORS
R=20 N=8 H(G)=54333333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
8	79128	3108105	0.0254	**	**
9	625563	6906900	0.0905	7.9057	3.5575
10	2467917	13123110	0.188	3.9451	2.0763
11	6387147	21474180	0.2974	2.588	1.5815
12	12026268	30421755	0.3953	1.8828	1.329
13	17308890	37442160	0.4622	1.4392	1.1693
>14	19496574	40116600	0.4859	1.1263	1.0512
15	17308890	37442160	0.4622	0.8877	0.9512
16	12026268	30421755	0.3953	0.6948	0.8551
17	6387147	21474180	0.2974	0.531	0.7523
18	2467917	13123110	0.188	0.3863	0.6322
19	625563	6906900	0.0905	0.2534	0.4816
20	79128	3108105	0.0254	0.1264	0.281

In all these examples and in the appendix we observe that

1) PI / BC increases for $i \leq \frac{1}{2}(M)$,

2) PR(I) decreases,

3) PR / BCR decreases.

Hence, we conjecture that if M is a matroid of size m , then

1) $\pi_i(M) \geq \frac{m-i+1}{i} \pi_{i-1}(M)$,

2) $[\pi_i(M)]^2 \geq \pi_{i-1}(M) \pi_{i+1}(M)$,

3) $[\pi_i(M)]^2 \geq \frac{m-i}{i+1} \cdot \frac{m-i+1}{i} \pi_{i-1}(M) \pi_{i+1}(M)$,

where $m - r(M) < i \leq r(M)$.

APPENDIX

M=24 V=18 P(G)=*123456 13 MINORS
R=17 N=7 H(G)=3335333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17136	346104	0.0495	**	**
8	113635	735471	0.1545	6.6313	3.1206
9	376614	1307504	0.288	3.3142	1.8642
10	812128	1961256	0.414	2.1563	1.4375
11	1253898	2496144	0.5023	1.5439	1.2131
>12	1443642	2704156	0.5338	1.1513	1.0627
13	1253898	2496144	0.5023	0.8685	0.9409
14	812128	1961256	0.414	0.6476	0.8243
15	376614	1307504	0.288	0.4637	0.6956
16	113635	735471	0.1545	0.3017	0.5364
17	17136	346104	0.0495	0.1507	0.3204

M=24 V=18 P(G)=*123456 13 MINORS
R=17 N=7 H(G)=4433333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17675	346104	0.051	**	**
8	117107	735471	0.1592	6.6255	3.1179
9	387509	1307504	0.2963	3.309	1.8613
10	834400	1961256	0.4254	2.1532	1.4354
11	1287008	2496144	0.5155	1.5424	1.2119
>12	1481242	2704156	0.5477	1.1509	1.0623
13	1287008	2496144	0.5155	0.8688	0.9412
14	834400	1961256	0.4254	0.6483	0.8251
15	387509	1307504	0.2963	0.4644	0.6966
16	117107	735471	0.1592	0.3022	0.5372
17	17675	346104	0.051	0.1509	0.3207

M=24 V=18 P(G)=*123456 13 MINORS
R=17 N=7 H(G)=4343333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17746	346104	0.0512	**	**
8	117740	735471	0.16	6.6347	3.1222
9	390160	1307504	0.2984	3.3137	1.8639
10	841116	1961256	0.4288	2.1558	1.4372
11	1298366	2496144	0.5201	1.5436	1.2128
>12	1494704	2704156	0.5527	1.1512	1.0626
13	1298366	2496144	0.5201	0.8686	0.941
14	841116	1961256	0.4288	0.6478	0.8245
15	390160	1307504	0.2984	0.4638	0.6957
16	117740	735471	0.16	0.3017	0.5364
17	17746	346104	0.0512	0.1507	0.3202

M=24 V=18 P(G)=*123456 13 MINORS
R=17 N=7 H(G)=4334333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17751	346104	0.0512	**	**
8	117807	735471	0.1601	6.6366	3.1231
9	390513	1307504	0.2986	3.3148	1.8646
10	842160	1961256	0.4293	2.1565	1.4376
11	1300312	2496144	0.5209	1.544	1.2131
>12	1497090	2704156	0.5536	1.1513	1.0627
13	1300312	2496144	0.5209	0.8685	0.9409
14	842160	1961256	0.4293	0.6476	0.8242
15	390513	1307504	0.2986	0.4637	0.6955
16	117807	735471	0.1601	0.3016	0.5363
17	17751	346104	0.0512	0.1506	0.3201

M=24 V=18 P(G)=*123456 13 MINORS
R=17 N=7 H(G)=4333433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17750	346104	0.0512	**	**
8	117792	735471	0.1601	6.6361	3.1229
9	390436	1307504	0.2986	3.3146	1.8644
10	841996	1961256	0.4293	2.1565	1.4377
11	1300134	2496144	0.5208	1.5441	1.2132
>12	1496936	2704156	0.5535	1.1513	1.0628
13	1300134	2496144	0.5208	0.8685	0.9409
14	841996	1961256	0.4293	0.6476	0.8242
15	390436	1307504	0.2986	0.4637	0.6955
16	117792	735471	0.1601	0.3016	0.5363
17	17750	346104	0.0512	0.1506	0.3202

M=24 V=18 P(G)=*123456 13 MINORS
R=17 N=7 H(G)=4333343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17731	346104	0.0512	**	**
8	117579	735471	0.1598	6.6312	3.1205
9	389581	1307504	0.2979	3.3133	1.8637
10	840064	1961256	0.4283	2.1563	1.4375
11	1297168	2496144	0.5196	1.5441	1.2132
>12	1493546	2704156	0.5523	1.1513	1.0628
13	1297168	2496144	0.5196	0.8685	0.9408
14	840064	1961256	0.4283	0.6476	0.8242
15	389581	1307504	0.2979	0.4637	0.6956
16	117579	735471	0.1598	0.3018	0.5365
17	17731	346104	0.0512	0.1508	0.3204

M=24 V=18 P(G)=*123456 13 MINORS
R=17 N=7 H(G)=4333334

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17466	346104	0.0504	**	**
8	115828	735471	0.1574	6.6316	3.1207
9	384024	1307504	0.2937	3.3154	1.8649
10	828668	1961256	0.4225	2.1578	1.4385
11	1280206	2496144	0.5128	1.5448	1.2138
>12	1474272	2704156	0.5451	1.1515	1.063
13	1280206	2496144	0.5128	0.8683	0.9407
14	828668	1961256	0.4225	0.6472	0.8238
15	384024	1307504	0.2937	0.4634	0.6951
16	115828	735471	0.1574	0.3016	0.5362
17	17466	346104	0.0504	0.1507	0.3204

M=24 V=18 P(G)=*123456 13 MINORS
R=17 N=7 H(G)=3443333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17955	346104	0.0518	**	**
8	118963	735471	0.1617	6.6256	3.1179
9	393453	1307504	0.3009	3.3073	1.8603
10	846688	1961256	0.4317	2.1519	1.4346
11	1305360	2496144	0.5229	1.5417	1.2113
>12	1502106	2704156	0.5554	1.1507	1.0622
13	1305360	2496144	0.5229	0.869	0.9414
14	846688	1961256	0.4317	0.6486	0.8255
15	393453	1307504	0.3009	0.4646	0.697
16	118963	735471	0.1617	0.3023	0.5375
17	17955	346104	0.0518	0.1509	0.3207

M=24 V=18 P(G)=*123456 13 MINORS
R=17 N=7 H(G)=3434333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18016	346104	0.052	**	**
8	119528	735471	0.1625	6.6345	3.1221
9	395838	1307504	0.3027	3.3116	1.8628
10	852764	1961256	0.4348	2.1543	1.4362
11	1315714	2496144	0.527	1.5428	1.2122
>12	1514424	2704156	0.56	1.151	1.0624
13	1315714	2496144	0.527	0.8687	0.9411
14	852764	1961256	0.4348	0.6481	0.8249
15	395838	1307504	0.3027	0.4641	0.6962
16	119528	735471	0.1625	0.3019	0.5368
17	18016	346104	0.052	0.1507	0.3202

M=24 V=18 P(G)=*123456 13 MINORS
R=17 N=7 H(G)=3433433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18019	346104	0.052	**	**
8	119567	735471	0.1625	6.6356	3.1226
9	396045	1307504	0.3029	3.3123	1.8631
10	853424	1961256	0.4351	2.1548	1.4365
11	1317040	2496144	0.5276	1.5432	1.2125
>12	1516098	2704156	0.5606	1.1511	1.0625
13	1317040	2496144	0.5276	0.8687	0.941
14	853424	1961256	0.4351	0.6479	0.8247
15	396045	1307504	0.3029	0.464	0.696
16	119567	735471	0.1625	0.3019	0.5367
17	18019	346104	0.052	0.1507	0.3202

M=24 V=18 P(G)=*123456 13 MINORS
R=17 N=7 H(G)=3433343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18000	346104	0.052	**	**
8	119356	735471	0.1622	6.6308	3.1204
9	395214	1307504	0.3022	3.3112	1.8625
10	851596	1961256	0.4342	2.1547	1.4365
11	1314306	2496144	0.5265	1.5433	1.2126
>12	1513008	2704156	0.5595	1.1511	1.0626
13	1314306	2496144	0.5265	0.8686	0.941
14	851596	1961256	0.4342	0.6479	0.8246
15	395214	1307504	0.3022	0.464	0.6961
16	119356	735471	0.1622	0.302	0.5368
17	18000	346104	0.052	0.1508	0.3204

M=24 V=18 P(G)=*123456 13 MINORS
R=17 N=7 H(G)=3344333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17975	346104	0.0519	**	**
8	119187	735471	0.162	6.6307	3.1203
9	394353	1307504	0.3016	3.3086	1.8611
10	848736	1961256	0.4327	2.1522	1.4348
11	1308536	2496144	0.5242	1.5417	1.2113
>12	1505754	2704156	0.5568	1.1507	1.0621
13	1308536	2496144	0.5242	0.869	0.9414
14	848736	1961256	0.4327	0.6486	0.8255
15	394353	1307504	0.3016	0.4646	0.6969
16	119187	735471	0.162	0.3022	0.5373
17	17975	346104	0.0519	0.1508	0.3204

M=24 V=18 P(G)=X123456 13 MINORS
R=17 N=7 H(G)=5333333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16686	346104	0.0482	**	**
8	110725	735471	0.1505	6.6358	3.1227
9	367432	1307504	0.281	3.3184	1.8666
10	793312	1961256	0.4044	2.159	1.4393
11	1225898	2496144	0.4911	1.5452	1.2141
>12	1411830	2704156	0.522	1.1516	1.063
13	1225898	2496144	0.4911	0.8683	0.9406
14	793312	1961256	0.4044	0.6471	0.8236
15	367432	1307504	0.281	0.4631	0.6947
16	110725	735471	0.1505	0.3013	0.5357
17	16686	346104	0.0482	0.1506	0.3202

M=24 V=18 P(G)=X123456 13 MINORS
R=17 N=7 H(G)=3533333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17104	346104	0.0494	**	**
8	113283	735471	0.154	6.6231	3.1167
9	375238	1307504	0.2869	3.3123	1.8632
10	809056	1961256	0.4125	2.1561	1.4374
11	1249162	2496144	0.5004	1.5439	1.2131
>12	1438202	2704156	0.5318	1.1513	1.0627
13	1249162	2496144	0.5004	0.8685	0.9409
14	809056	1961256	0.4125	0.6476	0.8243
15	375238	1307504	0.2869	0.4637	0.6956
16	113283	735471	0.154	0.3018	0.5367
17	17104	346104	0.0494	0.1509	0.3208

M=24 V=18 P(G)=X123456 13 MINORS
R=17 N=7 H(G)=3353333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17134	346104	0.0495	**	**
8	113605	735471	0.1544	6.6303	3.1201
9	376456	1307504	0.2879	3.3137	1.8639
10	811744	1961256	0.4138	2.1562	1.4375
11	1253290	2496144	0.502	1.5439	1.2131
>12	1442934	2704156	0.5335	1.1513	1.0627
13	1253290	2496144	0.502	0.8685	0.9409
14	811744	1961256	0.4138	0.6476	0.8243
15	376456	1307504	0.2879	0.4637	0.6956
16	113605	735471	0.1544	0.3017	0.5364
17	17134	346104	0.0495	0.1508	0.3204

M=24 V=18 P(G)=X123455 14 MINORS
R=17 N=7 H(G)=5333333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16514	346104	0.0477	XX	XX
8	109967	735471	0.1495	6.659	3.1336
9	365528	1307504	0.2795	3.3239	1.8697
10	789856	1961256	0.4027	2.1608	1.4405
11	1221062	2496144	0.4891	1.5459	1.2146
>12	1406434	2704156	0.5201	1.1518	1.0632
13	1221062	2496144	0.4891	0.8681	0.9405
14	789856	1961256	0.4027	0.6468	0.8232
15	365528	1307504	0.2795	0.4627	0.6941
16	109967	735471	0.1495	0.3008	0.5348
17	16514	346104	0.0477	0.1501	0.3191

M=24 V=18 P(G)=X123455 14 MINORS
R=17 N=7 H(G)=3533333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16928	346104	0.0489	XX	XX
8	112513	735471	0.1529	6.6465	3.1277
9	373322	1307504	0.2855	3.318	1.8663
10	805600	1961256	0.4107	2.1579	1.4386
11	1244342	2496144	0.4985	1.5446	1.2136
>12	1432830	2704156	0.5298	1.1514	1.0629
13	1244342	2496144	0.4985	0.8684	0.9408
14	805600	1961256	0.4107	0.6474	0.8239
15	373322	1307504	0.2855	0.4634	0.6951
16	112513	735471	0.1529	0.3013	0.5357
17	16928	346104	0.0489	0.1504	0.3197

M=24 V=18 P(G)=X123455 14 MINORS
R=17 N=7 H(G)=3353333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16962	346104	0.049	XX	XX
8	112879	735471	0.1534	6.6548	3.1316
9	374712	1307504	0.2865	3.3195	1.8672
10	808672	1961256	0.4123	2.1581	1.4387
11	1249062	2496144	0.5003	1.5445	1.2136
>12	1438242	2704156	0.5318	1.1514	1.0628
13	1249062	2496144	0.5003	0.8684	0.9408
14	808672	1961256	0.4123	0.6474	0.8239
15	374712	1307504	0.2865	0.4633	0.695
16	112879	735471	0.1534	0.3012	0.5355
17	16962	346104	0.049	0.1502	0.3193

M=24 V=18 P(G)=*123455 14 MINORS
R=17 N=7 H(G)=3335333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17024	346104	0.0491	**	**
8	113281	735471	0.154	6.6541	3.1313
9	376010	1307504	0.2875	3.3192	1.867
10	811360	1961256	0.4136	2.1578	1.4385
11	1253078	2496144	0.502	1.5444	1.2134
>12	1442814	2704156	0.5335	1.1514	1.0628
13	1253078	2496144	0.502	0.8684	0.9408
14	811360	1961256	0.4136	0.6474	0.824
15	376010	1307504	0.2875	0.4634	0.6951
16	113281	735471	0.154	0.3012	0.5355
17	17024	346104	0.0491	0.1502	0.3193

M=24 V=18 P(G)=*123455 14 MINORS
R=17 N=7 H(G)=3333533

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17496	346104	0.0505	**	**
8	115747	735471	0.1573	6.6156	3.1132
9	382830	1307504	0.2927	3.3074	1.8604
10	824416	1961256	0.4203	2.1534	1.4356
11	1271898	2496144	0.5095	1.5427	1.2121
>12	1463994	2704156	0.5413	1.151	1.0624
13	1271898	2496144	0.5095	0.8687	0.9411
14	824416	1961256	0.4203	0.6481	0.8249
15	382830	1307504	0.2927	0.4643	0.6965
16	115747	735471	0.1573	0.3023	0.5375
17	17496	346104	0.0505	0.1511	0.3212

M=24 V=18 P(G)=*123455 14 MINORS
R=17 N=7 H(G)=3333353

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16574	346104	0.0478	**	**
8	110341	735471	0.15	6.6574	3.1329
9	366776	1307504	0.2805	3.324	1.8697
10	792544	1961256	0.4041	2.1608	1.4405
11	1225130	2496144	0.4908	1.5458	1.2145
>12	1411062	2704156	0.5218	1.1517	1.0631
13	1225130	2496144	0.4908	0.8682	0.9405
14	792544	1961256	0.4041	0.6469	0.8233
15	366776	1307504	0.2805	0.4627	0.6941
16	110341	735471	0.15	0.3008	0.5348
17	16574	346104	0.0478	0.1502	0.3191

M=24 V=18 P(G)=*123455 14 MINORS
R=17 N=7 H(G)=4433333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17493	346104	0.0505	**	**
8	116309	735471	0.1581	6.6488	3.1288
9	385523	1307504	0.2948	3.3146	1.8644
10	830816	1961256	0.4236	2.155	1.4366
11	1282008	2496144	0.5135	1.543	1.2124
>12	1475670	2704156	0.5457	1.151	1.0625
13	1282008	2496144	0.5135	0.8687	0.9411
14	830816	1961256	0.4236	0.648	0.8248
15	385523	1307504	0.2948	0.464	0.696
16	116309	735471	0.1581	0.3016	0.5363
17	17493	346104	0.0505	0.1504	0.3196

M=24 V=18 P(G)=*123455 14 MINORS
R=17 N=7 H(G)=4343333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17566	346104	0.0507	**	**
8	116968	735471	0.159	6.6587	3.1335
9	388284	1307504	0.2969	3.3195	1.8672
10	837788	1961256	0.4271	2.1576	1.4384
11	1293766	2496144	0.5183	1.5442	1.2133
>12	1489592	2704156	0.5508	1.1513	1.0627
13	1293766	2496144	0.5183	0.8685	0.9409
14	837788	1961256	0.4271	0.6475	0.8241
15	388284	1307504	0.2969	0.4634	0.6951
16	116968	735471	0.159	0.3012	0.5355
17	17566	346104	0.0507	0.1501	0.3191

M=24 V=18 P(G)=*123455 14 MINORS
R=17 N=7 H(G)=4334333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17609	346104	0.0508	**	**
8	117289	735471	0.1594	6.6607	3.1344
9	389447	1307504	0.2978	3.3204	1.8677
10	840496	1961256	0.4285	2.1581	1.4387
11	1298192	2496144	0.52	1.5445	1.2135
>12	1494798	2704156	0.5527	1.1514	1.0628
13	1298192	2496144	0.52	0.8684	0.9408
14	840496	1961256	0.4285	0.6474	0.824
15	389447	1307504	0.2978	0.4633	0.695
16	117289	735471	0.1594	0.3011	0.5354
17	17609	346104	0.0508	0.1501	0.319

M=24 V=18 P(G)=¥123455 14 MINORS
R=17 N=7 H(G)=4333433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17944	346104	0.0518	**	**
8	119052	735471	0.1618	6.6346	3.1221
9	394390	1307504	0.3016	3.3127	1.8634
10	850060	1961256	0.4334	2.1553	1.4369
11	1312114	2496144	0.5256	1.5435	1.2127
>12	1510544	2704156	0.5586	1.1512	1.0626
13	1312114	2496144	0.5256	0.8686	0.941
14	850060	1961256	0.4334	0.6478	0.8245
15	394390	1307504	0.3016	0.4639	0.6959
16	119052	735471	0.1618	0.3018	0.5366
17	17944	346104	0.0518	0.1507	0.3202

M=24 V=18 P(G)=¥123455 14 MINORS
R=17 N=7 H(G)=4333343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17324	346104	0.05	**	**
8	115272	735471	0.1567	6.6538	3.1312
9	382912	1307504	0.2928	3.3218	1.8685
10	827008	1961256	0.4216	2.1597	1.4398
11	1278132	2496144	0.512	1.5454	1.2143
>12	1472048	2704156	0.5443	1.1517	1.0631
13	1278132	2496144	0.512	0.8682	0.9406
14	827008	1961256	0.4216	0.647	0.8235
15	382912	1307504	0.2928	0.463	0.6945
16	115272	735471	0.1567	0.301	0.5351
17	17324	346104	0.05	0.1502	0.3193

M=24 V=18 P(G)=¥123455 14 MINORS
R=17 N=7 H(G)=3443333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17773	346104	0.0513	**	**
8	118185	735471	0.1606	6.6496	3.1292
9	391571	1307504	0.2994	3.3132	1.8636
10	843360	1961256	0.43	2.1537	1.4358
11	1300768	2496144	0.5211	1.5423	1.2118
>12	1497006	2704156	0.5535	1.1508	1.0623
13	1300768	2496144	0.5211	0.8689	0.9413
14	843360	1961256	0.43	0.6483	0.8251
15	391571	1307504	0.2994	0.4642	0.6964
16	118185	735471	0.1606	0.3018	0.5365
17	17773	346104	0.0513	0.1503	0.3195

M=24 V=18 P(G)=*123455 14 MINORS
R=17 N=7 H(G)=3434333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17872	346104	0.0516	**	**
8	119004	735471	0.1618	6.6586	3.1334
9	394766	1307504	0.3019	3.3172	1.8659
10	851100	1961256	0.4339	2.1559	1.4373
11	1313602	2496144	0.5262	1.5434	1.2126
>12	1512144	2704156	0.5591	1.1511	1.0625
13	1313602	2496144	0.5262	0.8687	0.941
14	851100	1961256	0.4339	0.6479	0.8246
15	394766	1307504	0.3019	0.4638	0.6957
16	119004	735471	0.1618	0.3014	0.5359
17	17872	346104	0.0516	0.1501	0.3191

M=24 V=18 P(G)=*123455 14 MINORS
R=17 N=7 H(G)=3433433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18216	346104	0.0526	**	**
8	120847	735471	0.1643	6.6341	3.1219
9	400062	1307504	0.3059	3.3104	1.8621
10	861616	1961256	0.4393	2.1537	1.4358
11	1329210	2496144	0.5325	1.5426	1.2121
>12	1529922	2704156	0.5657	1.151	1.0624
13	1329210	2496144	0.5325	0.8688	0.9412
14	861616	1961256	0.4393	0.6482	0.825
15	400062	1307504	0.3059	0.4643	0.6964
16	120847	735471	0.1643	0.302	0.537
17	18216	346104	0.0526	0.1507	0.3203

M=24 V=18 P(G)=*123455 14 MINORS
R=17 N=7 H(G)=3433343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17587	346104	0.0508	**	**
8	117017	735471	0.1591	6.6536	3.1311
9	388467	1307504	0.2971	3.3197	1.8673
10	838412	1961256	0.4274	2.1582	1.4388
11	1295098	2496144	0.5188	1.5447	1.2136
>12	1491318	2704156	0.5514	1.1515	1.0629
13	1295098	2496144	0.5188	0.8684	0.9407
14	838412	1961256	0.4274	0.6473	0.8239
15	388467	1307504	0.2971	0.4633	0.695
16	117017	735471	0.1591	0.3012	0.5355
17	17587	346104	0.0508	0.1502	0.3193

M=24 V=18 P(G)=*123455 14 MINORS
R=17 N=7 H(G)=3344333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17833	346104	0.0515	**	**
8	118677	735471	0.1613	6.6549	3.1317
9	393335	1307504	0.3008	3.3143	1.8643
10	847200	1961256	0.4319	2.1538	1.4359
11	1306624	2496144	0.5234	1.5422	1.2117
>12	1503702	2704156	0.556	1.1508	1.0623
13	1306624	2496144	0.5234	0.8689	0.9413
14	847200	1961256	0.4319	0.6483	0.8252
15	393335	1307504	0.3008	0.4642	0.6964
16	118677	735471	0.1613	0.3017	0.5363
17	17833	346104	0.0515	0.1502	0.3193

M=24 V=18 P(G)=*123455 14 MINORS
R=17 N=7 H(G)=3343433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18232	346104	0.0526	**	**
8	121016	735471	0.1645	6.6375	3.1235
9	400662	1307504	0.3064	3.3108	1.8623
10	862748	1961256	0.4398	2.1533	1.4355
11	1330642	2496144	0.533	1.5423	1.2118
>12	1531416	2704156	0.5663	1.1508	1.0623
13	1330642	2496144	0.533	0.8688	0.9413
14	862748	1961256	0.4398	0.6483	0.8251
15	400662	1307504	0.3064	0.4644	0.6966
16	121016	735471	0.1645	0.302	0.5369
17	18232	346104	0.0526	0.1506	0.3201

M=24 V=18 P(G)=*123455 14 MINORS
R=17 N=7 H(G)=3343343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17608	346104	0.0508	**	**
8	117252	735471	0.1594	6.659	3.1336
9	389404	1307504	0.2978	3.321	1.8681
10	840496	1961256	0.4285	2.1584	1.4389
11	1298236	2496144	0.52	1.5446	1.2136
>12	1494872	2704156	0.5528	1.1514	1.0628
13	1298236	2496144	0.52	0.8684	0.9408
14	840496	1961256	0.4285	0.6474	0.8239
15	389404	1307504	0.2978	0.4633	0.6949
16	117252	735471	0.1594	0.3011	0.5353
17	17608	346104	0.0508	0.1501	0.3191

M=24 V=18 P(G)=*123455 14 MINORS
R=17 N=7 H(G)=3334433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18184	346104	0.0525	**	**
8	120563	735471	0.1639	6.6301	3.12
9	398718	1307504	0.3049	3.3071	1.8602
10	857696	1961256	0.4373	2.1511	1.434
11	1321882	2496144	0.5295	1.5412	1.2109
>12	1520922	2704156	0.5624	1.1505	1.062
13	1321882	2496144	0.5295	0.8691	0.9415
14	857696	1961256	0.4373	0.6488	0.8258
15	398718	1307504	0.3049	0.4648	0.6973
16	120563	735471	0.1639	0.3023	0.5375
17	18184	346104	0.0525	0.1508	0.3205

M=24 V=18 P(G)=*123455 14 MINORS
R=17 N=7 H(G)=3334343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17639	346104	0.0509	**	**
8	117453	735471	0.1596	6.6587	3.1335
9	389991	1307504	0.2982	3.3204	1.8677
10	841500	1961256	0.429	2.1577	1.4384
11	1299410	2496144	0.5205	1.5441	1.2132
>12	1496046	2704156	0.5532	1.1513	1.0627
13	1299410	2496144	0.5205	0.8685	0.9409
14	841500	1961256	0.429	0.6476	0.8242
15	389991	1307504	0.2982	0.4634	0.6951
16	117453	735471	0.1596	0.3011	0.5354
17	17639	346104	0.0509	0.1501	0.3191

M=24 V=18 P(G)=*123455 14 MINORS
R=17 N=7 H(G)=3333443

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17899	346104	0.0517	**	**
8	118659	735471	0.1613	6.6293	3.1197
9	392629	1307504	0.3002	3.3088	1.8612
10	845152	1961256	0.4309	2.1525	1.435
11	1303168	2496144	0.522	1.5419	1.2115
>12	1499642	2704156	0.5545	1.1507	1.0622
13	1303168	2496144	0.522	0.8689	0.9414
14	845152	1961256	0.4309	0.6485	0.8254
15	392629	1307504	0.3002	0.4645	0.6968
16	118659	735471	0.1613	0.3022	0.5372
17	17899	346104	0.0517	0.1508	0.3205

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=5333333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16502	346104	0.0476	**	**
8	109861	735471	0.1493	6.6574	3.1329
9	365248	1307504	0.2793	3.3246	1.8701
10	789344	1961256	0.4024	2.1611	1.4407
11	1220330	2496144	0.4888	1.546	1.2147
>12	1405622	2704156	0.5198	1.1518	1.0632
13	1220330	2496144	0.4888	0.8681	0.9405
14	789344	1961256	0.4024	0.6468	0.8232
15	365248	1307504	0.2793	0.4627	0.694
16	109861	735471	0.1493	0.3007	0.5347
17	16502	346104	0.0476	0.1502	0.3191

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=3533333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16920	346104	0.0488	**	**
8	112451	735471	0.1528	6.646	3.1275
9	373214	1307504	0.2854	3.3189	1.8668
10	805472	1961256	0.4106	2.1582	1.4388
11	1244202	2496144	0.4984	1.5446	1.2136
>12	1432698	2704156	0.5298	1.1514	1.0629
13	1244202	2496144	0.4984	0.8684	0.9408
14	805472	1961256	0.4106	0.6473	0.8239
15	373214	1307504	0.2854	0.4633	0.695
16	112451	735471	0.1528	0.3013	0.5356
17	16920	346104	0.0488	0.1504	0.3197

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=3353333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17014	346104	0.0491	**	**
8	113189	735471	0.1539	6.6526	3.1306
9	375744	1307504	0.2873	3.3196	1.8672
10	810848	1961256	0.4134	2.1579	1.4386
11	1252330	2496144	0.5017	1.5444	1.2135
>12	1441974	2704156	0.5332	1.1514	1.0628
13	1252330	2496144	0.5017	0.8684	0.9408
14	810848	1961256	0.4134	0.6474	0.824
15	375744	1307504	0.2873	0.4633	0.695
16	113189	735471	0.1539	0.3012	0.5355
17	17014	346104	0.0491	0.1503	0.3194

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=3335333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17524	346104	0.0506	**	**
8	116051	735471	0.1577	6.6224	3.1164
9	383994	1307504	0.2936	3.3088	1.8612
10	826976	1961256	0.4216	2.1536	1.4357
11	1275826	2496144	0.5111	1.5427	1.2121
>12	1468506	2704156	0.543	1.151	1.0624
13	1275826	2496144	0.5111	0.8687	0.9411
14	826976	1961256	0.4216	0.6481	0.8249
15	383994	1307504	0.2936	0.4643	0.6965
16	116051	735471	0.1577	0.3022	0.5372
17	17524	346104	0.0506	0.151	0.3208

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=3333533

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16566	346104	0.0478	**	**
8	110277	735471	0.1499	6.6568	3.1326
9	366656	1307504	0.2804	3.3248	1.8702
10	792416	1961256	0.404	2.1611	1.4407
11	1225002	2496144	0.4907	1.5459	1.2146
>12	1410934	2704156	0.5217	1.1517	1.0631
13	1225002	2496144	0.4907	0.8682	0.9405
14	792416	1961256	0.404	0.6468	0.8232
15	366656	1307504	0.2804	0.4627	0.694
16	110277	735471	0.1499	0.3007	0.5346
17	16566	346104	0.0478	0.1502	0.3192

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=3333353

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16984	346104	0.049	**	**
8	112865	735471	0.1534	6.6453	3.1272
9	374514	1307504	0.2864	3.3182	1.8665
10	808160	1961256	0.412	2.1578	1.4385
11	1248214	2496144	0.5	1.5445	1.2135
>12	1437246	2704156	0.5314	1.1514	1.0628
13	1248214	2496144	0.5	0.8684	0.9408
14	808160	1961256	0.412	0.6474	0.824
15	374514	1307504	0.2864	0.4634	0.6951
16	112865	735471	0.1534	0.3013	0.5357
17	16984	346104	0.049	0.1504	0.3197

M=24 V=18 P(G)=¥123446 14 MINORS
R=17 N=7 H(G)=3333335

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16506	346104	0.0476	**	**
8	109903	735471	0.1494	6.6583	3.1333
9	365408	1307504	0.2794	3.3248	1.8702
10	789728	1961256	0.4026	2.1612	1.4408
11	1220934	2496144	0.4891	1.546	1.2147
>12	1406306	2704156	0.52	1.1518	1.0632
13	1220934	2496144	0.4891	0.8681	0.9405
14	789728	1961256	0.4026	0.6468	0.8232
15	365408	1307504	0.2794	0.4627	0.694
16	109903	735471	0.1494	0.3007	0.5346
17	16506	346104	0.0476	0.1501	0.3191

M=24 V=18 P(G)=¥123446 14 MINORS
R=17 N=7 H(G)=4433333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17483	346104	0.0505	**	**
8	116227	735471	0.158	6.648	3.1284
9	385349	1307504	0.2947	3.3154	1.8649
10	830560	1961256	0.4234	2.1553	1.4368
11	1281680	2496144	0.5134	1.5431	1.2124
>12	1475322	2704156	0.5455	1.151	1.0625
13	1281680	2496144	0.5134	0.8687	0.9411
14	830560	1961256	0.4234	0.648	0.8247
15	385349	1307504	0.2947	0.4639	0.6959
16	116227	735471	0.158	0.3016	0.5362
17	17483	346104	0.0505	0.1504	0.3196

M=24 V=18 P(G)=¥123446 14 MINORS
R=17 N=7 H(G)=4343333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17594	346104	0.0508	**	**
8	117140	735471	0.1592	6.6579	3.1331
9	388920	1307504	0.2974	3.3201	1.8675
10	839196	1961256	0.4278	2.1577	1.4385
11	1295918	2496144	0.5191	1.5442	1.2133
>12	1492064	2704156	0.5517	1.1513	1.0627
13	1295918	2496144	0.5191	0.8685	0.9409
14	839196	1961256	0.4278	0.6475	0.8241
15	388920	1307504	0.2974	0.4634	0.6951
16	117140	735471	0.1592	0.3011	0.5354
17	17594	346104	0.0508	0.1501	0.3191

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=4334333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17959	346104	0.0518	**	**
8	119223	735471	0.1621	6.6386	3.124
9	395089	1307504	0.3021	3.3138	1.864
10	851632	1961256	0.4342	2.1555	1.437
11	1314472	2496144	0.5266	1.5434	1.2127
>12	1513202	2704156	0.5595	1.1511	1.0626
13	1314472	2496144	0.5266	0.8686	0.941
14	851632	1961256	0.4342	0.6478	0.8245
15	395089	1307504	0.3021	0.4639	0.6958
16	119223	735471	0.1621	0.3017	0.5364
17	17959	346104	0.0518	0.1506	0.32

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=4333433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17314	346104	0.05	**	**
8	115188	735471	0.1566	6.6528	3.1307
9	382728	1307504	0.2927	3.3226	1.8689
10	826764	1961256	0.4215	2.1601	1.4401
11	1277814	2496144	0.5119	1.5455	1.2143
>12	1471680	2704156	0.5442	1.1517	1.0631
13	1277814	2496144	0.5119	0.8682	0.9406
14	826764	1961256	0.4215	0.647	0.8234
15	382728	1307504	0.2927	0.4629	0.6943
16	115188	735471	0.1566	0.3009	0.535
17	17314	346104	0.05	0.1503	0.3194

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=4333343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17579	346104	0.0507	**	**
8	116977	735471	0.159	6.6543	3.1314
9	388331	1307504	0.297	3.3197	1.8673
10	838156	1961256	0.4273	2.1583	1.4389
11	1294730	2496144	0.5186	1.5447	1.2137
>12	1490886	2704156	0.5513	1.1515	1.0629
13	1294730	2496144	0.5186	0.8684	0.9407
14	838156	1961256	0.4273	0.6473	0.8239
15	388331	1307504	0.297	0.4633	0.6949
16	116977	735471	0.159	0.3012	0.5355
17	17579	346104	0.0507	0.1502	0.3193

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=4333334

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17276	346104	0.0499	**	**
8	114952	735471	0.1562	6.6538	3.1312
9	381856	1307504	0.292	3.3218	1.8685
10	824832	1961256	0.4205	2.16	1.44
11	1274884	2496144	0.5107	1.5456	1.2144
>12	1468336	2704156	0.5429	1.1517	1.0631
13	1274884	2496144	0.5107	0.8682	0.9406
14	824832	1961256	0.4205	0.6469	0.8234
15	381856	1307504	0.292	0.4629	0.6944
16	114952	735471	0.1562	0.301	0.5351
17	17276	346104	0.0499	0.1502	0.3193

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=3443333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17803	346104	0.0514	**	**
8	118371	735471	0.1609	6.6489	3.1289
9	392261	1307504	0.3	3.3138	1.864
10	844896	1961256	0.4307	2.1539	1.4359
11	1303120	2496144	0.522	1.5423	1.2118
>12	1499706	2704156	0.5545	1.1508	1.0623
13	1303120	2496144	0.522	0.8689	0.9413
14	844896	1961256	0.4307	0.6483	0.8251
15	392261	1307504	0.3	0.4642	0.6964
16	118371	735471	0.1609	0.3017	0.5364
17	17803	346104	0.0514	0.1504	0.3196

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=3434333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18228	346104	0.0526	**	**
8	120976	735471	0.1644	6.6368	3.1232
9	400530	1307504	0.3063	3.3108	1.8623
10	862492	1961256	0.4397	2.1533	1.4355
11	1330266	2496144	0.5329	1.5423	1.2118
>12	1530984	2704156	0.5661	1.1508	1.0623
13	1330266	2496144	0.5329	0.8688	0.9413
14	862492	1961256	0.4397	0.6483	0.8251
15	400530	1307504	0.3063	0.4643	0.6965
16	120976	735471	0.1644	0.302	0.5369
17	18228	346104	0.0526	0.1506	0.3201

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=3433433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17579	346104	0.0507	**	**
8	116955	735471	0.159	6.6531	3.1308
9	388365	1307504	0.297	3.3206	1.8678
10	838320	1961256	0.4274	2.1585	1.439
11	1294952	2496144	0.5187	1.5446	1.2136
>12	1491114	2704156	0.5514	1.1514	1.0629
13	1294952	2496144	0.5187	0.8684	0.9408
14	838320	1961256	0.4274	0.6473	0.8239
15	388365	1307504	0.297	0.4632	0.6948
16	116955	735471	0.159	0.3011	0.5353
17	17579	346104	0.0507	0.1503	0.3193

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=3433343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17848	346104	0.0515	**	**
8	118770	735471	0.1614	6.6545	3.1315
9	394040	1307504	0.3013	3.3176	1.8661
10	849840	1961256	0.4333	2.1567	1.4378
11	1312048	2496144	0.5256	1.5438	1.213
>12	1510524	2704156	0.5585	1.1512	1.0627
13	1312048	2496144	0.5256	0.8686	0.9409
14	849840	1961256	0.4333	0.6477	0.8243
15	394040	1307504	0.3013	0.4636	0.6954
16	118770	735471	0.1614	0.3014	0.5358
17	17848	346104	0.0515	0.1502	0.3193

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=3433334

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17541	346104	0.0506	**	**
8	116723	735471	0.1587	6.6542	3.1314
9	387521	1307504	0.2963	3.32	1.8675
10	836492	1961256	0.4265	2.1585	1.439
11	1292250	2496144	0.5176	1.5448	1.2138
>12	1488066	2704156	0.5502	1.1515	1.0629
13	1292250	2496144	0.5176	0.8684	0.9407
14	836492	1961256	0.4265	0.6473	0.8238
15	387521	1307504	0.2963	0.4632	0.6949
16	116723	735471	0.1587	0.3012	0.5354
17	17541	346104	0.0506	0.1502	0.3193

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=3344333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18199	346104	0.0525	**	**
8	120731	735471	0.1641	6.6339	3.1218
9	399393	1307504	0.3054	3.3081	1.8608
10	859232	1961256	0.4381	2.1513	1.4342
11	1324264	2496144	0.5305	1.5412	1.2109
>12	1523658	2704156	0.5634	1.1505	1.062
13	1324264	2496144	0.5305	0.8691	0.9415
14	859232	1961256	0.4381	0.6488	0.8257
15	399393	1307504	0.3054	0.4648	0.6972
16	120731	735471	0.1641	0.3022	0.5373
17	18199	346104	0.0525	0.1507	0.3203

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=3343433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17630	346104	0.0509	**	**
8	117376	735471	0.1595	6.6577	3.133
9	389812	1307504	0.2981	3.321	1.868
10	841244	1961256	0.4289	2.158	1.4387
11	1299086	2496144	0.5204	1.5442	1.2133
>12	1495688	2704156	0.5531	1.1513	1.0627
13	1299086	2496144	0.5204	0.8685	0.9409
14	841244	1961256	0.4289	0.6475	0.8241
15	389812	1307504	0.2981	0.4633	0.695
16	117376	735471	0.1595	0.3011	0.5353
17	17630	346104	0.0509	0.1502	0.3191

M=24 V=18 P(G)=*123446 14 MINORS
R=17 N=7 H(G)=3343343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17899	346104	0.0517	**	**
8	119181	735471	0.162	6.6585	3.1334
9	395435	1307504	0.3024	3.3179	1.8663
10	852636	1961256	0.4347	2.1561	1.4374
11	1315978	2496144	0.5272	1.5434	1.2126
>12	1514862	2704156	0.5601	1.1511	1.0625
13	1315978	2496144	0.5272	0.8687	0.9411
14	852636	1961256	0.4347	0.6479	0.8246
15	395435	1307504	0.3024	0.4637	0.6956
16	119181	735471	0.162	0.3013	0.5358
17	17899	346104	0.0517	0.1501	0.3191

M=24 V=18 P(G)=X123446 14 MINORS
R=17 N=7 H(G)=3343334

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17600	346104	0.0508	**	**
8	117212	735471	0.1593	6.6597	3.134
9	389268	1307504	0.2977	3.321	1.868
10	840240	1961256	0.4284	2.1585	1.439
11	1297868	2496144	0.5199	1.5446	1.2136
>12	1494440	2704156	0.5526	1.1514	1.0628
13	1297868	2496144	0.5199	0.8684	0.9408
14	840240	1961256	0.4284	0.6474	0.8239
15	389268	1307504	0.2977	0.4632	0.6949
16	117212	735471	0.1593	0.3011	0.5353
17	17600	346104	0.0508	0.1501	0.319

M=24 V=18 P(G)=X123446 14 MINORS
R=17 N=7 H(G)=3334433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17915	346104	0.0517	**	**
8	118843	735471	0.1615	6.6337	3.1217
9	393397	1307504	0.3008	3.3102	1.862
10	846944	1961256	0.4318	2.1528	1.4352
11	1305968	2496144	0.5231	1.5419	1.2115
>12	1502858	2704156	0.5557	1.1507	1.0622
13	1305968	2496144	0.5231	0.8689	0.9414
14	846944	1961256	0.4318	0.6485	0.8253
15	393397	1307504	0.3008	0.4644	0.6967
16	118843	735471	0.1615	0.302	0.537
17	17915	346104	0.0517	0.1507	0.3203

M=24 V=18 P(G)=X123446 14 MINORS
R=17 N=7 H(G)=3334343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18180	346104	0.0525	**	**
8	120523	735471	0.1638	6.6294	3.1197
9	398586	1307504	0.3048	3.3071	1.8602
10	857440	1961256	0.4371	2.1512	1.4341
11	1321506	2496144	0.5294	1.5412	1.2109
>12	1520490	2704156	0.5622	1.1505	1.062
13	1321506	2496144	0.5294	0.8691	0.9415
14	857440	1961256	0.4371	0.6488	0.8257
15	398586	1307504	0.3048	0.4648	0.6972
16	120523	735471	0.1638	0.3023	0.5375
17	18180	346104	0.0525	0.1508	0.3205

M=24 V=18 P(G)=X123446 14 MINORS
R=17 N=7 H(G)=3334334

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17956	346104	0.0518	**	**
8	119184	735471	0.162	6.6375	3.1235
9	394882	1307504	0.302	3.3132	1.8636
10	850972	1961256	0.4338	2.155	1.4366
11	1313146	2496144	0.526	1.5431	1.2124
>12	1511528	2704156	0.5589	1.151	1.0625
13	1313146	2496144	0.526	0.8687	0.9411
14	850972	1961256	0.4338	0.648	0.8247
15	394882	1307504	0.302	0.464	0.696
16	119184	735471	0.162	0.3018	0.5365
17	17956	346104	0.0518	0.1506	0.3201

M=24 V=18 P(G)=X123446 14 MINORS
R=17 N=7 H(G)=3333443

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17611	346104	0.0508	**	**
8	117161	735471	0.1593	6.6527	3.1306
9	388947	1307504	0.2974	3.3197	1.8673
10	839324	1961256	0.4279	2.1579	1.4386
11	1296130	2496144	0.5192	1.5442	1.2133
>12	1492278	2704156	0.5518	1.1513	1.0627
13	1296130	2496144	0.5192	0.8685	0.9409
14	839324	1961256	0.4279	0.6475	0.8241
15	388947	1307504	0.2974	0.4634	0.6951
16	117161	735471	0.1593	0.3012	0.5355
17	17611	346104	0.0508	0.1503	0.3194

M=24 V=18 P(G)=X123446 14 MINORS
R=17 N=7 H(G)=3333434

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17316	346104	0.05	**	**
8	115208	735471	0.1566	6.6532	3.1309
9	382800	1307504	0.2927	3.3226	1.869
10	826928	1961256	0.4216	2.1602	1.4401
11	1277996	2496144	0.5119	1.5454	1.2143
>12	1471824	2704156	0.5442	1.1516	1.063
13	1277996	2496144	0.5119	0.8683	0.9406
14	826928	1961256	0.4216	0.647	0.8235
15	382800	1307504	0.2927	0.4629	0.6943
16	115208	735471	0.1566	0.3009	0.535
17	17316	346104	0.05	0.1503	0.3193

M=24 V=18 P(G)=X123446 14 MINORS
R=17 N=7 H(G)=3333344

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17525	346104	0.0506	**	**
8	116517	735471	0.1584	6.6486	3.1287
9	386307	1307504	0.2954	3.3154	1.8649
10	832608	1961256	0.4245	2.1553	1.4368
11	1284776	2496144	0.5147	1.543	1.2124
>12	1478838	2704156	0.5468	1.151	1.0625
13	1284776	2496144	0.5147	0.8687	0.9411
14	832608	1961256	0.4245	0.648	0.8247
15	386307	1307504	0.2954	0.4639	0.6959
16	116517	735471	0.1584	0.3016	0.5362
17	17525	346104	0.0506	0.1504	0.3196

M=24 V=18 P(G)=¥123436 16 MINORS
R=17 N=7 H(G)=5333333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16494	346104	0.0476	**	**
8	109797	735471	0.1492	6.6567	3.1326
9	365128	1307504	0.2792	3.3254	1.8705
10	789216	1961256	0.4024	2.1614	1.4409
11	1220202	2496144	0.4888	1.546	1.2147
>12	1405494	2704156	0.5197	1.1518	1.0632
13	1220202	2496144	0.4888	0.8681	0.9405
14	789216	1961256	0.4024	0.6467	0.8231
15	365128	1307504	0.2792	0.4626	0.6939
16	109797	735471	0.1492	0.3007	0.5345
17	16494	346104	0.0476	0.1502	0.3192

M=24 V=18 P(G)=¥123436 16 MINORS
R=17 N=7 H(G)=3533333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16976	346104	0.049	**	**
8	112803	735471	0.1533	6.6448	3.1269
9	374406	1307504	0.2863	3.3191	1.867
10	808032	1961256	0.4119	2.1581	1.4387
11	1248074	2496144	0.5	1.5445	1.2136
>12	1437114	2704156	0.5314	1.1514	1.0628
13	1248074	2496144	0.5	0.8684	0.9408
14	808032	1961256	0.4119	0.6474	0.8239
15	374406	1307504	0.2863	0.4633	0.695
16	112803	735471	0.1533	0.3012	0.5356
17	16976	346104	0.049	0.1504	0.3197

M=24 V=18 P(G)=¥123436 16 MINORS
R=17 N=7 H(G)=3353333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17550	346104	0.0507	**	**
8	116325	735471	0.1581	6.6282	3.1191
9	385000	1307504	0.2944	3.3096	1.8617
10	829152	1961256	0.4227	2.1536	1.4357
11	1279146	2496144	0.5124	1.5427	1.2121
>12	1472310	2704156	0.5444	1.151	1.0624
13	1279146	2496144	0.5124	0.8688	0.9412
14	829152	1961256	0.4227	0.6482	0.8249
15	385000	1307504	0.2944	0.4643	0.6964
16	116325	735471	0.1581	0.3021	0.5371
17	17550	346104	0.0507	0.1508	0.3205

M=24 V=18 P(G)=*123436 16 MINORS
R=17 N=7 H(G)=4433333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17515	346104	0.0506	**	**
8	116435	735471	0.1583	6.6477	3.1283
9	386133	1307504	0.2953	3.3162	1.8654
10	832352	1961256	0.4243	2.1556	1.437
11	1284448	2496144	0.5145	1.5431	1.2124
>12	1478490	2704156	0.5467	1.151	1.0625
13	1284448	2496144	0.5145	0.8687	0.9411
14	832352	1961256	0.4243	0.648	0.8247
15	386133	1307504	0.2953	0.4639	0.6958
16	116435	735471	0.1583	0.3015	0.536
17	17515	346104	0.0506	0.1504	0.3196

M=24 V=18 P(G)=*123436 16 MINORS
R=17 N=7 H(G)=4343333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17970	346104	0.0519	**	**
8	119340	735471	0.1622	6.641	3.1252
9	395504	1307504	0.3024	3.314	1.8641
10	852380	1961256	0.4346	2.1551	1.4367
11	1315326	2496144	0.5269	1.5431	1.2124
>12	1514032	2704156	0.5598	1.151	1.0625
13	1315326	2496144	0.5269	0.8687	0.9411
14	852380	1961256	0.4346	0.648	0.8247
15	395504	1307504	0.3024	0.4639	0.6959
16	119340	735471	0.1622	0.3017	0.5364
17	17970	346104	0.0519	0.1505	0.3199

M=24 V=18 P(G)=*123436 16 MINORS
R=17 N=7 H(G)=4334333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17571	346104	0.0507	**	**
8	116915	735471	0.1589	6.6538	3.1312
9	388229	1307504	0.2969	3.3206	1.8678
10	838064	1961256	0.4273	2.1586	1.4391
11	1294584	2496144	0.5186	1.5447	1.2137
>12	1490682	2704156	0.5512	1.1514	1.0629
13	1294584	2496144	0.5186	0.8684	0.9408
14	838064	1961256	0.4273	0.6473	0.8239
15	388229	1307504	0.2969	0.4632	0.6948
16	116915	735471	0.1589	0.3011	0.5353
17	17571	346104	0.0507	0.1502	0.3193

M=24 V=18 P(G)=¥123436 16 MINORS
R=17 N=7 H(G)=4333433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17266	346104	0.0498	**	**
8	114868	735471	0.1561	6.6528	3.1307
9	381672	1307504	0.2919	3.3227	1.869
10	824588	1961256	0.4204	2.1604	1.4403
11	1274566	2496144	0.5106	1.5457	1.2144
>12	1467968	2704156	0.5428	1.1517	1.0631
13	1274566	2496144	0.5106	0.8682	0.9406
14	824588	1961256	0.4204	0.6469	0.8233
15	381672	1307504	0.2919	0.4628	0.6942
16	114868	735471	0.1561	0.3009	0.535
17	17266	346104	0.0498	0.1503	0.3194

M=24 V=18 P(G)=¥123436 16 MINORS
R=17 N=7 H(G)=3443333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18195	346104	0.0525	**	**
8	120691	735471	0.1641	6.6331	3.1215
9	399261	1307504	0.3053	3.3081	1.8608
10	858976	1961256	0.4379	2.1514	1.4342
11	1323888	2496144	0.5303	1.5412	1.2109
>12	1523226	2704156	0.5632	1.1505	1.062
13	1323888	2496144	0.5303	0.8691	0.9415
14	858976	1961256	0.4379	0.6488	0.8257
15	399261	1307504	0.3053	0.4648	0.6972
16	120691	735471	0.1641	0.3022	0.5373
17	18195	346104	0.0525	0.1507	0.3203

M=24 V=18 P(G)=¥123436 16 MINORS
R=17 N=7 H(G)=3434333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17872	346104	0.0516	**	**
8	118908	735471	0.1616	6.6533	3.1309
9	394502	1307504	0.3017	3.3177	1.8662
10	850716	1961256	0.4337	2.1564	1.4376
11	1313098	2496144	0.526	1.5435	1.2127
>12	1511568	2704156	0.5589	1.1511	1.0625
13	1313098	2496144	0.526	0.8686	0.941
14	850716	1961256	0.4337	0.6478	0.8245
15	394502	1307504	0.3017	0.4637	0.6955
16	118908	735471	0.1616	0.3014	0.5358
17	17872	346104	0.0516	0.1503	0.3193

M=24 V=18 P(G)=X123436 16 MINORS
R=17 N=7 H(G)=3433433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17571	346104	0.0507	**	**
8	116915	735471	0.1589	6.6538	3.1312
9	388229	1307504	0.2969	3.3206	1.8678
10	838064	1961256	0.4273	2.1586	1.4391
11	1294584	2496144	0.5186	1.5447	1.2137
>12	1490682	2704156	0.5512	1.1514	1.0629
13	1294584	2496144	0.5186	0.8684	0.9408
14	838064	1961256	0.4273	0.6473	0.8239
15	388229	1307504	0.2969	0.4632	0.6948
16	116915	735471	0.1589	0.3011	0.5353
17	17571	346104	0.0507	0.1502	0.3193

M=24 V=18 P(G)=¥123444 15 MINORS
R=17 N=7 H(G)=5333333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16146	346104	0.0466	**	**
8	108423	735471	0.1474	6.7151	3.16
9	361656	1307504	0.2766	3.3356	1.8762
10	782928	1961256	0.3991	2.1648	1.4432
11	1211382	2496144	0.4853	1.5472	1.2156
>12	1395666	2704156	0.5161	1.1521	1.0635
13	1211382	2496144	0.4853	0.8679	0.9402
14	782928	1961256	0.3991	0.6463	0.8225
15	361656	1307504	0.2766	0.4619	0.6928
16	108423	735471	0.1474	0.2997	0.5329
17	16146	346104	0.0466	0.1489	0.3164

M=24 V=18 P(G)=¥123444 15 MINORS
R=17 N=7 H(G)=3533333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16560	346104	0.0478	**	**
8	111033	735471	0.1509	6.7048	3.1552
9	369738	1307504	0.2827	3.3299	1.8731
10	799344	1961256	0.4075	2.1619	1.4412
11	1235718	2496144	0.495	1.5459	1.2146
>12	1423278	2704156	0.5263	1.1517	1.0631
13	1235718	2496144	0.495	0.8682	0.9405
14	799344	1961256	0.4075	0.6468	0.8232
15	369738	1307504	0.2827	0.4625	0.6938
16	111033	735471	0.1509	0.3003	0.5338
17	16560	346104	0.0478	0.1491	0.3169

M=24 V=18 P(G)=¥123444 15 MINORS
R=17 N=7 H(G)=3353333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16722	346104	0.0483	**	**
8	112167	735471	0.1525	6.7077	3.1565
9	373464	1307504	0.2856	3.3295	1.8728
10	807120	1961256	0.4115	2.1611	1.4407
11	1247382	2496144	0.4997	1.5454	1.2143
>12	1436562	2704156	0.5312	1.1516	1.063
13	1247382	2496144	0.4997	0.8683	0.9406
14	807120	1961256	0.4115	0.647	0.8235
15	373464	1307504	0.2856	0.4627	0.694
16	112167	735471	0.1525	0.3003	0.5339
17	16722	346104	0.0483	0.149	0.3167

M=24 V=18 P(G)=¥123444 16 MINORS
R=17 N=7 H(G)=3335333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17856	346104	0.0515	**	**
8	118081	735471	0.1605	6.6129	3.1119
9	390114	1307504	0.2983	3.3037	1.8583
10	839200	1961256	0.4278	2.1511	1.4341
11	1293822	2496144	0.5183	1.5417	1.2113
>12	1488894	2704156	0.5505	1.1507	1.0622
13	1293822	2496144	0.5183	0.8689	0.9413
14	839200	1961256	0.4278	0.6486	0.8255
15	390114	1307504	0.2983	0.4648	0.6972
16	118081	735471	0.1605	0.3026	0.5381
17	17856	346104	0.0515	0.1512	0.3213

M=24 V=18 P(G)=¥123444 15 MINORS
R=17 N=7 H(G)=3333533

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16274	346104	0.047	**	**
8	109191	735471	0.1484	6.7095	3.1574
9	364184	1307504	0.2785	3.3352	1.8761
10	788304	1961256	0.4019	2.1645	1.443
11	1219478	2496144	0.4885	1.5469	1.2154
>12	1404882	2704156	0.5195	1.152	1.0634
13	1219478	2496144	0.4885	0.868	0.9403
14	788304	1961256	0.4019	0.6464	0.8227
15	364184	1307504	0.2785	0.4619	0.6929
16	109191	735471	0.1484	0.2998	0.533
17	16274	346104	0.047	0.149	0.3167

M=24 V=18 P(G)=¥123444 15 MINORS
R=17 N=7 H(G)=4433333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17109	346104	0.0494	**	**
8	114741	735471	0.156	6.7064	3.1559
9	381699	1307504	0.2919	3.3266	1.8712
10	824112	1961256	0.4201	2.159	1.4393
11	1272744	2496144	0.5098	1.5443	1.2134
>12	1465398	2704156	0.5419	1.1513	1.0628
13	1272744	2496144	0.5098	0.8685	0.9409
14	824112	1961256	0.4201	0.6475	0.8241
15	381699	1307504	0.2919	0.4631	0.6947
16	114741	735471	0.156	0.3006	0.5344
17	17109	346104	0.0494	0.1491	0.3168

M=24 V=18 P(G)=*123444 15 MINORS
R=17 N=7 H(G)=4343333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17262	346104	0.0498	**	**
8	115920	735471	0.1576	6.7153	3.1601
9	386100	1307504	0.2952	3.3307	1.8735
10	834444	1961256	0.4254	2.1612	1.4408
11	1289502	2496144	0.5165	1.5453	1.2141
>12	1485000	2704156	0.5491	1.1516	1.063
13	1289502	2496144	0.5165	0.8683	0.9407
14	834444	1961256	0.4254	0.6471	0.8235
15	386100	1307504	0.2952	0.4627	0.694
16	115920	735471	0.1576	0.3002	0.5337
17	17262	346104	0.0498	0.1489	0.3164

M=24 V=18 P(G)=*123444 16 MINORS
R=17 N=7 H(G)=4334333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18077	346104	0.0522	**	**
8	120163	735471	0.1633	6.6472	3.1281
9	398243	1307504	0.3045	3.3141	1.8642
10	858256	1961256	0.4376	2.1551	1.4367
11	1324448	2496144	0.5305	1.5431	1.2125
>12	1524570	2704156	0.5637	1.151	1.0625
13	1324448	2496144	0.5305	0.8687	0.9411
14	858256	1961256	0.4376	0.648	0.8247
15	398243	1307504	0.3045	0.464	0.696
16	120163	735471	0.1633	0.3017	0.5364
17	18077	346104	0.0522	0.1504	0.3196

M=24 V=18 P(G)=*123444 15 MINORS
R=17 N=7 H(G)=4333433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16982	346104	0.049	**	**
8	113908	735471	0.1548	6.7075	3.1565
9	379784	1307504	0.2904	3.3341	1.8754
10	821740	1961256	0.4189	2.1637	1.4424
11	1271042	2496144	0.5092	1.5467	1.2153
>12	1464192	2704156	0.5414	1.1519	1.0633
13	1271042	2496144	0.5092	0.868	0.9404
14	821740	1961256	0.4189	0.6465	0.8228
15	379784	1307504	0.2904	0.4621	0.6932
16	113908	735471	0.1548	0.2999	0.5332
17	16982	346104	0.049	0.149	0.3168

M=24 V=18 P(G)=*123444 15 MINORS
R=17 N=7 H(G)=3443333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17469	346104	0.0504	**	**
8	117153	735471	0.1592	6.7063	3.1559
9	389475	1307504	0.2978	3.3244	1.87
10	840240	1961256	0.4284	2.1573	1.4382
11	1296864	2496144	0.5195	1.5434	1.2127
>12	1492830	2704156	0.552	1.1511	1.0625
13	1296864	2496144	0.5195	0.8687	0.9411
14	840240	1961256	0.4284	0.6479	0.8246
15	389475	1307504	0.2978	0.4635	0.6952
16	117153	735471	0.1592	0.3007	0.5347
17	17469	346104	0.0504	0.1491	0.3168

M=24 V=18 P(G)=*123444 16 MINORS
R=17 N=7 H(G)=3434333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18349	346104	0.053	**	**
8	121944	735471	0.1658	6.6458	3.1274
9	403787	1307504	0.3088	3.3112	1.8625
10	869340	1961256	0.4432	2.1529	1.4353
11	1340584	2496144	0.537	1.542	1.2116
>12	1542744	2704156	0.5705	1.1507	1.0622
13	1340584	2496144	0.537	0.8689	0.9413
14	869340	1961256	0.4432	0.6484	0.8253
15	403787	1307504	0.3088	0.4644	0.6967
16	121944	735471	0.1658	0.302	0.5368
17	18349	346104	0.053	0.1504	0.3197

M=24 V=18 P(G)=*123444 15 MINORS
R=17 N=7 H(G)=3433433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17245	346104	0.0498	**	**
8	115689	735471	0.1572	6.7085	3.1569
9	385511	1307504	0.2948	3.3323	1.8744
10	833520	1961256	0.4249	2.1621	1.4414
11	1288540	2496144	0.5162	1.5459	1.2146
>12	1484046	2704156	0.5488	1.1517	1.0631
13	1288540	2496144	0.5162	0.8682	0.9406
14	833520	1961256	0.4249	0.6468	0.8232
15	385511	1307504	0.2948	0.4625	0.6937
16	115689	735471	0.1572	0.3	0.5334
17	17245	346104	0.0498	0.149	0.3167

M=24 V=18 P(G)=*123444 16 MINORS
R=17 N=7 H(G)=3344333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18337	346104	0.0529	**	**
8	121815	735471	0.1656	6.6431	3.1261
9	403031	1307504	0.3082	3.3085	1.861
10	866880	1961256	0.442	2.1509	1.4339
11	1335784	2496144	0.5351	1.5409	1.2107
>12	1536786	2704156	0.5683	1.1504	1.0619
13	1335784	2496144	0.5351	0.8692	0.9416
14	866880	1961256	0.442	0.6489	0.8259
15	403031	1307504	0.3082	0.4649	0.6973
16	121815	735471	0.1656	0.3022	0.5373
17	18337	346104	0.0529	0.1505	0.3198

M=24 V=18 P(G)=*123444 15 MINORS
R=17 N=7 H(G)=3343433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17338	346104	0.05	**	**
8	116376	735471	0.1582	6.7121	3.1586
9	387788	1307504	0.2965	3.3321	1.8743
10	838140	1961256	0.4273	2.1613	1.4408
11	1295194	2496144	0.5188	1.5453	1.2141
>12	1491480	2704156	0.5515	1.1515	1.0629
13	1295194	2496144	0.5188	0.8683	0.9407
14	838140	1961256	0.4273	0.6471	0.8236
15	387788	1307504	0.2965	0.4626	0.694
16	116376	735471	0.1582	0.3001	0.5335
17	17338	346104	0.05	0.1489	0.3165

M=24 V=18 P(G)=*123444 16 MINORS
R=17 N=7 H(G)=3334433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18049	346104	0.0521	**	**
8	119879	735471	0.1629	6.6418	3.1255
9	396903	1307504	0.3035	3.3108	1.8623
10	854336	1961256	0.4356	2.1525	1.435
11	1317112	2496144	0.5276	1.5416	1.2113
>12	1515570	2704156	0.5604	1.1506	1.0621
13	1317112	2496144	0.5276	0.869	0.9414
14	854336	1961256	0.4356	0.6486	0.8255
15	396903	1307504	0.3035	0.4645	0.6968
16	119879	735471	0.1629	0.302	0.5369
17	18049	346104	0.0521	0.1505	0.3199

M=24 V=18 P(G)=*123444 15 MINORS
R=17 N=7 H(G)=3333443

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17054	346104	0.0492	**	**
8	114324	735471	0.1554	6.7036	3.1546
9	381240	1307504	0.2915	3.3347	1.8757
10	824700	1961256	0.4204	2.1632	1.4421
11	1275146	2496144	0.5108	1.5461	1.2148
>12	1468704	2704156	0.5431	1.1517	1.0631
13	1275146	2496144	0.5108	0.8682	0.9405
14	824700	1961256	0.4204	0.6467	0.8231
15	381240	1307504	0.2915	0.4622	0.6934
16	114324	735471	0.1554	0.2998	0.5331
17	17054	346104	0.0492	0.1491	0.3169

M=24 V=18 P(G)=*123336 15 MINORS
R=17 N=7 H(G)=5333333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16130	346104	0.0466	**	**
8	108303	735471	0.1472	6.7143	3.1597
9	361464	1307504	0.2764	3.3375	1.8773
10	782688	1961256	0.399	2.1653	1.4435
11	1211174	2496144	0.4852	1.5474	1.2158
>12	1395426	2704156	0.516	1.1521	1.0635
13	1211174	2496144	0.4852	0.8679	0.9402
14	782688	1961256	0.399	0.6462	0.8224
15	361464	1307504	0.2764	0.4618	0.6927
16	108303	735471	0.1472	0.2996	0.5326
17	16130	346104	0.0466	0.1489	0.3164

M=24 V=18 P(G)=*123336 15 MINORS
R=17 N=7 H(G)=3533333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16672	346104	0.0481	**	**
8	111681	735471	0.1518	6.6987	3.1523
9	371882	1307504	0.2844	3.3298	1.873
10	803808	1961256	0.4098	2.1614	1.4409
11	1242454	2496144	0.4977	1.5457	1.2144
>12	1430910	2704156	0.5291	1.1516	1.063
13	1242454	2496144	0.4977	0.8682	0.9406
14	803808	1961256	0.4098	0.6469	0.8233
15	371882	1307504	0.2844	0.4626	0.6939
16	111681	735471	0.1518	0.3003	0.5338
17	16672	346104	0.0481	0.1492	0.3172

M=24 V=18 P(G)=*123336 16 MINORS
R=17 N=7 H(G)=3353333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17880	346104	0.0516	**	**
8	118339	735471	0.1609	6.6185	3.1145
9	391086	1307504	0.2991	3.3047	1.8589
10	841312	1961256	0.4289	2.1512	1.4341
11	1297050	2496144	0.5196	1.5416	1.2113
>12	1492602	2704156	0.5519	1.1507	1.0622
13	1297050	2496144	0.5196	0.8689	0.9414
14	841312	1961256	0.4289	0.6486	0.8255
15	391086	1307504	0.2991	0.4648	0.6972
16	118339	735471	0.1609	0.3025	0.5379
17	17880	346104	0.0516	0.151	0.321

M=24 V=18 P(G)=¥123336 15 MINORS
R=17 N=7 H(G)=3335333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16254	346104	0.0469	**	**
8	109029	735471	0.1482	6.7078	3.1566
9	363832	1307504	0.2782	3.337	1.877
10	787680	1961256	0.4016	2.1649	1.4433
11	1218666	2496144	0.4882	1.5471	1.2156
>12	1403958	2704156	0.5191	1.152	1.0634
13	1218666	2496144	0.4882	0.868	0.9403
14	787680	1961256	0.4016	0.6463	0.8226
15	363832	1307504	0.2782	0.4619	0.6928
16	109029	735471	0.1482	0.2996	0.5327
17	16254	346104	0.0469	0.149	0.3167

M=24 V=18 P(G)=¥123336 15 MINORS
R=17 N=7 H(G)=4433333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17173	346104	0.0496	**	**
8	115125	735471	0.1565	6.7038	3.1547
9	383123	1307504	0.293	3.3278	1.8719
10	827232	1961256	0.4217	2.1591	1.4394
11	1277560	2496144	0.5118	1.5443	1.2134
>12	1470870	2704156	0.5439	1.1513	1.0627
13	1277560	2496144	0.5118	0.8685	0.9409
14	827232	1961256	0.4217	0.6475	0.8241
15	383123	1307504	0.293	0.4631	0.6947
16	115125	735471	0.1565	0.3004	0.5342
17	17173	346104	0.0496	0.1491	0.3169

M=24 V=18 P(G)=¥123336 16 MINORS
R=17 N=7 H(G)=4343333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18084	346104	0.0522	**	**
8	120248	735471	0.1634	6.6494	3.1291
9	398594	1307504	0.3048	3.3147	1.8645
10	858908	1961256	0.4379	2.1548	1.4365
11	1325178	2496144	0.5308	1.5428	1.2122
>12	1525272	2704156	0.564	1.1509	1.0624
13	1325178	2496144	0.5308	0.8688	0.9412
14	858908	1961256	0.4379	0.6481	0.8249
15	398594	1307504	0.3048	0.464	0.6961
16	120248	735471	0.1634	0.3016	0.5363
17	18084	346104	0.0522	0.1503	0.3195

M=24 V=18 P(G)=*123336 15 MINORS
R=17 N=7 H(G)=4334333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16964	346104	0.049	**	**
8	113768	735471	0.1546	6.7064	3.1559
9	379536	1307504	0.2902	3.336	1.8765
10	821424	1961256	0.4188	2.1642	1.4428
11	1270604	2496144	0.509	1.5468	1.2153
>12	1463696	2704156	0.5412	1.1519	1.0633
13	1270604	2496144	0.509	0.868	0.9404
14	821424	1961256	0.4188	0.6464	0.8227
15	379536	1307504	0.2902	0.462	0.693
16	113768	735471	0.1546	0.2997	0.5328
17	16964	346104	0.049	0.1491	0.3168

M=24 V=18 P(G)=*123336 15 MINORS
R=17 N=7 H(G)=4333343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17229	346104	0.0497	**	**
8	115617	735471	0.1572	6.7106	3.1579
9	385263	1307504	0.2946	3.3322	1.8743
10	833088	1961256	0.4247	2.1623	1.4415
11	1287876	2496144	0.5159	1.5459	1.2146
>12	1483326	2704156	0.5485	1.1517	1.0631
13	1287876	2496144	0.5159	0.8682	0.9405
14	833088	1961256	0.4247	0.6468	0.8232
15	385263	1307504	0.2946	0.4624	0.6936
16	115617	735471	0.1572	0.3	0.5335
17	17229	346104	0.0497	0.149	0.3166

M=24 V=18 P(G)=*123336 15 MINORS
R=17 N=7 H(G)=4333334

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16886	346104	0.0487	**	**
8	113316	735471	0.154	6.7106	3.1579
9	377896	1307504	0.289	3.3348	1.8758
10	817948	1961256	0.417	2.1644	1.4429
11	1265378	2496144	0.5069	1.547	1.2155
>12	1457792	2704156	0.539	1.152	1.0634
13	1265378	2496144	0.5069	0.868	0.9403
14	817948	1961256	0.417	0.6464	0.8226
15	377896	1307504	0.289	0.462	0.693
16	113316	735471	0.154	0.2998	0.533
17	16886	346104	0.0487	0.149	0.3166

M=24 V=18 P(G)=*123336 16 MINORS
R=17 N=7 H(G)=3443333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18328	346104	0.0529	**	**
8	121731	735471	0.1655	6.6418	3.1255
9	402782	1307504	0.308	3.3087	1.8611
10	866400	1961256	0.4417	2.151	1.434
11	1335082	2496144	0.5348	1.5409	1.2107
>12	1535994	2704156	0.568	1.1504	1.0619
13	1335082	2496144	0.5348	0.8691	0.9416
14	866400	1961256	0.4417	0.6489	0.8259
15	402782	1307504	0.308	0.4648	0.6973
16	121731	735471	0.1655	0.3022	0.5372
17	18328	346104	0.0529	0.1505	0.3199

M=24 V=18 P(G)=*123336 15 MINORS
R=17 N=7 H(G)=3434333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17299	346104	0.0499	**	**
8	116001	735471	0.1577	6.7056	3.1555
9	386603	1307504	0.2956	3.3327	1.8746
10	835740	1961256	0.4261	2.1617	1.4411
11	1291618	2496144	0.5174	1.5454	1.2143
>12	1487430	2704156	0.55	1.1516	1.063
13	1291618	2496144	0.5174	0.8683	0.9407
14	835740	1961256	0.4261	0.647	0.8235
15	386603	1307504	0.2956	0.4625	0.6938
16	116001	735471	0.1577	0.3	0.5334
17	17299	346104	0.0499	0.1491	0.3168

M=24 V=18 P(G)=*123336 15 MINORS
R=17 N=7 H(G)=3433343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17568	346104	0.0507	**	**
8	117864	735471	0.1602	6.709	3.1571
9	392346	1307504	0.3	3.3288	1.8724
10	847404	1961256	0.432	2.1598	1.4398
11	1308870	2496144	0.5243	1.5445	1.2135
>12	1507032	2704156	0.5573	1.1513	1.0628
13	1308870	2496144	0.5243	0.8685	0.9408
14	847404	1961256	0.432	0.6474	0.824
15	392346	1307504	0.3	0.4629	0.6944
16	117864	735471	0.1602	0.3004	0.534
17	17568	346104	0.0507	0.149	0.3167

M=24 V=18 P(G)=*123336 16 MINORS
R=17 N=7 H(G)=3344333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18059	346104	0.0521	**	**
8	120003	735471	0.1631	6.645	3.127
9	397461	1307504	0.3039	3.312	1.863
10	855648	1961256	0.4362	2.1527	1.4351
11	1319168	2496144	0.5284	1.5417	1.2113
>12	1517946	2704156	0.5613	1.1506	1.0621
13	1319168	2496144	0.5284	0.869	0.9414
14	855648	1961256	0.4362	0.6486	0.8255
15	397461	1307504	0.3039	0.4645	0.6967
16	120003	735471	0.1631	0.3019	0.5367
17	18059	346104	0.0521	0.1504	0.3197

M=24 V=18 P(G)=*123336 15 MINORS
R=17 N=7 H(G)=3334433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17034	346104	0.0492	**	**
8	114164	735471	0.1552	6.7021	3.1539
9	380920	1307504	0.2913	3.3366	1.8768
10	824220	1961256	0.4202	2.1637	1.4425
11	1274526	2496144	0.5105	1.5463	1.2149
>12	1468064	2704156	0.5428	1.1518	1.0632
13	1274526	2496144	0.5105	0.8681	0.9405
14	824220	1961256	0.4202	0.6466	0.823
15	380920	1307504	0.2913	0.4621	0.6932
16	114164	735471	0.1552	0.2997	0.5328
17	17034	346104	0.0492	0.1492	0.317

M=24 V=18 P(G)=*123333 19 MINORS
R=17 N=7 H(G)=3353333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18172	346104	0.0525	**	**
8	120225	735471	0.1634	6.6159	3.1133
9	396878	1307504	0.3035	3.3011	1.8568
10	852960	1961256	0.4349	2.1491	1.4327
11	1314262	2496144	0.5265	1.5408	1.2106
>12	1512126	2704156	0.5591	1.1505	1.062
13	1314262	2496144	0.5265	0.8691	0.9415
14	852960	1961256	0.4349	0.649	0.826
15	396878	1307504	0.3035	0.4652	0.6979
16	120225	735471	0.1634	0.3029	0.5385
17	18172	346104	0.0525	0.1511	0.3211

M=24 V=18 P(G)=*123333 18 MINORS
R=17 N=7 H(G)=4343333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18102	346104	0.0523	**	**
8	120828	735471	0.1642	6.6748	3.1411
9	400860	1307504	0.3065	3.3176	1.8661
10	864012	1961256	0.4405	2.1553	1.4369
11	1333038	2496144	0.534	1.5428	1.2122
>12	1534320	2704156	0.5673	1.1509	1.0624
13	1333038	2496144	0.534	0.8688	0.9412
14	864012	1961256	0.4405	0.6481	0.8249
15	400860	1307504	0.3065	0.4639	0.6959
16	120828	735471	0.1642	0.3014	0.5358
17	18102	346104	0.0523	0.1498	0.3183

M=24 V=18 P(G)=*123333 18 MINORS
R=17 N=7 H(G)=3443333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18369	346104	0.053	**	**
8	122463	735471	0.1665	6.6668	3.1373
9	405531	1307504	0.3101	3.3114	1.8626
10	872496	1961256	0.4448	2.1514	1.4343
11	1344420	2496144	0.5385	1.5408	1.2106
>12	1546722	2704156	0.5719	1.1504	1.0619
13	1344420	2496144	0.5385	0.8692	0.9416
14	872496	1961256	0.4448	0.6489	0.8259
15	405531	1307504	0.3101	0.4647	0.6971
16	122463	735471	0.1665	0.3019	0.5368
17	18369	346104	0.053	0.1499	0.3187

M=24 V=18 P(G)=*123333 18 MINORS
R=17 N=7 H(G)=3344333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18097	346104	0.0522	**	**
8	120687	735471	0.164	6.6688	3.1383
9	400103	1307504	0.306	3.3152	1.8648
10	861504	1961256	0.4392	2.1532	1.4354
11	1328200	2496144	0.5321	1.5417	1.2113
>12	1528290	2704156	0.5651	1.1506	1.0621
13	1328200	2496144	0.5321	0.869	0.9414
14	861504	1961256	0.4392	0.6486	0.8255
15	400103	1307504	0.306	0.4644	0.6966
16	120687	735471	0.164	0.3016	0.5362
17	18097	346104	0.0522	0.1499	0.3186

M=24 V=18 P(G)=*122446 15 MINORS
R=17 N=7 H(G)=5333333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16374	346104	0.0473	**	**
8	109413	735471	0.1487	6.6821	3.1445
9	364480	1307504	0.2787	3.3312	1.8738
10	788448	1961256	0.402	2.1632	1.4421
11	1219434	2496144	0.4885	1.5466	1.2152
>12	1404726	2704156	0.5194	1.1519	1.0633
13	1219434	2496144	0.4885	0.868	0.9404
14	788448	1961256	0.402	0.6465	0.8229
15	364480	1307504	0.2787	0.4622	0.6934
16	109413	735471	0.1487	0.3001	0.5336
17	16374	346104	0.0473	0.1496	0.318

M=24 V=18 P(G)=*122446 15 MINORS
R=17 N=7 H(G)=3533333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17368	346104	0.0501	**	**
8	115299	735471	0.1567	6.6385	3.124
9	382046	1307504	0.2921	3.3135	1.8638
10	823392	1961256	0.4198	2.1552	1.4368
11	1270762	2496144	0.509	1.5433	1.2126
>12	1462842	2704156	0.5409	1.1511	1.0626
13	1270762	2496144	0.509	0.8686	0.941
14	823392	1961256	0.4198	0.6479	0.8246
15	382046	1307504	0.2921	0.4639	0.6959
16	115299	735471	0.1567	0.3017	0.5365
17	17368	346104	0.0501	0.1506	0.32

M=24 V=18 P(G)=*122446 15 MINORS
R=17 N=7 H(G)=3335333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17404	346104	0.0502	**	**
8	115665	735471	0.1572	6.6458	3.1274
9	383318	1307504	0.2931	3.314	1.8641
10	826080	1961256	0.4211	2.155	1.4367
11	1274830	2496144	0.5107	1.5432	1.2125
>12	1467486	2704156	0.5426	1.1511	1.0625
13	1274830	2496144	0.5107	0.8687	0.9411
14	826080	1961256	0.4211	0.6479	0.8247
15	383318	1307504	0.2931	0.464	0.696
16	115665	735471	0.1572	0.3017	0.5364
17	17404	346104	0.0502	0.1504	0.3197

M=24 V=18 P(G)=*122446 15 MINORS
R=17 N=7 H(G)=3333533

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16378	346104	0.0473	**	**
8	109455	735471	0.1488	6.683	3.1449
9	364640	1307504	0.2788	3.3314	1.8739
10	788832	1961256	0.4022	2.1633	1.4422
11	1220038	2496144	0.4887	1.5466	1.2152
>12	1405410	2704156	0.5197	1.1519	1.0633
13	1220038	2496144	0.4887	0.8681	0.9404
14	788832	1961256	0.4022	0.6465	0.8228
15	364640	1307504	0.2788	0.4622	0.6933
16	109455	735471	0.1488	0.3001	0.5336
17	16378	346104	0.0473	0.1496	0.3179

M=24 V=18 P(G)=*122446 15 MINORS
R=17 N=7 H(G)=3333353

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16792	346104	0.0485	**	**
8	112035	735471	0.1523	6.6719	3.1397
9	372494	1307504	0.2848	3.3248	1.8702
10	804576	1961256	0.4102	2.1599	1.4399
11	1243258	2496144	0.498	1.5452	1.2141
>12	1431738	2704156	0.5294	1.1516	1.063
13	1243258	2496144	0.498	0.8683	0.9407
14	804576	1961256	0.4102	0.6471	0.8236
15	372494	1307504	0.2848	0.4629	0.6944
16	112035	735471	0.1523	0.3007	0.5347
17	16792	346104	0.0485	0.1498	0.3184

M=24 V=18 P(G)=*122446 15 MINORS
R=17 N=7 H(G)=3333335

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16310	346104	0.0471	**	**
8	109029	735471	0.1482	6.6847	3.1457
9	363216	1307504	0.2777	3.3313	1.8738
10	785760	1961256	0.4006	2.1633	1.4422
11	1215386	2496144	0.4869	1.5467	1.2153
>12	1400118	2704156	0.5177	1.1519	1.0633
13	1215386	2496144	0.4869	0.868	0.9403
14	785760	1961256	0.4006	0.6465	0.8228
15	363216	1307504	0.2777	0.4622	0.6933
16	109029	735471	0.1482	0.3001	0.5336
17	16310	346104	0.0471	0.1495	0.3178

M=24 V=18 P(G)=¥122446 15 MINORS
R=17 N=7 H(G)=4433333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17739	346104	0.0512	**	**
8	118019	735471	0.1604	6.653	3.1308
9	391301	1307504	0.2992	3.3155	1.865
10	843104	1961256	0.4298	2.1546	1.4364
11	1300560	2496144	0.521	1.5425	1.212
>12	1496826	2704156	0.5535	1.1509	1.0623
13	1300560	2496144	0.521	0.8688	0.9412
14	843104	1961256	0.4298	0.6482	0.825
15	391301	1307504	0.2992	0.4641	0.6961
16	118019	735471	0.1604	0.3016	0.5361
17	17739	346104	0.0512	0.1503	0.3194

M=24 V=18 P(G)=¥122446 15 MINORS
R=17 N=7 H(G)=4343333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17146	346104	0.0495	**	**
8	114484	735471	0.1556	6.677	3.1421
9	381144	1307504	0.2915	3.3292	1.8726
10	823964	1961256	0.4201	2.1618	1.4412
11	1273678	2496144	0.5102	1.5457	1.2145
>12	1466912	2704156	0.5424	1.1517	1.0631
13	1273678	2496144	0.5102	0.8682	0.9406
14	823964	1961256	0.4201	0.6469	0.8233
15	381144	1307504	0.2915	0.4625	0.6938
16	114484	735471	0.1556	0.3003	0.5339
17	17146	346104	0.0495	0.1497	0.3182

M=24 V=18 P(G)=¥122446 15 MINORS
R=17 N=7 H(G)=4334333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17839	346104	0.0515	**	**
8	118829	735471	0.1615	6.6611	3.1346
9	394479	1307504	0.3017	3.3197	1.8673
10	850844	1961256	0.4338	2.1568	1.4379
11	1313410	2496144	0.5261	1.5436	1.2128
>12	1511982	2704156	0.5591	1.1511	1.0626
13	1313410	2496144	0.5261	0.8686	0.941
14	850844	1961256	0.4338	0.6478	0.8244
15	394479	1307504	0.3017	0.4636	0.6954
16	118829	735471	0.1615	0.3012	0.5355
17	17839	346104	0.0515	0.1501	0.319

M=24 V=18 P(G)=¥122446 15 MINORS
R=17 N=7 H(G)=4333433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17156	346104	0.0495	**	**
8	114568	735471	0.1557	6.678	3.1425
9	381520	1307504	0.2917	3.33	1.8731
10	825008	1961256	0.4206	2.1624	1.4416
11	1275596	2496144	0.511	1.5461	1.2148
>12	1469264	2704156	0.5433	1.1518	1.0632
13	1275596	2496144	0.511	0.8681	0.9405
14	825008	1961256	0.4206	0.6467	0.8231
15	381520	1307504	0.2917	0.4624	0.6936
16	114568	735471	0.1557	0.3002	0.5338
17	17156	346104	0.0495	0.1497	0.3182

M=24 V=18 P(G)=¥122446 15 MINORS
R=17 N=7 H(G)=4333343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17419	346104	0.0503	**	**
8	116355	735471	0.1582	6.6797	3.1434
9	387125	1307504	0.296	3.3271	1.8714
10	836400	1961256	0.4264	2.1605	1.4403
11	1292512	2496144	0.5178	1.5453	1.2141
>12	1488474	2704156	0.5504	1.1516	1.063
13	1292512	2496144	0.5178	0.8683	0.9407
14	836400	1961256	0.4264	0.6471	0.8235
15	387125	1307504	0.296	0.4628	0.6942
16	116355	735471	0.1582	0.3005	0.5343
17	17419	346104	0.0503	0.1497	0.3181

M=24 V=18 P(G)=¥122446 15 MINORS
R=17 N=7 H(G)=4333334

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17114	346104	0.0494	**	**
8	114308	735471	0.1554	6.6792	3.1431
9	380568	1307504	0.291	3.3293	1.8727
10	822924	1961256	0.4195	2.1623	1.4415
11	1272494	2496144	0.5097	1.5463	1.2149
>12	1465760	2704156	0.542	1.1518	1.0632
13	1272494	2496144	0.5097	0.8681	0.9404
14	822924	1961256	0.4195	0.6467	0.823
15	380568	1307504	0.291	0.4624	0.6936
16	114308	735471	0.1554	0.3003	0.5339
17	17114	346104	0.0494	0.1497	0.3181

M=24 V=18 P(G)=¥122446 15 MINORS
R=17 N=7 H(G)=3434333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18408	346104	0.0531	**	**
8	122107	735471	0.166	6.6333	3.1215
9	403758	1307504	0.3088	3.3065	1.8599
10	868192	1961256	0.4426	2.1502	1.4335
11	1337610	2496144	0.5358	1.5406	1.2105
>12	1538826	2704156	0.569	1.1504	1.0619
13	1337610	2496144	0.5358	0.8692	0.9416
14	868192	1961256	0.4426	0.649	0.826
15	403758	1307504	0.3088	0.465	0.6975
16	122107	735471	0.166	0.3024	0.5376
17	18408	346104	0.0531	0.1507	0.3203

M=24 V=18 P(G)=¥122446 15 MINORS
R=17 N=7 H(G)=3433433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17816	346104	0.0514	**	**
8	118596	735471	0.1612	6.6567	3.1325
9	393678	1307504	0.301	3.3194	1.8672
10	849180	1961256	0.4329	2.157	1.438
11	1310906	2496144	0.5251	1.5437	1.2129
>12	1509120	2704156	0.558	1.1512	1.0626
13	1310906	2496144	0.5251	0.8686	0.941
14	849180	1961256	0.4329	0.6477	0.8244
15	393678	1307504	0.301	0.4635	0.6953
16	118596	735471	0.1612	0.3012	0.5355
17	17816	346104	0.0514	0.1502	0.3192

M=24 V=18 P(G)=¥122446 15 MINORS
R=17 N=7 H(G)=3433343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18088	346104	0.0522	**	**
8	120423	735471	0.1637	6.6576	3.1329
9	399368	1307504	0.3054	3.3163	1.8654
10	860700	1961256	0.4388	2.1551	1.4367
11	1327984	2496144	0.532	1.5429	1.2122
>12	1528506	2704156	0.5652	1.1509	1.0624
13	1327984	2496144	0.532	0.8688	0.9412
14	860700	1961256	0.4388	0.6481	0.8248
15	399368	1307504	0.3054	0.464	0.696
16	120423	735471	0.1637	0.3015	0.536
17	18088	346104	0.0522	0.1502	0.3191

M=24 V=18 P(G)=X122446 15 MINORS
R=17 N=7 H(G)=3433334

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17784	346104	0.0513	**	**
8	118418	735471	0.161	6.6586	3.1334
9	393080	1307504	0.3006	3.3194	1.8671
10	848048	1961256	0.4324	2.1574	1.4382
11	1309488	2496144	0.5246	1.5441	1.2132
>12	1507644	2704156	0.5575	1.1513	1.0627
13	1309488	2496144	0.5246	0.8685	0.9409
14	848048	1961256	0.4324	0.6476	0.8242
15	393080	1307504	0.3006	0.4635	0.6952
16	118418	735471	0.161	0.3012	0.5355
17	17784	346104	0.0513	0.1501	0.3191

M=24 V=18 P(G)=X122446 15 MINORS
R=17 N=7 H(G)=3334433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17765	346104	0.0513	**	**
8	118285	735471	0.1608	6.6583	3.1333
9	392243	1307504	0.2999	3.316	1.8652
10	845152	1961256	0.4309	2.1546	1.4364
11	1303688	2496144	0.5222	1.5425	1.212
>12	1500390	2704156	0.5548	1.1508	1.0623
13	1303688	2496144	0.5222	0.8688	0.9413
14	845152	1961256	0.4309	0.6482	0.825
15	392243	1307504	0.2999	0.4641	0.6961
16	118285	735471	0.1608	0.3015	0.5361
17	17765	346104	0.0513	0.1501	0.3191

M=24 V=18 P(G)=X122446 15 MINORS
R=17 N=7 H(G)=3334343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18028	346104	0.052	**	**
8	119961	735471	0.1631	6.6541	3.1313
9	397430	1307504	0.3039	3.3129	1.8635
10	855648	1961256	0.4362	2.1529	1.4353
11	1319230	2496144	0.5285	1.5417	1.2114
>12	1518030	2704156	0.5613	1.1506	1.0621
13	1319230	2496144	0.5285	0.869	0.9414
14	855648	1961256	0.4362	0.6485	0.8254
15	397430	1307504	0.3039	0.4644	0.6967
16	119961	735471	0.1631	0.3018	0.5366
17	18028	346104	0.052	0.1502	0.3193

M=24 V=18 P(G)=¥122446 15 MINORS
R=17 N=7 H(G)=3334334

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17804	346104	0.0514	**	**
8	118612	735471	0.1612	6.662	3.1351
9	393674	1307504	0.301	3.319	1.8669
10	849052	1961256	0.4329	2.1567	1.4378
11	1310666	2496144	0.525	1.5436	1.2128
>12	1508832	2704156	0.5579	1.1511	1.0626
13	1310666	2496144	0.525	0.8686	0.941
14	849052	1961256	0.4329	0.6478	0.8244
15	393674	1307504	0.301	0.4636	0.6954
16	118612	735471	0.1612	0.3012	0.5356
17	17804	346104	0.0514	0.1501	0.3189

M=24 V=18 P(G)=¥122446 15 MINORS
R=17 N=7 H(G)=3333443

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17413	346104	0.0503	**	**
8	116307	735471	0.1581	6.6793	3.1432
9	386897	1307504	0.2959	3.3265	1.8711
10	835740	1961256	0.4261	2.1601	1.44
11	1291210	2496144	0.5172	1.5449	1.2139
>12	1486818	2704156	0.5498	1.1514	1.0629
13	1291210	2496144	0.5172	0.8684	0.9408
14	835740	1961256	0.4261	0.6472	0.8237
15	386897	1307504	0.2959	0.4629	0.6944
16	116307	735471	0.1581	0.3006	0.5344
17	17413	346104	0.0503	0.1497	0.3181

M=24 V=18 P(G)=¥122446 15 MINORS
R=17 N=7 H(G)=3333434

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17116	346104	0.0494	**	**
8	114328	735471	0.1554	6.6795	3.1433
9	380640	1307504	0.2911	3.3293	1.8727
10	823088	1961256	0.4196	2.1623	1.4415
11	1272676	2496144	0.5098	1.5462	1.2148
>12	1465904	2704156	0.542	1.1518	1.0632
13	1272676	2496144	0.5098	0.8681	0.9405
14	823088	1961256	0.4196	0.6467	0.8231
15	380640	1307504	0.2911	0.4624	0.6936
16	114328	735471	0.1554	0.3003	0.5339
17	17116	346104	0.0494	0.1497	0.3181

M=24 V=18 P(G)=X122446 15 MINORS
R=17 N=7 H(G)=3333344

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17323	346104	0.05	XX	XX
8	115635	735471	0.1572	6.6752	3.1412
9	384149	1307504	0.2938	3.322	1.8686
10	828768	1961256	0.4225	2.1574	1.4382
11	1279456	2496144	0.5125	1.5438	1.2129
>12	1472922	2704156	0.5446	1.1512	1.0626
13	1279456	2496144	0.5125	0.8686	0.941
14	828768	1961256	0.4225	0.6477	0.8244
15	384149	1307504	0.2938	0.4635	0.6952
16	115635	735471	0.1572	0.301	0.5351
17	17323	346104	0.05	0.1498	0.3183

M=24 V=18 P(G)=¥122455 15 MINORS
R=17 N=7 H(G)=5333333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16402	346104	0.0473	**	**
8	109575	735471	0.1489	6.6805	3.1438
9	364856	1307504	0.279	3.3297	1.8729
10	789072	1961256	0.4023	2.1626	1.4417
11	1220278	2496144	0.4888	1.5464	1.215
>12	1405650	2704156	0.5198	1.1519	1.0633
13	1220278	2496144	0.4888	0.8681	0.9404
14	789072	1961256	0.4023	0.6466	0.8229
15	364856	1307504	0.279	0.4623	0.6935
16	109575	735471	0.1489	0.3003	0.5339
17	16402	346104	0.0473	0.1496	0.318

M=24 V=18 P(G)=¥122455 15 MINORS
R=17 N=7 H(G)=3533333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17320	346104	0.05	**	**
8	115005	735471	0.1563	6.64	3.1247
9	381050	1307504	0.2914	3.3133	1.8637
10	821280	1961256	0.4187	2.1553	1.4368
11	1267582	2496144	0.5078	1.5434	1.2126
>12	1459206	2704156	0.5396	1.1511	1.0626
13	1267582	2496144	0.5078	0.8686	0.941
14	821280	1961256	0.4187	0.6479	0.8246
15	381050	1307504	0.2914	0.4639	0.6959
16	115005	735471	0.1563	0.3018	0.5365
17	17320	346104	0.05	0.1506	0.32

M=24 V=18 P(G)=¥122455 15 MINORS
R=17 N=7 H(G)=3353333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16402	346104	0.0473	**	**
8	109575	735471	0.1489	6.6805	3.1438
9	364856	1307504	0.279	3.3297	1.8729
10	789072	1961256	0.4023	2.1626	1.4417
11	1220278	2496144	0.4888	1.5464	1.215
>12	1405650	2704156	0.5198	1.1519	1.0633
13	1220278	2496144	0.4888	0.8681	0.9404
14	789072	1961256	0.4023	0.6466	0.8229
15	364856	1307504	0.279	0.4623	0.6935
16	109575	735471	0.1489	0.3003	0.5339
17	16402	346104	0.0473	0.1496	0.318

M=24 V=18 P(G)=¥122455 15 MINORS
R=17 N=7 H(G)=3335333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16912	346104	0.0488	**	**
8	112923	735471	0.1535	6.677	3.1421
9	375398	1307504	0.2871	3.3243	1.8699
10	810576	1961256	0.4132	2.1592	1.4394
11	1252234	2496144	0.5016	1.5448	1.2138
>12	1441962	2704156	0.5332	1.1515	1.0629
13	1252234	2496144	0.5016	0.8684	0.9407
14	810576	1961256	0.4132	0.6473	0.8238
15	375398	1307504	0.2871	0.4631	0.6946
16	112923	735471	0.1535	0.3008	0.5347
17	16912	346104	0.0488	0.1497	0.3182

M=24 V=18 P(G)=¥122455 15 MINORS
R=17 N=7 H(G)=4433333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17717	346104	0.0511	**	**
8	117877	735471	0.1602	6.6533	3.1309
9	390731	1307504	0.2988	3.3147	1.8645
10	841792	1961256	0.4292	2.1544	1.4362
11	1298528	2496144	0.5202	1.5425	1.212
>12	1494486	2704156	0.5526	1.1509	1.0623
13	1298528	2496144	0.5202	0.8688	0.9412
14	841792	1961256	0.4292	0.6482	0.825
15	390731	1307504	0.2988	0.4641	0.6962
16	117877	735471	0.1602	0.3016	0.5363
17	17717	346104	0.0511	0.1503	0.3193

M=24 V=18 P(G)=¥122455 15 MINORS
R=17 N=7 H(G)=4343333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17174	346104	0.0496	**	**
8	114644	735471	0.1558	6.6754	3.1413
9	381480	1307504	0.2917	3.3275	1.8717
10	824460	1961256	0.4203	2.1612	1.4408
11	1274306	2496144	0.5105	1.5456	1.2144
>12	1467584	2704156	0.5427	1.1516	1.063
13	1274306	2496144	0.5105	0.8683	0.9406
14	824460	1961256	0.4203	0.6469	0.8234
15	381480	1307504	0.2917	0.4627	0.694
16	114644	735471	0.1558	0.3005	0.5342
17	17174	346104	0.0496	0.1498	0.3183

M=24 V=18 P(G)=*122455 15 MINORS
R=17 N=7 H(G)=4334333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17497	346104	0.0505	**	**
8	116931	735471	0.1589	6.6829	3.1449
9	388913	1307504	0.2974	3.326	1.8708
10	839820	1961256	0.4282	2.1594	1.4396
11	1297270	2496144	0.5197	1.5447	1.2136
>12	1493730	2704156	0.5523	1.1514	1.0628
13	1297270	2496144	0.5197	0.8684	0.9408
14	839820	1961256	0.4282	0.6473	0.8239
15	388913	1307504	0.2974	0.463	0.6946
16	116931	735471	0.1589	0.3006	0.5345
17	17497	346104	0.0505	0.1496	0.3179

M=24 V=18 P(G)=*122455 15 MINORS
R=17 N=7 H(G)=4333433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17800	346104	0.0514	**	**
8	118502	735471	0.1611	6.6574	3.1329
9	393336	1307504	0.3008	3.3192	1.867
10	848528	1961256	0.4326	2.1572	1.4381
11	1310176	2496144	0.5248	1.544	1.2131
>12	1508436	2704156	0.5578	1.1513	1.0627
13	1310176	2496144	0.5248	0.8685	0.9409
14	848528	1961256	0.4326	0.6476	0.8242
15	393336	1307504	0.3008	0.4635	0.6953
16	118502	735471	0.1611	0.3012	0.5355
17	17800	346104	0.0514	0.1502	0.3191

M=24 V=18 P(G)=*122455 15 MINORS
R=17 N=7 H(G)=4333343

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17182	346104	0.0496	**	**
8	114708	735471	0.1559	6.676	3.1416
9	381784	1307504	0.2919	3.3283	1.8721
10	825340	1961256	0.4208	2.1617	1.4411
11	1276042	2496144	0.5112	1.546	1.2147
>12	1469792	2704156	0.5435	1.1518	1.0632
13	1276042	2496144	0.5112	0.8681	0.9405
14	825340	1961256	0.4208	0.6467	0.8231
15	381784	1307504	0.2919	0.4625	0.6938
16	114708	735471	0.1559	0.3004	0.5341
17	17182	346104	0.0496	0.1497	0.3183

M=24 V=18 P(G)=*122455 15 MINORS
R=17 N=7 H(G)=3434333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18040	346104	0.0521	**	**
8	120045	735471	0.1632	6.6543	3.1314
9	397682	1307504	0.3041	3.3127	1.8634
10	856128	1961256	0.4365	2.1527	1.4351
11	1319926	2496144	0.5287	1.5417	1.2113
>12	1518822	2704156	0.5616	1.1506	1.0621
13	1319926	2496144	0.5287	0.869	0.9414
14	856128	1961256	0.4365	0.6486	0.8255
15	397682	1307504	0.3041	0.4645	0.6967
16	120045	735471	0.1632	0.3018	0.5366
17	18040	346104	0.0521	0.1502	0.3193

M=24 V=18 P(G)=*122455 15 MINORS
R=17 N=7 H(G)=3433433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18432	346104	0.0532	**	**
8	122320	735471	0.1663	6.6362	3.1229
9	404766	1307504	0.3095	3.309	1.8613
10	871132	1961256	0.4441	2.1521	1.4347
11	1343106	2496144	0.538	1.5417	1.2114
>12	1545576	2704156	0.5715	1.1507	1.0622
13	1343106	2496144	0.538	0.869	0.9414
14	871132	1961256	0.4441	0.6485	0.8254
15	404766	1307504	0.3095	0.4646	0.6969
16	122320	735471	0.1663	0.3021	0.5372
17	18432	346104	0.0532	0.1506	0.3202

M=24 V=18 P(G)=*122444 16 MINORS
R=17 N=7 H(G)=3333533

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16062	346104	0.0464	**	**
8	108261	735471	0.1471	6.7401	3.1718
9	361896	1307504	0.2767	3.3428	1.8803
10	784224	1961256	0.3998	2.1669	1.4446
11	1213818	2496144	0.4862	1.5477	1.2161
>12	1398582	2704156	0.5171	1.1522	1.0635
13	1213818	2496144	0.4862	0.8678	0.9402
14	784224	1961256	0.3998	0.646	0.8222
15	361896	1307504	0.2767	0.4614	0.6922
16	108261	735471	0.1471	0.2991	0.5318
17	16062	346104	0.0464	0.1483	0.3152

M=24 V=18 P(G)=*122444 16 MINORS
R=17 N=7 H(G)=4433333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17397	346104	0.0502	**	**
8	116757	735471	0.1587	6.7113	3.1582
9	388395	1307504	0.297	3.3265	1.8711
10	838224	1961256	0.4273	2.1581	1.4387
11	1293984	2496144	0.5183	1.5437	1.2129
>12	1489590	2704156	0.5508	1.1511	1.0626
13	1293984	2496144	0.5183	0.8686	0.941
14	838224	1961256	0.4273	0.6477	0.8244
15	388395	1307504	0.297	0.4633	0.695
16	116757	735471	0.1587	0.3006	0.5344
17	17397	346104	0.0502	0.149	0.3166

M=24 V=18 P(G)=*122444 16 MINORS
R=17 N=7 H(G)=4343333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16758	346104	0.0484	**	**
8	112932	735471	0.1535	6.7389	3.1712
9	377352	1307504	0.2886	3.3414	1.8795
10	817308	1961256	0.4167	2.1659	1.4439
11	1264482	2496144	0.5065	1.5471	1.2156
>12	1456704	2704156	0.5386	1.152	1.0633
13	1264482	2496144	0.5065	0.868	0.9403
14	817308	1961256	0.4167	0.6463	0.8226
15	377352	1307504	0.2886	0.4617	0.6925
16	112932	735471	0.1535	0.2992	0.532
17	16758	346104	0.0484	0.1483	0.3153

M=24 V=18 P(G)=*122444 16 MINORS
R=17 N=7 H(G)=5333333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16002	346104	0.0462	**	**
8	107919	735471	0.1467	6.744	3.1736
9	360792	1307504	0.2759	3.3431	1.8805
10	781920	1961256	0.3986	2.1672	1.4448
11	1210374	2496144	0.4848	1.5479	1.2162
>12	1394658	2704156	0.5157	1.1522	1.0636
13	1210374	2496144	0.4848	0.8678	0.9401
14	781920	1961256	0.3986	0.646	0.8222
15	360792	1307504	0.2759	0.4614	0.6921
16	107919	735471	0.1467	0.2991	0.5317
17	16002	346104	0.0462	0.1482	0.315

M=24 V=18 P(G)=*122444 16 MINORS
R=17 N=7 H(G)=3533333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17064	346104	0.0493	**	**
8	114237	735471	0.1553	6.6946	3.1504
9	379674	1307504	0.2903	3.3235	1.8695
10	819504	1961256	0.4178	2.1584	1.4389
11	1265598	2496144	0.507	1.5443	1.2134
>12	1457190	2704156	0.5388	1.1513	1.0628
13	1265598	2496144	0.507	0.8685	0.9408
14	819504	1961256	0.4178	0.6475	0.8241
15	379674	1307504	0.2903	0.4632	0.6949
16	114237	735471	0.1553	0.3008	0.5349
17	17064	346104	0.0493	0.1493	0.3174

M=24 V=18 P(G)=*122444 17 MINORS
R=17 N=7 H(G)=3335333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17728	346104	0.0512	**	**
8	117663	735471	0.1599	6.6371	3.1233
9	389366	1307504	0.2977	3.3091	1.8614
10	838176	1961256	0.4273	2.1526	1.4351
11	1292650	2496144	0.5178	1.5422	1.2117
>12	1487682	2704156	0.5501	1.1508	1.0623
13	1292650	2496144	0.5178	0.8689	0.9413
14	838176	1961256	0.4273	0.6484	0.8252
15	389366	1307504	0.2977	0.4645	0.6968
16	117663	735471	0.1599	0.3021	0.5372
17	17728	346104	0.0512	0.1506	0.3201

M=24 V=18 P(G)=*122444 17 MINORS
R=17 N=7 H(G)=4334333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17947	346104	0.0518	**	**
8	119727	735471	0.1627	6.6711	3.1393
9	397547	1307504	0.304	3.3204	1.8677
10	857340	1961256	0.4371	2.1565	1.4377
11	1323226	2496144	0.5301	1.5434	1.2126
>12	1523178	2704156	0.5632	1.1511	1.0625
13	1323226	2496144	0.5301	0.8687	0.9411
14	857340	1961256	0.4371	0.6479	0.8246
15	397547	1307504	0.304	0.4636	0.6955
16	119727	735471	0.1627	0.3011	0.5354
17	17947	346104	0.0518	0.1498	0.3185

M=24 V=18 P(G)=*122444 16 MINORS
R=17 N=7 H(G)=4333433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16804	346104	0.0485	**	**
8	113208	735471	0.1539	6.7369	3.1703
9	378416	1307504	0.2894	3.3426	1.8802
10	819744	1961256	0.4179	2.1662	1.4441
11	1268524	2496144	0.5081	1.5474	1.2158
>12	1461456	2704156	0.5404	1.152	1.0634
13	1268524	2496144	0.5081	0.8679	0.9403
14	819744	1961256	0.4179	0.6462	0.8224
15	378416	1307504	0.2894	0.4616	0.6924
16	113208	735471	0.1539	0.2991	0.5318
17	16804	346104	0.0485	0.1484	0.3154

M=24 V=18 P(G)=*122444 17 MINORS
R=17 N=7 H(G)=3434333

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	18544	346104	0.0535	**	**
8	123183	735471	0.1674	6.6427	3.1259
9	407378	1307504	0.3115	3.307	1.8602
10	875808	1961256	0.4465	2.1498	1.4332
11	1349086	2496144	0.5404	1.5403	1.2103
>12	1551906	2704156	0.5738	1.1503	1.0618
13	1349086	2496144	0.5404	0.8693	0.9417
14	875808	1961256	0.4465	0.6491	0.8262
15	407378	1307504	0.3115	0.4651	0.6977
16	123183	735471	0.1674	0.3023	0.5375
17	18544	346104	0.0535	0.1505	0.3198

M=24 V=18 P(G)=*122444 16 MINORS
R=17 N=7 H(G)=3433433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17512	346104	0.0505	**	**
8	117540	735471	0.1598	6.7119	3.1585
9	391514	1307504	0.2994	3.3309	1.8736
10	845820	1961256	0.4312	2.1603	1.4402
11	1306654	2496144	0.5234	1.5448	1.2138
>12	1504512	2704156	0.5563	1.1514	1.0628
13	1306654	2496144	0.5234	0.8684	0.9408
14	845820	1961256	0.4312	0.6473	0.8238
15	391514	1307504	0.2994	0.4628	0.6943
16	117540	735471	0.1598	0.3002	0.5337
17	17512	346104	0.0505	0.1489	0.3165

M=24 V=18 P(G)=*122444 17 MINORS
R=17 N=7 H(G)=3334433

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	17887	346104	0.0516	**	**
8	119265	735471	0.1621	6.6676	3.1377
9	395609	1307504	0.3025	3.317	1.8658
10	852288	1961256	0.4345	2.1543	1.4362
11	1314472	2496144	0.5266	1.5422	1.2117
>12	1512702	2704156	0.5593	1.1508	1.0622
13	1314472	2496144	0.5266	0.8689	0.9413
14	852288	1961256	0.4345	0.6483	0.8252
15	395609	1307504	0.3025	0.4641	0.6962
16	119265	735471	0.1621	0.3014	0.5359
17	17887	346104	0.0516	0.1499	0.3187

M=24 V=18 P(G)=*122444 16 MINORS
R=17 N=7 H(G)=3333443

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
7	16834	346104	0.0486	**	**
8	113364	735471	0.1541	6.7342	3.169
9	378920	1307504	0.2898	3.3425	1.8801
10	820620	1961256	0.4184	2.1656	1.4437
11	1269526	2496144	0.5085	1.547	1.2155
>12	1462464	2704156	0.5408	1.1519	1.0633
13	1269526	2496144	0.5085	0.868	0.9404
14	820620	1961256	0.4184	0.6463	0.8226
15	378920	1307504	0.2898	0.4617	0.6926
16	113364	735471	0.1541	0.2991	0.5318
17	16834	346104	0.0486	0.1484	0.3155

M=127 V=114 P(G)=*12345678910111213

R=113 N=14 H(G)=99999999999999

I	PI(I)	BC(I)	PI/BC	PR(I)	PR/BCR
14	8.76437087E+13	1.54883332E+18	00	**	**
15	4.48664988E+15	1.16678777E+19	03E-04	51.1919	6.7953
16	1.17280452E+17	8.16751436E+19	01.4E-03	26.1398	3.7342
17	2.08193988E+18	5.33290643E+20	03.9E-03	17.7518	2.7187
18	2.81752045E+19	3.25899838E+21	08.6E-03	13.5331	2.2145
19	3.09487714E+20	1.86963591E+22	0.0165	10.9843	1.9147
20	2.86958266E+21	1.00960339E+23	0.0284	9.272	1.717
21	2.30675171E+22	5.14416966E+23	0.0448	8.0386	1.5776
22	1.63901769E+23	2.47855447E+24	0.0661	7.1053	1.4746
23	1.04447954E+24	1.131514E+25	0.0923	6.3725	1.3959
24	6.03786894E+24	4.90322733E+25	0.1231	5.7807	1.334
25	3.19504567E+25	2.02012966E+26	0.1581	5.2916	1.2843
26	1.55917352E+26	7.92512405E+26	0.1967	4.8799	1.2439
27	7.05993565E+26	2.96458344E+27	0.2381	4.5279	1.2104
28	2.98151005E+27	1.0587798E+28	0.2815	4.2231	1.1824
29	1.17952386E+28	3.61445518E+28	0.3263	3.9561	1.1588
30	4.38782658E+28	1.18072203E+29	0.3716	3.7199	1.1387
31	1.53986807E+29	3.69451731E+29	0.4167	3.5094	1.1215
32	5.11267321E+29	1.10835519E+30	0.4612	3.3202	1.1067
33	1.61002318E+30	3.19071949E+30	0.5045	3.149	1.0938
34	4.81946195E+30	8.82140095E+30	0.5463	2.9934	1.0827
35	1.37405042E+31	2.34397225E+31	0.5862	2.851	1.0729
36	3.73772848E+31	5.99015131E+31	0.6239	2.7202	1.0644
37	9.71622473E+31	1.47325343E+32	0.6595	2.5994	1.0569
38	2.41705569E+32	3.48928444E+32	0.6927	2.4876	1.0503
39	5.76141422E+32	7.96272604E+32	0.7235	2.3836	1.0445
40	1.31742403E+33	1.75179973E+33	0.752	2.2866	1.0393
41	2.89287434E+33	3.71723357E+33	0.7782	2.1958	1.0348
42	6.10593941E+33	7.61147826E+33	0.8022	2.1106	1.0307
43	1.23984713E+34	1.50459454E+34	0.824	2.0305	1.0272
44	2.42390999E+34	2.87240775E+34	0.8438	1.955	1.024
45	4.5656956E+34	5.29799653E+34	0.8617	1.8836	1.0212
46	8.2912734E+34	9.44425468E+34	0.8779	1.8159	1.0187
47	1.45249882E+35	1.62762687E+35	0.8924	1.7518	1.0165
48	2.45597511E+35	2.71271145E+35	0.9053	1.6908	1.0145
49	4.01012464E+35	4.3735552E+35	0.9169	1.6328	1.0127
50	6.32572135E+35	6.82274611E+35	0.9271	1.5774	1.0111
51	9.64394918E+35	1.03010088E+36	0.9362	1.5245	1.0097
52	1.42151011E+36	1.50553206E+36	0.9441	1.4739	1.0085
53	2.02645477E+36	2.1304699E+36	0.9511	1.4255	1.0073
54	2.79474853E+36	2.91953282E+36	0.9572	1.3791	1.0063
55	3.72975584E+36	3.87501629E+36	0.9625	1.3345	1.0054
56	4.8178101E+36	4.9821638E+36	0.967	1.2917	1.0046
57	6.02473228E+36	6.20585316E+36	0.9708	1.2505	1.0039
58	7.29490749E+36	7.48982278E+36	0.9739	1.2108	1.0032
59	8.55380575E+36	8.75928426E+36	0.9765	1.1725	1.0026
60	9.71424298E+36	9.92718882E+36	0.9785	1.1356	1.002
61	1.06858751E+37	1.09036336E+37	0.98	1.1	1.0015
62	1.13865915E+37	1.16070939E+37	0.981	1.0655	1.0009
>63	1.17538582E+37	1.1975573E+37	0.9814	1.0322	1.0004
>64	1.17538582E+37	1.1975573E+37	0.9814	1	1

BIBLIOGRAPHY

[75] Andrews, G., A theorem on reciprocal polynomials with applications to permutations and compositions. American Mathematical Monthly 82 (1975), 830-833.

[67] Birkhoff, G., Lattice Theory, A.M.S. Colloquium Publication (3rd ed.), Providence (1967).

[12] Birkhoff, G.D., A determinantal formula for the number of ways of colouring a map. Annals of Math. (2) 14 (1912), 42-46.

[71] Brylawski, T.H., A combinatorial model for series-parallel networks. Transaction of the American Mathematical Society 154 (1971), 1-22.

[74] Comtet, L., Advanced Combinatorics. D. Reidel Publishing Company (1974) 268-271.

[70] Crapo, H.H. and Rota, G.C., On the Foundations of Combinatorial Theory: Combinatorial Geometries. M.I.T. Press Cambridge, Mass. (1970).

[52] Dirac, G.A., A property of 4-chromatic graphs and some remarks on critical graphs. Journal of the London Mathematical Society 27 (1952), 85-92.

[80] Dowling, T.A., On the independent set numbers of a finite matroid. Annals of Discrete Mathematics 8 (1980), 21-28.

[74] Dowling, T.A. and Wilson, R.M., The slimmest geometric lattices. Transactions of the American Mathematical Society 196 (1974).

[75] ____________ Whitney number inequalities for geometric lattices. Proceedings of the American Mathematical Society. 47 (1975), 504-512.

[65] Duffin, R.J., Topology of series-parallel networks. J. Math Anal. Appl. 10 (1965), 303-310.

[65] Edmonds, J. and Fulkerson, D.R., Transversals and matroid partition. J. Res. Nat. Bur. Stand. 69B (1965), 147-153.

[53] Erdos, P., On a conjecture of Hammersley. J. London M.S., 28 (1953) 232-6.

[48] Hall, M., (Jr.), Distinct representatives of subsets. Bull. Amer. Math. Soc. 51 (1948), 922-926.

[69] Harary, F., Graph Theory. Addison Wesley (1969).

[52] Hardy, G.H., Littlewood, J., Polya, G., Inequalities. Cambridge University Press (1952).

[67] Harper, L.H., Stirling Behavior is asymptotically normal. Ann. Math. Statist. 38 (1967), 410-414.

[72] Heron, A.P., Matroid polynomials. Combinatorics (Institute of Math. and Appl.) (D.J.A. Welsh and D.R. Woodall, eds) (1972), 164-203.

[68] Lieb, E.H., Concavity properties and a generating function for Stirling numbers. Journal of Combinatorial Theory 5 (1968), 203-206.

[72] Mason, J.H., Matroids: unimodal conjectures and Motzkin's theorem. Combinatorics (Institute of Math. and Appl.) (D.J.A. Welsh and D.R. Woodall, eds) (1972), 164-203.

[66] Minty, G.J., On the axiomatic foundations of the theories of directed linear graphs, electrical networks and network programming. Journ. Math. Mech. 15 (1966), 485-520.

[68] Read, R.C., An introduction to chromatic polynomials. J. Combinatorial Theory 4 (1968), 52-71.

[64] Rota, G.C., On the foundations of combinatorial theory I. Z. Wahrsch. 2 (1964), 340-368.

[75] Seymour, P.D., A unimodality problem in matroids. (D. Phil. thesis) Oxford (1975), 66-86.

[82] ______________ On the points-lines-planes conjecture, J. Combinatorial Theory, Series B 33 (1982), 17-26.

[81] Stanley, R.P., Two combinatorial applications of the Aleksandrov-Fenchel inequalities. J. Combinatorial Theory, Series A 31 (1981), 56-65.

[75] Stonesifer, J.R., Logarithmic concavity for edge lattices of graphs. J. Comb. Th. Series A, 18 (1975), 36-46.

[66] Tutte, W.T., Connectivity in matroids. Canad. J. Math. 18 (1966), 1301-24.

[71] Welsh, D.J.A., Combinatorial problems in matroid theory. Combinatorial Mathematics and its Applications. Academic Press (1971), 291-307.

[76] ____________ Matroid Theory. Academic Press (1976).

[32a] Whitney, H., A logical expansion in mathematics. Bull. Amer. Math. Soc. 38 (1932), 572-579.

[32b] ____________ The coloring of graphs. Annals of Math. 33 (1932), 688-718.