

Rate of convergence for the Euclidean minimum spanning tree limit law

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Let N_n be the number of points of a Poisson point process of intensity n times the Lebesgue measure over $[0,1]^2$, and let $L_{\text{MST}}(N_n)$ be the length of the optimal spanning tree connecting these N_n points. It is well-known that there is a constant $0 < \beta_{\text{MST}} < \infty$ such that $\lim_{n \rightarrow \infty} \text{E}L_{\text{MST}}(N_n)/\sqrt{n} = \beta_{\text{MST}}$. In this paper we give the exact rate of convergence for this limiting behavior.

Euclidean minimum spanning tree; asymptotic analysis; rate of convergence

1. Introduction

The minimum spanning tree problem (MSTP) in the plane requires finding the length of the shortest tree spanning n points of \mathbf{R}^2 . We are concerned here with stochastic versions of the problem. First let X_i , $1 \leq i < \infty$, be uniformly and independently distributed random variables in $[0,1]^2$ and let $L_{\text{MST}}(n)$ be the length of the shortest tree spanning $\{X_1, X_2, \dots, X_n\}$. Steele [9] proved that $L_{\text{MST}}(n)$ is asymptotic to $\beta_{\text{MST}}\sqrt{n}$ with probability one (the same being true in expectation). In fact this result is valid for any uniform i.i.d. random variables with compact support of measure one in \mathbf{R}^d , $d \geq 2$, provided \sqrt{n} is replaced by $n^{(d-1)/d}$, the constant depending only on the dimension of the space and not on the shape of the compact support. Similar results had previously been obtained for the traveling salesman problem, the weighted matching problem, and the Steiner tree problem (Beardwood et al.

[2] and Steele [8]). Questions about rates of convergence for these limit laws have been raised many times in the literature (see for example [2,5,8,9]). There are in fact two issues concerning information on rates of convergence (let P be a generic symbol representing any of the problems pre-cited):

1. What is the asymptotic size of $L_P(n) - \text{E}L_P(n)$?
2. What can be said about the rate of convergence of the normalized means $\text{E}L_P(n)/\sqrt{n}$ to β_P ?

With respect to the first question, Rhee and Talagrand [7] proved that, for the TSP, there is a constant k such that $\|L_{\text{TSP}}(n) - \text{E}L_{\text{TSP}}(n)\|_p \leq k\sqrt{p}$ for each p for all n . This interesting result indicates that $L_{\text{TSP}}(n)$ is quite concentrated around its mean. With respect to the second question, some partial results have been obtained in Jaillet [4], where it is proved that $|\text{E}L_P(n)/\sqrt{n} - \beta_P| = O(1/\sqrt{n})$ (see also Alexander [1]). However, the important question remained open: Is $1/\sqrt{n}$ the exact rate of convergence, or does $\text{E}L_P(n)/\sqrt{n}$ go faster to β_P ?

In this paper, we consider the case of the MST and give an answer to this question for a Poisson point process π_n of intensity n times the Lebesgue measure over $[0,1]^2$. More precisely, let N_n be a

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Poisson random variable with parameter n representing the number of points of this process in $[0,1]^2$. We prove that $|\mathbf{E}L_{\text{MST}}(N_n)/\sqrt{n} - \beta_{\text{MST}}| = \Theta(1/\sqrt{n})$.

The paper is organized as follows. In Section 2, we state, in the context of a Poisson point process, some simple results of [4]. In Section 3, we prove our main result. Finally in Section 4, we give some final remarks.

2. Background results

If one follows the usual subadditivity argument (such as in [2,8]), one can go one step further and show that $\mathbf{E}L_{\text{MST}}(N_n) \geq \beta_{\text{MST}}\sqrt{n} - k_1$ for a positive constant k_1 . Also, adapting a classical argument given in [2] for the TSP, one can show that $\mathbf{E}L_{\text{MST}}(N_n) \leq \beta_{\text{MST}}\sqrt{n} + k_2$ for a positive constant k_2 . Let us present these arguments (adapted from [4]), and combine the results in one lemma.

Lemma 1. *Let N_n be the number of points of a Poisson point process π_n of intensity n times the Lebesgue measure over $[0,1]^2$, and let $L_{\text{MST}}(N_n)$ be the length of the minimum spanning tree connecting these N_n points. Then we have*

$$|\mathbf{E}L_{\text{MST}}(N_n)/\sqrt{n} - \beta_{\text{MST}}| = o(1/\sqrt{n}). \tag{1}$$

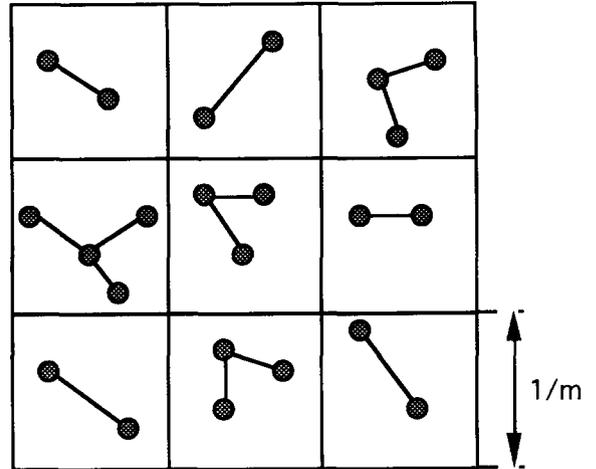
Proof Let us first prove that there exists a constant k_1 such that

$$\mathbf{E}L_{\text{MST}}(N_n) \geq \beta_{\text{MST}}\sqrt{n} - k_1. \tag{2}$$

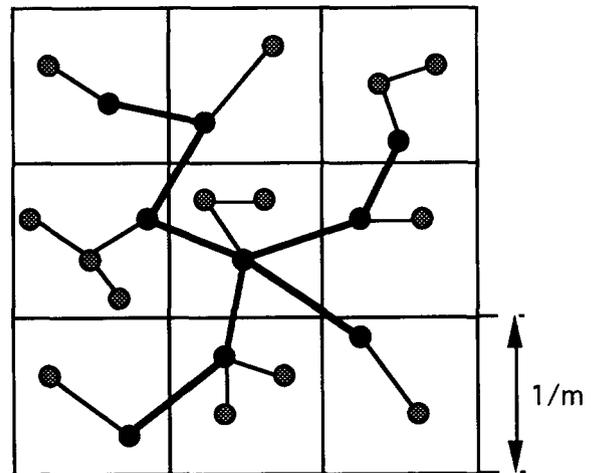
Consider $x = \{x_i : 1 \leq i < \infty\}$ to be an arbitrary infinite sequence of points in $[0,1]^2$, and let $x^{(n)} = \{x_1, x_2, \dots, x_n\}$ be its first n points. Let $\{Q_i : 1 \leq i \leq m^2\}$ be a partition of $[0,1]^2$ into m^2 squares with edges parallel to the axes and of side length $1/m$. Then there exists a constant k_1 such that

$$L_{\text{MST}}(x^{(n)}) \leq \sum_{i=1}^{m^2} L_{\text{MST}}(x^{(n)} \cap Q_i) + k_1 m. \tag{3}$$

The argument, classical, has its origin in [2, Lemma 1] and has been used subsequently in many papers. Consider the following tree construction connecting $x^{(n)}$ (see Figure 1 for an illustration): first construct optimal trees connecting $x^{(n)} \cap Q_i$ for $1 \leq i \leq m^2$. Then, in each square Q_i where $x^{(n)} \cap Q_i$ is not empty, choose one point as a representative and finally construct an



(i) Find MST trees in the subsquares



(ii) Patch the MST trees together by connecting their representatives

Fig. 1. Tree construction on $x^{(n)}$ in $[0,1]^2$

optimal tree connecting the set S of all representatives (at most m^2 points). The combination of the small trees together with the large tree gives a spanning tree connecting $x^{(n)}$ of length $\sum_{i=1}^{m^2} L_{\text{MST}}(x^{(n)} \cap Q_i) + L_{\text{MST}}(S)$. Now it is easy to show (see [3]) that there exists a constant k_1 such that $L_{\text{MST}}(S) \leq k_1\sqrt{|S|}$, and this establishes (3). From (3) we then have, starting with a Poisson point process $\pi_{m^2 n}$ on $[0,1]^2$ and using an obvious scaling property, the following subadditive inequality:

$$\mathbf{E}L_{\text{MST}}(N_{m^2 n}) \leq m\mathbf{E}L_{\text{MST}}(N_n) + k_1 m. \tag{4}$$

Dividing both sides of (4) by $\sqrt{m^2 n}$, letting m go to infinity, and using the fact that $EL_{MST}(N_{m^2 n})/\sqrt{m^2 n}$ goes to β_{MST} when m goes to infinity, we finally obtain (2). Now let us prove that there exists a constant k_2 such that

$$EL_{MST}(N_n) \leq \beta_{MST}\sqrt{n} + k_2. \quad (5)$$

From the arguments given above, it will be enough to prove that there exists a constant k_2 such that

$$L_{MST}(x^{(n)}) \geq \sum_{i=1}^{m^2} L_{MST}(x^{(n)} \cap Q_i) - k_2 m. \quad (6)$$

Indeed, one would then have the following inequality:

$$EL_{MST}(N_{m^2 n}) \geq m EL_{MST}(N_n) - k_2 m, \quad (7)$$

which can be analyzed in the same manner as inequality (4). The argument for proving (6) is also classical and is adapted from [2, Lemma 2] (see Figure 2 for an illustration). Let T^* be an optimal tree through $x^{(n)}$ and let us suppose that $x^{(n)} \cap Q_i$ is not empty. Let $T_i^* = T^* \cap Q_i$ and let T_{ij} for $1 \leq j \leq \mu_i$ ($\mu_i \leq |x^{(n)} \cap Q_i|$) be the connected pieces of T_i^* which contain at least an element of $x^{(n)}$. Let l_i be the total length of all these connected pieces. By using some portion of the perimeter of Q_i one can connect endpoints of these pieces (lying on the perimeter of Q_i) in order to form a tree spanning $x^{(n)} \cap Q_i$. Now, the additional points, used for this connection, lie outside of the convex hull of $x^{(n)} \cap Q_i$. Hence we have $L_{MST}(x^{(n)} \cap Q_i) \leq l_i + \text{per}(Q_i) = l_i + 4/m$. By summing both sides for all i , we get the validity of (6), with $k_2 = 4$. \square

3. The main result

The main result of this paper is the following theorem:

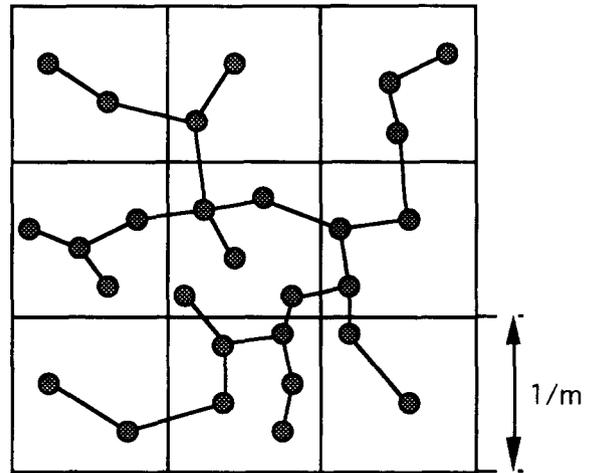
Theorem 1. *Let N_n be the number of points of a Poisson point process π_n of intensity n times the Lebesgue measure over $[0,1]^2$, and let $L_{MST}(N_n)$ be the length of the optimal spanning tree connecting these N_n points. Then*

$$|EL_{MST}(N_n)/\sqrt{n} - \beta_{MST}| = \Theta(1/\sqrt{n}). \quad (8)$$

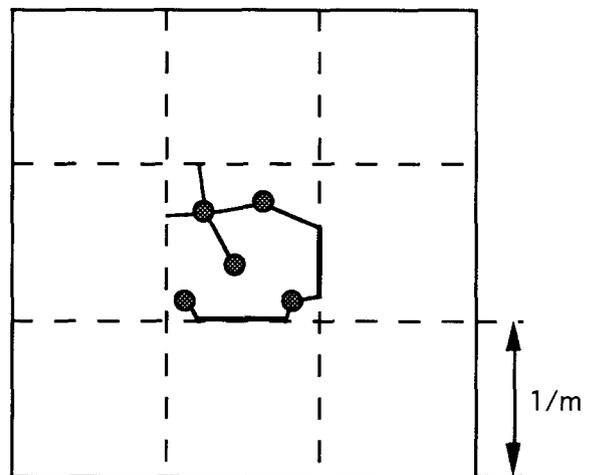
Proof. From Lemma 1, it suffices to prove that there exists a positive constant c such that

$$EL_{MST}(N_n)/\sqrt{n} \geq \beta_{MST} + c/\sqrt{n}. \quad (9)$$

Let us first replace the traditional *partitioning and patching* way of getting the subadditivity inequality (as in Figure 1) by a *recursive* way. We divide $[0,1]^2$ into four squares with edges parallel to the axes and of side length $\frac{1}{2}$ and we solve the MSTP in each of them. Then we select in each (not empty) quadrant the point closest to the



(i) Find a MST tree spanning all points



(ii) Connect the components in a subsquare, using part of the perimeter

Fig. 2. Tree construction on $x^{(n)} \cap Q_i$ in $[0,1]^2$

center of $[0,1]^2$, and finally we construct a tree connecting these points (see Figure 3, parts (i) and (ii)). Starting with the Poisson point process π_{4n} in $[0,1]^2$, we obtain

$$EL_{MST}(N_{4n}) \leq 2EL_{MST}(N_n) + k/\sqrt{n}, \quad (10)$$

where k is a positive constant. By using this inequality recursively we get

$$EL_{MST}(N_{4^m n}) \leq 2^m EL_{MST}(N_n) + 2^{m-1} \left(1 - \left(\frac{1}{4}\right)^m\right) 4k/3\sqrt{n}. \quad (11)$$

Dividing each side by $\sqrt{4^m n}$, and letting m go to infinity, we get

$$\frac{EL_{MST}(N_n)}{\sqrt{n}} \geq \beta_{MST} - \frac{2k}{3n}, \quad (12)$$

which is already an improvement on (2). Can we do better? Yes. The main idea is to improve the feasible solution, obtained from the connection of the 4 trees, by considering potential savings along the borderline of two given adjacent subsquares. Figure 3 part (iii) illustrates such savings. Note that there will be savings each time there exists a point in one of the subsquares which has its closest point (among π_{4n}) that is located in another subsquare. In order to evaluate the size and likelihood of these savings, let us refer to Figure 4, where we show two concentrated balls centered on the borderline of two subsquares, of radius r and $4r$, respectively. Now consider the following event \mathcal{H} :

There is exactly one point in region A , no point in region B , and at least one point in region C .

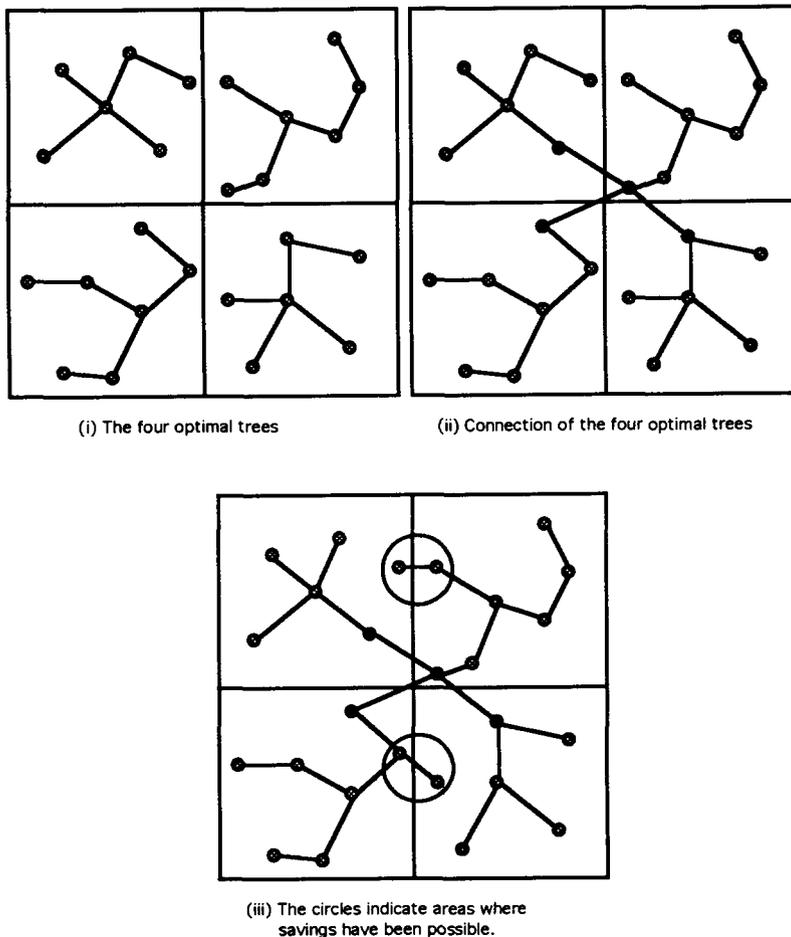
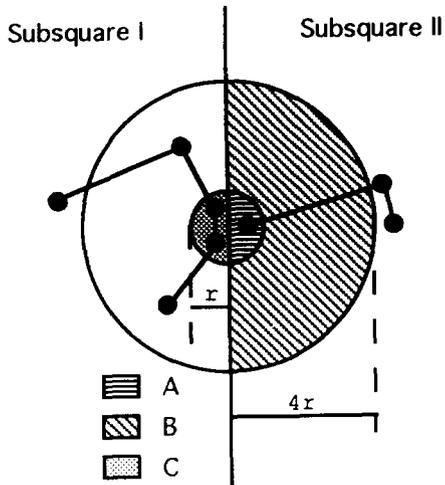
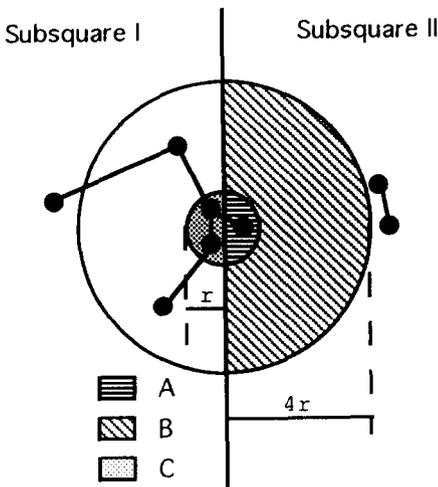


Fig. 3. Construction of a feasible solution and 'post-savings'



(i) Two concentric squares centered on the border of two subsquares



(ii) Exchange of two edges with savings of at least r

Fig. 4. Conditions for savings

If such an event is true, then, by connecting the point of A to one of the points of C , one gets savings of at least $3r - 2r = r$ (see Figure 4 (ii)). What is the probability of this event when one considers a Poisson point process π_{4n} in $[0,1]^2$? For $r = \alpha/\sqrt{n}$, where α is any positive constant, one can always find n large enough, so that this probability is greater than or equal to a positive constant, say a (indeed, if $r = \alpha/\sqrt{n}$, the number of points in region A , B , and C are all given by

independent Poisson random variables with constant parameters). Along the side of two adjacent squares (of length $\frac{1}{2}$), one can pack at least $\lfloor \sqrt{n}/(8\alpha) \rfloor$ non-overlapping, and thus independent, such combination of two concentric balls. The expected total savings will then be bounded from below by

$$a(\alpha/\sqrt{n})(\lfloor \sqrt{n}/(8\alpha) \rfloor) \geq c_1, \tag{13}$$

where c_1 is a positive constant. Instead of (10), we now have

$$EL_{MST}(N_{4n}) \leq 2EL_{MST}(N_n) + k/\sqrt{n} - c_1. \tag{14}$$

This in turn implies that, for any positive constant $c < c_1$, there exists $n(c)$ large enough such that for all $n \geq n(c)$,

$$EL_{MST}(N_{4n}) \leq 2EL_{MST}(N_n) - c. \tag{15}$$

By using this inequality recursively we get

$$EL_{MST}(N_{4^m n}) \leq 2^m EL_{MST}(N_n) - c(2^m - 1). \tag{16}$$

Dividing each side by $\sqrt{4^m n}$, and letting m go to infinity, we finally get the desired result. \square

4. Concluding remarks

The techniques and results developed in this paper remain valid for other functionals of geometric probability. For the Steiner tree problem and the minimum weight matching problem, the arguments are in fact almost identical. However, for the traveling salesman problem, difficulties may arise. Indeed, going back to Figure 4 (i), and assuming that event \mathcal{H} is true, then, if one decides to visit the unique point of region A by a double link from one of the points of region C , savings will not necessarily occur with certainty (the point in A could indeed be very close to the line joining its predecessor and successor along the previous TSP tour). In a first version of this paper, we conjectured that these savings would occur with a constant probability, still implying a rate of $\Theta(1/\sqrt{n})$ for the TSP. In fact, Rhee [6] has recently and independently proved this result. The basic idea of her proof is identical to ours, although it involves solving a number of significant technical problems in order to insure that

such savings indeed occur with a constant probability.

Finally, for all these problems, it is natural to expect that $|\mathbf{E}L_p(n)/\sqrt{n} - \beta_p| = \mathcal{O}(1/\sqrt{n})$ remains true under the uniform fixed sample sized model. However, as pointed out in [6], this does not seem to be an easy consequence of the corresponding result stated under the Poisson model. The usual way to link the two models (see [4] for details) is to prove that $|\mathbf{E}L_p(k+1) - \mathbf{E}L_p(k)| = \mathcal{O}(1/\sqrt{k})$, which then implies that $|\mathbf{E}L_p(N_n) - \mathbf{E}L_p(n)| = \mathcal{O}(1)$. This last relationship is however not sufficient here and a deeper understanding of how $|\mathbf{E}L_p(k+m) - \mathbf{E}L_p(k)|$ behaves as a function of m and k seems to be necessary for closing the gap.

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