Pattern Recognition Letters 6 (1987) 251–259
North-Holland

A general approach to parameter evaluation in fuzzy digital pictures

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Received 26 January 1987

Abstract: A general approach to the evaluation of parameters from fuzzy regions is outlined. The main idea is to consider a fuzzy subset of an image as the nested family of its level-cuts, and interpret this family as a body of evidence in the sense of Shafer. Any intrinsic parameter can then be calculated as a mathematical expectation based on a probability density function. Fuzzy-valued parameters can also be derived. The approach encompasses recent proposals by Rosenfeld for specific parameters such as perimeter, diameter, etc., as well as the cardinality of a fuzzy set. It is also extended to relational parameters between fuzzy regions in the image.

Key words: parameter extraction, fuzzy sets, theory of evidence.

1. Introduction and motivation

Classical segmentation methods (Pavlidis, 1982) aim at sharing an image into precisely bounded regions. As a consequence all information regarding the imprecision or noise pervading the boundaries of the corresponding objects is lost. The apparent precision of a segmented contour is arbitrary in the sense that changing the segmentation technique results in producing a new contour which, although hopefully close to the first one, may be distinct from it. This type of problem especially occurs in unsupervised environments. In order to deal with such types of errors in segmentation, the use of fuzzy set theory (Zadeh, 1965; Dubois and Prade, 1980) has been proposed by several authors (Jain, 1983; Nakagawa and Rosenfeld, 1978; Pal and King, 1983; Huntsberger et al., 1985; Goetcherian, 1980). In such approaches, regions are viewed as fuzzy subsets of the image, obtained by assigning to pixels membership grades belonging to [0, 1].

The construction of fuzzy regions requires a working definition of the membership function, in order to be able to compute membership grades from actual data. Most authors view the membership grades as reflecting the gray levels. Such a definition, although being natural, only accounts for a small part of the available information which enables a proper discrimination among regions. Recently, Huntsberger et al. (1985) applied Bezdek (1981)'s fuzzy c-means algorithm to perform a segmentation by integrating various features of the image. In their approach the membership grades are the results of a clustering procedure. Lastly, Dubois and Jaulent (1986) propose another interpretation of membership grades in a fuzzy region, obtained by pooling the contours derived from the parallel application of several classical segmentation methods. The idea is to retrieve the information about the contour imprecision, by comparing and merging several crisp representations of a region.

If the imprecision pervading the segmentation process can be captured under the form of fuzzy regions, this imprecision must be carried over to the parameters which describe the various features

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of the region. This question of imprecise parameter extraction is considered in a research project about man-machine communication with a scene analyser. The idea is to be able to automatically retrieve objects in a 2-D scene, from a verbal imprecise description of the objects (Farreny and Prade, 1984; Jaulent, 1986). This problem is not usual in classical pattern recognition where a learning stage provides a precise description of objects to be retrieved, and discriminating features can simplify the recognition step. In the case of man-machine communication, the only available information is the verbal description, where parameter values (e.g. diameter; width...) are imprecisely specified, or even omitted ('find the rectangle on the left of the screen'). The recognition step boils down to a pattern matching process between the description of objects provided by the vision system and the query provided by the human operator. There is a qualitative difference between these two items of information: The former is arbitrarily precise and the latter is likely to be imprecise. As a consequence the grades of compatibility between objects and the query do not convey as much information as they could. Especially objects can be rejected as being not compatible while they would still be possible candidates if imprecision were taken into account. So far, the implemented scene analyser assumes precise contour of objects are available (see Dubois and Jaulent (1985), Jaulent (1985) for the shape analysis procedure). The integration of imprecision at the segmentation stage would thus be an improvement.

The question of parameter evaluation from fuzzy regions has been addressed in several recent papers by Rosenfeld (1984a,b, 1986), Rosenfeld and Haber (1985), each devoted to a particular parameter, e.g. diameter, distance and perimeter. In this paper, we propose a general approach, where Rosenfeld's results appear as a particular case. The results presented here rely on Shafer (1976)'s theory of belief functions, and its links with fuzzy set and possibility theory (Zadeh, 1978; Dubois and Prade, 1985). Such links make it possible to come up with a statistical interpretation of membership functions (Dubois and Prade, 1986a), which proves useful for application to the synthesis of fuzzy regions (Dubois and Jaulent, 1986) as well as their analysis through parameter evaluation.

2. The representation of fuzzy regions

There are three main representations of a fuzzy set F defined on a referential set Ω, supposedly finite:

– the membership function $\mu_F : \Omega \to [0,1]$ which assigns to each $w \in \Omega$ its membership grade $\mu_F(w)$. $S(F) = \{w \mid \mu_F(w) > 0\}$ is called the support of F, $I(F) = \{w \mid \mu_F(w) = 1\}$ the core of F.

– the set of α-cuts $C(F) = \{F(\alpha) \mid \alpha \in [0,1]\}$ where $F(\alpha) = \{w \mid \mu_F(w) \geq \alpha\}$. Note that $C(F)$ contains $I(F)$, which is empty as soon as the fuzzy set is subnormalized, i.e. $\mu_F(w) < 1 \ \forall w$.

– a convex combination of sets, i.e. a pair $(\mathcal{F}, m)$ where $\mathcal{F}$ is attached a positive weight $m(A)$, and

$$\sum_{A \in \mathcal{F}} m(A) = 1. \tag{1}$$

m is called a basic probability assignment, and $A \in \mathcal{F}$ a focal subset.

Membership functions and α-cuts were first introduced by Zadeh (1965). The last representation is more in the spirit of random set theory (Matheron, 1975; Goodman and Nguyen, 1985) or evidence theory (Shafer, 1976). $(\mathcal{F}, m)$ can be called a random set, and $m(A)$ is the probability that A is the 'true' representative of $(\mathcal{F}, m)$. The membership function is recovered from the set of α-cuts via the representation theorem (Zadeh, 1971).

$$\mu_F(w) = \sup\{\alpha \mid w \in F(\alpha)\}. \tag{2}$$

Note that the α-cuts are nested in the sense that

$$\alpha \leq \alpha' \quad \Rightarrow \quad F(\alpha) \supseteq F(\alpha'). \tag{3}$$

A membership function μ_F can be obtained from a convex combination of characteristic functions μ_A of sets A in $\mathcal{F}$ as:

$$\mu_F(w) = \sum_{A \subseteq \Omega} m(A)\mu_A(w) = \sum_{A \in \mathcal{F}, w \in A} m(A). \tag{4}$$

It is easy to check that two different random sets $(\mathcal{F}, m), (\mathcal{F}', m')$ may lead to the same membership function. However if we restrict ourselves to *consonant* random sets, i.e. $\mathcal{F}$ a *nested* family of sets $\{A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n\}$, then μ_F is equivalent to a single consonant random set. Namely, let $M(F) = \mu_F(\Omega) - \{0\} = \{\alpha_1 > \alpha_2 > \cdots > \alpha_n\}$ be the set of positive membership grades for F.

Proposition 1 (Dubois and Prade, 1982). *Given a membership function μ_F, the only consonant random set such that (4) holds is defined by $\mathscr{F} = C(F) = \{F(\alpha_1) \subseteq F(\alpha_2) \subseteq \cdots \subseteq F(\alpha_n)\}$ and $\forall A$,*

$$\begin{aligned} m(A) &= \alpha_i - \alpha_{i+1} \quad \text{if } A = F(\alpha_i) \\ &= 0 \qquad\qquad \text{otherwise}, \end{aligned} \tag{5}$$

with the convention $\alpha_{m+1} = 0$.

In other words, the focal sets are the α-cuts of F and if $\{\alpha_1 > \alpha_2 > \cdots > \alpha_n\}$ is the set of positive membership grades defining F then m is built from the difference of successive α_i's. Especially, if $w \in F(\alpha_i)$ and w is not in $F(\alpha_{i-1})$, then (4) reads

$$\mu_F(w) = \sum_{j=i,n} m(F(\alpha_j)). \tag{6}$$

Note that when μ_F is subnormalized ($I(F) = \emptyset$) then $\emptyset \in \mathscr{F}$.

These representations can be extended to the infinite case (Nguyen, 1984; Goodman and Nguyen, 1985) and are very useful to the parameter extraction problem. A fuzzy region is a fuzzy subset R of a digital image denoted Ω; it is supposed to be connected, that is, for all $\alpha \in [0,1]$ the α-cuts $R(\alpha)$ are connected, consistently with Rosenfeld (1979, 1983).

3. Evaluation of average intrinsic parameters

In this paragraph we are concerned with measurable properties of a single region. If R is a connected set of pixels, a property f of R is measured by a real number denoted $f(R)$: $f(R)$ can be the diameter, perimeter, surface, etc., of R. When R is a fuzzy region, with membership function μ_R, we view it as a nested uncertain region, under the form of a finite set $\{R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n\}$ of regions together with a basic probability assignment m defined from μ_R by inversion of (4), i.e.

$$m(R_i) = \alpha_i - \alpha_{i+1} \tag{7}$$

where $\alpha_1 = 1$, $\alpha_i = \mu_R(x)$ for any $x \in R_i \setminus R_{i-1}$ and $\alpha_{n+1} = 0$. In other words the core $I(R)$ is not empty in the following. R_i is short for $R(\alpha_i)$.

The ill-definition of the boundary of R translates into some uncertainty regarding the value of $f(R)$. Very naturally:

Definition 1. The property f measured on a fuzzy region R yields a random number defined by the probability allocation on the reals: $\forall r \in R$

$$\begin{aligned} p_f(r) &= \sum \{m(R_i) \mid f(R_i) = r\} \\ &= 0 \quad \text{if } r \text{ is not in } \{f(R_i) \mid i = 1, n\}. \end{aligned}$$

Remark. This definition is valid for any uncertain region in the sense of (Dubois and Jaulent, 1986), i.e. does not presupposes that the R_i's are nested.

The expected value $\underline{f}(R)$ of $f(R)$ is easily evaluated as:

$$\underline{f}(R) = \sum_{i=1,n} m(R_i) \cdot f(R_i). \tag{8}$$

This expected value has already been proposed in the literature of fuzzy sets in order to measure some features of fuzzy sets, as proved in the following examples:

Definition 2. The area of a fuzzy region R is the scalar cardinality of R, defined by

$$a(R) = \sum_{w \in \Omega} \mu_R(w). \tag{9}$$

This definition is originally due to De Luca and Termini (1972) and applied by Rosenfeld and Haber (1985) to fuzzy regions.

Proposition 2. *$a(R)$ is the expected area of R in the sense of (8), i.e. $a(R) = \underline{a}(R)$.*

Proof. Note that when R is a crisp region then $a(R) = \underline{a}(R)$ is the area of R. Now

$$\begin{aligned} \underline{a}(R) &= \sum_{i=1,n} m(R_i) a(R_i) = \sum_{i=1}^{n} (\alpha_i - \alpha_{i+1}) a(R_i) \\ &= \sum_{i=2}^{n} \alpha_i (a(R_i) - a(R_{i-1})) + a(R_1). \end{aligned}$$

$a(R_i) - a(R_{i-1}) = a(R_i \setminus R_{i-1})$ where $R_i \setminus R_{i-1} = \{w \mid \mu_R(w) = \alpha_i\}$ is the α_i-section of R. Hence $a(R_i) - a(R_{i-1})$ is the number of pixels w such that $\mu_R(w) = \alpha_i$ for $i > 1$. Hence the result. □

Definition 3 (Rosenfeld, 1984). The height of a fuzzy region along the y-axis is $h(R) = \sum_y \max_x \mu_R(x, y)$ where (x, y) denotes the coordinate of pixel w.

In the above definition Ω is viewed as a rectangular array of pixels, with a Cartesian coordinate system.

Proposition 3. *The height of R along the y-axis is equal to the expected height of R.*

Proof. The projection of R along the y-axis is defined by (Zadeh, 1975). It is $P_y(R)$ such that $\mu_{P_y(R)}(y) = \max_x \mu_R(x, y)$. Moreover the α-cuts of $P_y(R)$ are the projections of the α-cuts R_i. Hence the expected height is

$$\underline{h}(R) = \sum_{i=1,n} m(R_i) \cdot L(P_y(R_i))$$

where L denotes the length of a connected subpart of the y-axis, i.e. the number of elements in $P_y(R_i)$. Hence

$$\underline{h}(R) = \sum_y \mu_{P_y(R_i)}(y),$$

using the result on cardinality. □

The expected perimeter of a fuzzy region is $\underline{\mathrm{pe}}(R) = \sum_{i=1,n} m(R_i) \cdot \mathrm{pe}(R_i)$, i.e. Rosenfeld and Haber's (1985) definition exactly, due to (7).

Definition 1 and (8) have the advantage of being very general, in the sense that any parameter which can be extracted from a region has a natural meaning for a fuzzy, or uncertain region R. For instance the coordinates of the center of gravity, the diameter, the orientation, the compactness etc. can be defined this way. Note that (8) is not *always* equivalent to definitions based on the membership function μ_R. For instance, Rosenfeld (1984) defines the extrinsic diameter of a fuzzy region R as

$$E(R) = \max_u h_u(R) \tag{10}$$

where h_u denotes the expected height along direction u, while the expected extrinsic diameter of R is the sense of (8) would be

$$\underline{e}(R) = \sum_{i=1,n} m(R_i) \cdot E(R_i). \tag{11}$$

The following inequality is easily established:

$$\underline{e}(R) \geq E(R). \tag{12}$$

Proof. $E(R) = \max_u \sum_{i=1,n} m(R_i) \cdot h_u(R_i)$ from the definition of h_u as seen earlier, while $\underline{e}(R) = \sum_{i=1,n} m(R_i) \cdot \max_u h_u(R_i)$. The equality would hold when the diameters of the R_i's are along the same direction u^*, so that $E(R_i) = h_{u^*}(R_i)\ \forall i$. □

Similarly the intrinsic diameter of a connected region R is (Rosenfeld, 1984)

$$\mathrm{ID}(R) = \max_{w, w'} \min_{P_{w,w'}} L(P_{w,w'}) \tag{13}$$

where $P_{w,w'}$ is any rectifiable path between two pixels w and w' in R and $P_{w,w'}$ is contained in R. L denotes the length of a path (number of pixels).

When R is a fuzzy region, $L(P_{w,w'})$ in (13) is changed into the cardinality of the fuzzy path $P_{w,w'}$ i.e.

$$L(P_{w,w'}) = \sum_{w'' \in P_{w,w'}} \mu_R(w'').$$

Contrastedly the expected intrinsic diameter of R would be the weighted sum of the intrinsic diameters of the R_i, say $\mathrm{id}(R)$. Rosenfeld (1984) proved that for a crisp and convex region R, $E(R) = \mathrm{ID}(R)$ but for a convex fuzzy region only the inequality $E(R) \geq \mathrm{ID}(R)$ is valid. Using $\underline{\mathrm{id}}$ and $\underline{e}$ as definitions of intrinsic and extrinsic diameters, what is true with crisp regions is still true for fuzzy regions, namely:

Proposition 4. *For any fuzzy connected region $\underline{e}(R) \leq \underline{\mathrm{id}}(R)$; moreover, $\underline{e}(R) = \underline{\mathrm{id}}(R)$ if the fuzzy region is convex (i.e. the R_i's are convex).*

Proof. If the R_i's are connected then $E(R_i) \leq \mathrm{ID}(R_i)$. Hence $\sum m(R_i) \cdot E(R_i) \leq \sum m(R_i) \cdot \mathrm{ID}(R_i)$. In the convex case, the R_i's are convex so that the result applies because it applies to the α-cuts of R. □

N.B. This proposition indicates that, in the convex case, $\underline{\mathrm{id}}(R) = \underline{e}(R) \geq \mathrm{ID}(R)$ since Rosenfeld (1984) proves that $E(R) \geq I(R)$ and (12) holds generally.

Lastly the expected measure $\underline{f}(R)$ is monotonic with respect to fuzzy region inclusion as soon as it is monotonic with respect to usual inclusion, since $R \subseteq R'$ implies that each α-cut of R is included in the α-cut of R'.

Proposition 5. *If for crisp regions $R, R', R \subseteq R' \Rightarrow f(R) \leq f(R')$ then for fuzzy regions $\mu_R \leq \mu_{R'} \Rightarrow \underline{f}(R) \leq \underline{f}(R')$.*

Proof. Let $\{\gamma_1=1>\gamma_2>\cdots>\gamma_k\}=M(R)\cup M(R')$. Clearly, $\underline{f}(R)=\sum_{i=1,k}(\gamma_i-\gamma_{i+1})f(R(\gamma_i))$ where $R(\gamma_i)$ is the γ_i-cut of R, and $\gamma_{k+1}=0$. Indeed this summation is also $\sum_{i=1,k}\gamma_i\cdot(f(R(\gamma_i))-f(R(\gamma_{i-1})))$, and if γ_i is not in $M(R)$ then $R(\gamma_i)=R(\gamma_{i-1})$ and γ_i vanishes from the summation. Similarly $\underline{f}(R')=\sum_{i=1,k}(\gamma_i-\gamma_{i+1})f(R'(\gamma_i))$. Now because $R\subseteq R'$, $R(\gamma_i)\subseteq R'(\gamma_i)$ $\forall i=1,m$. Hence the result holds. □

4. Fuzzy evaluation of intrinsic parameters

In the preceding lines we have been interested in scalar evaluations of the properties of fuzzy regions. In the scope of the pattern matching problem, described at the beginning of this paper, between imprecise verbal descriptions of objects, and fuzzy regions which describe the boundaries of objects in a picture, the expected value may be considered as not sufficient. Basically one would like to extract fuzzy parameter values from fuzzy regions in order to match items of information of the same nature, as proposed in Farreny and Prade (1984). One may suggest three ways of achieving this purpose.

(a) A rough description of the imprecision pervading $f(R)$ could be obtained as a fuzzy number $f(R)$ (Dubois and Prade, 1980) whose support could be the interval $[\inf_i f(R_i), \sup_i f(R_i)]$ and modal value $\underline{f}(R)$.

(b) A more rigourous definition of this fuzzy interval could be obtained by transforming the probability measure associated to $f(R)$ by definition 1, into a possibility distribution $\pi=\mu_{F(R)}$ consistent with the probability measure in the sense that the grades of possibility $\Pi(f(R)\in A)$ and necessity $N(f(R)\in A)$ act as bounds on the probability $P(f(R)\in A)$ (Dubois and Prade, 1986a). Such transformations were proposed in previous papers (Dubois and Prade, 1982, 1986a,b).

Given a possibility distribution $\pi:\Omega\to[0,1]$, such that $\max_{w\in\Omega}\pi(w)=1$, the possibility and the necessity of $A\subseteq\Omega$ are respectively defined by

$$\Pi(A)=\max\{\Pi(w)\mid w\in A\},\tag{14}$$

$$N(A)=1-\Pi(A^c),\tag{15}$$

where A^c is the complement of A. Given a probability allocation p on the reals (a finite one here, for simplicity), let $S(p)=\{r\mid p(r)>0\}$ be the (finite) support of p, and $C(r)=\{r'\in S(p)\mid p(r')\le p(r)\}$. Consider the following possibility distribution

$$\forall r\in S(r),\ \pi^*(r)=\sum_{r'\in C(r)}p(r').\tag{16}$$

Note that $\exists r$, $\pi^*(r)=1$ (choosing r as a mode of p, for instance). Given two possibility distributions π and π', π is said to be more *specific* (Yager, 1982) than π' if and only if $\pi\le\pi'$. This inequality means that π specifies a smaller range of possible values than π'. Then the probability/possibility transformation (16) has the following optimal property:

Proposition 6 (Dubois and Prade, 1982). *π^* is the most specific possibility distribution consistent with p, i.e.*

$$\forall r,r'\ p(r)\ge p(r')\ \text{ if and only if } \pi^*(r)\ge\pi^*(r').$$

and such that p and π^* define the same ordering on R, i.e.

$$\forall r,r'\ p(r)\ge p(r')\ \text{ if and only if } \pi^*(r)\ge\pi^*(r').$$

N.B. Another (suboptimal) transformation is motivated in (Dubois and Prade, 1983) and such that

$$\pi(r)=\sum_{r'\in S(p)}\min(p(r),p(r')).$$

However $\pi>\pi^*$, generally.

(c) Given a fuzzy region R, it is possible to directly define a fuzzy restriction $F(R)$ on the value of $f(R)$ by stating,

$$\mu_{F(R)}(r)=\sup\{\alpha\mid f(R(\alpha))=r\}.$$

This idea is used to define the fuzzy cardinality of a fuzzy set, for instance (Dubois and Prade, 1980, 1985). When f is a monotonic set-function with regard to inclusion ($R\subseteq R'\Rightarrow f(R)\le f(R')$) it is easy to see that

$$\mu_{F(R)}(f(R(\alpha)))=\sum_{r\ge f(R(\alpha))}p(r)$$

where p is built according to Definition 1. Denoting Φ the distribution function associated with p, ($\Phi(r)=P((-\infty,r))$) it is clear that $\mu_{F(R)}(f(R(\alpha)))=1-\Phi(f(R(\alpha)))$ i.e. the fuzzy number $F(R)$ is then, in essence, the probability distribution function of $f(R)$ on the support of p.

On the whole, method b looks the most attractive, since remaining close to the original data; method c is just another way of expressing the probability measure of the parameter under evaluation.

Example. Consider the fuzzy region defined in Figure 1 on a 7×7 pixel array, and let us calculate the area of the fuzzy region.

	1	2	3	4	5	6	7
1	×	×	×	×	×	5	5
2	×	×	×	×	×	5	5
3	×	×	×	×	×	5	5
4	8	8	×	×	×	3	3
5	8	8	×	×	×	3	3
6	1	1	1				
7	1	1	1				

Figure 1. Fuzzy region R (x: element of the core; $1\le i\le 9$: $\mu_R(w)=i\cdot10^{-1}$).

– Probability density of the area: $p(21)=0.2$; $p(25)=0.3$; $p(31)=0.2$; $p(35)=0.2$; $p(41)=0.1$, since $a(R(1))=21$; $a(R(0.8))=25$; $a(R(0.5))=31$; $a(R(0.3))=35$; $a(R(0.1))=41$.
– Expected area $=\underline{a}(R)=29$.
– Fuzzy area: *method a*: a triangular fuzzy number with support $[21,41]$ and mean value 29.
– Fuzzy area: *method b*: $\pi_f^*(21)=0.7$; $\pi_f^*(25)=1$; $\pi_f^*(31)=0.7$; $\pi_f^*(35)=0.7$; $\pi_f^*(41)=0.1$; of course one may construct a continuous approximation of this discrete distribution.

5. Evaluation of relational parameters

In this paragraph, we are concerned with relational properties between two connected regions R and R'. A relational property f, between R and R' is evaluated by a real number denoted $f(R,R')$. Definition 1 can be extended to a property f relating two fuzzy regions by:

Definition 4. The property f relating two fuzzy regions $R=(\mathcal{F},m)$ and $R'=(\mathcal{F}',m')$ is evaluated by means of a random number $f(R,R')$ defined by the probability assignment p_f such that $\forall r\in N$,

$$p_f(r)=\sum_i\sum_j\{m(R_i)\cdot m'(R_j')\mid f(R_i,R_j')=r\},$$
$$=0\quad\text{if } r \text{ is not in } \{f(R_i,R_j')\mid i=1,\dots,n;\ j=1,\dots,m\}.$$

This definition is also valid for uncertain (not nested) regions, more generally. $f(R,R')$ can be some distance between R and R' or some relative position parameter. The expected value $\underline{f}(R,R')$ of $f(R,R')$ is easily evaluated as:

$$\underline{f}(R,R')=\sum_i\sum_j m(R_i)\cdot m'(R_j')\cdot f(R_i,R_j')$$

This has interesting applications to elementary relational properties as inclusion and overlapping.
– The property of inclusion is defined as: $\forall A,B\subseteq\Omega$,

$$f(A\subseteq B)=1\quad\text{if } A\subseteq B,$$
$$=0\quad\text{otherwise.}$$

In that case, $p_f(1)$ evaluates to what extent $R\subseteq R'$:

$$p_f(1)=\sum_j\Big(\sum_{R_i\subseteq R_j'}m(R_i)\Big)\cdot m'(R_j')$$
$$=\sum_i\Big(\sum_{R_i\subseteq R_j'}m'(R_j')\Big)\cdot m(R_i')=\underline{f}(R\subseteq R')$$

where $\sum_{R_i\subseteq R_j'}m(R_i)=\underline{f}(R\subseteq R_j')$ is the degree to which R_j' contains R and $\sum_{R_i\subseteq R_j'}m'(R_j')=\underline{f}(R_i\subseteq R')$ is the degree to which R' contains R_i. These indices were already proposed in Dubois and Jaulent (1986). The value $\underline{f}(R\subseteq R')$ satisfies:

$$\underline{f}(R\subseteq R')=\sum_j(\underline{f}(R\subseteq R_j'))\cdot m'(R_j')$$
$$=\sum_i(\underline{f}(R_i\subseteq R'))\cdot m(R_i)\tag{17}$$

and generalizes these two indices to the case of two uncertain or fuzzy regions.
– The property of overlapping is defined as: $\forall A,B\subseteq D$,

$$f(A\cap B)=1\quad\text{if } A\cap B\neq\emptyset,$$
$$=0\quad\text{else.}$$

In that case, $p_f(1)$ evaluates to what extent $R\cap R'\neq\emptyset$:

$$p_f(1) = \sum_j \Big(\sum_{R_i \cap R'_j \neq \emptyset} m(R_i) \Big) \cdot m'(R'_j)$$
$$= \sum_i \Big(\sum_{R_i \cap R'_j \neq \emptyset} m'(R'_j) \Big) \cdot m(R_i)$$

where $\sum_{R_i \cap R'_j \neq \emptyset} m(R_i) = f(R \cap R'_j)$ is the degree to which R'_j and R overlap and $\sum_{R_i \cap R'_j \neq \emptyset} m'(R'_j) = f(R_i \cap R')$ is the degree to which R' and R_i overlap. The overlapping index is also suggested in (Dubois and Jaulent, 1986). It is generalized for two fuzzy regions R and R' into:

$$f(R \cap R') = \sum_j (f(R \cap R'_j)) \cdot m'(R'_j)$$
$$= \sum_i (f(R_i \cap R')) \cdot m(R_i). \tag{18}$$

It is important to notice that the two indices $f(R \subseteq R')$ and $f(R \cap R')$ are also generalizations of grades of 'belief', commonality and plausibility in the sense of Shafer (1976)'s theory of evidence. Namely $f(R \subseteq R')$ is formally a grade of 'belief' when R' is a crisp region, a grade of commonality when R is a crisp region, and $f(R \cap R')$ is a grade of plausiblity when any of R or R' is crisp.

Definition 2 underlies an assumption of statistical independence between the segmentation processes which yield R and R'. Namely if R_i is obtained as a representation of R, then any R'_j can be simultaneously obtained as representation of R'. For instance if $R = R'$ in (17), we do not get $f(R \subseteq R) = 1$ generally, since for a fuzzy region:

$$f(R \subseteq R) = \sum_{i=1,n} m(R_i) \cdot \Big(\sum_{j=1,n} m(R_j) \Big) < 1 \tag{19}$$

because $i > j \Rightarrow R_i \subset R_j$. Indeed, in that case, we consider two independent segmentation processes having by chance produced the same result R. However the fact that $\exists R_i, R_j$ such that $R_i \not\subset R_j$ prevents $f(R \subseteq R)$ from being equal to 1.

Another possible assumption is the complete dependency between the segmentation processes S and S' yielding fuzzy regions R and R' defined as follows:

S produces $R(\alpha)$ if and only if
S' produces $R'(\alpha)$. (19)

Then, let $M(R) \cup M(R') = \{1 > \gamma_2 > \cdots > \gamma_k\}$ as in the proof of Proposition 5. (19) implies ($\forall i = 1, k$, $m(R(\gamma_i)) = m'(R'(\gamma_i)) = \gamma_i - \gamma_{i+1}$). ($\gamma_{m+1} = 0$), the reason is that we consider only the joint occurrences $\{(R(\gamma_i), R'(\gamma_i)) \mid i = 1, k\}$. The probability density $f(R, R')$ is then obtained as follows:

Definition 5. In the case of complete dependency of the measurement processes, $f(R, R')$ is a random number defined by the probability assignment: $\forall r$,

$$p_f(r) = \sum \{\gamma_i - \gamma_{i+1} \mid f(R(\gamma_i), R'(\gamma_i)) = r\},$$
$$= 0 \quad \text{otherwise}, \tag{20}$$

and then the expected value is $f^*(R, R') = \sum_i (\gamma_i - \gamma_{i+1}) f(R(\gamma_i), R'(\gamma_i))$.

Proposition 7. (a) *When $f(R, R') = f(R \subseteq R')$, it holds that $f^*(R \subseteq R) = 1$.*

(b) *Moreover if $R = A$ (crisp region) then*

$$f(R \subseteq A) = f^*(R \subseteq A), \quad f(A \subseteq R) = f^*(A \subseteq R),$$
$$f(A \cap R) = f^*(A \cap R).$$

Proof. (a) If $R = R'$, then $M(R) = M(R') = \{1 > \alpha_2 > \cdots > \alpha_n\}$. Then,

$$f^*(R \subseteq R) = \sum_{i=1,n} (\alpha_i - \alpha_{i+1}) f(R(\alpha_i) \subseteq R(\alpha_i))$$
$$= \sum_{i=1,n} (\alpha_i - \alpha_{i+1}) = \alpha_1 = 1.$$

(b) If $R' = A$, then $M(A) = \{1\}$ while $M(R) = \{1 > \alpha_1 > \alpha_2 > \cdots > \alpha_n\}$. Hence,

$$f^*(R \subseteq A) = \sum_{i=1,n} (\alpha_i - \alpha_{i+1}) f(R_i \subseteq A)$$
$$= \sum \{m(R_i) \mid R_i \subseteq A\} = f(R \subseteq A).$$

The same proof applies to other indices. □

Deriving fuzzy-valued relational parameters can be achieved using the same techniques as for intrinsic parameters (see the previous section).

Concepts of distances (Hausdorff distance, minimal distance, maximal distance, etc.) could be extended by this approach. For instance if A and A' are two convex regions, define the maximal distance between A and A' as the diameter of the convex hull of $A \cup A'$, i.e. $\text{dist}(A, A') = E([A \cup A'])$ where [ ] denotes the convex hull. Using Definitions 4 and 5, we can easily prove: $\text{dist}^*(R, R) = e(R)$, while $\underline{\text{dist}}(R, R) \neq e(R)$ generally. Similarly

using a Hausdorff distance $H(A, A')$ (e.g. Matheron (1975)), we have $H^*(R, R) = 0$ (because $H(A, A') = 0$) while $\underline{H}(R, R) \neq 0$, generally.

Rosenfeld (1985) has introduced a concept of shortest distance between two fuzzy regions R and R' as a fuzzy set $\Delta(R, R')$ of real numbers such that

$$\mu_{\Delta(R,R')}(r) = \sup_{w, w' : d(w,w') \leq r} \min(\mu_R(w), \mu_{R'}(w')). \tag{22}$$

It is easy to check that if $r \leq r'$ then $\mu_{\Delta(R,R')}(r) \leq \mu_{\Delta(R,R')}(r')$ so that $\mu_{\Delta(R,R')}$ has the shape of a probability distribution function. Then this definition is compatible with definition 5 in the following sense.

Proposition 8. *Definition 5 applied to the shortest distance Δ produces the probability density function whose distribution is $\mu_{\Delta(R,R')}$.*

Proof. Let $f = \Delta$, the shortest distance between two regions A and B, defined by $\Delta(A, B) = \min\{d(w, w') \mid w \in A, w' \in B\}$. If R and R' are two fuzzy regions, it is easy to check that

$$\alpha \leq \alpha' \Rightarrow \Delta(R(\alpha), R'(\alpha)) \leq \Delta(R(\alpha'), R'(\alpha')). \tag{23}$$

Let $r_i = \Delta(R(\gamma_i), R'(\gamma_i))$ where γ_i is defined as earlier. Clearly, $r_1 > r_2 > \cdots > r_k$. Hence the PDF associated to the pdf p_Δ is defined by

$$\Phi_\Delta(r) = \gamma_i \quad \text{if } r \in [r_i, r_{i+1}), i = 1, k-1,$$
$$= 0 \quad \text{if } r < r_k,$$
$$= 1 \quad \text{if } r \geq r_1.$$

Note that if $r \geq r_i$ then $\exists w \in R$, $w' \in R'$ such that $d(w, w') \leq r$ (choose w and w' such that $d(w, w') = \Delta(I(R), I(R'))$. Moreover if $r < r_k$ then $\nexists w \in S(R)$, $w' \in S(R')$ such that $d(w, w') \leq r$ so that $\mu_{\Delta(R,R')}$ coincides with Φ_Δ when r is not in $[r_k, r_1)$. Assume that $r \in [r_i, r_{i+1})$ then choosing w and w' such that $d(w, w') \leq r$, $w \in R(\gamma_i)$, $w' \in R'(\gamma_i)$ is possible since $\Delta(R(\gamma_i), R'(\gamma_i)) = r_i \leq r$. Hence $\mu_{\Delta(R,R')}(r) \geq \gamma_i$. Now if $\mu_{\Delta(R,R')}(r) \geq \gamma_i + \varepsilon$, with $\varepsilon > 0$ then $\exists w, w'$, $d(w, w') \leq r < r_{i+1}$, $\mu_R(w) \geq \gamma_i + \varepsilon$, $\mu_{R'}(w') \geq \gamma_i + \varepsilon$. But we know that $\mu_{\Delta(R,R')}$ only takes values in $M(R) \cup M(R')$ i.e. $\varepsilon = \gamma_{i+1} - \gamma_i$. Hence $\exists w, w'$, $d(w, w') < \Delta(R(\gamma_{i+1}), R'(\gamma_{i+1}))$ and $w \in R(\gamma_{i+1})$, $w' \in R'(\gamma_{i+1})$. This is a contradiction and the result $\mu_{\Delta(R,R')} = \Phi_\Delta$ is right. □

Hence Rosenfeld (1985)'s definition of the fuzzy minimum distance between fuzzy regions is a particular case of our definition of relational parameters, under the strong dependency condition. Note that the density derived from Definition 3 is constructed by Rosenfeld from the membership function $\mu_{\Delta(R,R')}$ because he notices its shape of PDF.

Remark. In the scope of the man-machine communication, if f is defined as a distance between R and R', one difficulty is to interpret the verbal notion of distance. Indeed, the notion of distance included in verbal description of a scene is ambiguously given by the human operator who may mean the distance between the two centers of gravity, the least distance between the two regions, etc. So the notion of distance (and relative location too), are naturally fuzzy notions and the property f is, in that case, a fuzzy property in the sense that even applied to crisp regions, it may return fuzzy values, due to ill-definition.

Conclusion

In this paper a general approach to the definition of characteristic parameters of fuzzy subsets of images has been proposed. It generalizes current definitions for standard subsets consistently with both fuzzy set and evidence theories. The basic idea is to view a fuzzy region as a consonant random set. We have shown that our proposal is in good agreement with specific suggestions made by Rosenfeld. Our approach is easy to implement because the parameter evaluation comes down to several classical parameter evaluation steps, as patent from definitions. One may expect some applications of our methodology to the analysis of images with imprecise contours, especially in the scope of man-vision system communication.

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