AN APPLICATION OF SUM COMPOSITION:
A SELF ORTHOGONAL LATIN SQUARE OF ORDER TEN

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BU-393-M November, 1971

ABSTRACT

A latin square is said to be self orthogonal if it is orthogonal to its own transpose. Self orthogonal latin squares form a very interesting and useful family. (a) Many different experimental designs can be constructed via these squares which cannot be constructed from arbitrary pairs of orthogonal latin squares. (b) They are also useful for efficient cataloguing pairs of orthogonal latin squares in the sense that one square is sufficient for each order. Trivially there are no self orthogonal latin squares of order 2 or 6. It is also easy to establish that there is no self orthogonal latin square of order 3. The following is a self orthogonal latin square of order 4

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{array}.$$

Mendelsohn, Horton, and Mullin and Nemeth have discovered many results in this area. However, order 10 is the smallest order that one cannot produce a self orthogonal latin square using the results of these authors. In this note we utilize the sum composition technique developed by Hedayat and Seiden, to produce a self orthogonal latin square of order 10.

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1. Introduction and Background

Let Σ be an n-set. A transversal (directrix) of a latin square L on Σ is a collection of n cells such that the entries of these cells exhaust the set Σ, and every row and column of L is represented in this collection. Two transversals in L are said to be parallel if they have no common cell. A collection of n cells is said to form a common transversal for a set of t latin squares on Σ if the collection is a transversal for each of these t latin squares. A collection of r transversals is said to be a set of r common parallel transversals for a set of t latin squares on Σ if each transversal is a common transversal and these r transversals have no cell in common.

Let L_1 and L_2 be two latin squares of order n on Σ. We use the notation $L_1 \perp L_2$ if L_1 and L_2 are orthogonal and $L_1 \not\perp L_2$ if they are not orthogonal. A latin square is said to be self orthogonal if it is orthogonal to its own transpose. Self orthogonal latin squares form a very interesting and useful family. (a) Many different experimental designs can be constructed via these squares which cannot be constructed from arbitrary pairs of orthogonal latin squares. (b) They are also useful for efficient cataloguing pairs of orthogonal latin squares in the sense that one square is sufficient for each order. Therefore it is desirable to study these squares. Trivially there are no self orthogonal latin squares of order 2 or 6. It is also easy to establish that there is no self orthogonal latin square of order 3. The following is a self orthogonal latin square of order 4

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{array}$$

Now a natural question is: For what orders do self orthogonal latin squares exist? This question has been partially answered by

(1) Mendelsohn [6] who proved that if $n \not\equiv 2 \bmod 4$, or $n \not\equiv 3, 6 \bmod 9$ then a self orthogonal latin square of order n exists.

(2) Horton [5] who proved that if there exists a self orthogonal latin square of order n_1 and a self orthogonal latin square of order n_2 with a subsquare of order n_3, then there is a self orthogonal latin square of order $n = n_1e + n_3$ where $e = n_2 - n_3$ unless $e = 2$ or 6. Sixteen is the smallest order that one can produce a self orthogonal latin square by Horton's result. Horton's result also produces some orders which cannot be generated by Mendelsohn's result. In the family of $n \equiv 2 \bmod 4$ the order 22 is the smallest one. In the family of $n \equiv 3, 6 \bmod 9$ the orders 21 and 33 are the smallest orders that can be generated.

(3) Mullin and Nemeth [7] also gives a construction method for a family of self orthogonal latin squares of order $n = 2m + 1$ provided that an Abelian group of order n having certain properties exists. For instance, whenever n is a prime power not of the form $2^k + 1$ then such a group exists.

Order 10 is the first smallest case which one cannot construct a self orthogonal latin square using the preceding results. The purpose of this note is to utilize the sum composition technique of Hedayat and Seiden [2,3] and produce a self orthogonal latin square of order 10. It should be mentioned that pairs of orthogonal latin squares of order 10 constructed by Bose, Shrikhande, and Parker [1], Hedayat and Seiden [2,3], and Hedayat, Parker, and Federer [4] do not have this property.

2. Construction of a Self Orthogonal Latin Square of Order Ten via Sum Composition Technique

We shall compose a special pair of latin squares of order 9 and a trivial pair of latin squares of order unity to obtain a pair of orthogonal latin squares L_1 and L_2 of order 10 such that L_2 is the transpose of L_1. In order to utilize the sum composition idea of Hedayat and Seiden [2,3] we need three latin squares A, B and C of orders 9 having at least the following properties:

(a) $A \perp B$, $B \perp C$, i.e. A and C should have 9 common parallel transversals.

(b) $A \not\perp C$.

(c) The entries of the first row and column of A should satisfy $a_{1t} = a_{t1}$ if and only if $t = 1$.

(d) The main diagonal of A should form a transversal for A.

(e) $C = A^t$, where A^t denotes the transpose of A.

The following A, B and C not only have the above properties but in addition have sufficient combinatorial structures to be utilized for the sum composition technique.

$$A = \begin{array}{ccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & \underline{7} & 8 \\
3 & 6 & 5 & 4 & 0 & 8 & 7 & 1 & \underline{2} \\
\underline{4} & 0 & 1 & 7 & 8 & 6 & 3 & 2 & 5 \\
7 & \underline{8} & 6 & 5 & 2 & 3 & 1 & 0 & 4 \\
8 & 2 & \underline{3} & 1 & 7 & 4 & 0 & 5 & 6 \\
1 & 3 & 4 & \underline{0} & 5 & 2 & 8 & 6 & 7 \\
2 & 5 & 0 & 8 & \underline{6} & 7 & 4 & 3 & 1 \\
5 & 4 & 7 & 6 & 3 & \underline{1} & 2 & 8 & 0 \\
6 & 7 & 8 & 2 & 1 & 0 & \underline{5} & 4 & 3
\end{array}
\qquad
B = \begin{array}{ccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 & 5 \\
5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 & 4 \\
4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 & 3 \\
3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 & 2 \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 0 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 0
\end{array}$$

$$C = \begin{array}{ccccccccc}
0 & 3 & \underline{4} & 7 & 8 & 1 & 2 & 5 & 6 \\
1 & 6 & 0 & \underline{8} & 2 & 3 & 5 & 4 & 7 \\
2 & 5 & 1 & 6 & \underline{3} & 4 & 0 & 7 & 8 \\
3 & 4 & 7 & 5 & 1 & \underline{0} & 8 & 6 & 2 \\
4 & 0 & 8 & 2 & 7 & 5 & \underline{6} & 3 & 1 \\
5 & 8 & 6 & 3 & 4 & 2 & 7 & \underline{1} & 0 \\
6 & 7 & 3 & 1 & 0 & 8 & 4 & 2 & \underline{5} \\
\underline{7} & 1 & 2 & 0 & 5 & 6 & 3 & 8 & 4 \\
8 & \underline{2} & 5 & 4 & 6 & 7 & 1 & 0 & 3
\end{array}$$

Note that the underlined cells in A and C form two common parallel transversals for A and C. Denote the entries of these underlined cells in A and C by a^*_{ij} and c^*_{ij} respectively. Now consider two 10×10 squares say $X = (x_{ij})$ and $Y = (y_{ij})$ with A and C in their top left 9×9 subsquares respectively. Project the underlined transversal in A on the 10^{th} row and column of X, i.e.

$$x_{10,j} = a^*_{i,j} \qquad j = 1,2,\cdots,9$$

$$x_{i,10} = a^*_{i,j} \qquad i = 1,2,\cdots,9 \; .$$

Replace all the entries of the underlined transversal in A embedded in X by integer 9. Also put 9 in the cell $x_{10,10}$. Denote the resulting square by L_1. Carry similar operations on the entries of Y and denote the resulting square by L_2. These squares are exhibited below:

$$L_1 = \begin{array}{ccccccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 9 & 8 & 7 \\
3 & 6 & 5 & 4 & 0 & 8 & 7 & 1 & 9 & 2 \\
9 & 0 & 1 & 7 & 8 & 6 & 3 & 2 & 5 & 4 \\
7 & 9 & 6 & 5 & 2 & 3 & 1 & 0 & 4 & 8 \\
8 & 2 & 9 & 1 & 7 & 4 & 0 & 5 & 6 & 3 \\
1 & 3 & 4 & 9 & 5 & 2 & 8 & 6 & 7 & 0 \\
2 & 5 & 0 & 8 & 9 & 7 & 4 & 3 & 1 & 6 \\
5 & 4 & 7 & 6 & 3 & 9 & 2 & 8 & 0 & 1 \\
6 & 7 & 8 & 2 & 1 & 0 & 9 & 4 & 3 & 5 \\
\hline
4 & 8 & 3 & 0 & 6 & 1 & 5 & 7 & 2 & 9
\end{array}
\qquad
L_2 = \begin{array}{ccccccccc|c}
0 & 3 & 9 & 7 & 8 & 1 & 2 & 5 & 6 & 4 \\
1 & 6 & 0 & 9 & 2 & 3 & 5 & 4 & 7 & 8 \\
2 & 5 & 1 & 6 & 9 & 4 & 0 & 7 & 8 & 3 \\
3 & 4 & 7 & 5 & 1 & 9 & 8 & 6 & 2 & 0 \\
4 & 0 & 8 & 2 & 7 & 5 & 9 & 3 & 1 & 6 \\
5 & 8 & 6 & 3 & 4 & 2 & 7 & 9 & 0 & 1 \\
6 & 7 & 3 & 1 & 0 & 8 & 4 & 2 & 9 & 5 \\
9 & 1 & 2 & 0 & 5 & 6 & 3 & 8 & 4 & 7 \\
8 & 9 & 5 & 4 & 6 & 7 & 1 & 0 & 3 & 2 \\
\hline
7 & 2 & 4 & 8 & 3 & 0 & 6 & 1 & 5 & 9
\end{array}$$

The reader can verify for himself that L_1 and L_2 are both latin squares of order 10 with the desired properties viz, $L_1 \perp L_2$ and $L_2 = L_1^t$.

Conclusion: We gave a detailed method for the construction of a self orthogonal latin square of order 10 via sum composition technique with the hope that one can generalize this idea to cover the unsettled cases. The next three interesting cases are 12, 14 and 15.

References

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