LINEAR SUBDIVISION IS STRICTLY A POLYNOMIAL PHENOMENON

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Linear Subdivision is Strictly a Polynomial Phenomenon

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Abstract

In this paper we give an elementary proof that polynomial curves are the only differentiable curves which permit subdivision by standard linear techniques. Subdivision methods for rational polynomial curves are also discussed.

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1. Introduction

Subdivision has many applications in computer aided geometric design. A single subdivision operation will trim a curve or surface. Successive subdivisions can be used to generate linear approximations which can then be applied to provide efficient plotting and display algorithms [3]. When combined with the convex hull property, subdivision yields accurate, robust, intersection algorithms [3].

Because of the importance of subdivision in computer aided geometric design, we would like to know precisely which curves and surfaces permit subdivision. In this paper we will restrict our attention to the simplest, most common, type of subdivision, namely linear subdivision, and we shall show that linear subdivision is strictly a polynomial phenomenon.

2. Polynomial Curves Admit Linear Subdivision

Given a collection of control points $P = (P_0,\ldots,P_N)$ and a collection of continuous, linearly independent, blending functions $B(t) = (B_0(t),\ldots,B_N(t))$, we can define a continuous, non-degenerate, parametric curve $B[P](t)$ by setting

$$B[P](t) = \sum_k B_k(t)P_k \qquad 0 \leq t \leq 1$$

In order for this curve to be coordinate-free, the blending functions must satisfy the additional condition

$$\sum_k B_k(t) = 1 \qquad 0 \leq t \leq 1$$

From here on we shall assume without further comment that this condition is always satisfied.

Now such a curve is said to permit linear subdivision if and only if for each parameter pair (u_0,u_1) there exist control points $P(u_0,u_1) = (P_0(u_0,u_1),\ldots,P_N(u_0,u_1))$ such that

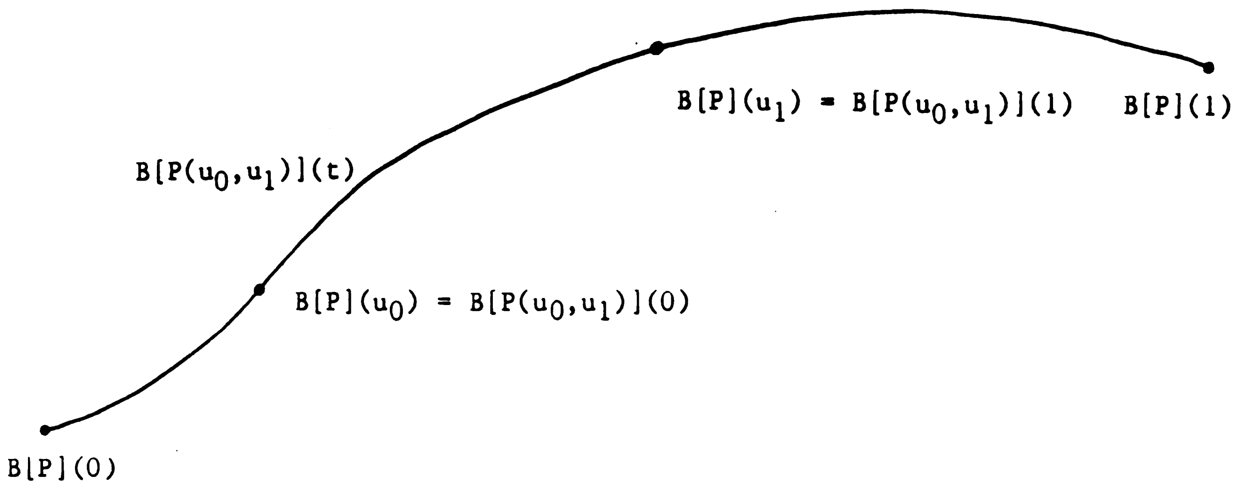
$$(*)\ B[P(u_0,u_1)](t) = B[P][(1-t)u_0 + tu_1]$$

In this case

$$B[P(u_0,u_1)](0) = B[P](u_0)$$

$$B[P(u_0,u_1)](1) = B[P](u_1)$$

Hence $B[P(u_0,u_1)](t)$ is equivalent to the segment of the curve $B[P](t)$ lying between $B[P](u_0)$ and $B[P](u_1)$. Therefore we say that the control points $P(u_0,u_1)$ subdivide the curve $B[P](t)$ from u_0 to u_1 (see diagram).



The curves $B[P](t)$ and $B[P(u_0,u_1)](t)$

The linearity in the definition refers to the linear function $(1-t)u_0 + tu_1$ inside the brackets on the right hand side of (*). Non-linear subdivision techniques will be discussed briefly only in the final section of this paper.

We will now show that if the blending functions $B(t)$ form a polynomial basis, then the curves $B[P](t)$ always admit linear subdivision. We begin with a proposition which characterizes the curves which admit linear subdivision in terms of conditions on their blending functions.

Let $\mathrm{Span}[B(t)]$ denote the space of all functions which can be written as linear combinations of the functions $B_0(t),\ldots,B_N(t)$. That is

$$\mathrm{Span}[B(t)] = \{f(t) \mid f(t) = \sum c_k B_k(t)\}$$

With the standard definitions of addition and scalar multiplication for functions, $\mathrm{Span}[B(t)]$ is a finite dimensional vector space over the real numbers. Moreover, we have the following important result.

Proposition 2.1: The curves $B[P](t)$ admit linear subdivision if and only if

$$B_j[(1-t)u_0 + tu_1] \in \mathrm{Span}[B(t)]$$

for all j,u_0,u_1. Moreover

$$B_j[(1-t)u_0+tu_1] = \sum_k c_{jk}B_k(t) \implies P_k(u_0,u_1) = \sum_j c_{jk}P_j$$

Proof: Suppose that for all j,u_0,u_1

$$B_j[(1-t)u_0 + tu_1] \in \mathrm{Span}[B(t)]$$

Then there exist constants $c_{jk} = c_{jk}(u_0,u_1)$ such that

$$B_j[(1-t)u_0 + tu_1] = \sum_k c_{jk}B_k(t)$$

Let

$$P_k(u_0,u_1) = \sum_j c_{jk}P_j \qquad k = 0,1,\ldots,N$$

$$P(u_0,u_1) = (P_0(u_0,u_1),\ldots,P_N(u_0,u_1))$$

Then

$$B[P(u_0,u_1)](t) = \sum_k B_k(t)P_k(u_0,u_1)$$

$$= \sum_k [\sum_j c_{jk}P_j]B_k(t)$$

$$= \sum_j [\sum_k c_{jk}B_k(t)]P_j$$

$$= \sum_j B_j[(1-t)u_0 + tu_1]P_j$$

$$= B[P][(1-t)u_0+tu_1]$$

Hence the control points $P(u_0,u_1)$ subdivide the curve $B[P](t)$ from u_0 to u_1. Thus the curves $B[P](t)$ admit linear subdivision.

Conversely suppose that the curves $B[P](t)$ admit linear subdivision. Then for each parameter pair (u_0,u_1) there exist control points $P(u_0,u_1) = (P_0(u_0,u_1),\ldots,P_N(u_0,u_1))$ such that

$$B[P(u_0,u_1)](t) = B[P][(1-t)u_0 + tu_1]$$

Let v be a unit vector, and select the control points P so that

$$P_k = P_0 \qquad k \neq j$$

$$= P_0 + v \qquad k = j$$

Then since $\sum B_k(t) = 1$, it follows that

$$P_0 + B_j[(1-t)u_0 + tu_1]v = \sum B_k[(1-t)u_0 + tu_1]P_k$$

$$= B[P][(1-t)u_0 + tu_1]$$

$$= B[P(u_0,u_1)](t)$$

$$= \sum B_k(t)P_k(u_0,u_1)$$

Therefore subtracting P_0 from both sides and then dotting with v, we obtain

$$B_j[(1-t)u_0+tu_1] = \sum [(P_k(u_0,u_1)-P_0).v]B_k(t) \in \mathrm{Span}[B(t)]$$

QED

Corollary 2.2: If the blending functions B(t) form a basis for all polynomials of degree N in t, then the curves B[P](t) admit linear subdivision.

Proof: If the blending functions B(t) are polynomials of degree $\leq$ N in t, then for each parameter pair (u_0,u_1) the functions $B[(1-t)u_0+tu_1]$ are clearly also polynomials of degree $\leq$ N in t. Hence if the blending functions B(t) form a basis for all polynomials of degree N in t, then certainly

$$B_j[(1-t)u_0 + tu_1] \in \mathrm{Span}[B(t)]$$

Therefore by proposition 2.1 the curves B[P](t) admit linear subdivision.

QED

The most important polynomial curves in computer aided geometric design are the Bezier curves. The blending functions for these curves are the Bernstein polynomials

$$B_j^N(t) = \binom{N}{j} t^j (1-t)^{N-j} \qquad j = 0,1,\ldots,N$$

For Bezier curves the following explicit subdivision formulas are known [2]:

$$B_j^N[(1-t)u_0+tu_1] = \sum_k \Big[\sum_{h+i=j} B_h^{N-k}(u_0)B_i^k(u_1)\Big]B_k^N(t)$$

$$P_k(u_0,u_1) = \sum_j \Big[\sum_{h+i=j} B_h^{N-k}(u_0)B_i^k(u_1)\Big]P_j$$

3. Linear Subdivision is Strictly a Polynomial Pheonomenon

Consider again a collection of control points $P = (P_0,\ldots,P_N)$ and a collection of continuous, linearly independent, blending functions $B(t) = (B_0(t),\ldots,B_N(t))$. By proposition 2.1 the curves B[P](t) admit linear subdivision if and only if

$$B_j[(1-t)u_0 + tu_1] \in \mathrm{Span}[B(t)]$$

for all j,u_0,u_1. We shall now show that if the blending functions are differentiable, then this condition implies that the blending functions are polynomials.

In the following table we summarize our assumptions on the blending functions B(t) together with the immediate consequences these conditions have for the space of functions Span[B(t)].

Assumptions on B(t)		Consequences for Span[B(t)]
A1. $B_0(t),\ldots,B_N(t)$ are linearly independent	==>	C1. Dim(Span[B(t)]) = N+1
A2. $B_j(t) \in C^1[0,1]$	==>	C2. Span[B(t)] $\subseteq C^1[0,1]$
A3. $B_j[(1-t)u_0+tu_1] \in$ Span[B(t)]	==>	C3. Span[$B((1-t)u_0+tu_1)$] $\subseteq$ Span[B(t)]

Now these conditions on the function space Span[B(t)] in turn imply that the functions f(t) in Span[B(t)] have the following properties:

P1. $f_0(t),\ldots,f_{N+1}(t) \in$ Span[B(t)] ==> there exist constants $c_0,\ldots,c_{N+1}$, not all zero, such that $\sum c_k f_k(t) = 0$.

P2. $f(t) \in$ Span[B(t)] ==> $f(t) \in C^1[0,1]$

P3. $f(t) \in$ Span[B(t)] ==> $f[(1-t)u_0+tu_1] \in$ Span[B(t)]

To show that the blending functions are indeed polynomials, we shall use these 3 properties to demonstrate that

$f(t) \in$ Span[B(t)] ==> f(t) is a polynomial of degree $\leq$ N

To get started, we need a somewhat technical result. Let $\{f_n(t)\}$ be a sequence of functions. We shall write $f_n(t)$ --> f(t) if and only if $f_n(t)$ approaches f(t) pointwise. That is,

$$f_n(t) \dashrightarrow f(t) \iff \lim_{n\to\infty} f_n(t_0) = f(t_0) \text{ for all } t_0$$

Lemma 3.1: $f_n(t) \in$ Span[B(t)] and $f_n(t)$ --> f(t) ==> $f(t) \in$ Span[B(t)]

Proof: Since a rigorous proof of this result is a bit technical, we defer the proof to the Appendix (see proposition A.2).

It follows immediately from the definition of Span[B(t)] and lemma 3.1 that the set Span[B(t)] is closed under the following operations:

1. addition

2. subtraction

3. scalar multiplication

4. pointwise limits

We shall use these closure properties to prove many of our subsequent results.

Proposition 3.2: $f(t) \in$ Span[B(t)] ==> $(1-t)f'(t) \in$ Span[B(t)]

Proof: Let $f(t) \in$ Span[B(t)] and define

$$g_n(t) = \frac{f[(1-t)/n + t] - f(t)}{1/n}$$

Then by P3, $g_n(t) \in$ Span[B(t)], and by P2 and L'Hopital's rule

$$\lim_{n \to \infty} g_n(t) = \lim_{n \to \infty} \frac{f[(1-t)/n + t] - f(t)}{1/n}$$

$$= \lim_{h \to 0} \frac{f[(1-t)h + t] - f(t)}{h}$$

$$= (1-t)f'(t)$$

Hence $g_n(t) \to (1-t)f'(t)$. Therefore by lemma 3.1 $(1-t)f'(t) \in \mathrm{Span}[B(t)]$

QED

Proposition 3.3: $f(t) \in \mathrm{Span}[B(t)] \implies tf'(t) \in \mathrm{Span}[B(t)]$

Proof: We proceed as in proposition 3.2. Let $f(t) \in \mathrm{Span}[B(t)]$ and define

$$g_n(t) = \frac{f[(1-1/n)t] - f(t)}{-1/n}$$

Again by P3, $g_n(t) \in \mathrm{Span}[B(t)]$, and by P2 and L'Hopital's rule

$$\lim_{n \to \infty} g_n(t) = \lim_{n \to \infty} \frac{f[(1-1/n)t] - f(t)}{-1/n}$$

$$= \lim_{h \to 0} \frac{f[(1-h)t] - f(t)}{-h}$$

$$= tf'(t)$$

Hence $g_n(t) \to tf'(t)$. Therefore by lemma 3.1 $tf'(t) \in \mathrm{Span}[B(t)]$

QED

Corollary 3.4: $f(t) \in \mathrm{Span}[B(t)] \implies f'(t) \in \mathrm{Span}[B(t)]$

Proof: This result follows immediately from propositions 3.2,3.3 since

$$f'(t) = (1-t)f'(t) + tf'(t)$$

Corollary 3.5: $f(t) \in \mathrm{Span}[B(t)] \implies f^{(k)}(t) \in \mathrm{Span}[B(t)]$

Proof: This result follows immediately from corollary 3.4 by induction on k.

Notice that corollary 3.5 implies that

$$f(t) \in \mathrm{Span}[B(t)] \implies f(t) \in C^{\infty}[0,1]$$

In particular, the blending functions themselves must be infinitely differentiable. Moreover we can extend these results even further.

Proposition 3.6: $f(t) \in Span[B(t)] \implies t^j f^{(k)}(t) \in Span[B(t)] \quad 0 \le j \le k$

Proof: By induction on k. From proposition 3.3 and corollary 3.4 this result is clearly true for k=1. Now suppose it is true for k=n; we shall show it is true for k=n+1. By corollary 3.4 and the inductive hypothesis

$$t^j f^{(n+1)}(t) = t^j (f')^{(n)}(t) \in Span[B(t)] \quad 0 \le j \le n$$

Hence we need only show that $t^{n+1} f^{(n+1)}(t) \in Span[B(t)]$. But by the inductive hypothesis

$$t^n f^{(n)}(t) \in Span[B(t)]$$

Hence by proposition 3.3

$$t^{n+1} f^{(n+1)}(t) = t[t^n f^{(n)}(t)]' - nt^n f^{(n)}(t) \in Span[B(t)]$$

QED

Proposition 3.7: $f(t) \in Span[B(t)] \implies f^{(N+1)}(t) = 0$

Proof: By proposition 3.6

$$t^j f^{(N+1)}(t) \in Span[B(t)] \qquad j = 0,1,\ldots,N+1$$

Hence by P1 there exist constants $c_0,\ldots,c_{N+1}$, not all zero, such that

$$\sum c_j t^j f^{(N+1)}(t) = 0$$

Thus for all t

$$[\sum c_j t^j] f^{(N+1)}(t) = 0$$

But $\sum c_j t^j$ is a non-zero polynomial of degree $\le$ N+1; hence $\sum c_j t^j$ has at most N+1 roots. Therefore $f^{(N+1)}(t) = 0$ except possibly at N+1 isolated points. Hence by continuity $f^{(N+1)}(t) = 0$ for all t.

QED

Corollary 3.8: $f(t) \in Span[B(t)] \implies f(t)$ is a polynomial of degree $\le$ N.

Proof: This result is an immediate consequence of proposition 3.7.

Corollary 3.9: The blending functions $B(t) = (B_0(t),\ldots,B_N(t))$ are polynomials of degree $\leq N$, and they form a basis for all polynomials of degree $\leq N$.

Proof: The blending functions must be polynomials of degree $\leq N$ by corollary 3.8, and they must form a basis for all polynomials of degree $\leq N$ since there are N+1 of them and by assumption A1 they are linearly independent.

Corollary 3.10: $f(t) \in \text{Span}[B(t)] \iff f(t)$ is a polynomial of degree $\leq N$.

Proposition 3.11: The curves $B[P](t)$ admit linear subdivision if and only if the blending functions $B(t)$ form a polynomial basis. Thus linear subdivision is strictly a polynomial phenomenon.

Proof: This result follows immediately from corollaries 2.2, 3.9.

4. Linear Subdivision and Rational Polynomial Curves

We can generalize the notion of linear subdivision to rational curves in the following manner. Given

$P = (P_0,\ldots,P_N)$ = control points

$w = (w_0,\ldots,w_N)$ = scalar weights

$B(t) = (B_0(t),\ldots,B_N(t))$ = continuous, linearly independent, blending functions

we define a rational parametric curve $B[P,w](t)$ by setting

$$B[P,w](t) = \frac{\sum B_k(t) w_k P_k}{\sum w_k B_k(t)} \qquad 0 \leq t \leq 1$$

We say that a rational parametric curve $B[P,w](t)$ admits linear subdivision if and only if for each parameter pair (u_0,u_1) there exist control points $P(u_0,u_1)$ and weights $w(u_0,u_1)$

$$P(u_0,u_1) = (P_0(u_0,u_1),\ldots,P_N(u_0,u_1))$$

$$w(u_0,u_1) = (w_0(u_0,u_1),\ldots,w_N(u_0,u_1))$$

such that

$$B[P(u_0,u_1),w(u_0,u_1)](t) = B[P,w][(1-t)u_0+tu_1]$$

Proposition 4.1: The rational curves $B[P,w](t)$ admit linear subdivision if

$$B_j[(1-t)u_0+tu_1] \in \text{Span}[B(t)]$$

for all j,u_0,u_1. Moreover if

$$B_j[(1-t)u_0+tu_1] = \sum_k c_{jk}B_k(t)$$

then

$$w_k(u_0,u_1) = \sum_j c_{jk}w_j$$

$$P_k(u_0,u_1) = \frac{\sum_j c_{jk}w_jP_j}{w_k(u_0,u_1)}$$

Proof: Same as proposition 2.1.

Notice that we do not claim that the converse of proposition 4.1 is valid. Nevertheless we still have the following result.

Corollary 4.2: If the blending functions $B(t)$ form a basis for all polynomials of degree N in t, then the rational curves $B[P,w](t)$ admit linear subdivision.

Proof: Same as corollary 2.2.

Again we do not claim that the converse of corollary 4.2 is valid. However we can prove the following partial converse.

Proposition 4.3: Suppose that

1. $B_j(t) \in C^1[0,1]$
2. $B[P,w](t)$ admits linear subdivision
3. $w = (1,\ldots,1) \Longrightarrow w(u_0,u_1) = (1,\ldots,1)$

Then the blending functions $B(t)$ form a basis for all polynomials of degree N in t.

Proof: From 2,3 it follows that the integral curves $B[P](t)$ admit linear subdivision. Hence this result follows immediately from proposition 3.11.

If we insist that the blending functions are differentiable and that integral curves are used to subdivide integral curves, then by the preceding proposition the blending functions must form a polynomial basis. Thus under these assumptions the converse of corollary 4.2 is valid. However if we allow rational curves to subdivide integral curves -- that is, if we can alter unit weights -- we do not know whether the converse of corollary 4.2 is valid. Whether differentiable, rational, non-polynomial curves could admit linear subdivision is still an open question.

We close this section by showing that, in general, it is not possible to subdivide a rational polynomial curve without altering the weights.

Proposition 4.4: Suppose that $w_i \neq w_j$ for some i,j. Then, in general, it is not possible to subdivide the rational polynomial curves $B[P,w](t)$ without changing the weights.

Proof: Define rational polynomial blending functions b(t) by setting

$$b_k(t) = \frac{w_k B_k(t)}{\sum w_k B_k(t)}$$

$$b(t) = (b_0(t),\ldots,b_N(t))$$

Then by construction

$$B[P,w](t) = \frac{\sum B_k(t) w_k P_k}{\sum w_k B_k(t)} = b[P](t)$$

Now if we could subdivide the rational polynomial curves $B[P,w](t)$ without altering the weights, then clearly the curves $b[P](t)$ would admit linear subdivision. But, in general, the functions b(t) are not polynomials so this subdivision property of $b[P](t)$ would violate proposition 3.11. Thus, in general, it is not possible to subdivide the rational polynomial curves $B[P,w](t)$ without changing the weights.

QED

5. Conclusions and Questions

The main result of this paper is that linear subdivision is strictly a polynomial phenomenon. Thus if we insist on the property of linear subdivision, we are restricted to the set of polynomial curves. However, many simple curves such as the circle are not polynomial curves. Thus if we want to include these curves, we must generalize our notion of subdivision. Fortunately the notion of linear subdivision can be extended quite naturally to rational polynomial curves. In this extended sense most simple curves do admit linear subdivision.

Do any other curves admit linear subdivision? Are there any differentiable, rational, non-polynomial curves which admit linear subdivision?

What about non-linear subdivision techniques? Let $H(u_0,u_1,t)$ be a continuous (differentiable) function such that:

a. $H(u_0,u_1,0) = u_0$

b. $H(u_0,u_1,1) = u_1$

c. $H(0,1,t) = t$

d. $H(u_0,u_1,t)$ is monotonic in t

We say that a curve $B[P](t)$ admits subdivision relative to the function $H(u_0,u_1,t)$ if and only if for each parameter pair (u_0,u_1) there exist control points $P(u_0,u_1)$ such that

$$B[P(u_0,u_1)](t) = B[P][H(u_0,u_1,t)]$$

Are there any differentiable curves other than polynomials which admit subdivision in this broader non-linear sense?

Finally, we have not touched at all on subdivision of spline curves via knot insertion. These subdivision techniques are discussed in detail in [1],[3]. The interested reader may wish to consult these papers for this alternate approach to subdivision for polynomial spline curves.

Appendix: Pointwise Convergence in Span[B(t)]

Let $B(t) = (B_0(t),\ldots,B_N(t))$ be a collection of linearly independent functions. In this Appendix we shall prove lemma 3.1, namely that Span[B(t)] is closed under pointwise limits. That is, we shall show that

$$f_n(t) \in \text{Span}[B(t)] \quad \text{and} \quad f_n(t) \dashrightarrow f(t) \implies f(t) \in \text{Span}[B(t)]$$

Notice that we require no additional assumptions about the functions B(t). In fact, they need not even be continuous for this result to be valid.

We shall adopt the following notation. Let $c = (c_0,\ldots,c_N)$, then

$$|c| = \sqrt{c_0^2 + \ldots + c_N^2}$$

Thus $|c|$ is just the standard norm on R^{N+1}.

Proposition A.1: Suppose that

$$f_n(t) = \sum c_{nk}B_k(t)$$

$$f_n(t) \dashrightarrow f(t)$$

and let $c_n = (c_{n0},\ldots,c_{nN})$. Then $\{c_n\}$ is bounded in R^{N+1}.

Proof: Suppose not. Then there is a subsequence $\{c_m\}$ such that $|c_m| > m$. Let

$$d_m = \frac{c_m}{|c_m|}$$

Then $|d_m| = 1$. Now since $\{d_m\}$ is a bounded sequence in R^{N+1} it has a convergent subsequence $\{d_p\}$. Let

$$d = \text{Lim } d_p$$

then

$$|d| = |\text{Lim } d_p| = \text{Lim } |d_p| = 1$$

so certainly

$$d \neq 0$$

Now consider the sequence of functions $\{f_p(t)\}$. By assumption

$$f_p(t) = \sum c_{pk}B_k(t)$$

$$f_p(t) \dashrightarrow f(t)$$

Dividing $f_p(t)$ by $|c_p|$ and recalling that $\{|c_p|\}$ is unbounded, we obtain

$$\frac{f_p(t)}{|c_p|} = \sum \frac{c_{pk}}{|c_p|} B_k(t) = \sum d_{pk} B_k(t)$$

$$\frac{f_p(t)}{|c_p|} \dashrightarrow \frac{f(t)}{|c_p|} \dashrightarrow 0$$

Therefore since $d = \text{Lim } d_p$

$$\sum d_{pk} B_k(t) \dashrightarrow \sum d_k B_k(t)$$

$$\sum d_{pk} B_k(t) \dashrightarrow 0$$

Hence

$$\sum d_k B_k(t) = 0$$

But $d \neq 0$. Therefore the functions $B_0(t),\ldots,B_N(t)$ are linearly dependent, contrary to assumption. Thus $\{c_n\}$ must be bounded in R^{N+1}.

QED

Proposition A.2: $f_n(t) \in \text{Span}[B(t)]$ and $f_n(t) \dashrightarrow f(t) \Longrightarrow f(t) \in \text{Span}[B(t)]$

Proof: Let $f_n(t) = \sum c_{nk} B_k(t)$. Then by proposition A.1 $\{c_n\}$ is a bounded sequence in R^{N+1}. Therefore it has a convergent subsequence $\{c_p\}$. Let $c = \text{Lim } c_p$, and consider the sequence of functions $\{f_p(t)\}$. By assumption

$$f_p(t) = \sum c_{pk} B_k(t)$$

$$f_p(t) \dashrightarrow f(t)$$

Moreover since $c = \text{Lim } c_p$ it follows that

$$\sum c_{pk} B_k(t) \dashrightarrow \sum c_k B_k(t)$$

$$\sum c_{pk} B_k(t) \dashrightarrow f(t)$$

Hence

$$f(t) = \sum c_k B_k(t) \in \text{Span}[B(t)]$$

QED

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