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Semi-proximity continuous functions in digital images

Longin Latecki [a,*], Frank Prokop [b]

[a] Department of Computer Science, University of Hamburg, Vogt-Kölln-Str. 30, 22527 Hamburg, Germany
[b] Department of Mathematics, University of Wollongong, Wollongong, N.S.W. 2522, Australia

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Abstract

Starting with the intuitive concept of "nearness" as a binary relation, semi-proximity spaces (sp-spaces) are defined. The restrictions on semi-proximity spaces are weaker than on topological proximity spaces. Thus, semi-proximity spaces generalize classical topological spaces. Moreover, it is possible to describe all digital pictures used in computer vision and computer graphics as non-trivial semi-proximity spaces, which is not possible in classical topology. Therefore, we use semi-proximity spaces to establish a formal relationship between the "topological" concepts of digital image processing and their continuous counterparts in $\mathbb{R}^n$.

Especially interesting are continuous functions in semi-proximity spaces which are called "semi-proximity" continuous functions. They can be used for characterizing well-behaved operations on digital images such as thinning. It will be shown that the deletion of a simple point can be treated as a semi-proximity continuous function. These properties and the fact that a variety of nearness relations can be defined on digital pictures indicate that semi-proximity continuous functions are a useful tool in the difficult task of shape description.

Keywords: Proximity; Nearness; Digital topology; Image processing

0. Introduction

Topology has been developed to formulate and treat continuity. This can be demonstrated by the following quotation from the translation (Alexandroff, 1961) of (Alexandroff, 1932, p. 8, no. 8):

"*A topological space is nothing other than a set of arbitrary elements (called points of the space) in which a concept of continuity is defined. Now this concept of continuity is based on the existence of relations, which may be defined as local or neighborhood relations – it is precisely these relations which are preserved in a continuous mapping of one figure onto another.*"

A basic property of continuous functions is that the continuous image of a connected set is connected. This property can be useful in digital image processing, since most transformations of digital images must preserve connectedness. If transformations of digital images are represented by continuous functions, then we are

* Corresponding author. Email: latecki@informatik.uni-hamburg.de

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guaranteed that they preserve connectedness. In image processing, there is a natural well-defined concept of connectedness that is based on the local neighborhood relations of digital images. Thus, the concept of digital continuity should also be based on the concept of local neighborhood relations.

The most useful transformations of digital images are those which preserve connectedness in both directions, since they do not split or merge different components of an image. (Formally, we will say that a function preserves connectedness in both directions, if both the image and the inverse image of a connected set are connected.) If transformations of digital images were represented by functions continuous in both directions, then they would preserve connectedness in both directions. However, in order to define continuity in the inverse direction in classical topology, the inverse function must exist, and thus the original function must be one-to-one. So in classical topology a definition of continuity in the inverse direction does not exist for functions that are not invertible. Since most transformations of digital images are not one-to-one, the definition of continuity in the inverse direction for functions which need not be one-to-one would be a useful concept in digital image processing.

Rosenfeld (1986) uses the natural metrics of digital images in the framework of classical topology to define continuous functions between digital images (see also Boxer, 1994). Thus, a function in Rosenfeld's approach must also be a homeomorphism, and thus one-to-one, in order to be guaranteed to preserve connectedness in both directions. On the other hand, if a function on $\mathbb{Z}^2$ is one-to-one, then it is continuous iff it is a composition of the basic transformations on $\mathbb{Z}^2$, i.e., translation, rotation by $\pm 90°$ or by $180°$, and vertical, horizontal, or diagonal reflection (Rosenfeld, 1986). This result is a consequence of the fact that a function is continuous in Rosenfeld's sense iff two points with distance one are mapped onto points with distance zero or one. Thus, this definition seems to be too restrictive for many applications in digital image processing. This limitation on the types of continuous functions available also applies to topologically continuous functions on $\mathbb{Z}^2$ which can be defined in the topology described in (Khalimsky et al., 1990) and (Kovalevsky, 1989).

In this paper, a semi-proximity structure is used to define continuous functions on digital images. Such functions can map points with distance one onto points with distance two or more. Thus, semi-proximity continuous functions are more flexible than metric or topologically continuous functions, yet they still preserve the usual kinds of connectedness of digital images. A function between semi-proximity spaces can be defined in a natural way to be continuous in both directions (bicontinuous), even if the function is not one-to-one. The key property is the fact that bicontinuous functions between semi-proximity spaces preserve connectedness in both directions. This fact and the variety of semi-proximity relations which can be defined on digital images indicate that semi-proximity continuous functions can be used to characterize well-behaved operations on digital images and to assist in the difficult task of shape description.

Moreover, semi-proximity spaces establish a formal relationship between the "topological" concepts of digital image processing and their continuous counterparts in $\mathbb{R}^k$ (Section 5). Since $\mathbb{R}^k$ with the usual topology is a semi-proximity space for every $k = 1, 2, \ldots$ and every digital image can be described as a semi-proximity space in such a way that digital connectedness and semi-proximity connectedness are equivalent, it makes sense to define semi-proximity continuous functions between $\mathbb{R}^k$ and digital images. It is impossible using classical topology, to define continuous functions between $\mathbb{R}^k$ and digital images, since the digital images which are most commonly used in applications cannot be described as topological spaces. For example, in (Chassery, 1979) it is shown that there is no topology on $(\mathbb{Z}^2, 8)$ in which connectedness is equivalent to 8-connectedness (see also (Latecki, 1993) for a much shorter proof, which requires only the consideration of a four-point subset of $\mathbb{Z}^2$).

In Sections 1 and 2, semi-proximity spaces, connectedness, and continuity are defined. It is also proved that a semi-proximity bicontinuous function, which need not be one-to-one, preserves connectedness in both directions. In Sections 3 and 4, some basic definitions of digital topology are reviewed, digital metric continuity is defined, and some useful descriptions of digital images as semi-proximity spaces are given. In Section 6, some examples of semi-proximity continuous functions on digital images are given. It is shown in Section 7 that semi-proximity continuity can be used to describe connectivity preserving thinning. Finally, in Section 8, properties of semi-proximity and other digital continuous functions are discussed.

1. Semi-proximity spaces

Riesz (1909) introduced proximity structures in his "theory of enchantment". In the early fifties Efremovič rediscovered the subject (see Naimpally and Warrack, 1970; Engelking, 1977). The axioms for Riesz and Efremovič proximity structures were developed to axiomatize the properties of the relationship between sets A and B in a metric space which could be defined by stating that "A is close to B" (i.e. $A \delta B$) iff $D(A, B) = \inf\{d(x, y): x \in A, y \in B\} = 0$. The five axioms of Efremovič are now widely accepted as a definition of a proximity structure. The topologies generated by proximity structures are always completely regular. So, in defining a "nearness" relation which is suitable for studying problems associated with digital images, where the topological spaces considered will not be completely regular, we will consider a generalization of a proximity structure, called a semi-proximity, which uses only four of the axioms for a proximity structure. Further, semi-proximity spaces generalize Herrlich's definition of nearness spaces (see Herrlich, 1974). Finally, we will show that underlying each semi-proximity space there is a Čech closure mapping and, conversely, that Čech closure mappings can be used to construct semi-proximity spaces. It should be noted that Čech (1966) refers to semi-proximity relations which arise from a semi-uniformity as proximity relations.

Definition 1.1. A relation δ on the power set of X, denoted $P(X)$, which $(\forall A, B, C \in P(X))$ satisfies:

(p_1) $(A \cup B)\ \delta\ C \Leftrightarrow A\ \delta\ C$ or $B\ \delta\ C$.

(p_2) $A\ \delta\ B \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$,

(p_3) $A \cap B \neq \emptyset \Rightarrow A\ \delta\ B$,

(p_4) $A\ \delta\ B \Rightarrow B\ \delta\ A$.

will be called a *(symmetric) semi-proximity* on $P(X)$.

If $A\ \delta\ B$ will say that A and B are *near*. We will write $A\ \not\delta\ B$ if A and B are not near. Finally, if X is a set and δ is a semi-proximity relation on $P(X)$, then (X, δ) will be called a *semi-proximity space*.

For completeness, we note that ${}^c : P(X) \to P(X)$ is a *Čech closure* mapping if $\emptyset^c = \emptyset$; $(\forall A \in P(X))$ $A \subseteq A^c$; and $(\forall A, B \in P(X))(A \cup B)^c = A^c \cup B^c$ (Čech, 1966, Chapter 14, p. 237). Theorem 1.2 establishes the relationship between semi-proximity and Čech closure spaces. It also follows from Theorem 1.2 that every topological space is a semi-proximity space, since every topological space is a Čech closure space.

Theorem 1.2. *If δ is a semi-proximity relation on $P(X)$, then ${}^c : P(X) \to P(X)$, given by $A^c = \{x \in X: A\ \delta\ \{x\}\}$ is a Čech closure mapping on $P(X)$. Conversely, if ${}^c : P(X) \to P(X)$ is a Čech closure mapping and δ_1 and δ_2 are given by $(\forall A, B \in P(X))$*

(i) $A\ \delta_1\ B \Leftrightarrow A^c \cap B^c \neq \emptyset$ *and* (ii) $A\ \delta_2\ B \Leftrightarrow (A \cap B^c \neq \emptyset$ *or* $A^c \cap B \neq \emptyset)$,

then (X, δ_1) and (X, δ_2) are semi-proximity spaces.

Proof. Let δ be a semi-proximity relation on $P(X)$. By (p_2), $\emptyset^c = \emptyset$. By (p_3), $A \subseteq A^c$. Finally,

$$(A \cup B)^c = \{x \in X: A \cup B\ \delta\ \{x\}\} = (\text{by } (p_1))\{x \in X: A\ \delta\ \{x\} \text{ or } B\ \delta\ \{x\}\}$$
$$= \{x \in X: A\ \delta\ \{x\}\} \cup \{x \in X: B\ \delta\ \{x\}\} = A^c \cup B^c.$$

Let now ${}^c : P(X) \to P(X)$ be a Čech closure mapping. Then (p_1), (p_2), and (p_4) are easily verified for δ_1, while (p_3) follows from $A \cap B \subseteq (A \cap B)^c \subseteq A^c \cap B^c$.

For δ_2, we show first (p_1): $(A \cup B)\ \delta_2\ C \Leftrightarrow (A \cup B) \cap C^c \neq \emptyset$ or $(A \cup B)^c \cap C \neq \emptyset \Leftrightarrow (A \cap C^c \neq \emptyset$ or $B \cap C^c \neq \emptyset)$ or $(A^c \cup B^c) \cap C \neq \emptyset \Leftrightarrow (A \cap C^c \neq \emptyset$ or $A^c \cap C \neq \emptyset)$ or $(B \cap C^c \neq \emptyset$ or $B^c \cap C \neq \emptyset) \Leftrightarrow (A\ \delta_2\ C$

or $B\ \delta_2\ C$), which implies (p_1). (p_2) follows easily by considering cases. (p_3) follows from the fact that $A \cap B \subseteq A \cap B^c$ and $A \cap B \subseteq A^c \cap B$. Clearly, ($p_4$) holds for δ_2. □

2. Connectedness and continuity in semi-proximity spaces

In this section we define the concepts of connectedness and continuity in semi-proximity spaces. Initially we will precede each analogous definition with "sp" for semi-proximity. It should be noted that for T_1 topological spaces, sp-connectivity and sp-continuity agree with their topological counterparts.

Definition 2.1. Let (X, δ) be a semi-proximity space and let $Z \subseteq X$. A pair of non-empty subsets A, B of X such that $Z = A \cup B$ and $A\ \not\delta\ B$ is called an *sp-separation* of Z. Z is *sp-connected* in (X, δ) if there is no sp-separation of Z in (X, δ), otherwise Z is *sp-disconnected.* In particular, a semi-proximity space X is sp-connected if there is no sp-separation of X.

Let $(X, {}^c)$ be a topological space, where ^{c} is the closure operator generating the topology. If we consider the sp-relation defined by $A\ \delta\ B \Leftrightarrow A \cap B^c \neq \emptyset$ or $A^c \cap B \neq \emptyset$, then sp-connectedness agrees with the usual definition of connectedness in topological spaces. Further, if $(X, {}^c)$ is a Čech closure space, then for the sp-relation defined by δ, sp-connectedness agrees with Čech connectedness.

We will adopt the convention that if we are working in a semi-proximity space, we will write "A, B is a separation for Z" instead of the technically correct statement "A, B is an sp-separation for Z". A similar convention will be adopted for all "sp" terms defined in this paper.

Definition 2.2. Let (X, δ_1) and (Y, δ_2) be semi-proximity spaces. A function $f: X \to Y$ is *sp-continuous* if $(\forall A, B \in P(X))\ (A\ \delta_1\ B \Rightarrow f(A)\ \delta_2\ f(B))$. A function $f: X \to Y$ is *inverse sp-continuous* if (i) $(\forall A, B \in P(X))(f(A)\ \delta_2\ f(B) \Rightarrow f^{-1}(f(A))\ \delta_1\ f^{-1}(f(B)))$ and (ii) the inverse image of every point in $f(X)$ is connected. A function $f: X \to Y$ will be called *sp-bicontinuous* if it is continuous and inverse continuous [1].

Note that if $f: X \to Y$ is a function from X *onto* Y, then f is *inverse sp-continuous* iff (i) $(\forall C, D \in P(Y))(C\ \delta_2\ D \Rightarrow f^{-1}(C)\ \delta_1\ f^{-1}(D))$ and (ii) the inverse image of every point in Y is connected. Further, the definition of sp-continuity agrees with the usual definition of continuity in proximity spaces.

The most important property of an sp-bicontinuous function, which is proved in Theorem 2.6, is that it preserves connectedness in both directions. It is this property of bicontinuous functions along with the fact that they need not be one-to-one which make them an interesting tool in digital image processing.

The following example shows that an sp-bicontinuous function between topological spaces need not be an open mapping. Let $f: [0, 2] \to [0, 1]$, $f(x) = \min(x, 1)$, be a function between closed intervals of real numbers. We treat the set of intervals as sp-spaces with δ defined by $A\ \delta\ B \Leftrightarrow A \cap B^c \neq \emptyset$ or $A^c \cap B \neq \emptyset$, where ^{c} is the usual closure operator. Then f is sp-bicontinuous, but it is not open (any open set in $(1, 2]$ is mapped onto $\{1\}$).

Theorem 2.4 will show that the continuous image of a connected semi-proximity space is connected. Theorem 2.5 will show that an inverse continuous function preserves connectedness in the inverse direction.

[1] Let $(X, {}^c)$ be a topological space, where ^{c} is the closure operator generating the topology. If the sp-relation is defined by $A\ \delta\ B \Leftrightarrow A \cap B^c \neq \emptyset$ or $A^c \cap B \neq \emptyset$, then for T_1 topological spaces, sp-continuity agrees with the usual definition of continuity. Moreover, if $f: X \to Y$ is an sp-bicontinuous bijection between T_1 topological spaces, f is a topological homeomorphism. This is actually true for R_0 Čech closure spaces, where a closure space is R_0 if ^{c} satisfies the condition (R_0) $(\forall a, b \in X)(a \in \{b\}^c \Leftrightarrow b \in \{a\}^c)$.

Lemma 2.3. *Let (X, δ) be a semi-proximity space. Then*

$$(\forall A, B \in P(X))(A\ \delta\ B \text{ and } A \subseteq C\ \&\ B \subseteq D \Rightarrow C\ \delta\ D).$$

Proof. By (p1), $A\ \delta\ B \Rightarrow (A \cup C)\ \delta\ B \Rightarrow C\ \delta\ B$. By (p1) and (p4), $C\ \delta\ B \Rightarrow C\ \delta\ (B \cup D) \Rightarrow C\ \delta\ D$. □

Theorem 2.4. *Let (X, δ_1) and (Y, δ_2) be semi-proximity spaces and $f: X \to Y$ be a continuous function. If X is connected, then $f(X)$ is connected.*

Proof. Let X be connected. $f(X)$ disconnected $\Rightarrow \exists C, D \subseteq Y, C \neq \emptyset, D \neq 0$ such that $f(X) = C \cup D$ and $C\ \not\delta_2\ D \Rightarrow X \subseteq f^{-1}(f(X)) = f^{-1}(C) \cup f^{-1}(D)$. $f^{-1}(C)$ and $f^{-1}(D)$ non-empty and X is connected $\Rightarrow$ $f^{-1}(C)\ \delta_1\ f^{-1}(D)$. By continuity of f, $f(f^{-1}(C))\ \delta_2\ f(f^{-1}(D))$. However, by Lemma 2.3, $f(f^{-1}(C)) \subseteq C$ and $f(f^{-1}(D)) \subseteq D \Rightarrow C\ \delta_2\ D$, which is a contradiction. □

Theorem 2.5. *Let (X, δ_1) and (Y, δ_2) be semi-proximity spaces and $f: X \to Y$ be inverse continuous. If $f(X)$ is connected, then X is connected.*

Proof. Let $f(X)$ be connected and X be disconnected. So, there exist $B \neq \emptyset$, $B' \neq \emptyset$ subsets of X such that $X = B \cup B'$ and $B\ \not\delta_1\ B'$. Since $f(X)$ is connected and $f(X) = f(B) \cup f(B')$, we obtain $f(B)\ \delta_2\ f(B')$. By inverse continuity of f, $f(B)\ \delta_2\ f(B') \Rightarrow f^{-1}(f(B))\ \delta_1\ f^{-1}(f(B'))$.

If $f^{-1}(f(B)) = B$ and $f^{-1}(f(B')) = B'$, we have a contradiction. So, at least one of these equalities does not hold. For example, let B be properly contained in $f^{-1}(f(B))$. This means that there exists $y \in f(B)$ such that $f^{-1}(\{y\}) \cap B' \neq \emptyset$. Since $y \in f(B)$, we also know that $f^{-1}(\{y\}) \cap B \neq \emptyset$. The fact that $f(X) = B \cup B'$ implies that $f^{-1}(\{y\}) = (B \cap f^{-1}(\{y\})) \cup (B' \cap f^{-1}(\{y\}))$. From Lemma 2.3 it follows that $B \cap f^{-1}(\{y\})\ \not\delta_1\ B' \cap f^{-1}(\{y\})$, which means that $f^{-1}(\{y\})$ is disconnected. This is a contradiction. □

As a simple consequence of Theorems 2.4 and 2.5, we obtain that if f is bicontinuous, then f preserves connectedness in both directions.

Theorem 2.6. *Let (X, δ_1) and (Y, δ_2) be semi-proximity spaces and $f: X \to Y$ be bicontinuous. Then, X is connected $\Leftrightarrow$ $f(X)$ is connected.*

Theorem 2.7 follows easily from the definitions and the properties of the image and inverse image functions.

Theorem 2.7. *Let (X, δ_1), (Y, δ_2) and (Z, δ_3) be semi-proximity spaces and $f: X \to Y$ and $g: Y \to Z$ be functions.*
(i) *If f and g are continuous, then $g \circ f$ is continuous.*
(ii) *If f and g are inverse continuous, then $g \circ f$ is inverse continuous.*
(iii) *If f and g are bicontinuous, then $g \circ f$ is bicontinuous.*

3. Metric continuity

The following definition of metric continuity is taken from (Rosenfeld, 1986). Rosenfeld has adopted the standard metric definition of continuity for digital pictures. However, to define metric continuity, Rosenfeld considers only metrics which fulfill the following conditions:
(i) for all $x, y \in \mathbb{Z}^n$, such that $x \neq y$, $d(x, y) \geqslant 1$, and
(ii) if $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, then $d(x, y) = 1$ iff there exists j, $1 \leqslant j \leqslant n$, such that $|x_j - y_j| = 1$ and $x_i = y_i$ for all $i \neq j$.

Obviously, the familiar Euclidean metric on $\mathbb{Z}^n$ and 4-distance which correspond to 4-adjacency on $\mathbb{Z}^2$ satisfy these conditions. However, the 8-distance based on 8-adjacency on $\mathbb{Z}^2$ does not satisfy condition (ii). Since digital pictures are equipped with natural metrics satisfying conditions (i) and (ii), metric continuity is a useful but restrictive property. Let r and s be natural numbers and let d_r and d_s be two metrics on $\mathbb{Z}^r$ and $\mathbb{Z}^s$, respectively, which fulfill conditions (i) and (ii). A function $f:(\mathbb{Z}^r, d_r) \to (\mathbb{Z}^s, d_s)$ is *metric continuous* at a point $p \in \mathbb{Z}^r$ if $(\forall \varepsilon \geqslant 1)(\exists \delta \geqslant 1)(\forall q \in \mathbb{Z}^r)(d_r(p, q) \leqslant \delta \Rightarrow d_s(f(p), f(q)) \leqslant \varepsilon)$. A function $f:(\mathbb{Z}^r, d_r) \to (\mathbb{Z}^s, d_s)$ is *metric continuous* if it is metric continuous at every point $p \in \mathbb{Z}^r$.

Although Rosenfeld refers to such functions as continuous, we will use the term metric continuity to avoid possible confusion with sp-continuity. The definition of metric continuity is analogous to the familiar epsilon–delta definition of continuity for real-valued functions. It is easy to show (see Rosenfeld, 1986) that the definition of metric continuity is equivalent to the following one.

A function $f:(\mathbb{Z}^r, d_r) \to (\mathbb{Z}^s, d_s)$ is *metric continuous at a point* $p \in \mathbb{Z}^r$ if

$$(\forall q \in \mathbb{Z}^r)(d_r(p, q) \leqslant 1 \;\Rightarrow\; d_s(f(p), f(q)) \leqslant 1).$$

A simple consequence of this property is that metric continuous functions preserve metric connectedness, where a set X is *metric connected* in $(\mathbb{Z}^r, d_r)$ iff $(\forall p, q \in X)$ there exists a sequence of points $p = p_1, p_2, \dots, p_n = q$ such that $d_r(p_i, p_{i-1}) \leqslant 1$, $1 < i \leqslant n$. For the 4-distance on $\mathbb{Z}^2$, the corresponding metric connectedness is equivalent to 4-connectedness. Theorem 3.1, which along with its proof is found in (Rosenfeld, 1986), indicates that metric continuity may be too restrictive for many applications in digital image processing.

Theorem 3.1. *If $f:\mathbb{Z}^2 \to \mathbb{Z}^2$ is metric continuous and one-to-one, then f is a translation, possibly combined with a vertical, horizontal, or diagonal reflection or with a rotation by* $\pm 90°$ *or by* $180°$.

4. Semi-proximity relations on digital pictures

We will now show that there is a natural way to define a Čech closure operator on $\mathbb{Z}^k$, and that connectivity in the corresponding semi-proximities agrees with the usual connectivity of digital pictures. First we recall some basic concepts of digital topology: The *4-neighbors* of a point (x, y) in $\mathbb{Z}^2$ are its four horizontal and vertical neighbors $(x+1, y)$, $(x-1, y)$ and $(x, y+1)$, $(x, y-1)$. The *8-neighbors* of a point (x, y) in $\mathbb{Z}^2$ are its four horizontal and vertical neighbors together with its four diagonal neighbors $(x+1, y+1)$, $(x+1, y-1)$, $(x-1, y+1)$, and $(x-1, y-1)$. For $n = 4$ or 8, the *n-neighborhood* of a point (x, y) in $\mathbb{Z}^2$ is the set $N_n((x, y))$ consisting of (x, y) and its n-neighbors. In an analogous way, one can define 6-, 18-, and 26-neighbors and -neighborhoods in $\mathbb{Z}^3$, and so on for $\mathbb{Z}^k$. If two points x and y are n-neighbors or $x = y$ we say that they are *n-adjacent* and write n-adj(x, y) or simply adj(x, y). By an *n-path* from p to q we mean a sequence of points $p = p_1, p_2, \dots, p_k = q$ such that p_i is an n-neighbor of p_{i-1}, $1 < i \leqslant k$. A set X is *n-connected* it for every pair of points p, q in X, there is an n-path contained in X connecting p with q.

Let $N(x)$ be some m-neighborhood of $x \in \mathbb{Z}^k$. If we define ^{c} on $P(\mathbb{Z}^k)$ by

$$(\forall Y \in P(\mathbb{Z}^k))(Y^c = \{x \in \mathbb{Z}^k : N(x) \cap Y \neq \emptyset\}),$$

then ^{c} is a Čech closure operator on $\mathbb{Z}^k$. Thus, by Theorem 1.2, ^{c} determines two semi-proximity relations δ_1 and δ_2 on $\mathbb{Z}^k$ given by

$$\text{(i) } A\ \delta_1\ B \Leftrightarrow A^c \cap B^c \neq \emptyset \quad \text{and} \quad \text{(ii) } A\ \delta_2\ B \Leftrightarrow (A \cap B^c \neq \emptyset \text{ or } A^c \cap B \neq \emptyset).$$

Observe that, for δ_i with $i = 1$ or 2, we have

$$(\forall A, B \in P(\mathbb{Z}^k))(A\ \delta_i\ B \Leftrightarrow (\exists \in A, b \in B)\ \{a\}\ \delta_i\ \{b\}).$$

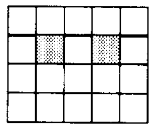

Fig. 1.

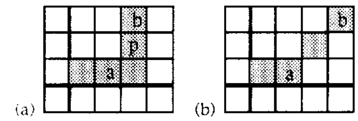

Fig. 2.

Thus, δ_1 has the following simple characterization in terms of the neighborhoods of points:

$$(\forall A,\ B \in P(\mathbb{Z}^k))(A\ \delta_1\ B \ \Leftrightarrow\ ((\exists a \in A,\ b \in B)\ N(a) \cap N(b) \neq \emptyset).$$

Similarly, δ_2 has a point-wise characterization given by

$$(\forall A,\ B \in P(\mathbb{Z}^k))(A\ \delta_2\ B \ \Leftrightarrow\ ((\exists a \in A,\ b \in B)\ a \in N(b) \text{ or } b \in N(a))).$$

Since in digital pictures $a \in N(b) \Leftrightarrow b \in N(a) \Leftrightarrow m\text{-adj}(a,\ b)$, we obtain a simple equivalence for δ_2 in terms of adjacency given by

$$(\forall A,\ B \in P(\mathbb{Z}^k))(A\ \delta_2\ B \ \Leftrightarrow\ (\exists a \in A,\ b \in B)\ m\text{-adj}(a,\ b)).$$

Thus, δ_2 is just an extension of m-adjacency to sets. Using this definition, it is easy to observe that the sp-connectedness induced by δ_2 is exactly the m-connectedness on $\mathbb{Z}^k$ for every m-adjacency relation. This is an important property, since m-connectedness is an intuitive concept of connectedness in digital pictures. If we define δ_1 using 4- or 8-neighborhoods, then δ_1-connectedness does not induce m-connectedness on $\mathbb{Z}^k$: If we consider the two-point set in Fig. 1, then it is δ_1-connected but not m-connected.

The most commonly used digital pictures are $(\mathbb{Z}^k,\ m,\ n,\ B)$ pictures with changeable $(m,\ n)$-connectedness, where B is the set of black points (see Kong and Rosenfeld, 1989). We will now define an sp-relation which is suitable for describing such pictures. We will begin by defining a relation γ for points in B of $(\mathbb{Z}^k,\ m,\ n,\ B)$:

$$\begin{aligned}(\forall x,\ y \in B)(x\ \gamma\ y) \text{ iff } (\exists p \in B)((&m\text{-adj}(x,\ p) \text{ and } n\text{-adj}(p,\ y)) \\ \text{or } (&n\text{-adj}(x,\ p) \text{ and } m\text{-adj}(p,\ y))).\end{aligned} \qquad (\gamma)$$

Note that γ is reflexive, since if $x = y$, then $x\ \gamma\ y$, because p can equal either x or y. This follows from the fact that for any k-adjacency, if $a = b$, then k-adj$(a,\ b)$. For example, if B is the set of black points given in Fig. 2(a) with 8-adjacency in $(\mathbb{Z}^2,\ 8,\ 4,\ B)$ and $a,\ b \in B$, then $a\ \gamma\ b$, since there is $p \in B$ such that 8-adj$(a,\ p)$ and 4-adj$(p,\ b)$. This is clearly not the case for a and b in Fig. 2(b). The extension of γ to the sets of black points is straightforward:

$$(\forall A,\ C \in P(B))(A\ \gamma\ C) \text{ iff } (\exists x \in A,\ y \in C)(x\ \gamma\ y).$$

By checking (p_1)–(p_4) of Definition 1.1, it can easily be shown that $(B,\ \gamma)$ is a semi-proximity space.

Theorem 4.1. *Let $A \subseteq \mathbb{Z}^k$ with m-adjacency in $(\mathbb{Z}^k,\ m,\ n,\ A)$ and let $(A,\ \gamma)$ be a semi-proximity space defined by (γ). Then, A is sp-connected $\Leftrightarrow$ A is m-connected.*

Proof. First we will show that sp-connectedness implies m-connectedness. Let A be m-disconnected and let C, D be an m-separation of A. This implies that there is no m-path in A joining any point in C to some point in D. Thus, C, D is also an sp-separation of A. Assume now that A is sp-disconnected and C, D is an sp-separation of A. It is clear that $(\forall x \in C,\ y \in D)(x\ \bar{\gamma}\ y)$. Thus, C, D is also an m-separation of A. If this were not the case, then $(\exists x \in C,\ y \in D)(\exists m\text{-path}(x,\ y) \subseteq A)$. Since an m-path is a finite sequence of points, we obtain $(\exists c \in C,\ d \in D)(c,\ d \in m\text{-path}(x,\ y)$ and $m\text{-adj}(c,\ d))$. But this implies that $C\ \gamma\ D$, a contradiction. □

5. A semi-proximity relationship between $\mathbb{R}^N$ and $\mathbb{Z}^N$

Using semi-proximity spaces we can establish a formal relationship between the "topological" concepts of digital image processing and their continuous counterparts. For example, a set $\alpha \subseteq \mathbb{Z}^k$ is a digital arc if α is an sp-bicontinuous image of a closed interval with the usual topology. Similar characterizations can be given for digital simple curves and digital surfaces. This is impossible in classical topology, since the digital images which are most commonly used in applications cannot be described as topological spaces. Chassery (1979) showed that there is no topology on $(\mathbb{Z}^2, 8)$ in which connectedness is equivalent to 8-connectedness (see also (Latecki, 1993) for a much shorter proof, which requires only the consideration of a four-point subset of $\mathbb{Z}^2$).

By the results in Section 1, $(\mathbb{R}^n, \delta)$ is a semi-proximity space having the usual topology of $\mathbb{R}^n$ if we define $(\forall A, B \in P(\mathbb{R}^n))(A \ \delta \ B$ iff $A \cap B^c \neq \emptyset$ or $A^c \cap B \neq \emptyset)$, where ^{c} is the usual closure operator defining the topology of $\mathbb{R}^k$. By the results in Section 4, $\mathbb{Z}^k$ with m-adjacency is a semi-proximity space if we define $(\forall A, B \in P(\mathbb{Z}^k))(A \ \delta_m \ B$ iff $(\exists a \in A, b \in B)$ m-adj$(a, b))$. Therefore, it makes sense to define sp-continuous functions between $(\mathbb{R}^n, \delta)$ and $(\mathbb{Z}^k, \delta_m)$. Note that this is the first formal link between the most commonly used digital pictures and $\mathbb{R}^n$. In the approach presented in (Khalimsky et al., 1990) and (Kovalevsky, 1989), $\mathbb{Z}^k$ can be treated as a T_0 topological space. However, the structure of $\mathbb{Z}^k$ obtained this way is not the one most commonly used in digital pictures with 4-, 8-, 6-, or 26-adjacency relations. Theorem 5.2 will show that a digital arc can be described as the sp-bicontinuous image of a closed interval with the usual topology. A digital arc is commonly used in computer vision and computer graphics, and is defined as follows.

Definition 5.1 (Rosenfeld, 1979). A finite set $\alpha \subseteq \mathbb{Z}^k$ is an *m-arc* connecting p with q iff α is m-connected and each point in $\alpha - \{p, q\}$ has exactly two m-neighbors in α, while the endpoints p and q have exactly one. A finite set $C \subseteq \mathbb{Z}^k$ is a *simple closed m-curve* iff C is m-connected and all points in C have exactly two m-neighbors in C. To rule out degenerate cases the usual restriction on the minimal number of points in a simple closed curve is assumed: a 4-curve has at least eight points and an 8-curve at least four points, and so on for every m-adjacency relation.

Theorem 5.2. *A finite set $\alpha \subseteq \mathbb{Z}^k$ is an m-arc $\Leftrightarrow$ α is an sp-bicontinuous image of a closed interval with the usual topology, i.e. $(\exists f:(I, \delta) \to (\mathbb{Z}^k, \delta_m))(f(I) = \alpha$ and f is sp-bicontinuous), where $(I, \delta) \subseteq (\mathbb{R}, \delta)$ is a closed interval.*

Proof. " $\Rightarrow$ " An m-arc α connecting p with q can be regarded as a sequence of points $p = p_1, p_2, \ldots, p_n = q$ such that p_i is an m-neighbor of p_{i-1}, $1 < i \leqslant n$ (Rosenfeld, 1979). Let $([0, n], \delta) \subseteq (\mathbb{R}, \delta)$ be an interval and let $f:([0, n], \delta) \to (\mathbb{Z}^k, \delta_m)$ be defined by $f([0, 1]) = p = p_1$ and $f((i, i+1]) = p_i$ for $i = 1$ to $n-1$, where $(i, i+1]$ is a half open, half closed interval. Clearly $f([0, n]) = \alpha$. All points in $\alpha - \{p, q\}$ have exactly two m-neighbors in α, while the endpoints p and q have exactly one, and we have a similar situation for the relation δ among the intervals $[0, 1], (i, i+1]$, where $i = 1$ to $i - 1$. So, it is easy to see that f is sp-bicontinuous.

" $\Leftarrow$ " Let $f:(I, \delta) \to (\mathbb{Z}^k, \delta_m))$ be an sp-bicontinuous function such that $f(I) = \alpha$, where $I \subseteq (\mathbb{R}, \delta)$ is a closed interval. Since f is an sp-bicontinuous function, $f(I) = \alpha$ is m-connected. Note that $(\forall x \in \alpha)(f^{-1}(x)$ is connected); therefore $f^{-1}(x)$ is a subinterval of I (we treat points in $\mathbb{R}$ as (degenerate) intervals).

Now, $I = \bigcup_{x \in \alpha} f^{-1}(x)$ is the union of a collection of pair-wise disjoint intervals; every interval in this collection is δ-near exactly two other intervals in I; except for exactly the two subintervals of I which contain the endpoints of I, say these are $f^{-1}(p)$ and $f^{-1}(q)$. These two subintervals are δ-near exactly one other subinterval. By the sp-bicontinuity of f, we obtain that all points in $\alpha - \{p, q\}$ have exactly two m-neighbors in α, while p and q have exactly one. $\square$

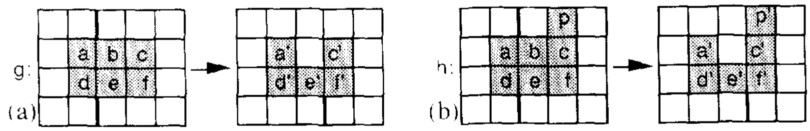

Fig. 3.

Theorem 5.3. *A finite set $C \subseteq \mathbb{Z}^k$ satisfying the restriction on the minimal number of points is a simple closed m-curve* $\Leftrightarrow$ *C is an sp-bicontinuous image of a unit circle with the usual topology, i.e.*

$$(\exists f:(S^1, \delta) \to (\mathbb{Z}^k, \delta_m))(f(S^1) = C \text{ and } f \text{ is sp-bicontinuous}),$$

where $(S^1, \delta) \subseteq (\mathbb{R}^2, \delta)$ *is a unit circle.*

Proof. The proof is very similar to the proof of Theorem 5.2. □

6. Examples of semi-proximity continuous functions on digital pictures

We give some examples of sp-continuous functions on digital sets. The digital sets considered in these examples are subsets of $(\mathbb{Z}^2, 8, 4, B)$. We will illustrate them as grey-colored squares in the following figures. For any $X \subseteq \mathbb{Z}^2$ with 8-adjacency, δ_2 is defined by

$$(\forall A, B \in P(X))(A\ \delta_2\ B \text{ iff } (\exists a \in A, b \in B)(8\text{-adj}(a, b))),$$

and γ is defined by $(\forall A, C \in P(B))(A\ \gamma\ C)$ iff $(\exists x \in A, y \in C)(x\ \gamma\ y)$, where

$$(\forall x, y \in X)(x\ \gamma\ y) \text{ iff } (\exists p \in X)((8\text{-adj}(x, p) \text{ and } 4\text{-adj}(p, y))$$
$$\text{or } (4\text{-adj}(x, p) \text{ and } 8\text{-adj}(p, y))).$$

Let g be the function between the two digital pictures given in Fig. 3(a), i.e. $g(x) = x'$ if $x \neq b$, and $g(b) = e'$. If we use δ_2 for both pictures, then g is sp-continuous, since it is clear that $u\ \delta_2\ v \Rightarrow f(u)\ \delta_2\ f(v)$. Similarly, if we use γ for both pictures, then g is sp-continuous.

Now let h be the function between the two digital pictures given in Fig. 3(b), i.e. $h(x) = x'$, for every $x \neq b$, and $h(b) = e'$. If δ_2 is used in both pictures, then h is not sp-continuous, since $b\ \delta_2\ p$, but $h(b)\ \not\delta_2\ h(p)$, i.e., $e'\ \not\delta_2\ p'$. However, if we use γ for both pictures, then h is sp-continuous, since $b\ \gamma\ p$ and $h(b)\ \gamma\ h(p)$. Note that in this case h maps points with 8-distance one onto points with 8-distance two and it still preserves the 8-connectivity of the digital image.

The function in Fig. 4 defined by $k(x) = x'$, for every $x \neq b$ and $k(b) = e'$, is not sp-continuous for either δ_2 or γ, since it does not preserve δ_2 or γ-connectedness, which are both equivalent to 8-connectedness.

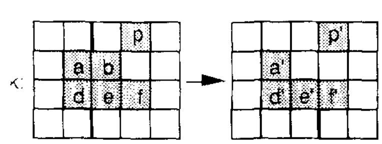

Fig. 4

7. Thinning and semi-proximity continuous functions

Many operations on 2D and 3D digital images are required to preserve connectedness in both directions, that is, there is a one-to-one correspondence between (black and white) components of the input and output image and their structure. For example, this requirement must be satisfied by any preprocessing step for character recognition, since an object with the structure of "8" should not be transformed into an object with the structure of "o" or "i". Thinning (or shrinking) is a kind of transformation where connectedness must be preserved in both directions. Thinning is a useful operation in digital image processing, since in many applications it is computationally easier to recognize the structure of a "thinner" image, provided that the thinning algorithm did not change the structure of connected components in the image. Thinning a set B of black points means deleting a subset $A \subseteq B$, i.e. changing the color of points in A from black to white. This suggests that every thinning transformation $T: B \to B - A$ should preserve automatically connectedness in the inverse direction, i.e., if X is a connected subset of $B - A$, then $T^{-1}(X)$ should be a connected subset of B. Therefore, to show that a thinning transformation preserves connectedness in both directions, it is enough to show that it maps connected sets to connected sets.

According to (Kong, 1993), a thinning algorithm (parallel or sequential) preserves connectedness (i.e. maps connected sets to connected sets) iff every set deleted by this algorithm can be ordered in a sequence such that every point is simple after all previous points are deleted. Intuitively, a point is simple if its deletion does not change locally the connected components of black and white points. Thus, this definition reduces the global problem of connectedness preservation to a local one. We show that deleting a simple point (i.e. turning its color from black to white) in a digital image $(\mathbb{Z}^2, 8, 4, Y)$ can be regarded as an sp-continuous function. Let δ_2 and γ be as defined in Section 6. Since thinning can be described as a recursive deletion of simple points, we can characterize a thinning algorithm transforming a black set Y to its subset X as a sequence of sp-continuous functions $f_n : (Y_n, \delta_2) \to (Y_{n+1}, \gamma)$ for $n = 1, \ldots, k-1$ such that $Y_1 = Y$ and $Y_k = X$ and $Y_n = Y_{n+1} - \{p_n\}$, where p_n is a simple point in Y_n. Since sp-connectedness relations induced by δ_2 and by γ are both equivalent to 8-connectedness, it follows that this thinning algorithm preserves 8-connectedness. An advantage of this approach to thinning is that the connectedness preservation of a thinning algorithm follows automatically from sp-continuity of functions f_n. The following results are stated for 2D images with 8-connectedness relation. However, analogous results can also be proved for other connectedness relations and 3D images. We do not present them here, since their proofs require a complicated pattern analysis and our main goal is to demonstrate the advantages of the approach of viewing a thinning algorithm as a sequence of sp-continuous functions. Theorem 7.2 shows that sp-continuity can be used to characterize the deletion of simple points defined as follows.

Definition 7.1. A black point p in $(\mathbb{Z}^2, 8, 4, B)$ is said to be *simple* iff
(C1) p is 8-adjacent to only one black 8-component in $N_8(p) - \{p\}$, and
(C2) p is 4-adjacent to only one white 4-component in $N_8(p)$.

Theorem 7.2. *Let Y be a set of black points in $(\mathbb{Z}^2, 8, 4, Y)$ and $X = Y - \{p\}$, where $p \in Y$. If p is simple in $(\mathbb{Z}^2, 8, 4, Y)$, then there exists an sp-continuous function $f:(Y, \delta_2) \to (X, \gamma)$, where δ_2 and γ are as defined in Section 6.*

Proof. Recall that δ_2 is just an extension of 8-adjacency to sets, i.e. $(\forall a, b \in Y)(a\ \delta_2\ b \Leftrightarrow \text{8-adj}(a, b))$. Thus, we have $(\forall y \in Y)(p\ \delta_2\ y \Leftrightarrow y \in Y \cap N_8(p))$. Let p be simple in $(\mathbb{Z}^2, 8, 4, Y)$. Then, there exists a white 4-neighbor w of p. Assume first that w is such that there also exists a black 4-neighbor q of p such that w and q lie on the opposite sides of p (see Fig. 5(a), where the dotted squares denote points of either color). We show that the function defined by $f(p) = q$ and $(\forall y \in Y - \{p\})(f(y) = y)$ is continuous. Since $(\forall y \in Y)(p\ \delta_2\ y \Leftrightarrow$

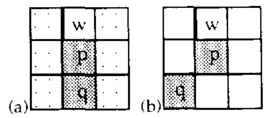

Fig. 5.

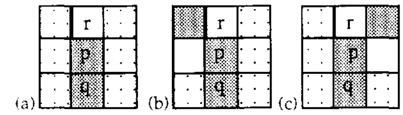

Fig. 6.

$y \in Y \cap N_8(p)$), it remains to show that $(\forall y \in Y \cap N_8(p))(f(y)\ \gamma\ f(p))$. This follows from the fact that $(\forall x \in X \cap N_8(p))(x\ \gamma\ q)$, which can be easily checked.

If there is no pair of 4-neighbors q and w of p such that w and q lie on the opposite sides of p and w is white while q is black, then, by the simplicity of p, all 4-neighbors of p are white and there is exactly one black 8-neighbor q of p (see Fig. 5(b)). In this case the function $f(p) = q$ and $(\forall y \in Y - \{p\})(f(y) = y)$ is clearly continuous. □

All thinning algorithms do not delete a simple point that does not have a black 4-neighbor, since such a point has only one black 8-neighbor, and therefore it is an endpoint. Thus, only simple points that have a black 4-neighbor need be considered. The following equivalence characterizes the simplicity of such points.

Theorem 7.3. *Let $p \in Y$ in $(\mathbb{Z}^2, 8, 4, Y)$ be such that $(\exists q \in Y - \{p\})$(4-adj$(p, q)$). p is simple in $(\mathbb{Z}^2, 8, 4, B)$ iff the function $f:(Y, \delta_2) \to (Y - \{p\}, \gamma)$, $f(p) = q$ and $(\forall y \in Y - \{p\})(f(y) = y)$, is sp-continuous.*

Proof. "$\Rightarrow$" It is just the proof of Theorem 7.2.

"$\Leftarrow$" If the 4-neighbor of p opposite to q, say r, were black (i.e. $r \in Y$), then f would not be sp-continuous, since $r\ \delta_2\ p$, but the equivalence $(f(r)\ \gamma\ f(p) \Leftrightarrow r\ \gamma\ q)$ is false (see Fig. 6(a), where the dotted squares denote points of either color). Therefore, r must be white (i.e. $r \notin Y$). If there were two black 8-components or two white 4-components, then one of the configurations (b) and (c) in Fig. 6 would occur in $N_8(p)$. In both cases, f would not be sp-continuous. Thus there is exactly one black 8-component and one white 4-component in $N_8(p)$. This implies that p is simple. □

Theorems 7.2 and 7.3 establish a relation between sp-continuous functions and thinning algorithms, which gives a new view of thinning as a sequence of sp-continuous mappings, so that the connectivity preservation of a thinning algorithm follows from the sp-continuity.

8. Semi-proximity and other digital continuous functions

The example given in Fig. 7 shows that semi-proximity continuous functions on digital images are more flexible than metric and topologically continuous functions. Consider the set of black points X on the left side in Fig. 7. The deletion of a simple point $p \in X$ cannot be described as a continuous function in Rosenfeld's

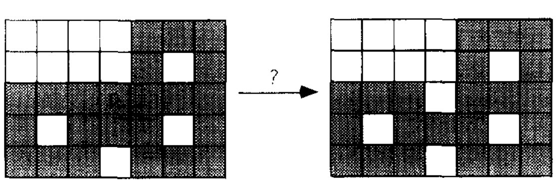

Fig. 7.

sense, since the definition of continuity in (Rosenfeld, 1986) requires that points with distance one are mapped onto points with distance at most one and Rosenfeld's restrictions on the distance imply that only 4-neighbors are allowed to have distance one. So, two 8-neighbors must have distance greater than one. Yet the point p is at distance one to its three black 4-neighbors, and there is no other black point (onto which p could be mapped) which is at distance $\leqslant 1$ to these three 4-neighbors. Thus, Theorem 7.2 (and therefore Theorem 7.3) is not true if we use the definition of metric continuous function given in (Rosenfeld, 1986).

This example also shows that the deletion of a simple point p cannot be described as a digital retraction as defined in (Boxer, 1994), since Boxer's definition of digital retraction is based on Rosenfeld's definition of metric continuity: A function $r: A \to B$, where $B \subseteq A$, is a *digital retraction* if r is metric continuous and $r(b) = b$ for all $b \in B$. For the same reason, the sets X and $X - \{p\}$ in Fig. 7 are also not homotopy equivalent in the digital sense as defined in (Boxer, 1994): As shown above, there is no metric continuous function between X and $X - \{p\}$, and there is no other set contained in $X - \{p\}$ which is digital homotopy equivalent to X and $X - \{p\}$, because there is no proper subset of $X - \{p\}$ onto which the $X - \{p\}$ could be metric continuously mapped. If we base Boxer's definitions of digital retraction and homotopy on sp-continuity, then $X - \{p\}$ will become a digital retract of X and the two sets will become homotopy equivalent: The function that maps p onto its south 4-neighbor and does not move the other points in X is sp-continuous if we treat X and $X - \{p\}$ as digital images described in Theorem 7.2.

The definition of metric continuous functions cannot be extended by allowing greater distance between the image points, e.g. points with distance one are mapped onto points with distance at most two, since such a function would not preserve connectedness. In particular, a metric continuous function $f: (\mathbb{Z}^2, d_4) \to (\mathbb{Z}^2, d_8)$ does not preserve 8-connectedness. On the other hand, the deletion of the simple point p in Fig. 7 cannot be described as a metric continuous function $f: (\mathbb{Z}^2, d_8) \to (\mathbb{Z}^2, d_8)$, since p must be mapped by f onto one of its black 8-neighbors, but then $f(p)$ cannot be at 8-distance one to the other black 8-neighbors of p. These limitations also apply to continuous functions in the T_0 topology on $\mathbb{Z}^2$ described in (Khalimsky et al., 1990) and (Kovalevsky, 1989). In this topology, $x \in (y)^c \Rightarrow d_8(x, y) \leqslant 1$, and therefore a topologically continuous function maps points with 8-distance one to the points with 8-distance at most one.

9. Concluding remarks

Starting with the intuitive concept of "nearness" as a binary relation, semi-proximity spaces have been defined. It was shown that all digital images used in computer vision and computer graphics can be treated as non-trivial semi-proximity spaces. Examples of different nearness relations on digital images have been given which induce the usual connectedness on digital images. Since the nearness relation on digital images shows many properties of the spatial concept "near", a digital structure with this relation can be useful in spatial reasoning, in particular, for pictorial inferences based on digital picture generating and inspection processes.

It was shown that semi-proximity continuous functions can be more flexible than metric and topologically continuous functions on digital images while still preserving the usual connectedness. Useful examples of a semi-proximity continuous function were given which can map points with 8-distance one onto points with 8-distance two and which still preserve the 8-connectedness of the digital image.

The semi-proximity presented in this paper establishes a formal relation between $\mathbb{R}^n$ and digital images as sp-bicontinuous functions. This is possible, since $\mathbb{R}^n$ with the usual topology and digital images with their usual structure are sp-spaces. This relation guarantees a complete equivalence of connected components, since sp-bicontinuous functions preserve connectedness in both directions. These properties and a great variety of sp-relations on digital images implies that sp-continuous functions can be used to divide digital images into classes, which can be useful in the difficult task of shape description.

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