ELSEVIER

Signal Processing 52 (1996) 323–341

SIGNAL PROCESSING

Time and frequency split Zak transform for finite Gabor expansion

Soo-Chang Pei*, Min-Hung Yeh

Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan, ROC

Received 21 July 1994; revised 21 March 1996

Abstract

The relationship between finite discrete Zak transform and finite Gabor expansion are well discussed in this paper. In this paper, we present two DFT-based algorithms for computing Gabor coefficients. One is based upon the time-split Zak transform, the other is based upon the frequency-split Zak transform. These two methods are time and frequency dual pairs. With the help of Zak transform, the closed-form solutions for analysis basis can also be derived while the oversampling ratio is an integer. Moreover, we extend the relationship between finite discrete Zak transform and Gabor expansion to the 2-D case and compute 2-D Gabor expansion coefficients through 2-D discrete Zak transform and 4-D DFT. Four methods can be applied in the 2-D case. They are time–time-split, time–frequency-split, frequency–time-split and frequency–frequency-split.

Zusammenfassung

In diesem Beitrag werden die Beziehungen zwischen der endlichen diskreten Zaktransformation und der endlichen Gaborentwicklung gründlich diskutiert. Wir präsentieren zwei Algorithmen auf DFT-Basis zur Berechnung von Gaborkoeffizienten. Einer beruht auf der zeitlich, der andere auf der frequenzmäßig zerlegten Zaktransformation. Diese Methoden sind dual bezüglich Zeit- und Frequenzbereich. Mit Hilfe der Zaktransformation kann man auch eine geschlossene Lösung für die Analysebasis ableiten, wenn um einen ganzzahligen Faktor überabgetastet wird. Darüberhinaus erweitern wir die Beziehung zwischen der endlichen diskreten Zaktransformation und der Gaborentwicklung auf den 2D-Fall und berechnen 2D-Gaborentwicklungs-Koeffizienten mittels einer 2D-Zaktransformation und einer 4D-DFT. Vier Methoden sind im 2D-Fall anwendbar. Sie beruhen auf Zeit–Zeit-, Zeit–Frequenz-, Frequenz–Zeit- und Frequenz–Frequenz-Zerlegungen.

Résumé

La relation existant entre la transformation de Zak discrète finie et l'expansion de Gabor finie est discutée en profondeur dans cet article. Nous présentons deux algorithmes basés sur la DFT pour le calcul des coefficients de Gabor. L'un est basé sur la transformation de Zak par partage de temps, l'autre sur la transformation de Zak par partage de fréquence. Ces deux méthodes constituent une paire duale temps–fréquence. A l'aide de la transformation de Zak, les solutions analytiques pour la base d'analyse peuvent également être dérivées si le rapport de sur-échantillonnage est un entier. De

* Corresponding author. E-mail: pei@cc.ee.ntu.edu.tw.

PII S0165-1684(96)00068-0

plus, nous étendons la relation entre la transformation de Zak discrète finie et l'expansion de Gabor au cas bi-dimensionnel et calculons les coefficients de l'expansion de Gabor 2-D via la transformation de Zak discrète 2-D et la DFT 4-D. Quatre méthodes peuvent être appliquées dans le cas 2-D. Ce sont les méthodes partage temps–temps, partage temps–fréquence, partage fréquence–temps et partage fréquence–fréquence.

Keywords: Zak transform; Gabor expansion

1. Introduction

Nonstationary signals exhibit time-varying properties and are often encountered in various areas such as audio signals, sonar, and synthetic aperture data. Several methods have been proposed to analyze the time-varying signals. Wigner distribution [5], short time Fourier transform [6] and Gabor transform [9] are widely used in time-varying signal analysis. In 1946, Gabor proposed a method to express signals. The Gabor expansion means that a signal can be expressed as a weighted summation of a basis after shift and modulation.

$$f(t) = \sum_{m,n=-\infty}^{\infty} C_{m,n} h(t - mT) \mathrm{e}^{\mathrm{j}n\Omega t}, \tag{1}$$

where T and Ω represent time and frequency sampling interval, respectively. The condition for Gabor expansion to be existed is $T\Omega \leqslant 2\pi$. It is called critical sampling when $T\Omega = 2\pi$ or oversampling when $T\Omega < 2\pi$. In 1980, Bastiaans [4] extended the Gabor expansion from a Gaussian window function to a more general expansion of a signal into a discrete set of window functions shifted in time and in frequency domain. He presented an algorithm to compute the expansion coefficients from sampled values of the complex spectrogram. Wexler and Raz have developed the discrete Gabor expansion for finite or periodic signals [17]. They derived the theories to compute the analysis basis and Gabor coefficients for discrete signals. The Gabor expansion can exhibit the time–frequency characterization of signal, but the computation load for Gabor coefficients is very expensive. A time–frequency mapping, Zak transform, has been used to calculate Gabor coefficients efficiently in critical sampling [2, 12, 13]. Recently, Zibulski and Zeevi have proposed a method which is based upon Zak transform and frame concept [7] to calculate the Gabor coefficients in the oversampling case. The work of this paper is extending the theories proposed by Zibulski and Zeevi to discrete case and developing DFT-based algorithms for computing Gabor coefficients efficiently in the oversampling scheme. One is based upon the time-split Zak transform, the other is based upon the frequency-split Zak transform. The time-split algorithm is the same as that proposed in [21], but it is independently developed. The Gabor expansion is not only suitable to the 1-D signals but also 2-D signals. Porat and Zeevi have extended the Gabor expansion to the two-dimensional case. The two-dimensional Gabor expansion has been widely used in image analysis and compression [14, 8]. But the problem of computation burden is a more serious case. In this paper, we present four DFT-based algorithms for computing 2-D Gabor coefficients to compute Gabor coefficients in oversampling case through Zak transform. With the help of Zak transform, the closed-form solutions for analysis basis can be derived while the oversampling ratio is an integer.

The rest of this paper is organized as follows. In Section 2, we review the 1-D finite Zak transform and 1-D finite Gabor expansion. In Section 3, the relationships between finite discrete Zak transform (FZT) and finite Gabor expansion are discussed for both critical and oversampling schemes. The closed forms of analysis bases are obtained through the aids of Zak transform when the oversampling ratio is an integer. The time and frequency-split algorithm for computing Gabor coefficients are presented. In Section 4, we extended the theories of Gabor expansion and Zak transform to two-dimensional case and derive four DFT-based algorithms for computing Gabor coefficients in oversampling scheme. Finally, the conclusions are drawn in Section 5.

2. Review of Gabor expansion and finite discrete Zak transform

2.1. Finite discrete 1-D Gabor expansion

The Gabor expansion of continuous signal $f(t)$ is defined as

$$f(t) = \sum_{m,n=-\infty}^{\infty} C_{m,n} h_{m,n}(t) \tag{2}$$

$$= \sum_{m,n=-\infty}^{\infty} C_{m,n} h(t - mT)\mathrm{e}^{\mathrm{j}n\Omega t}, \tag{3}$$

where $h_{m,n}(t) = h(t - mT)\mathrm{e}^{\mathrm{j}n\Omega t}$. T and Ω represent the time and frequency sampling intervals, respectively. The grid $(t_m, f_n) = (mT, n\Omega)$ in the time–frequency plane is called Gabor lattice. The existence of Eq. (2) has been found to be possible for arbitrary $f(t)$ only for $T\Omega \leqslant 2\pi$. The case, $T\Omega = 2\pi$, is called critical sampling and the case, $T\Omega < 2\pi$, is oversampling.

Bastiaans has introduced an auxiliary function $\gamma(t)$ [4] for computing the Gabor coefficients. The coefficients in Gabor expansion can be evaluated through using the biorthogonal function $\gamma(t)$:

$$C_{m,n} = \int_{-\infty}^{\infty} f(t)\gamma^*_{m,n}(t)\,\mathrm{d}t, \tag{4}$$

where $\gamma_{m,n}(t) = \gamma(t - mT)\mathrm{e}^{\mathrm{j}n\Omega t}$. $\gamma(t)$ is the biorthogonal function of $h(t)$.

In [17], Wexler and Raz have developed the finite discrete Gabor expansion for finite or discrete signals. The periodic signal $\tilde{f}(i)$ with period L is defined as

$$\tilde{f}(i) = f(i + k \cdot L), \tag{5}$$

where k can be any integer. The discrete version for the finite or periodic sequences is defined as

$$\tilde{f}(i) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} C_{m,n}\tilde{h}_{m,n}(i), \tag{6}$$

where

$$C_{m,n} = \sum_{i=0}^{L-1} \tilde{f}(i)\tilde{\gamma}^*_{m,n}(i),$$

$$\tilde{h}_{m,n}(i) = \tilde{h}(i - m\Delta M)W_L^{n\Delta Ni},$$

$$\tilde{\gamma}_{m,n}(i) = \tilde{\gamma}(i - m\Delta M)W_L^{n\Delta Ni},$$

$$W_L^{n\Delta Ni} = \mathrm{e}^{\mathrm{j}2\pi n\Delta Ni/L}.$$

where $\tilde{f}(i)$, $\tilde{h}(i)$ and $\tilde{\gamma}(i)$ indicate the periodic extensions of $f(i)$, $h(i)$ and $\gamma(i)$, respectively. L is the signal length of the original finite signal or the period of the periodic signal. M is the number of sampling points in the time domain. ΔM is the time sampling interval. N is the number of sampling points in the frequency domain. ΔN is the frequency sampling interval. $M\Delta M = L$, $N\Delta N = L$. The condition $\Delta M \times \Delta N \leqslant L$ must be satisfied for a stable reconstruction. The critical sampling occurs when $\Delta M \cdot \Delta N = M \cdot N = L$. In critical sampling case, the number of coefficients $C_{m,n}$ is equal to the number of time samples in $\tilde{f}(i)$. $\Delta M \cdot \Delta N < L$ (or $MN > L$) is oversampling case. Define $\alpha = MN/L = N/\Delta M = M/\Delta N = L/\Delta M\Delta N = q/p$, where α is called the **oversampling ratio**. The two integers, p and q, are relatively prime numbers. The values, $\Delta M/p$

and $\Delta N/p$, are both integers. Wexler and Raz have proved that the biorthogonality between synthesis basis $\tilde{h}(i)$ and analysis basis $\tilde{\gamma}(i)$ in the finite discrete case is equivalent to

$$\sum_{i=0}^{L-1} \tilde{h}(i+mN)W_L^{-nMi}\tilde{\gamma}^*(i) = \delta(m)\delta(n), \quad 0 \leqslant m < \Delta N,\ 0 \leqslant n < \Delta M. \tag{7}$$

In [17], it have been proved that the analysis basis $\gamma(i)$ can be obtained by solving the following equation:

$$H\gamma = \mu, \tag{8}$$

where $\mu = (1,0,\ldots,0)^{\mathrm{T}}$, and the matrix H is a $(\Delta M \cdot \Delta N) \times L$ matrix. It has been stated in [17] that matrix H can be constructed as

$$H(m\Delta M + n, i) = \tilde{h}(i+mN)W_L^{-nMi}, \quad 0 \leqslant m < M,\ 0 \leqslant n < N. \tag{9}$$

The matrix H is a block Hankel matrix, and it can be solved efficiently [1]. For the critical sampling case, $\Delta M \Delta N = L$, $\tilde{\gamma}(i)$ is unique if the matrix H is nonsingular. For the oversampling scheme, $\Delta M \Delta N < L$, the linear system given by (8) is underdetermined and the solution is not unique. An optimal solution proposed in [16] is based upon the criterion.

$$\Gamma : \min_{\tilde{\gamma}: H\gamma^* = u} \left\| \frac{\tilde{\gamma}(i)}{\|\tilde{\gamma}(i)\|} - \tilde{h}(i) \right\|^2. \tag{10}$$

The goal of Eq. (10) is to try to find an analysis basis, γ, which is similar to the synthesis basis as similar as possible. In general, the error Γ decreases as the oversampling ratio, α, increases. In the critical sampling case, the error Γ is quite large.

The above-defined finite discrete Gabor expansion is called (M,N)-point Gabor expansion in our following discussion.

2.2. 1-D finite discrete Zak transform

The Zak transform of continuous signal is defined as [2]

$$\hat{f}(\tau,\Omega) = \sum_{k=-\infty}^{\infty} f[\lambda(\tau+k)]\mathrm{e}^{-\mathrm{j}2\pi k\Omega}. \tag{11}$$

The Zak transform is a signal transform that maps $\mathbf{L}^2(R)$ onto $\mathbf{L}^2([0,1))$. Generally, the variable τ is treated as the time variable and the variable Ω is regarded as the frequency variable. The Zak transform has many desirable properties and can also be treated as a time–frequency distribution. The signal can be recovered from its transform domain by using the inverse transform formula.

$$f(x) = \lambda^{-1/2}\int_0^1 \hat{f}\left(\frac{x}{\lambda},\Omega\right)\mathrm{d}\Omega. \tag{12}$$

The discrete Zak transform for discrete signal is defined as [1]

$$\hat{f}\left(\frac{a}{A},\frac{b}{B}\right) = \sum_{r=-\infty}^{\infty} f\left(\frac{a}{A}+r\right)\mathrm{e}^{-\mathrm{j}2\pi(rb/B)}, \quad 0 \leqslant b < B,\ 0 \leqslant a < A. \tag{13}$$

For convenience of our further discussion, the index in definition of discrete Zak transform has been changed into integers in this paper. Then it is defined as

$$\hat{f}(a,b) = \sum_{r=-\infty}^{\infty} f(a+Ar)\mathrm{e}^{-\mathrm{j}2\pi(rb/B)}, \quad 0 \leqslant b < B,\ 0 \leqslant a < A. \tag{14}$$

If the signal is $L = A \times B$ periodic or finite with length L, its definition becomes

$$\hat{f}_{(A,B)}(a,b) = \sum_{r=0}^{B-1} f(a + Ar)\mathrm{e}^{-\mathrm{j}2\pi(rb/B)}, \quad 0 \leqslant b < B,\ 0 \leqslant a < A \tag{15}$$

$$= \sum_{r=0}^{B-1} f(a + Ar)W_L^{-rbA}, \quad 0 \leqslant b < B,\ 0 \leqslant a < A, \tag{16}$$

where $W_L = \mathrm{e}^{\mathrm{j}2\pi/L}$. We call this transform to be (A,B)-point finite discrete Zak transform (FZT) in this paper. The discrete signal $f(i)$ performed by (A,B)-FZT is denoted by $\hat{f}_{(A,B)}$ in the following discussions. Similar to the continuous case, discrete signal $f(i)$ can be recovered from the inverse finite discrete Zak transform (IFZT).

$$f(a + Ar) = \frac{1}{B}\sum_{b=0}^{B-1} \hat{f}_{(A,B)}(a,b)\mathrm{e}^{\mathrm{j}2\pi(rb/B)}, \quad 0 \leqslant a < A,\ 0 \leqslant r < B. \tag{17}$$

Fig. 1 shows the data spread sheet for signal $f(i)$ while processing (A,B)-point FZT in the time domain. The data in each column is processed by a discrete Fourier transform to get a time slice in Zak transform domain. Fig. 2 shows the data spread sheet for signal in $f(i)$ in the Zak transform domain. The original signal can be recovered from an inverse DFT for the data in Zak transform domain using Eq. (17).

Two shift properties that are useful for our further development will be introduced as follows:

$$g(i) = f(i-1), \quad 0 \leqslant i < L,$$
$$h(i) = f(i)W_L^i, \quad 0 \leqslant i < L,$$

then

$$\hat{g}_{(A,B)}(a,b) = \hat{f}_{(A,B)}(a-1,b), \tag{18}$$

$$\hat{h}_{(A,B)}(a,b) = \hat{f}_{(A,B)}(a,b-1)W_L^a, \tag{19}$$

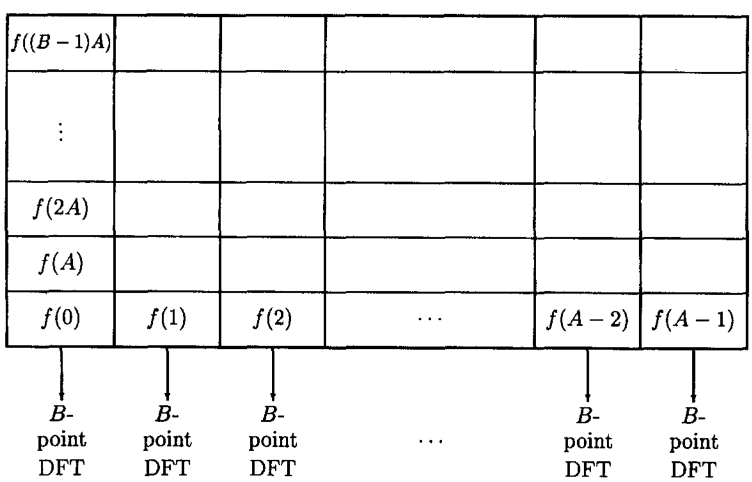

Fig. 1. The data spread sheet of (A,B)-point FZT.

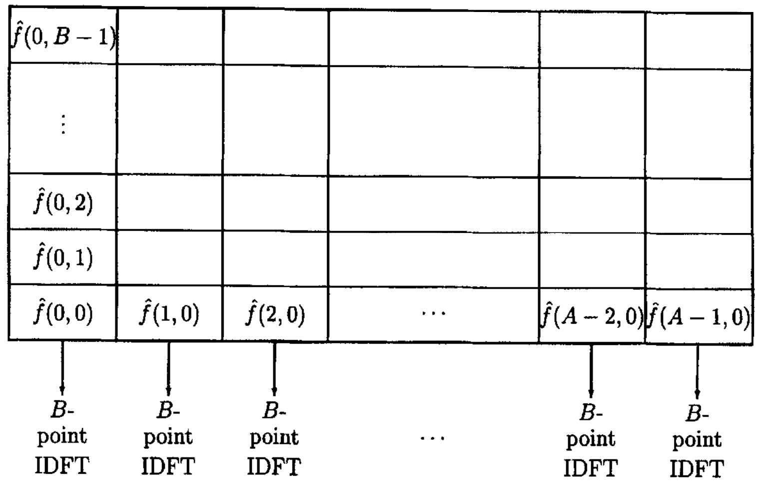

Fig. 2. The data spread sheet in Zak transform domain.

where $W_L = \mathrm{e}^{\mathrm{j}2\pi/L}$. Two periodic properties are also used for further development:

$$\hat{f}_{(A,B)}(a-A,b) = \hat{f}_{(A,B)}(a,b)W_L^{-bA}, \tag{20}$$

$$\hat{f}_{(A,B)}(a,b+B) = \hat{f}_{(A,B)}(a,b). \tag{21}$$

3. Relationship between 1-D discrete Gabor expansion and 1-D FZT

3.1. Critical sampling scheme

We will review the discrete Gabor expansion obtained through FZT [2]. In the critical sampling case, the discrete (M,N)-point Gabor expansion can be obtained through (N,M)-point 1-D FZT. In computing the finite Gabor coefficients by FZT, the numbers of time and frequency samples are interchanged for Gabor expansion and FZT. The FZT of analysis basis function is

$$\hat{\gamma}_{(N,M)}(a,b) = \frac{1}{N\hat{h}^*_{(N,M)}(a,b)}. \tag{22}$$

By taking the inverse Zak transform of Eq. (22), the closed form of analysis basis function is derived.

$$\gamma(k) = \frac{1}{L}\sum_{b=0}^{M-1} \frac{W_M^{bp}}{\sum_{a=0}^{M-1} h(aN+q)W_M^{ab}}, \tag{23}$$

where $k = pN+q$, $0 \leqslant p < M$, $0 \leqslant q < N$, $0 \leqslant k < L$, $W_M = \mathrm{e}^{\mathrm{j}2\pi/M}$. As mentioned above, the solution of analysis basis function in critical sampling is unique. This analysis basis function calculated from Eq. (23) is exactly the same as that derived from Eq. (8). It can be proved easily by replacing the solution in (23) into Eq. (8) or solve Eq. (8) directly. The Gabor coefficients can be evaluated through the following equation:

$$C_{m,n} = \frac{1}{L}\sum_{a=0}^{N-1}\sum_{b=0}^{M-1} \frac{\hat{f}_{(N,M)}(a,b)}{\hat{h}_{(N,M)}(a,b)} W_L^{-anM+bmN}, \tag{24}$$

where $0 \leqslant m < M$, $0 \leqslant n < N$. Eq. (24) indicates the Gabor coefficients can be evaluated through a 2-D discrete Fourier transform from distribution in the Zak transform domain of analysis basis and signal.

3.2. Oversampling scheme

Applying the frame operator [7, 21], Zibulski and Zeevi have developed an algorithm for computing Gabor coefficients of continuous signal in oversampling case from continuous Zak transform domain. By introducing the frame operator S. The following equation has been shown in [7, 21]:

$$\gamma = S^{-1}h. \tag{25}$$

Now two methods for computing discrete Gabor analysis basis function and coefficients will be introduced for the oversampling scheme. The discrete (M,N)-point Gabor expansion can be evaluated by two methods: $(N,\Delta N)$ and $(\Delta M, M)$-point FZT. The time-split method is to utilize $(N,\Delta N)$-point FZT which is corresponding to the continuous case $\lambda = 2\pi/\Omega$.

The time-split method has been presented in [21]. Here we only review the results based upon the mathematical notation in this paper. The closed-form solution of the analysis basis function while the oversampling ratio is an integer.

$$\gamma(k) = \frac{q}{N}\sum_{b=0}^{\Delta N-1} \frac{\sum_{a=0}^{\Delta N-1} h(aN+v)W_{\Delta N}^{ab}}{\sum_{l=0}^{q-1}\sum_{a=0}^{\Delta N-1} |h(aN+t+l\Delta M)W_{\Delta N}^{ab}|^2} W_{\Delta N}^{bu}, \tag{26}$$

where $k = uN + v$, $0 \leqslant u < \Delta N$, $0 \leqslant v < N$, t is the remainder of k divided by ΔM. The Gabor expansion coefficients in general case are

$$C_{m,n} = \frac{1}{\Delta N}\sum_{a=0}^{N-1}\sum_{b=0}^{\Delta N-1} \hat{f}_{(N,\Delta N)}(a,b)\hat{\gamma}^*_{(N,\Delta N)}(a-m\Delta M,b)W_N^{-na} \tag{27}$$

$$= \frac{1}{\Delta N}\sum_{a=0}^{N-1}\sum_{b=0}^{(\Delta N/p)-1}\left[\sum_{d=0}^{p-1} \hat{f}_{(N,\Delta N)}\left(a, b+d\frac{\Delta N}{p}\right)\hat{\gamma}^*_{(N,\Delta N)}\left(a - v\Delta M, b + d\frac{\Delta N}{p}\right)\right] W_{L/p}^{buN-na\Delta N/p}. \tag{28}$$

Analysis algorithm of the time-split method is listed as follows:

Step 1: Compute the $(N,\Delta N)$-point FZT of analysis basis $\gamma(i)$.
Step 2: Compute the $(N,\Delta N)$-point FZT of signal $f(i)$.
Step 3: for $s = 0$ to $q-1$
Step 4: $\quad \Gamma(a,b) = 0$, $0 \leqslant a < N$, $0 \leqslant b < \frac{\Delta N}{p}$
Step 5: $\quad$ for $t = 0$ to $p-1$
Step 6: $\quad\quad \Gamma(a,b) = \Gamma(a,b) + \hat{f}_{(N,\Delta N)}(a, b + t\frac{\Delta N}{p})\hat{\gamma}^*_{(N,\Delta N)}(a - s\Delta M, b + t\frac{\Delta N}{p})$
$\quad\quad\quad 0 \leqslant a < N$, $0 \leqslant b < \frac{\Delta N}{p}$
Step 7: $\quad$ end
Step 8: $\quad$ Compute an $(N,\Delta N/p)$-point 2-D DFT of the results $\Gamma(a,b)$
Step 9: end

The reconstruction of function from its Gabor coefficients is given by

$$\hat{f}_{(N,\Delta N)}\left(a,b+t\frac{\Delta N}{p}\right)=\sum_{s=0}^{q-1}\hat{h}_{(N,\Delta N)}\left(a-s\Delta M,b+t\frac{\Delta N}{p}\right)\sum_{n=0}^{N-1}\sum_{m=0}^{(\Delta N/p)-1}C_{mq+s,n}W_{L/p}^{an(\Delta N/p)-bm\,\Delta N}$$

$$0\leqslant a<N,\quad 0\leqslant b<\frac{\Delta N}{p},\quad 0\leqslant t<p. \tag{29}$$

Synthesis algorithm of the time-split method is listed as follows:

Step 1: Compute the $(N,\Delta N)$-point FZT of synthesis basis $h(i)$.
Step 2: for $s=0$ to $q-1$
Step 3: Compute the 2-D DFT of $C_{mq+s,n}$, $F(a,b)=fft2(C_{mq+s,n})_{(m,n)->(a,b)}$
$0\leqslant m<\frac{\Delta N}{p}$, $0\leqslant n<N$
Step 4: $\Gamma(a,b)=0$, $0\leqslant a<N$, $0\leqslant b<\frac{\Delta N}{p}$
Step 5: for $t=0$ to $p-1$
Step 6: $\Gamma(a,b+t\frac{\Delta N}{p})=\Gamma(a,b+\frac{\Delta N}{p})+\hat{h}_{(N,\Delta N)}(a-s\Delta M,b+t\frac{\Delta N}{p})F(a,b)$
$0\leqslant a<N$, $0\leqslant b<\frac{\Delta N}{p}$
Step 7: end
Step 8: end
Step 9: Compute $(N,\Delta N)$-point IFZT of $\Gamma(a,b)$ to obtain the reconstructed signal $f(i)$.

The frequency-split method is to utilize $(\Delta M,M)$-point FZT. It is corresponding to the continuous case $\lambda=T$. The Zak transform of analysis basis function for $p=1$ in frequency-split method is

$$\hat{\gamma}_{(\Delta M,M)}(a,b)=\frac{q}{\Delta M}\frac{\hat{h}_{(\Delta M,M)}(a,b)}{\sum_{l=0}^{q-1}|\hat{h}_{(\Delta M,M)}(a,b+l\,\Delta N)|^2},\quad 0\leqslant a<\Delta M,\ 0\leqslant b<M. \tag{30}$$

Fig. 3 illustrates an example for the summation terms in denominator of Eq. (30). This example is the same as that of Fig. 3, but (2,8)-point FZT and the algorithm of frequency-split method are applied. The terms with the same patterns will be added together for computing the denominator of the Zak transform of analysis basis function. The analysis basis function, $\gamma(k)$, can be obtained by IFZT of Eq. (30). Then its closed form is

$$\gamma(k)=\frac{q}{\Delta M}\sum_{b=0}^{M-1}\frac{\sum_{a=0}^{M-1}h(a\,\Delta M+v)W_M^{ab}}{\sum_{l=0}^{q-1}\sum_{a=0}^{M-1}|h(a\,\Delta M+t+lN)W_M^{ab}|^2}W_M^{bu}, \tag{31}$$

where $k=u\Delta M+v$, $0\leqslant u<M$, $0\leqslant v<\Delta M$, $0\leqslant k<L$, t is the remainder of k divided by N. The Gabor expansion coefficients in general case are

$$C_{m,n}=\frac{1}{M}\sum_{a=0}^{\Delta M-1}\sum_{b=0}^{M-1}\hat{f}_{(\Delta M,M)}(a,b)\hat{\gamma}^*_{(\Delta M,M)}(a-m\Delta M,b-n\Delta N)W_N^{-na} \tag{32}$$

$$=\frac{1}{M}\sum_{a=0}^{(\Delta M/p)-1}\sum_{b=0}^{M-1}\left[\sum_{d=0}^{p-1}\hat{f}_{(\Delta M,M)}\left(a+d\frac{\Delta M}{p},b+v\,\Delta N\right)\hat{\gamma}^*_{(\Delta M,M)}(a,b)W_{L/p}^{-av\,\Delta N}\right]W_L^{bm\,\Delta M/p-auM}, \tag{33}$$

where $0\leqslant m<M$, $0\leqslant n<N$, $n=u\cdot q+v$, $0\leqslant v<q$ and $0\leqslant u<\Delta M/p$. The proof of Eq. (33) is listed in Appendix A. Eq. (33) indicates that the Gabor coefficients in frequency-split method can be calculated through q amount of operations, which are $(\Delta M/p,M)$-point 2-D DFT. The vth 2-D DFT is to compute the $(u\cdot q+v)$th frequency slice of Gabor coefficients ($0\leqslant u<\Delta M/p$, $0\leqslant v<q$).

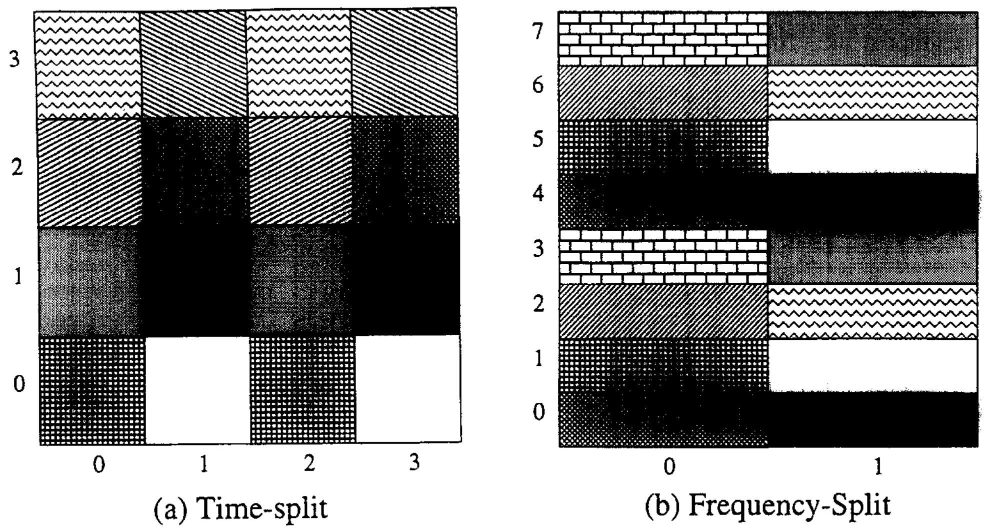

Fig. 3. The summation term denominator in Zak transform domain for time-split and frequency-split methods, $L = 16$, $M = 8$, $N = 4$.

Analysis algorithm of the frequency-split method is listed as follows:
Step 1: Compute the $(\Delta M, M)$-point FZT of analysis basis $\gamma(i)$.
Step 2: Compute the $(\Delta M, M)$-point FZT of signal $f(i)$.
Step 3: for $s = 0$ to $q-1$
Step 4: $\Gamma(a,b) = 0,\ 0 \leqslant a < \frac{\Delta M}{p},\ 0 \leqslant b < M$
Step 5: for $t = 0$ to $p-1$
Step 6: $\Gamma(a,b) = \Gamma(a,b) + \hat{f}_{(\Delta M,M)}(a + t\frac{\Delta M}{p}, b)\hat{\gamma}^*_{(\Delta M,M)}(a + \frac{\Delta M}{p}, b + s\Delta N)$
$0 \leqslant a < \frac{\Delta M}{p},\ 0 \leqslant b < M$
Step 7: end
Step 8: Compute an $(\Delta M/p, M)$-point 2-D DFT of the results $\Gamma(a,b)$
Step 9: end

The reconstruction of signal $f(i)$ can be obtained from the Gabor coefficients.

$$\hat{f}_{(\Delta M,M)}(a,b) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} C_{m,n}\hat{h}_{(\Delta M,M)}(a - m\Delta M, b - n\Delta N)W_N^{an}, \qquad 0 \leqslant a < \Delta M,\ 0 \leqslant b < M, \tag{34}$$

$$\hat{f}_{(\Delta M,M)}\left(d\frac{\Delta M}{p} + a, b\right) = \sum_{v=0}^{q-1}\left[\sum_{m=0}^{M-1}\sum_{u=0}^{\Delta M/p-1} C_{m,uq+v}W_{L/p}^{-(b-v\,\Delta N)m\,\Delta M/p+auM}\right] \times \hat{h}_{(\Delta M,M)}\left(d\frac{\Delta M}{p} + a, b - v\,\Delta N\right)W_L^{av\Delta N},$$

$$0 \leqslant a < \frac{\Delta M}{p},\ 0 \leqslant b < M,\ 0 \leqslant d < p. \tag{35}$$

The proof of Eq. (35) is also listed in Appendix A.

Synthesis algorithm of the frequency-split method is listed as follows:
Step 1: Compute the $(\Delta M, M)$-point FZT of synthesis basis $h(i)$.
Step 2: for $s = 0$ to $q - 1$
Step 3: Compute the 2-D DFT of $C_{m,uq+s}$, $F(a,b) = fft2(C_{m,uq+s})_{(m,u)->(a,b)}$
$0 \leqslant m < M,\ 0 \leqslant u < \frac{\Delta M}{p}$
Step 4: $\Gamma(a,b) = 0,\quad 0 \leqslant a < \Delta M,\ 0 \leqslant b < M$
Step 5: for $t = 0$ to $p - 1$
Step 6: $\Gamma(a + t\frac{\Delta M}{p}, b) = \Gamma(a + \frac{\Delta M}{p}, b) + \hat{h}_{(\Delta M,M)}(a + t\frac{\Delta M}{p}, b - s\Delta N)F(a,b)W_L^{as\Delta N}$
$0 \leqslant a < \frac{\Delta M}{p},\ 0 \leqslant b < M$
Step 7: end
Step 8: end
Step 9: Compute $(\Delta M, M)$-point IFZT of $\Gamma(a,b)$ to obtain the reconstructed signal $f(i)$.

Example. $L = 64$, $M = 16$, $N = 8$, $\Delta M = 4$, $\Delta N = 8$. The oversampling ratio in this example is 2. The Gabor coefficients in time and frequency-split methods can be obtained through two 2-D DFT. In time-split method, (8,8)-point DFT and FZT are used. The first DFT compute $C_{0,0}, C_{0,1}, C_{0,2}, \ldots, C_{0,7}, C_{2,0}, C_{2,1}, \ldots, C_{14,7}$. The second DFT compute $C_{1,0}, C_{1,1}, C_{1,2}, \ldots, C_{1,7}, C_{3,0}, C_{3,1}, \ldots, C_{15,7}$. In frequency-split method (4,16)-point DFT and FZT are used. The first DFT compute $C_{0,0}, C_{1,0}, C_{2,0}, \ldots, C_{15,0}, C_{0,2}, C_{1,2}, \ldots, C_{15,6}$. The second DFT compute $C_{0,1}, C_{1,1}, C_{2,1}, \ldots, C_{15,1}, C_{0,3}, C_{1,3}, \ldots, C_{15,7}$.

3.3. Discussion

As mentioned in Section 2, the analysis basis is uniquely existed. So the synthesis basis and Gabor coefficients obtained from solving Eq. (8) and the Zak transform method are identical.

In the oversampling scheme, although the processes of the time and frequency-split methods are different, the results in these two methods are identical. The equivalent results can be proved by checking Eqs. (26)

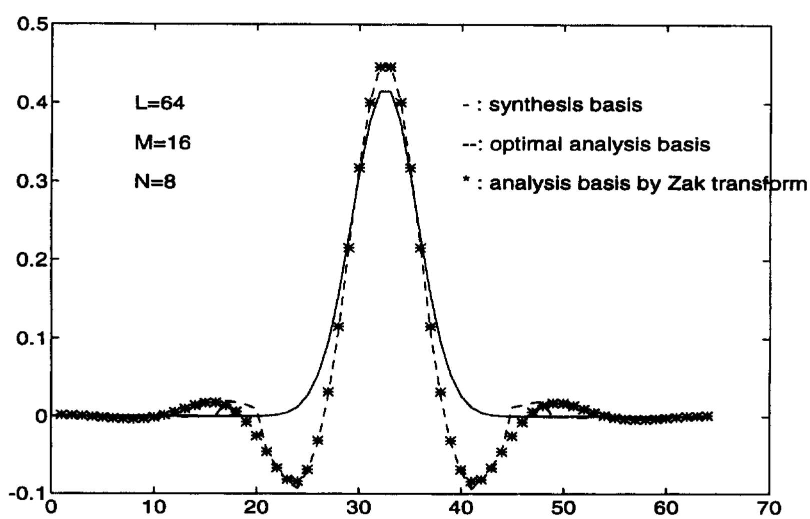

Fig. 4. An example of analysis bases obtained from time and frequency-split methods and the optimal solution, $L = 64$, $M = 16$, $N = 8$.

and (31). Both these two methods are based upon the least-squares norm criterion. The optimal solution obtained from Eq. (10) is to find an analysis basis which is similar to the synthesis basis as similar as possible. Fig. 4 shows the bases obtained from time and frequency-split methods and Eq. (10). It can be found that the analysis obtained from optimal solution and the Zak transform methods are different.

The critical sampling is only a special case for oversampling. The oversampling ratio for the critical scheme is equal to one. In critical sampling scheme, the time sampling interval ΔM is equal to N and the frequency sampling interval is equal to M. The Gabor coefficients in critical sampling can be obtained through (N,M)-point FZT and it is exactly a special case for the oversampling scheme, which needs $(N,\Delta N/p)$ or $(\Delta M/p,M)$-point FZT. The computation of Gabor coefficients in critical scheme needs one DFT calculation and in oversampling schemes needs q DFT calculation. This fact can be used to verify that the critical sampling is a special case for oversampling scheme again.

The algorithms presented in the previous subsection are time-split and frequency-split. So in 2-D case, four methods can be used to calculate Gabor coefficient by Zak transform in oversampling scheme. They are time–time-split, time–frequency-split, frequency–time-split and frequency–frequency-split.

4. An extension to two-dimensional case for Zak transform and Gabor expansion

The two methods that are proposed in the previous section for computing Gabor coefficients is suitable for finite signals. Image data is an example of two-dimensional finite signal. Now we will extend the above theories to the two-dimensional case and use 2-D Gabor expansion as a tool for image analysis.

4.1. 2-D discrete Gabor expansion

The definition of 2-D Gabor expansion for continuous signal is

$$f(x,y) = \sum_{m_x,n_x,m_y,n_y} C_{m_x n_x m_y n_y} h_{m_x n_x m_y n_y}(x,y) \tag{36}$$

$$= \sum_{m_x,n_y,m_y,n_y} C_{m_x n_x m_y n_y} h(x - m_x T_x, y - m_y T_y) \mathrm{e}^{\mathrm{j}(x\Omega_x n_x + y\Omega_y n_y)}, \tag{37}$$

where $h(x,y)$ is the 2-D synthesis basis function. $T_x\Omega_x \leqslant 2\pi$ and $T_y\Omega_y \leqslant 2\pi$ are the existence condition for Eq. (37). We can define 2-D discrete Gabor expansion for finite and periodical signals. The signal $f(x,y)$ is assumed to be finite with length L_x and L_y in the x and y directions or finite with periods L_x and L_y in the x and y directions. Thus, the 2-D finite discrete Gabor expansion is defined as

$$\tilde{f}(x,y) = \sum_{m_x,n_x,m_y,n_y} C_{m_x n_x m_y n_y} h_{m_x n_x m_y n_y}(x,y) \tag{38}$$

$$= \sum_{m_x,n_x,m_y,n_y} C_{m_x n_x m_y n_y} h(x - m_x \Delta M_x, y - m_y \Delta M_y) W_L^{(n_x x + n_y y)}, \tag{39}$$

where $\tilde{f}(x,y)$ is periodic extension of $f(x,y)$, $h(x,y)$ is the synthesis basis. $0 \leqslant m_x < M_x$, $0 \leqslant m_y < M_y$, $0 \leqslant n_x < N_x$ and $0 \leqslant n_y < N_y$. M_x is the number of sampling points in time domain of x direction. M_y is the number of sampling points in time domain of y direction. N_x is the number of sampling points in frequency domain of x direction. N_y is the number of sampling points in frequency domain of y direction. ΔM_x is the time sampling interval in x direction. ΔM_y is the time sampling interval in y direction. ΔN_x is frequency sampling interval in x direction. ΔN_y is frequency sampling interval in y direction. $M_x \cdot \Delta M_x = L_x$, $M_y \cdot \Delta M_y = L_y$, $N_x \cdot \Delta N_x = L_x$, $N_y \cdot \Delta N_y = L_y$. The conditions, $\Delta M_x \Delta N_x \leqslant L_x$ and $\Delta M_y \Delta N_y \leqslant L_y$, must be satisfied for a stable reconstruction. The critical sampling case occurs when $M_x N_x = L_x$ and $M_y N_y = L_y$.

Define $\alpha_x = M_xN_x/L_x = q_x/p_x$ to be the oversampling ratio in x direction, and $\alpha_y = M_yN_y/L_y = q_y/p_y$ to be the oversampling ratio in y direction.

4.2. 2-D FZT

2-D continuous Zak transform can be extended from the definition of 1-D continuous Zak transform directly. Its definition is defined as

$$\hat{f}(\tau_x,\tau_y,\Omega_x,\Omega_y) = \sum_p \sum_q f[\lambda_x(\tau_x+p),\lambda_y(\tau_y+q)]\mathrm{e}^{-\mathrm{j}2\pi(p\Omega_x+q\Omega_y)}\,. \tag{40}$$

Several properties about 2-D continuous Zak transform have been listed in [19]. Furthermore, we can define 2-D FZT. The 2-D FZT is defined as

$$\hat{f}\left(\frac{a_x}{A_x},\frac{a_y}{A_y},\frac{b_x}{B_x},\frac{b_y}{B_y}\right) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f\left(\frac{a_x}{A_x}+k,\frac{a_y}{A_y}+l\right)\mathrm{e}^{-\mathrm{j}2\pi(kb_x/B_x+lb_y/B_y)},$$
$$0\leqslant b_x<B_x,\ 0\leqslant b_y<B_y,\ 0\leqslant a_x<A_x,\ 0\leqslant a_y<A_y. \tag{41}$$

For convenience of our discussion, we change the indices into integers.

$$\hat{f}(a_x,a_y,b_x,b_y) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f(a_x+kA_x,a_y+lA_y)\mathrm{e}^{-\mathrm{j}2\pi(kb_x/B_x+lb_y/B_y)},$$
$$0\leqslant b_x<B_x,\ 0\leqslant b_y<B_y,\ 0\leqslant a_x<A_x,\ 0\leqslant a_y<A_y\,. \tag{42}$$

If the signal is finite or periodic with length $L_x = A_xB_x$ points in x direction and $L_y = A_yB_y$ points in y direction, then

$$\hat{f}(a_x,a_y,b_x,b_y) = \sum_{k=0}^{B_x-1}\sum_{l=0}^{B_y-1} f(a_x+kA_x,a_y+lA_y)\mathrm{e}^{-\mathrm{j}2\pi(kb_x/B_x+lb_y/B_y)},$$
$$0\leqslant b_x<B_x,\ 0\leqslant b_y<B_y,\ 0\leqslant a_x<A_x,\ 0\leqslant a_y<A_y\,. \tag{43}$$

This 2-D FZT is called (A_x,A_y,B_x,B_y)-point 2-D FZT, and it is denoted by $\hat{f}_{(A_x,A_y,B_x,B_y)}$.

4.3. Relationship between 2-D finite Gabor expansion and 2-D FZT

4.3.1. Critical sampling scheme

In [19], it has been shown that 2-D Gabor coefficients can be obtained from 2-D Zak transform

$$C_{m_x,n_x,m_y,n_y} = \int_0^1\int_0^1\int_0^1\int_0^1 \frac{\hat{f}(\tau_x,\tau_y,\Omega_x,\Omega_y)}{\hat{h}(\tau_x,\tau_y,\Omega_x,\Omega_y)}\mathrm{e}^{-\mathrm{j}2\pi(n_x\tau_x+n_y\tau_y-m_x\Omega_x-m_y\Omega_y)}\,\mathrm{d}\tau_x\,\mathrm{d}\tau_y\,\mathrm{d}\Omega_x\,\mathrm{d}\Omega_y\,, \tag{44}$$

where $\hat{f}(\tau_x,\tau_y,\Omega_x,\Omega_y)$ is the 2-D Zak transform of signal $f(x,y)$, and $\hat{h}(\tau_x,\tau_y,\Omega_x,\Omega_y)$ is the 2-D Zak transform of synthesis basis $h(x,y)$. The coefficients can be obtained by a 4-D continuous Fourier transform. In the discrete case, (N_x,N_y,M_x,M_y) point FZT can be used to calculate the 2-D Gabor coefficients.

$$C_{m_x,n_x,m_y,n_y} = \frac{1}{L_xL_y}\sum_{a_x=0}^{N_x-1}\sum_{a_y=0}^{N_y-1}\sum_{b_x=0}^{M_x-1}\sum_{b_y=0}^{M_y-1}\frac{\hat{f}_{(N_x,N_y,M_x,M_y)}(a_x,a_y,b_x,b_y)}{\hat{h}_{(N_x,N_y,M_x,M_y)}(a_x,a_y,b_x,b_y)}W_{L_x}^{-a_xn_xM_x+b_xm_xN_x}W_{L_y}^{-a_yn_yM_y+b_ym_yN_y}, \tag{45}$$

where $W_{L_x} = \mathrm{e}^{\mathrm{j}2\pi/L_x}$, $W_{L_y} = \mathrm{e}^{\mathrm{j}2\pi/L_y}$, $\hat{h}_{(N_x,N_y,M_x,M_y)}$ is the 2-D FZT of synthesis basis. From Eq. (45), the 2-D Gabor coefficients can be derived through a 4-D DFT operation in critical sampling case.

4.3.2. Oversampling scheme

In 1-D case, two methods based upon Zak transform for computing finite Gabor coefficients in oversampling scheme are presented. Thus, there exists four methods based on Zak transform for computing 2-D Gabor coefficients in oversampling scheme: $(N_x, N_y, \Delta N_x, \Delta N_y)$, $(N_x, \Delta M_y, \Delta N_x, M_y)$, $(\Delta M_x, N_y, M_x, \Delta N_y)$ and $(\Delta M_x, \Delta M_y, M_x, M_y)$-point 2-D discrete Zak transform. They are named time–time-split, time–frequency-split, frequency–time-split and frequency–frequency-split, respectively. Their analysis bases and Gabor coefficients can be attained through the following equations:

– *time–time-split method*: $(N_x, N_y, \Delta N_x, \Delta N_y)$-point Zak transform

If $p_x = p_y = 1$, the analysis basis function can be obtained from Eq. (46).

$$\hat{\gamma}_{(N_x,N_y,\Delta N_x,\Delta N_y)}(a_x,a_y,b_x,b_y) = \frac{q_x q_y}{N_x N_y} \cdot \frac{\hat{h}_{(N_x,N_y,\Delta N_x,\Delta N_y)}(a_x,a_y,b_x,b_y)}{\sum_{k=0}^{q_x-1}\sum_{l=0}^{q_y-1} |\hat{h}_{(N_x,N_y,\Delta N_x,\Delta N_y)}(a_x - k\,\Delta M_x, b - l\,\Delta M_y, b_x, b_y)|^2}, \tag{46}$$

where $0 \leqslant a_x < N_x$, $0 \leqslant a_y < N_y$, $0 \leqslant b_x < \Delta N_x$, $0 \leqslant b_y < \Delta N_y$, $0 \leqslant u < \Delta N_x/p_x$, $0 \leqslant s < \Delta N_y/p_y$.
The Gabor coefficients in general case is calculated through Eq. (47).

$$C_{m_x,n_x,m_y,n_y} = \frac{1}{\Delta N_x \Delta N_y} \sum_{a_x=0}^{N_x-1} \sum_{a_y=0}^{N_y-1} \sum_{b_x=0}^{(\Delta N_x/p_x)-1} \sum_{b_y=0}^{(\Delta N_y/p_y)-1}$$
$$\left[\sum_{d_x=0}^{p_x-1} \sum_{d_y=0}^{p_y-1} \hat{f}_{(N_x,N_y,\Delta N_x,\Delta N_y)}(a_x, a_y, b_x + d_x(\Delta N_x/p_x), b_y + d_y(\Delta N_y/p_y)) \right.$$
$$\left. \times \hat{\gamma}^*_{(N_x,N_y,\Delta N_x,\Delta N_y)}(a_x, -v\Delta M_x, b - t\Delta M_y, b_x + d_x(\Delta N_x/p_x), b_y + d_y(\Delta N_y/p_y)) \right]$$
$$\times W_{L/p}^{b_x u N_x L_y - a_x n_x (\Delta N_x/p_x) L_y + b_y s N_y L_x - a_y n_y (\Delta N_y/p_y) L_x}, \tag{47}$$

where $L = L_x \cdot L_y$, $p = p_x p_y$, $m_x = uq_x + v$, $m_y = sq_y + t$, $0 \leqslant v < q_x$, $0 \leqslant t < q_y$. These Gabor coefficients can be obtained through $(q_x \cdot q_y)$ operations, which are $(N_x, N_y, \Delta N_x/p_x, \Delta N_y/p_y)$-point 4-D DFT calculation.

Analysis algorithm of the time–time-split method is listed as follows:

Step 1: Compute the $(N_x, N_y, \Delta N_x, \Delta N_y)$-point 2-D FZT of analysis basis $\gamma(x,y)$.
Step 2: Compute the $(N_x, N_y, \Delta N_x, \Delta N_y)$-point 2-D FZT of signal $f(x,y)$.
Step 3: for $s_x = 0$ to $q_x - 1$
Step 4: for $s_y = 0$ to $q_y - 1$
Step 5: $\Gamma(a_x,a_y,b_x,b_y) = 0$, $0 \leqslant a_x < N_x$, $0 \leqslant a_y < N_y$, $0 \leqslant b_x < \frac{\Delta N_x}{p_x}$, $0 \leqslant b_y < \frac{\Delta N_y}{p_y}$
Step 6: for $t_x = 0$ to $p_x - 1$
Step 7: for $t_y = 0$ to $p_y - 1$
Step 8: $\Gamma(a_x,a_y,b_x,b_y) = \Gamma(a_x,a_y,b_x,b_y) + \hat{f}_{(N_x,N_y,\Delta N_x,\Delta N_y)}(a_x, a_y, b_x + t_x\frac{\Delta N_x}{p_x}, b_y + t_y\frac{\Delta N_y}{p_y})$

$$\hat{\gamma}^*_{(N_x,N_y,\Delta N_x,\Delta N_y)}(a_x - s_x\Delta M_x, a_y - s_y\Delta M_y, b_x + t_x\tfrac{\Delta N_x}{p_x}, b_y + t_y\tfrac{\Delta N_y}{p_y}),$$

$0 \leqslant a_x < N_x$, $0 \leqslant a_y < N_y$, $0 \leqslant b_x < \frac{\Delta N_x}{p_x}$, $0 \leqslant b_y < \frac{\Delta N_y}{p_y}$

Step 9: end
Step 10: end
Step 11: Compute an $(N_x, N_y, \Delta N_x/p_x, \Delta N_y/p_y)$-point 4-D DFT of the results $\Gamma(a_x,a_y,b_x,b_y)$

Step 12: end
Step 13: end

Synthesis algorithm of the time–time-split method is listed as follows:
Step 1: Compute the $(N_x, N_y, \Delta N_x, \Delta N_y)$-point 2-D FZT of synthesis basis $h(x, y)$.
Step 2: for $s_x = 0$ to $q_x - 1$
Step 3: for $s_y = 0$ to $q_y - 1$
Step 4: Compute the 4-D DFT of $C_{m_x q_x + s_x, m_y q_y + s_y, n_x, n_y}$,
$F(a_x, a_y, b_x, b_y) = fft4(C_{m_x q_x + s_x, m_y q_y + s_y, n_x, n_y})_{(m_x, m_y, n_x, n_y) -> (a_x, a_y, b_x, b_y)}$
$0 \leqslant m_x < \frac{\Delta N_x}{p_x},\ 0 \leqslant m_y < \frac{\Delta N_y}{p_y},\ 0 \leqslant n_x < N_x,\ 0 \leqslant n_y < N_y$
Step 5: $\Gamma(a_x, a_y, b_x, b_y) = 0, \quad 0 \leqslant a_x < N_x,\ 0 \leqslant a_y < N_y,\ 0 \leqslant b_x < \frac{\Delta N_x}{p_x},\ 0 \leqslant b_y < \frac{\Delta N_y}{p_y}$
Step 6: for $t_x = 0$ to $p_x - 1$
Step 7: for $t_y = 0$ to $p_y - 1$
Step 8: $\Gamma(a_x, a_y, b_x + t_x \frac{\Delta N_x}{p_x}, b_y + t_y \frac{\Delta N_y}{p_y}) = \Gamma(a_x, a_y, b_x + \frac{\Delta N_x}{p_x}, b_y + \frac{\Delta N_y}{p_y})$
$+\hat{h}_{(N_x, N_y, \Delta N_x, \Delta N_y)}(a_x - s_x \Delta M_x, a_y - s_y \Delta M_y, b_x + t_x \frac{\Delta N_x}{p_x}, b_y + t_y \frac{\Delta N_y}{p_y}) \cdot F(a_x, a_y, b_x, b_y)$,
$0 \leqslant a_x < N_x,\ 0 \leqslant a_y < N_y,\ 0 \leqslant b_x < \frac{\Delta N_x}{p_x},\ 0 \leqslant b_y < \frac{\Delta N_y}{p_y}$

Step 9: end
Step 10: end
Step 11: end
Step 12: end
Step 13: Compute $(N_x, N_y, \Delta N_x, \Delta N_y)$-point IFZT of $\Gamma(a_x, a_y, b_x, b_y)$ to obtain the reconstructed signal $f(x, y)$.

– *time–frequency-split method*: $(N_x, \Delta M_y, \Delta N_x, M_y)$-point Zak transform
If $p_x = p_y = 1$, the analysis basis function can be obtained from Eq. (48):

$$\hat{\gamma}_{(N_x, \Delta M_y, \Delta N_x, M_y)}(a_x, a_y, b_x, b_y) = \frac{q_x q_y}{N_x \Delta M_y} \cdot \frac{\hat{h}_{(N_x, \Delta M_y, \Delta N_x, M_y)}(a_x, a_y, b_x, b_y)}{\sum_{k=0}^{q_x-1} \sum_{l=0}^{q_y-1} |\hat{h}_{(N_x, \Delta M_y, \Delta N_x, M_y)}(a - k\Delta M_x, a_y, b_x, d + l\Delta N_y)|^2}, \tag{48}$$

where $0 \leqslant a_x < N_x,\ 0 \leqslant a_y < \Delta M_y,\ 0 \leqslant b_x < \Delta N_x,\ 0 \leqslant b_y < M_y$. The Gabor coefficients in general case is calculated through Eq. (49):

$$\begin{aligned} C_{m_x, n_x, m_y, n_y} = {} & \frac{1}{\Delta N_x M_y} \sum_{a_x=0}^{N_x-1} \sum_{a_y=0}^{(\Delta M_y/p_y)-1} \sum_{b_x=0}^{(\Delta N_x/p_x)-1} \sum_{b_y=0}^{M_y-1} \\ & \times \left[\sum_{d_x=0}^{p_x-1} \sum_{d_y=0}^{p_y-1} \hat{f}_{(N_x, \Delta M_y, \Delta N_x, M_y)} \left(a_x, a_y + d_y \frac{\Delta M_y}{p_y}, b_x + d_x \frac{\Delta N_y}{p_x}, b_y \right) \right. \\ & \left. \times \hat{\gamma}_{(N_x, \Delta M_y, \Delta N_x, M_y)} \left(a_x - v\Delta M_x, a_y + d_y \frac{\Delta M_y}{p_y}, b_x + d_x \frac{\Delta M_x}{p_x}, b_y + t\,\Delta N_y \right) W_{L_y}^{-t a_y \Delta N_y} \right] \\ & \times W_{L/p}^{b_x u N_x L_y - a_x n_x (\Delta N_x/p_x) L_y + b_y m_y (\Delta M_y/p_y) L_x - a_y s M_y L_x}, \end{aligned} \tag{49}$$

where $L = L_x \cdot L_y, p = p_x p_y,\ m_x = uq_x + v,\ n_y = sq_y + t,\ 0 \leqslant v < q_x,\ 0 \leqslant t < q_y,\ 0 \leqslant u < \Delta N_x/p_x$, $0 \leqslant s < \Delta M_y/p_y$. These Gabor coefficients can be obtained through $(q_x \cdot q_y)$ operations, which are $(N_x, \Delta M_y/p_y, \Delta N_x/p_x, M_y)$-point 4-D DFT calculations.

The analysis and synthesis algorithms for time–frequency-split method can easily be derived based upon the algorithms listed in the previous discussions.

frequency–time-split method: $(\Delta M_x, N_y, M_x, \Delta N_y)$-point Zak transform

If $p_x = p_y = 1$, the analysis basis function can be obtained from Eq. (50):

$$\hat{\gamma}_{(\Delta M_x,N_y,M_x,\Delta N_y)}(a_x,a_y,b_x,b_y) = \frac{q_x q_y}{\Delta M_x N_y} \cdot \frac{\hat{h}_{(\Delta M_x,N_y,M_x,\Delta N_y)}(a_x,a_y,b_x,b_y)}{\sum_{k=0}^{q_x-1}\sum_{l=0}^{q_y-1} |\hat{h}_{(\Delta M_x,N_y,M_x,\Delta N_y)}(a_x,a_y-l\Delta M_y,b_x+k\Delta N_x,b_y)|^2}, \tag{50}$$

where $0 \leqslant a_x < \Delta M_x,\ 0 \leqslant a_y < N_y,\ 0 \leqslant b_x < M_x,\ 0 \leqslant b_y < \Delta N_y$. The Gabor coefficients in general case is calculated through Eq. (51):

$$\begin{aligned} C_{m_x,n_x,m_y,n_y} = {} & \frac{1}{M_x \Delta N_y} \sum_{a_x=0}^{(\Delta M_x/p_x)-1} \sum_{a_y=0}^{N_y-1} \sum_{b_x=0}^{M_x-1} \sum_{b_y=0}^{\Delta N_y/p_y-1} \\ & \times \Bigg[\sum_{d_x=0}^{p_x-1}\sum_{d_y=0}^{p_y-1} \hat{f}_{(\Delta M_x,N_y,M_x,\Delta N_y)}\left(a_x + d_x\frac{\Delta M_x}{p_x}, a_y, b_x, b_y + d_y\frac{\Delta N_y}{p_y}\right) \\ & \times \hat{\gamma}^*_{(\Delta M_x,N_y,M_x,\Delta N_y)}\left(a_x + d_x\frac{\Delta M_x}{p_x}, a_y - t\Delta M_y, b_x + v\Delta N_x, b_y + d_y\frac{\Delta N_y}{p_y}\right) W_{L_x}^{-v\Delta N_x}\Bigg] \\ & \times W_{L/p}^{b_x m \frac{\Delta M_x}{p_x} L_y - a_x u M_x L_y + b_y s N_y L_x - n_y a_y \frac{\Delta N_y}{p_y} L_x}, \end{aligned} \tag{51}$$

where $L = L_x \cdot L_y,\ p = p_x p_y, n_x = uq_x + v,\ m_y = sq_y + t,\ 0 \leqslant v < q_x,\ 0 \leqslant t < q_y, 0 \leqslant u < \Delta M_x/p_x$, $0 \leqslant s < \Delta N_y/p_y$. These Gabor coefficients can be obtained through $(q_x \cdot q_y)$ operations, which can be $(\Delta M_x/p_x, N_y, M_x, \Delta N_y/p_y)$-point 4-D DFT calculations.

The analysis and synthesis algorithms for frequency–time-split method are easily derived based upon the algorithms listed in the previous discussions.

frequency–frequency-split method: $(\Delta M_x, \Delta M_y, M_x, M_y)$-point Zak transform

If $p_x = p_y = 1$, the analysis basis function can be obtained from Eq. (52).

$$\hat{\gamma}_{(\Delta M_x,\Delta M_y,M_x,M_y)}(a_x,a_y,b_x,b_y) = \frac{q_x q_y}{\Delta M_x \Delta M_y} \cdot \frac{\hat{h}_{(\Delta M_x,\Delta M_y,M_x,M_y)}(a_x,a_y,b_x,b_y)}{\sum_{k=0}^{q_x-1}\sum_{l=0}^{q_y-1} |\hat{h}_{(\Delta M_x,\Delta M_y,M_x,M_y)}(a_x,a_y,b_x+k\Delta N_x,b_y+l\Delta N_y)|^2}, \tag{52}$$

where $0 \leqslant a_x < \Delta M_x,\ 0 \leqslant a_y < \Delta M_y,\ 0 \leqslant b_x < M_x,\ 0 \leqslant b_y < M_y$. The Gabor coefficients in general case is calculated through Eq. (53):

$$\begin{aligned} C_{m_x,n_x,m_y,n_y} = {} & \frac{1}{M_x M_y} \sum_{a_x=0}^{(\Delta M_x/p_x)-1} \sum_{a_y=0}^{(\Delta M_y/p_y)-1} \sum_{b_x=0}^{M_x-1} \sum_{b_y=0}^{M_y-1} \\ & \Bigg[\sum_{d_x=0}^{p_x-1}\sum_{d_y=0}^{p_y-1} \hat{f}_{(\Delta M_x,\Delta M_y,M_x,M_y)}\left(a_x + d_x\frac{\Delta M_x}{p_x}, a_y + \frac{\Delta M_y}{p_y}, b_x, b_y\right) \end{aligned}$$

$$\times \hat{\gamma}^*_{(\Delta M_x, \Delta M_y, M_x, M_y)} \left(a_x + d_x \frac{\Delta M_x}{p_x}, a_y + \frac{\Delta M_y}{p_y}, b_x + v\Delta N_x, b_y + t\Delta N_y \right)$$

$$\times W_{L_x}^{-v\Delta N_x} W_{L_y}^{-t\Delta N_y} \Big] W_{L/p}^{b_x m \frac{\Delta M_x}{p_x} L_y - a_x u M_x L_y + b_y M_y \frac{\Delta M_y}{p_y} L_x - a_y s M_y L_x}, \qquad (53)$$

where $L = L_x \cdot L_y$, $p = p_x p_y$, $n_x = uq_x + v$, $n_y = sq_y + t$, $0 \leqslant v < q_x$, $0 \leqslant t < q_y$, $0 \leqslant u < \Delta M_x / p_x$, $0 \leqslant s < \Delta M_y / p_y$. These Gabor coefficients can be obtained through $(q_x \cdot q_y)$ operations, which are $(\Delta M_x/p_x, \Delta M_y/p_y, M_x, M_y)$-point 4-D DFT calculations.

The analysis and synthesis algorithms for frequency–frequency-split method are easily derived based upon the algorithms listed in the previous discussions.

Similar to the 1-D case, the above four methods will result the same analysis basis function and Gabor coefficients.

Gabor expansion can localize the time–frequency characteristics of signals. It is a very useful tool for signal processing. 2-D Gabor expansion has been widely used in image analysis and compression. Fig. 5 shows an example for image energy presented in the various signal bands. It is obtained by 2-D FZT in Gabor oversampling scheme. The size of image in this example is 64 × 64 pixels. The number of time sampling point in x direction is 16. The number of time sampling point in y direction is 16. The number of frequency sampling point in x direction is 8. The number of frequency sampling point in y direction is 8.

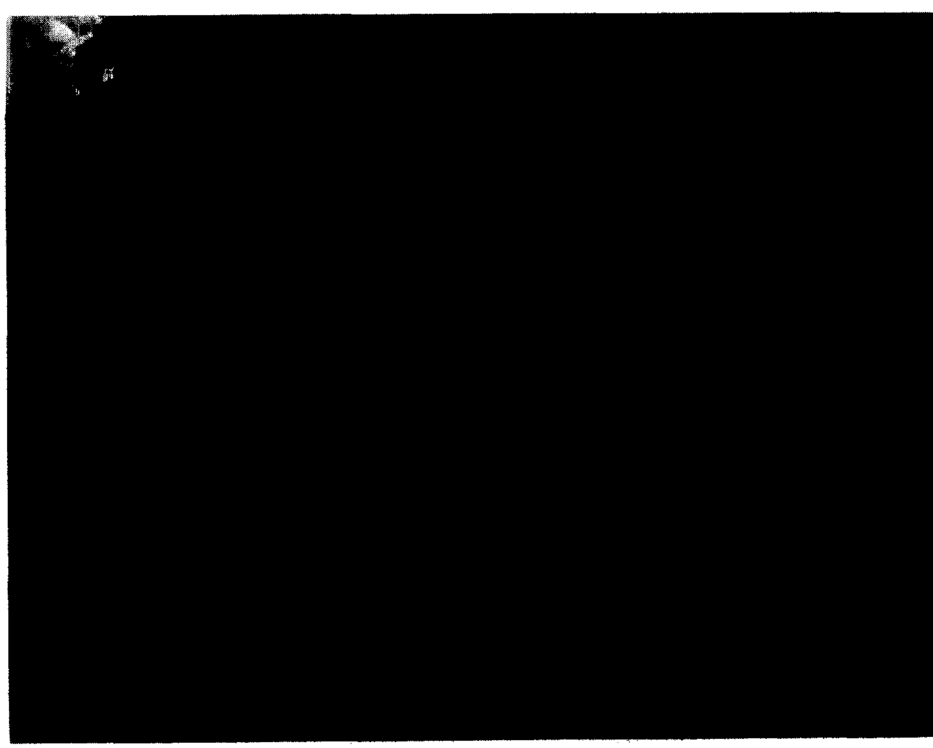

Fig. 5. The signal distribution for the various bands in the oversampling scheme.

5. Conclusion

According to the above discussion, several conclusions can be made. First, the (M,N)-point finite Gabor expansion can be obtained by using (N,M)-point FZT in critical sampling case. In oversampling case, (M,N)-point finite Gabor expansion can be obtained by two methods. One is calculated by $(N,\Delta N)$-point Zak transform, the other is obtained by $(\Delta M,M)$-point Zak transform. These two methods will result the same analysis bases and Gabor coefficients. If the oversampling ratio increases, the analysis bases will be more similar to the synthesis bases. This phenomenon is the same as that of optimal basis in [16]. But their analysis bases are different to the bases generated in [16]. The Gabor coefficients in the proposed two methods can be achieved through q 2-D DFT operations.

2-D Gabor expansion has been widely used in image analysis and compression. We can extend the theories from 1-D case to the 2-D case and compute 2-D Gabor coefficients efficiently. Four DFT-based algorithms for computing 2-D Gabor coefficients are attained to compute them efficiently.

Appendix A

At first, we will prove Eq. (33):

$$\begin{aligned} C_{m,n} &= \frac{1}{M}\sum_{a=0}^{\Delta M-1}\sum_{b=0}^{M-1}\hat{f}_{(\Delta M,M)}\hat{\gamma}^*_{(\Delta M,M)}(a-m\Delta M, b-n\Delta N)W_L^{-an\Delta N} \\ &= \frac{1}{M}\sum_{a=0}^{\Delta M-1}\sum_{b=0}^{M-1}\hat{f}_{(\Delta M,M)}\hat{\gamma}^*_{(\Delta M,M)}(a, b-n\Delta N)W_L^{m\Delta M(b-n\Delta N)}W_L^{-an\Delta N}, \end{aligned}$$

$0 \leqslant m < M,\ 0 \leqslant n < N$. Let $n = u\cdot q + v,\ 0 \leqslant v < q,\ 0 \leqslant u < \Delta M/p$.

$$\begin{aligned} C_{m,n} &= \frac{1}{M}\sum_{a=0}^{\Delta M-1}\sum_{b=0}^{M-1}\hat{f}_{(\Delta M,M)}(a,b)\hat{\gamma}^*_{(\Delta M,M)}(a, b-uq\Delta N - v\Delta N)W_L^{bm\Delta M}W_L^{-muq\Delta M\Delta N} \\ &\qquad \times W_L^{-mv\Delta M\Delta N}W_L^{-auq\Delta N}W_L^{-av\Delta N} \\ &= \frac{1}{M}\sum_{a=0}^{\Delta M-1}\sum_{b=0}^{M-1}\hat{f}_{(\Delta M,M)}(a,b)\hat{\gamma}^*_{(\Delta M,M)}(a, b-v\Delta N)W_L^{bm\Delta M}W_L^{-mv\Delta M\Delta N}W_L^{-aupM}W_L^{-av\Delta N}. \end{aligned}$$

Replace a by $a = d\Delta M/p + a',\ 0 \leqslant a' < \Delta M/p,\ 0 \leqslant d < p$.

$$\begin{aligned} C_{m,n} &= \sum_{a'=0}^{\Delta M/p-1}\sum_{b=0}^{M-1}\sum_{d=0}^{p-1}\hat{f}_{(\Delta M,M)}\left(d\frac{\Delta M}{p}+a', b\right)\hat{\gamma}^*_{(\Delta M,M)}\left(d\frac{\Delta M}{p}+a', b-v\Delta N\right)W_L^{-(d\Delta M/p+a')v\Delta N} \\ &\qquad \times W_{L/p}^{m(b-v\Delta N)\frac{\Delta M}{p}-a'uM} \\ &= \sum_{a=0}^{\Delta M/p-1}\sum_{b=0}^{M-1}\sum_{d=0}^{p-1}\hat{f}_{(\Delta M,M)}\left(\frac{d\Delta M}{p}+a, b+v\Delta N\right)\hat{\gamma}^*_{(\Delta M,M)}\left(\frac{d\Delta M}{p}+a, b\right)W_L^{-av\Delta N}W_{L/p}^{+mb\Delta M/p-auM}. \end{aligned}$$

Eq. (33) has been proved over.

Next, we will prove Eq. (35).

$$\hat{f}_{(\Delta M,M)}(a,b) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} C_{m,n}\,\hat{h}_{(\Delta M,M)}(a - m\Delta M, b - n\Delta N) W_N^{an}$$
$$= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} C_{m,n}\,\hat{h}_{(\Delta M,M)}(a, b - n\Delta N) W_L^{-m\Delta M(b-n\Delta N)+an\Delta N}.$$

Let $n = u\cdot q + v,\ 0 \leqslant v < q,\ 0 \leqslant u < \Delta M/p$.

$$\hat{f}_{(\Delta M,M)}(a,b) = \sum_{m=0}^{M-1}\sum_{u=0}^{\Delta M/p-1}\sum_{v=0}^{q-1} C_{m,uq+v}\,\hat{h}_{(\Delta M,M)}(a, b - uq\Delta N - v\Delta N)$$
$$\times W_L^{-m\Delta M(b-uq\Delta N-v\Delta N)} W_L^{auPM} W_L^{av\Delta N}$$
$$= \sum_{m=0}^{M-1}\sum_{u=0}^{\Delta M/p-1}\sum_{v=0}^{q-1} C_{m,uq+v}\,\hat{h}_{(\Delta M,M)}(a, b - v\Delta N) W_L^{-m\Delta M(b-v\Delta N)} W_L^{auPM} W_L^{av\Delta N}.$$

Replace a by $a = d\Delta M/p + a',\ 0 \leqslant a' < \Delta M/p,\ 0 \leqslant d < p$.

$$\hat{f}_{(\Delta M,M)}\left(d\frac{\Delta M}{p} + a', b\right)$$
$$= \sum_{m=0}^{M-1}\sum_{u=0}^{\Delta M/p-1}\sum_{v=0}^{q-1} C_{m,uq+v}\,\hat{h}_{(\Delta M,M)}\left(d\frac{\Delta M}{p} + a', b - v\Delta N\right) W_L^{-m\Delta M(b-v\Delta N)} W_L^{a'upM} W_L^{av\Delta N}.$$

So

$$\hat{f}_{(\Delta M,M)}\left(d\frac{\Delta M}{p} + a, b\right) =$$
$$\sum_{v=0}^{q-1}\left[\sum_{m=0}^{M-1}\sum_{u=0}^{\Delta M/p-1} C_{m,uq+v} W_{L/p}^{-(b-v\Delta N)m\Delta M/p+auM}\right]\hat{h}_{(\Delta M,M)}\left(d\frac{\Delta M}{p} + a, b - v\Delta N\right) W_L^{av\Delta N},$$
$$0 \leqslant a < \frac{\Delta M}{p},\quad 0 \leqslant b < M,\quad 0 \leqslant d < p.$$

The proof of Eq. (35) is completed.

References

[1] H. Akaike, "Block Toeplitz matrix inversion", *SIAM Appl. Math.*, Vol. 24, March 1973.

[2] L. Auslander, I.C. Gertner and R. Tolimieri, "The discrete Zak transform application to time-frequency analysis and synthesis of nonstationary signals", *IEEE Trans. Signal Process.*, Vol. 39, April 1991, pp. 825–835.

[3] R. Balart, "Matrix reformulation of the Gabor transform", *Opt. Engrg.*, Vol. 30, 1992, pp. 1235–1242.

[4] M.J. Bastiaans, "Gabor's expansion of a signal into Gaussian elementary signals", *Proc. IEEE*, Vol. 68, 1980, pp. 538–539.

[5] T.A.C.M. Claasen and W.F.G. Mecklenbrauker, "The Wigner distribution – A tool for time-frequency signal analysis, Part I: Continuous-time signals", *Philips J. Res.*, Vol. 35, 1980, pp. 217–250.

[6] L. Cohen, "Time-frequency distributions – A review", *Proc. IEEE*, Vol. 77, July 1989, pp. 941–981.

[7] I. Daubechies, "The Wavelet transform, time-frequency localization and signal analysis", *IEEE Trans. Inform. Theory*, Vol. 36, September 1990, pp. 961–1005.

[8] T. Ebrahimi and M. Kunt, "Image compression by Gabor expansion", *Opt. Engrg.*, Vol. 30, July 1991, pp. 873–880.

[9] D. Gabor, "Theory of communication", *J. Inst. Elec. Eng.*, Vol. 93, November 1946, pp. 429–439.

[10] G.E. Heil and D.F. Walnut, "Continuous and discrete Wavelet transforms", *SIAM Rev.*, Vol. 31, December 1988, pp. 628–666.

[11] A.J.E.M. Janssen, "Gabor representation of generalized functions", *J. Math. Appl.*, Vol. 80, 1981, pp. 377–394.
[12] A.J.E.M. Janssen, "The Zak transform: A signal transform for sampled time-continuous signals", *Philips J. Res.*, Vol. 43, 1988, pp. 23–69.
[13] R.S. Orr, "The order of computation for finite discrete Gabor transforms", *IEEE Trans. Signal Process.*, Vol. 41, January 1993, pp. 122–130.
[14] M. Porat and Y. Zeevi, "The generalized Gabor scheme of image representation in biological and machine vision", *IEEE Trans. Pattern Anal. Mach. Intell.*, Vol. 40, July 1988, pp. 452–468.
[15] M.R. Portnoff, "Time–frequency representation of digital signals", *IEEE Trans. Acoust. Speech Signal Process.*, Vol. ASSP-28, February 1980, pp. 55–69.
[16] S. Qian and D. Chen, "Optimal biorthogonal analysis window function for discrete Gabor transform", *IEEE Trans. Signal Process.*, Vol. 42, March 1994, pp. 694–697.
[17] J. Wexler and S. Raz, "Discrete Gabor expansions", *Signal Processing*, Vol. 21, No. 3, November 1990, pp. 207–220.
[18] J. Yao, "Complete Gabor transformation for signal representation", *IEEE Trans. Image Process.*, April, 1993, pp. 152–159.
[19] Y. Zeevi and Gerner, "The finite Zak transform: An efficient tool for image representation, analysis", *J. Visual Commun. Image Representation*, Vol. 3, March 1992, pp. 13–23.
[20] M. Zibulski and Y.Y. Zeevi, "Oversampling in the Gabor scheme", *IEEE Trans. Signal Process.*, Vol. 41, August 1993, pp. 2679–2687.
[21] M. Zibulski and Y. Zeevi, "Frame analysis of the discrete Gabor-scheme", *IEEE Trans. Signal Process.*, Vol. 42, April 1994, pp. 942–945.