

# **BLOWING UP THE POWER OF A SINGULAR CARDINAL**

**Moti Gitik**

School of Mathematics  
Raymond and Beverly Sackler  
Faculty of Exact Sciences  
Tel Aviv University  
Tel Aviv 69978 Israel

## Introduction

Suppose that  $\kappa$  is a singular cardinal of cofinality  $\omega$ . We like to blow up its power. Overlapping extenders were used for this purpose in [Git-Mag2]. On the other hand, it is shown in [Git-Mit] that it is necessary to have for every  $n < \omega$  unboundedly many  $\alpha$ 's in  $\kappa$  with  $o(\alpha) \geq \alpha^{+n}$ . The aim of the present paper is show that this assumption is also sufficient. Ideas of [Git hid. et] will be extended in order to produce  $\kappa^{++}$   $\omega$ -sequences. In [Git hid. ext] an  $\omega$ -sequence corresponding to two different sequences of measures was constructed. Here we would like to construct a lot of  $\omega$ -sequences corresponding to the same sequence of measures.

The first stage will be to force with a forcing which produces  $\kappa^{++}$  Prikry sequences but the cost is that  $\kappa^{++}$  is collapsed. Then a projection of this forcing will be defined such that the resulting forcing will still have  $\kappa^{++}$  Prikry sequences but also satisfy  $\kappa^{++}$ -c.c. and preserve  $\kappa$  strong limit cardinal.

### 1. Preparation Forcing– the first try

Let us assume GCH. Suppose that  $\kappa_\omega = \bigcup_{n < \omega} \kappa_n$  and  $o(\kappa_n) = \kappa_n^{+n+2} + 1$ . We will define a forcing which will combine ideas of [Git-Mag2] and [Git hid. ext]. In contrast to [Git hid. ext] we like to produce lots of Prikry sequences even by the cost of collapsing cardinals. The main feature of this forcing will be the Prikry condition. Splitting it above and below  $\kappa_n$  ( $n < \omega$ ) we will be able to conclude that the part above  $\kappa_n$  does not add new subsets to  $\kappa_n$  and the part below does not effect cardinals above  $\kappa_n$ . The problematic cardinal will be  $\kappa_\omega^{++}$ . In order to prevent it from collapsing we construct a projection of the forcing which will satisfy  $\kappa_\omega^{++}$ -c.c.

For every  $n < \omega$ . Let us fix a nice system  $\mathbb{U}_n = \langle \langle \mathcal{U}_{n,\alpha} \mid \alpha < \kappa_n^{+n+2} \rangle, \langle \pi_{n,\alpha,\beta} \mid \alpha, \beta < \kappa_n^{+n+2}, \mathcal{U}_{n,\alpha} \triangleleft \mathcal{U}_{n,\beta} \rangle \rangle$ . We refer to [Git-Mag1] for the basic definitions. Actually an extender of the length  $\kappa_n^{+n+2}$  will be fine for our purpose as well.

For every  $n < \omega$ , let us first define a forcing notion  $\langle Q_n, \leq_n \rangle$  and then use it as the level  $n$  in the main forcing.

Fix  $n < \omega$ . We like to define a forcing  $\langle Q_n, \leq_n \rangle$ . Let us drop the lower index  $n$  for a while.

$Q$  will be the union of two sets  $Q^0$  and  $Q^1$  defined below.

**Definition 1.1.** Set  $Q^1$  to be the product of  $\{p \mid p \text{ is a partial function from } \kappa^{+n+2} \text{ to } \kappa^{+n+2} \text{ such that } \text{dom } p \text{ is an ordinal less than } \kappa^{+n+2}\}$  and  $\{q \mid q \text{ is a partial function from } \kappa_\omega^{++} \text{ to } \kappa^{+n+2} \text{ of cardinality less than } \kappa_\omega^+\}$ .

The ordering on  $Q^1$  is an inclusion. I.e.  $Q^1$  is the product of the product of two Cohen forcings: for adding a new subset to  $\kappa^{+n+2}$  and for adding  $\kappa_\omega^{++}$  new subsets to  $\kappa_\omega^+$ .

**Definition 1.2.** A set  $Q^0$  consists of triples  $\langle p, a, f \rangle$  where

- (1)  $p = \langle \{ \langle \gamma, p^\gamma \rangle \mid \gamma < \delta \}, g, T \rangle$  where
  - (1a)  $g \subseteq \kappa^{+n+2}$  of cardinality  $< \kappa$ .
  - (1b)  $\delta < \kappa^{+n+2}$
  - (1c)  $o \in g$  and every initial segment of  $g$  (including  $g$  itself) has the least upper bound in  $g$ .
  - (1d)  $\delta > \max(g)$
  - (1e) for every  $\gamma \in g$   $p^\gamma$  is the empty sequence
  - (1f)  $T \in \mathcal{U}_{\max(g)}$
  - (1h) for every  $\gamma \in \delta \setminus g$   $p^\gamma$  is an ordinal below  $\kappa_\omega^{++}$ .

Further we shall denote  $g$  by  $\text{supp}(p)$ , the maximal element of  $g$  by  $mc(p)$ ,  $\delta$  by  $\delta(p)$  and  $T$  by  $T(p)$ . Let us refer to ordinals below  $\delta(p)$  as coordinates. We will frequently confuse between an ordinal  $\gamma$  and one element sequence  $\langle \gamma \rangle$ .

- (2)  $a$  is a partial one to one order preserving function between  $\kappa_\omega^{++}$  and  $\delta(p)$  of cardinality less than  $\kappa$ . Also every  $\gamma \in \text{dom } a$  is below  $mc(p)$  in sense of the ordering of extender  $\mathbb{U}$ .
- (3)  $f$  is a partial function from  $\kappa_\omega^{++}$  to  $\kappa^{+n+2}$  of cardinality less than  $\kappa_\omega^+$  and such that  $\text{dom } f \cap \text{dom } a = \emptyset$ .

Let us give some intuitive motivation for the definition of  $Q^0$ . Basically we like to add  $\kappa_\omega^{++}$ . Prikry sequences (actually a one element sequence).

The length of the extender used is only  $\kappa^{+n+2}$ . A typical element of  $Q^0$  consists of a triple  $\langle p, a, f \rangle$ . The first part of it  $p$  is as a condition of [Git-Mag1] with slight changes need for mainly technical reasons. The idea is to assign ordinals  $< \kappa_\omega^{++}$  to the coordinates of such  $p$ 's.  $a$  is responsible for this assignment. Basically, if for some  $\alpha < \kappa_\omega^{++}$ ,  $\beta < \kappa^{+n+2}$   $a(\alpha) = \beta$ , then  $\alpha$ -th sequence will be read from the  $\beta$ -th Prikry sequence. Clearly, we do not want to allow this assignment to grow into the one to one correspondence between  $\kappa^{+n+2}$  and  $\kappa_\omega^{++}$ . The third part  $f$  and mainly the definition of the ordering below is designed to prevent such correspondence.

**Definition 1.3.**  $Q = Q^0 \cup Q^1$ .

Let us turn to the definition of the order over  $Q$ . First we define  $\leq^*$  the pure extension.

**Definition 1.4.** Let  $t, s \in Q$ . Then  $t \leq^* s$  if either

- (1)  $t, s \in Q^1$  and  $t$  is weaker than  $s$  in the ordering of  $Q^1$  or
- (2)  $t, s \in Q^0$  and the following holds:

let  $t = \langle p, a, f \rangle$ ,  $s = \langle q, b, g \rangle$  (2a)  $p \leq^* q$  in sense of [G2t-Mag1] with only addition in (v):

- (i)  $\delta(p) \leq \delta(q)$
- (ii)  $\text{supp}(p) \subseteq \text{supp}(q)$
- (iii) for every  $\gamma < \delta(p)$   $p^\gamma = q^\gamma$
- (iv)  $\pi_{mc(q), mc(p)}$  projects  $T(q)$  into  $T(p)$
- (v) for every  $\gamma \in \text{supp}(p) \cup \text{dom } a$  and  $\nu \in T(q)$

$$\pi_{mc(q), \gamma}(\nu) = \pi_{mc(p), \gamma}(\pi_{mc(q), mc(p)}(\nu)) .$$

$$(2b) \quad a \subseteq b$$

$$(2c) \quad f \subseteq g.$$

Notice that in contrast to [Git-Mag1], the commutativity in (2a)(v) does not cause a special problem since the number of coordinates  $\text{supp}(p) \cup \text{dom } a$  has cardinality  $< \kappa$ , i.e. below the degree of completeness of ultrafilters in the extender used here.

**Definition 1.4.1.** Let  $s, t \in Q$ . We say that  $s$  extends  $t$  if  $t \leq^* s$  or  $t \in Q^0$ ,  $s \in Q^1$  and the conditions below following hold.

Let  $t = \langle p, a, f \rangle$  and  $s = \langle q, h \rangle$ .

- (1)  $\delta(p) \leq \text{dom } q$  (recall that by 1.1,  $\text{dom } q$  is an ordinal  $< \kappa^{+n+2}$ ).
- (2) for every  $\gamma \in \delta(p) \setminus \text{supp}(p)$  if  $p^\gamma < \kappa^{+n+2}$  then  $p^\gamma = q(\gamma)$  otherwise  $q(\gamma) = \kappa$ .
- (3)  $q(mc(p)) \in T(p)$
- (4) for every  $\gamma \in \text{supp}(p)$   $q(\gamma) = \pi_{mc(p), \gamma}(q(mc(p)))$
- (5)  $h \supseteq f$
- (6)  $\text{dom } h \supseteq \text{dom } a$
- (7) for every  $\beta \in \text{dom } a$   $h(\beta) = q(a(\beta))$ , if  $a(\beta) \in \text{supp}(p)$  or  $h(\beta) = \pi_{mc(p), a(\beta)}(q(mc(p)))$ , otherwise.

The conditions (1) to (4) are as in [Git-Mag 1] with only change in (2) in case  $p^\gamma \geq \kappa^{+n+2}$ . Then it is replaced by  $\kappa$ . The idea behind this is to remove unnecessary information a condition may have in order to prevent collapses of cardinals above  $\kappa^{+n+2}$ . The conditions (5) to (7) are the heard of the matter. Our purpose is to forbid the assignment  $a$  from growing into a 1–1 function from  $\kappa_\omega^{++}$  to  $\kappa^{+n+2}$  but to still produce  $\kappa_\omega^{++}$ -sequences. What actually happens in the definition is a switch from Prikry type harmful forcing to a nice Cohen type forcing. The only essential information from  $a$  is put into  $h$ . The actual place of the sequence  $\beta(\beta \in \text{dom } a)$  is hidden after passing from  $t$  to  $s$ .

**Lemma 1.5.**  $Q^1$  is dense in  $Q$ .

The proof follows from Definition 1.4.1.

**Lemma 1.6.**  $\langle Q, \leq \rangle$  does not collapse cardinals or blows up their powers.

Follows from 1.5.

**Lemma 1.7.**  $\langle Q, \leq, \leq^* \rangle$  satisfies the Prikry condition.

The proof of the parallel statement of [Git-Mag 1] applies here without essential changes.

Now let us put all  $Q_n$ 's defined above together.

**Definition 1.8.** A set of forcing conditions  $\mathcal{P}$  consists of all elements  $p$  of the form  $\langle p_n \mid n < \omega \rangle$  so that

- (1) for every  $n < \omega$   $p_n \in Q_n$
- (2) there exists  $\ell < \omega$  such that for every  $n \geq \ell$   $p_n \in Q_n^0$ .

Let us denote further the least such  $\ell$  by  $\ell(p)$ .

**Definition 1.9.** Let  $p = \langle p_n \mid n < \omega \rangle, q = \langle q_n \mid n < \omega \rangle \in \mathcal{P}$ . We say that  $p$  extends  $q$  ( $p \geq q$ ) if for every  $n < \omega$   $p_n$  extends  $q_n$  in the ordering of  $Q_n$ .

**Definition 1.10.** Let  $p, q \in \mathcal{P}$ . We say that  $p$  is a direct or pure extension  $q$  iff  $p \geq q$  and  $\ell(p) = \ell(q)$ .

**Lemma 1.11.**  $\langle \mathcal{P}, \leq, \leq^* \rangle$  satisfies the Prikry condition.

**Sketch of the Proof.** Let  $\sigma$  be a statement of the forcing language and  $p \in \mathcal{P}$ . We are looking for  $q \geq^* p$  deciding  $\sigma$ . Assume for simplicity that  $\ell(p) = 0$ . As in [Git-Mag 1] we extend  $p$  level by level trying to decide  $\sigma$ . Suppose that we passed level 0 and are now on level 1. We have here basically two new points. The first to our advantage is that the measures on the level 1 are  $\kappa_1$ -complete and  $\kappa_1 > \kappa_0$ . So we can always shrink sets of measure 1 in order to have the same condition in  $Q_0^0$  on the level 0. The second point is that the cardinality of  $Q_0^1$  is big. However let us then use the completeness of  $Q_0^1$ . Recall that  $Q_0^1$  is  $\kappa_\omega^+$ -closed forcing.

The rest of the proof is parallel to [Git-Mag 1]. □

Let  $G$  be a generic subset of  $\mathcal{P}$ . For  $\beta < \kappa_\omega^{++}$  let  $G(\beta) : \omega \rightarrow \kappa_\omega$  be the function defined as follows.  $G(\beta)(n) = \nu$  iff there is  $\langle p_k \mid k < \omega \rangle \in G$  such that  $\beta \in \text{dom } p_{n,2}$   $p_{n,2}(\beta) = \nu$ , where  $p_{n,2}$  is the second coordinate of  $p_n \in Q_n^1$ .

Notice that we cannot claim  $G(\beta)$ 's are increasing with  $\beta$ . Actually, lots of them will be old sequences and also they may be equal or reverse the order. But the following is still true.

**Lemma 1.12.** For every  $\gamma < \kappa_\omega^{++}$  there is  $\beta, \gamma < \beta < \kappa_\omega^{++}$  such that  $G(\beta)$  is above every  $G(\beta')$  with  $\beta' < \beta$ .

**Proof:** Work in  $V$ . Let  $p \in \mathcal{P}$ . Suppose for simplicity that  $\ell(p) = 0$ . Otherwise work above the level  $\ell(p) - 1$ . Let  $p = \langle p_n \mid n < \omega \rangle$  and  $p_n = \langle p_{n0}, p_{n1}, p_{n2} \rangle$  ( $n < \omega$ ). Pick some  $\beta, \gamma < \beta < \kappa_\omega^{++}$  which above everything appears in  $p$ , i.e.  $\beta > \cup\{\delta(p_{n0}) \cup \sup(\text{dom } p_{n1} \cup \text{dom } p_{n2}) \mid n < \omega\}$ . Extend  $p$  to a condition  $q = \langle q_n \mid n < \omega \rangle$ ,  $q_n = \langle q_{n0}, q_{n1}, q_{n2} \rangle$  such that  $q_{n1} = p_{n1}, q_{n2} = p_{n2}$  and  $mc(q_{n0}) > mc(p_{n0})$  for every  $n < \omega$ . Extend now  $q$  to  $r = \langle r_n \mid n < \omega \rangle$ ,  $r_n = \langle r_{n0}, r_{n1}, r_{n2} \rangle$  by adding the pair  $\langle \beta, mc(q_{n0}) \rangle$  to  $q_{n1}$  for every  $n < \omega$ .

We claim that

$$r \Vdash \left( G(\beta) >_{\sim} G(\beta') \text{ for every } \beta' < \beta \right).$$

Fix  $\beta' < \beta$  and let  $s \geq r$ . W.l. of  $g$ .  $\ell(s) = \ell(r) = 0$ . Since otherwise we repeat the same argument above  $\ell(s)$ . Let  $s = \langle s_n \mid n < \omega \rangle$  and  $s_n = \langle s_{n0}, s_{n1}, s_{n2} \rangle$  for every  $n < \omega$ . Denote by  $A$  the set of all  $n$ 's such that  $\beta' \in \text{dom } s_{n1}$ . For every  $n \in \omega \setminus A$  extend  $s_n$  by adding there pair  $\langle \beta', 0 \rangle$  to  $s_{n2}$ . Let us still denote the resulting condition by  $s$ . Then the function  $G(\beta') \upharpoonright \omega \setminus A$  will be forced by  $s$  to be an old function. Hence  $G(\beta) \upharpoonright \omega \setminus A$  is above it.

Now let  $n \in A$ . Then, since  $\beta' < \beta$ ,  $\beta', \beta \in \text{dom } s_{n1}$  and  $s_{n1}$  is order preserving, the coordinate assigned to  $\beta'$  by  $s_{n1}$  is below the one assigned to  $\beta$ . Hence  $s$  forces that  $G(\beta) \upharpoonright A$  is above  $G(\beta') \upharpoonright A$  and we are done.  $\square$

For  $n < \omega$  let us split  $\mathcal{P}$  into  $\mathcal{P} \upharpoonright n$  and  $\mathcal{P} \setminus n$  as follows:

$$\begin{aligned} \mathcal{P} \upharpoonright n &= \{p \upharpoonright n \mid p \in \mathcal{P}\} \\ \mathcal{P} \setminus n &= \{p \setminus n \mid p \in \mathcal{P}\}. \end{aligned}$$

The following lemma is routine

**Lemma 1.13.** *For every  $n < \omega$  the forcing with  $\mathcal{P}$  is the same as the forcing with  $(\mathcal{P} \setminus n) \times (\mathcal{P} \upharpoonright n)$ .*

**Lemma 1.14.**  *$\langle \mathcal{P}, \leq \rangle$  preserves the cardinals  $\leq \kappa_\omega^+$  and GCH holds below  $\kappa_\omega$  in a generic extension by  $\mathcal{P}$ .*

**Proof:** For every  $n < \omega$   $\kappa_{n+1}$  is preserved since  $\mathcal{P}$  splits as 1.13 into a forcing  $\mathcal{P} \setminus n$  and  $\mathcal{P} \upharpoonright n$ . By analogous of 1.11 for  $\mathcal{P} \setminus n$ ,  $\mathcal{P} \setminus n$  does add new bounded subsets of  $\kappa_{n+1}$ . By 1.6,

$\mathcal{P} \restriction n$  preserves cardinals. Therefore, nothing below  $\kappa_\omega$  is collapsed. Now if  $\kappa_\omega^+$  is collapsed then  $|\kappa_\omega^+| = \kappa_\omega$  which is impossible by the Weak Covering Lemma [Mit-St-Sch] or just directly using arguments like those of [Git-Mag 1], Lemma 1.11.  $\square$

Unfortunately,  $\kappa_\omega^{++}$  is collapsed by  $\mathcal{P}$  as it is shown in the next lemma.

**Lemma 1.15.** *In  $V[G]$   $|(\kappa_\omega^{++})^V| = \kappa_\omega^+$ .*

**Proof:** Work in  $V$ . The cardinality of the set  $\prod_{n < \omega} \kappa_n^{+n+2} / \text{finite}$  is  $\kappa_\omega^+$ . Fix some enumeration  $\langle g_i \mid i < \kappa_\omega^+ \rangle$  of it.

Now in  $V[G]$ , let  $p = \langle p_n \mid n < \omega \rangle \in G$ ,  $p_n = \langle p_{n0}, p_{n1}, p_{n2} \rangle$  ( $n < \omega$ ),  $\beta < \kappa_\omega^{++}$  and starting with some  $n_0 < \omega$   $\beta \in \text{dom } p_{n_1}$ . Find  $i < \kappa_\omega^+$  s.t. the function  $\{\langle n, p_{n1}(\beta) \rangle \mid n \geq n_0\}$  belongs to the equivalence class  $g_i$ . Set then  $i \mapsto \beta$ . Using genericity of  $G$  it is easy to see that this defines a function from  $\kappa_\omega^+$  unboundedly into  $\kappa_\omega^{++}$ .  $\square$

We would like to project the forcing  $\mathcal{P}$  to a forcing preserving  $\kappa_\omega^{++}$ . The idea is to make it impossible to read from the sequence  $G(\beta)$  ( $\beta < \kappa_\omega^{++}$ ) the sequence of coordinates (mod finite) which produces  $G(\beta)$  in sense of 1.15. The methods of [Git] will be used for this purpose. But first the forcing  $\mathcal{P}$  should be fixed slightly. The point is that we like to have much freedom in moving  $\beta$ 's from the beginning.  $\mathcal{P}$  is quite rigid in this sense. Thus, for example, if some  $\beta < \kappa_\omega^{++}$  corresponds to a sequence of coordinates  $g$  in  $\prod_{n < \omega} \kappa_n^+$ , then using  $G(\beta)$  only it is easy to reconstruct  $g$  modulo finite.

## 2. The Preparation Forcing

Suppose that  $n < \omega$  is fixed. For every  $k \leq n$  we consider a language  $\mathcal{L}_{n,k}$  containing a constant  $c_\alpha$  for every  $\alpha < \kappa_n^{+k}$  and a structure

$$\mathfrak{a}_{n,k} = \langle H(\lambda^{+k}), \in, \lambda, 0, 1, \dots, \alpha, \dots, \mid \alpha < \kappa_n^{+k} \rangle$$

in this language, where  $\lambda$  is a regular cardinal big enough. For an ordinal  $\xi < \lambda$  (usually  $\xi$  will be below  $\kappa_n^{+n+2}$ ) we denote by  $tp_{n,k}(\xi)$  the  $\mathcal{L}_{n,k}$ -type realized by  $\xi$  in  $\mathfrak{a}_{n,k}$ . Let  $\delta < \lambda$ .  $\mathcal{L}_{n,k,\delta}$  will be the language obtained from  $\mathcal{L}_{n,k}$  by adding a new constant  $c$ .  $\mathfrak{a}_{n,k,\delta}$  will be  $\mathcal{L}_{n,k,\delta}$ -structure obtained from  $\mathfrak{a}_{n,k}$  by interpreting  $c$  as  $\delta$ . The type  $tp_{n,k}(\delta, \xi)$  is defined in the obvious fashion. Further we shall freely identify types with ordinals corresponding



to them in some fixed well ordering of the power sets of  $\kappa_n^{+k}$ 's. The following is an easy statement proved in [Git].

**Lemma 2.0.** *Suppose that  $\alpha_0, \alpha_1 < \kappa_n^{+n+2}$  are realizing the same  $\mathcal{L}_{n,k,\rho}$ -type for some  $\rho < \min(\alpha_0, \alpha_1)$  and  $n \geq k > 0$ . Then for every  $\beta$ ,  $\alpha_0 \leq \beta < \kappa_n^{+n+2}$  there is  $\gamma$ ,  $\alpha_1 \leq \gamma < \kappa_n^{+n+2}$  such that the  $k-1$ -type realized by  $\beta$  over  $\alpha_0$  (i.e.  $\mathcal{L}_{n,k-1,\alpha_0}$ -type) is the same as those realized by  $\gamma$  over  $\alpha_1$ .*

**Lemma 2.1.** *Let  $\gamma < \kappa_n^{+n+2}$ . Then there is  $\alpha < \kappa_n^{+n+2}$  such that for every  $\beta \in (\alpha, \kappa_n^{+n+2})$  the type  $tp_{n,n}(\gamma, \beta)$  appears (is realized) unboundedly often in  $\kappa_n^{+n+2}$ .*

**Proof:** The total number of such types is  $\kappa_n^{+n+1}$ . Let  $\langle t_i \mid i < \kappa_n^{+n+1} \rangle$  be an enumeration of all of them. For each  $i < \kappa_n^{+n+1}$  set  $A_i$  to be the subset of  $\kappa_n^{+n+2}$  consisting of all the ordinals realizing  $t_i$ . Define  $\alpha$  to be the supremum of  $\{\cup A_i \mid i < \kappa_n^{+n+1} \text{ and } A_i \text{ is bounded in } \kappa_n^{+n+2}\}$ .  $\square$

**Lemma 2.2.** *Let  $\gamma < \kappa_n^{+n+2}$ . Then there is a club  $C \subseteq \kappa_n^{+n+2}$  such that for every  $\beta \in C$  the type  $tp_{n,n}(\gamma, \beta)$  is realized stationary many times in  $\kappa_n^{+n+2}$ .*

**Proof:** Similar to 2.1.  $\square$

**Lemma 2.3.** *The set  $C = \{\beta < \kappa_n^{+n+2} \mid \text{for every } \gamma < \beta \text{ } tp_{n,n}(\gamma, \beta) \text{ is realized stationary often in } \kappa_n^{+n+2}\}$  containing a club.*

**Proof:** Suppose otherwise. Let  $S = \kappa_n^{+n+2} \setminus C$ . Then

$$S = \{\beta < \kappa_n^{+n+2} \mid \exists \gamma < \beta \text{ } tp_{n,n}(\gamma, \beta) \text{ appears only nonstationary often in } \kappa_n^{+n+2}\}$$

and it is stationary. Find  $S' \subseteq S$  stationary and  $\gamma' < \kappa_n^{+n+2}$  such that for every  $\beta \in S'$   $tp_{n,n}(\gamma', \beta)$  appears only nonstationary often in  $\kappa_n^{+n+2}$ . But this contradicts 2.2. Contradiction.  $\square$

For  $\ell \leq k \leq n$  and  $\mathcal{L}_{n,k}$ -type  $t$  let us denote by  $t \upharpoonright \ell$  the reduction of  $t$  to  $\mathcal{L}_{n,\ell}$ , i.e. the  $\mathcal{L}_{n,\ell}$ -type obtained from  $t$  by removing formulas not in  $\mathcal{L}_{n,\ell}$ .

**Lemma 2.4.** Let  $0 < k, \ell \leq n$ ,  $\gamma < \beta < \kappa_n^{+n+2}$  and  $t$  be a  $\mathcal{L}_{n,\ell,\gamma}$ -type realized above  $\gamma$ . Suppose that  $tp_{n,k}(\gamma, \beta)$  is realized unboundedly often in  $\kappa_n^{+n+2}$ . Then there is  $\delta$ ,  $\gamma < \delta < \beta$  realizing  $t \upharpoonright \min(k-1, \ell)$ .

**Proof:** Pick some  $\alpha$ ,  $\gamma < \alpha < \kappa_n^{+n+2}$  realizing  $t$ . Let  $\rho > \max(\beta, \alpha)$  be an ordinal realizing  $tp_{n,k}(\gamma, \beta)$ . Then  $\rho$  satisfies in  $H(\lambda^{+k})$  the following formula of  $\mathcal{L}_{n,k,\gamma}$ :

$$\exists y (c < y < x) \wedge (H(\lambda^{+k-1}) \text{ satisfies } \psi(y) \text{ for every } \psi \text{ in the set of formulas coded by } c \upharpoonright \min(k-1, \ell)).$$

Hence the same formula is satisfied by  $\beta$ . Therefore, there is  $\delta$ ,  $\gamma < \delta < \beta$  realizing  $t \upharpoonright \min(k-1, \ell)$ .  $\square$

The above lemma will be used for proving  $\kappa_\omega^{++}$ -c.c. of the final forcing via  $\Delta$ -system argument.

Let us specify now ordinals which will be allowed further to produce Prikry sequences.

**Definition 2.5.** Let  $k \leq n$  and  $\beta < \kappa_n^{+n+2}$ .  $\beta$  is called  $k$ -good iff

(1) for every  $\gamma < \beta$   $tp_{n,k}(\gamma, \beta)$  is realized unboundedly many times in  $\kappa_n^{+n+2}$  and

$$(2) \quad cf\beta \geq \kappa_n^{++}.$$

$\beta$  is called good iff for some  $k \leq n$   $\beta$  is  $k$ -good.

By Lemma 2.3, there are stationary many  $n$ -good ordinals. Also it is obvious that  $k$ -goodness implies  $\ell$ -goodness for every  $\ell \leq k \leq n$ .

**Lemma 2.5.1.** Suppose that  $n \geq k > 0$  and  $\beta$  is  $k$ -good. Then there are arbitrarily large  $k-1$ -good ordinals below  $\beta$ .

**Proof:** Let  $\gamma < \beta$ . Pick some  $\alpha > \beta$  realizing  $tp_{n,k}(\gamma, \beta)$ . The fact that  $\gamma < \beta < \alpha$  and  $\beta$  is  $k-1$ -good can be expressed in the language  $\mathcal{L}_{n,k,\gamma}$  as in Lemma 2.4. So they are in  $tp_{n,k}(\gamma, \beta)$ . Hence there is  $\delta$ ,  $\gamma < \delta < \beta$  which is  $k-1$ -good.  $\square$

Let us now turn to fixing of the forcings introduced in Section 1. We are going to use on the level  $n$  a forcing notion  $Q_n^*$ . It is defined as  $Q_n$  was with only one addition that each ordinal in the range of assignment functions is good.

**Definition 2.6.** A set  $Q_n^*$  is the subset of  $Q_n$  consisting of  $Q_n^1$  and all the triples  $\langle p, a, f \rangle$  of  $Q_n^0$  such that every  $\alpha \in \text{rng } a$  is good. The ordering of  $Q_n^*$  is just the restriction of the ordering of  $Q_n$ .

Lemma 1.5, 1.6 and 1.7 hold easily with  $Q_n$  replaced by  $Q_n^*$ . Let us show few additional properties of  $Q_n^*$  which are slightly more involved.

**Lemma 2.7.** Suppose  $\langle p, a, f \rangle \in Q_n^*$  and  $\kappa_\omega^{++} > \beta > \sup(\text{dom } a \cup \text{dom } f)$ . Then there is a condition  $\langle q, b, f \rangle \geq^* \langle p, a, f \rangle$  such that  $\beta \in \text{dom } b$  and  $b(\beta)$  is  $n$ -good.

**Proof:** Using Lemma 2.3 find some  $\xi < \kappa_n^{+n+2}$  above  $mc(p)$  which is  $n$ -good. Now extend  $p$  to  $q$  such that  $\xi \in \text{supp}(q)$ . Let  $b = a \cup \{\langle \beta, \xi \rangle\}$ . Then  $\langle q, b, f \rangle$  is as desired.  $\square$

**Lemma 2.8.** Suppose that  $\langle p, a, f \rangle, \langle q, b, g \rangle \in Q_n^*$ ,  $\beta \in \text{dom } a$  it is  $k$ -good for  $k > 1$ ,  $\{\gamma_i \mid i < \mu\} \subseteq (\beta \cap \text{dom } b) \setminus \text{dom } f$ ,  $\gamma_0 > \sup(\beta \cap \text{dom } a)$  and  $b(\gamma_0) > \sup a''(\beta \cap \text{dom } a)$ . Then there is  $\langle p^*, a^*, f \rangle$  a direct extension of  $\langle p, a, f \rangle$  such that

- (1)  $\{\gamma_i \mid i < \mu\} \subseteq \text{dom } a^*$ .
- (2) for every  $i < \mu$   $a^*(\gamma_i)$  and  $b(\gamma_i)$  are realizing the same  $k - 1$ -type
- (3) for every  $i < \mu$ , if  $b(\gamma_i)$  is  $\ell$ -good ( $\ell \leq n$ ) then  $a^*(\gamma_i)$  is  $\min(\ell, k - 1)$ -good.
- (4) if  $t$  is the  $n$ -type over  $\sup(a''(\beta \cap \text{dom } a))$  realized by the ordinal coding  $\{b(\gamma_i) \mid i < \mu\}$ , then the code of  $\{a^*(\gamma_i) \mid i < \mu\}$  realizes  $t \upharpoonright k - 1$ .

**Proof:** Denote  $\sup(a''(\beta \cap \text{dom } a))$  by  $\rho$ . Let  $t$  be the  $n$ -type over  $\rho$  realized by the ordinal coding  $\{b(\gamma_i) \mid i < \mu\}$ . By Lemma 2.4, there is  $\delta, \rho < \delta < \beta$  realizing  $t \upharpoonright k - 1$ . Let  $\langle \xi_i \mid i < \mu \rangle$  be the sequence coded by  $\delta$ . Define

$$a^* = a \cup \{\langle \gamma_i, \xi_i \rangle \mid i < \mu\}, \quad p^* = p$$

and  $f^* = f$ . Then  $\langle p^*, a^*, f^* \rangle$  is as required.  $\square$

**Lemma 2.8.1.** Suppose that  $\langle p, a, f \rangle, \langle q, b, g \rangle \in Q_n^*$  and  $\beta \in \text{dom } a$ ,  $\gamma \in \text{dom } b$  are such that

- (1)  $\beta$  is  $k$ -good for some  $k \geq 2$
- (2)  $\beta \cap \text{dom } a = \gamma \cap \text{dom } b$  and for every  $\delta \in \beta \cap \text{dom } a$   $a(\delta) = b(\delta)$
- (3)  $\beta > \sup(\text{dom } b)$ .

Then there direct extensions  $\langle p^*, a^*, f \rangle \geq^* \langle p, a, f \rangle$  and  $\langle q^*, b^*, g \rangle \geq^* \langle q, b, g \rangle$  such that

- (a)  $\text{dom } a^* = \text{dom } b^* = \text{dom } a \cup \text{dom } b$
- (b) for every  $\delta \in \text{dom } a^*$   $a^*(\delta)$  and  $b^*(\delta)$  are realizing the same  $k - 2$ -type over  $\rho =_{df} \sup a''((\beta \cap \text{dom } a))$
- (c) for every  $\delta \in \text{dom } b$  if  $b(\delta)$  is  $\ell$ -good then  $a^*(\delta)$  is  $\min(\ell, k - 2)$ -good
- (d) for every  $\delta \in \text{dom } a$  if  $a(\delta)$  is  $\ell$ -good then  $b^*(\delta)$  is  $\min(\ell, k - 2)$ -good
- (e)  $mc(p^*)$  and  $mc(q^*)$  are realizing the same  $k - 2$ -type over  $\rho$ , more over for every  $\delta \in \text{dom } a \cup \text{dom } b$  the way  $mc(p^*)$  projects to  $a^*(\delta)$  is the same as  $mc(q^*)$  projects to  $b^*(\delta)$ .

**Proof:** Let  $s$  denotes the  $k - 1$ -type realized by  $mc(q)$  over  $\rho = \sup(a''(\beta \cap \text{dom } a))$ . By Lemma 2.4, there is  $\delta, \rho < \delta < \beta$  realizing  $s$ . For every  $\eta \in \text{dom } b$  let  $\tilde{\eta}$  be the ordinal projecting from  $\delta$  exactly the same way as  $b(\eta)$  projects from  $mc(q)$ . Notice that for  $\eta \in \text{dom } b \cap \text{dom } a$   $\tilde{\eta} = b(\eta) = a(\eta) < \rho$ . Also,  $\tilde{\eta}$  and  $b(\eta)$  are realizing the same  $k - 1$ -type over and if  $b(\eta)$  is  $\ell$ -good then  $\tilde{\eta}$  is  $\min(\ell, k - 1)$ -good, for every  $\eta \in \text{dom } b$ .

Pick  $p^*$  to be a direct extension of  $p$  with  $mc(p^*)$  above  $mc(p)$ ,  $\delta$ . Set  $a^* = a \cup \{\langle \eta, \tilde{\eta} \rangle \mid \eta \in \text{dom } b\}$ . Now we should define the condition  $\langle q^*, b^*, g \rangle$ . Since  $\delta$  and  $mc(q)$  are realizing the same  $k - 1$ -type, by Lemma 2.0 there exists  $\nu$  realizing over  $mc(q)$  the same  $k - 2$ -type as  $mc(p^*)$  is realizing over  $\delta$ . For  $\eta \in \text{dom } a$  define  $\tilde{\eta}$  as above only using  $mc(p^*)$  and  $\nu$  instead of  $\delta$  and  $mc(q)$ . Set  $b^* = b \cup \{\langle \eta, \tilde{\eta} \rangle \mid \eta \in \text{dom } a\}$ . Let  $q^*$  be the condition obtained from  $q$  by adding  $\nu$  as a new maximal coordinate. Then  $\langle q^*, b^*, g \rangle$  is as desired.  $\square$

Let us now define the forcing  $\mathcal{P}^*$ .

**Definition 2.9.** A set of forcing conditions  $\mathcal{P}^*$  consists of all elements  $p = \langle p_n \mid n < \omega \rangle \in \mathcal{P}$  such that for every  $n < \omega$

- (1)  $p_n \in Q_n^*$
- (2) if  $n \geq \ell(p)$  then  $\text{dom } p_{n,1} \subseteq \text{dom } p_{n+1,1}$  where  $p_n = \langle p_{n0}, p_{n1}, p_{n2} \rangle$
- (3) if  $n \geq \ell(p)$  and  $\beta \in \text{dom } p_{n,1}$  then for some nondecreasing converging to infinity sequence of natural numbers  $\langle k_m \mid \omega > m \geq n \rangle$  for every  $m \geq n$   $p_{m,1}(\beta)$  is  $k_m$ -good.

The ordering of  $\mathcal{P}^*$  is as that of  $\mathcal{P}$ .

The intuitive meaning of (3) is that we are trying to make the places assigned to the  $\beta$ -th sequence more and more indistinguishable while climbing to higher and higher levels.

The following lemma is crucial for transferring the main properties of  $\mathcal{P}$  to  $\mathcal{P}^*$ .

**Lemma 2.10.**  $\langle \mathcal{P}^*, \leq^* \rangle$  is  $\kappa_0$ -closed.

**Proof:** Let  $\langle p(\alpha) \mid \alpha < \mu < \kappa_0 \rangle$  be a  $\leq^*$ -increasing sequence of conditions of  $\mathcal{P}^*$ . Let for each  $\alpha < \mu$   $p(\alpha) = \langle p(\alpha)_n \mid n < \omega \rangle$  and for each  $n < \omega$   $p(\alpha)_n = \langle p(\alpha)_{n0}, p(\alpha)_{n1}, p(\alpha)_{n2} \rangle$ . For every  $n < \omega$  find  $q_{n0} \in Q_n^{0*}$  such that  $q_{n0} \geq^* p(\alpha)_{n0}$  for every  $\alpha < \mu$ . Set  $q_{n1} = \bigcup_{\alpha < \mu} p(\alpha)_{n,1}$  and  $q_{n2} = \bigcup_{\alpha < \mu} p(\alpha)_{n,2}$  for every  $n < \omega$ . Set  $q_n = \langle q_{n0}, q_{n1}, q_{n2} \rangle$  ( $n < \omega$ ) and  $q = \langle q_n \mid n < \omega \rangle$ . Then  $q \in \mathcal{P}^*$ . Let us check the condition (3) of Definition 2.9. Suppose that  $\beta \in \text{dom } q_{n,1}$  for some  $n < \omega$ . Then there is  $\alpha < \mu$  such that  $\beta \in \text{dom } p(\alpha)_{n,1}$ . But now the sequence  $\langle k_m \mid \omega > m \geq n \rangle$  witnessing (3) for  $p(\alpha)$  will be fine also for  $q$ .  $\square$

Analogous of Lemmas 1.11, 1.13 and 1.14 hold for  $\mathcal{P}^*$ . We define  $\mathcal{P}^* \restriction n$  and  $\mathcal{P}^* \setminus n$  from  $\mathcal{P}^*$  exactly as  $\mathcal{P} \restriction n$  and  $\mathcal{P} \setminus n$  were defined from  $\mathcal{P}$ .

**Lemma 2.11.**  $\langle \mathcal{P}^*, \leq, \leq^* \rangle$  satisfies the Prikry condition.

**Lemma 2.12.** For every  $n < \omega$  the forcing with  $\mathcal{P}^*$  is the same as the forcing with  $(\mathcal{P}^* \setminus n) \times (\mathcal{P}^* \restriction n)$ .

**Lemma 2.13.**  $\langle \mathcal{P}^*, \leq \rangle$  preserves the cardinals below  $\kappa_\omega$  and GCH below  $\kappa_\omega$  still holds in a generic extension by  $\mathcal{P}^*$ .

Let us show that  $\mathcal{P}^*$  adds lot of Prikry sequence. Let  $G$  be a generic subset of  $\mathcal{P}$ . For  $\beta < \kappa_\omega^{++}$  we define  $G(\beta) : \omega \rightarrow \kappa_\omega$  as in Section 1, i.e.  $G(\beta)(n) = \nu$  iff there is  $\langle p_k \mid k < \omega \rangle \in G$  such that  $\beta \in \text{dom } p_{n,2}$  and  $p_{n,2}(\beta) = \nu$  where  $p_n = \langle p_{n1}, p_{n2} \rangle \in Q_n^{1*}$ .

We claim that for unboundedly many  $\beta$ 's  $G(\beta)$  will be a Prikry sequence and  $G(\beta)$  will be bigger (modulo finite) than  $G(\beta')$  for every  $\beta' < \beta$ . The next lemma proves even slightly more.

**Lemma 2.14.** Suppose  $p = \langle p_k \mid k < \omega \rangle \in \mathcal{P}^*$ ,  $p_k = \langle p_{k0}, p_{k1}, p_{k2} \rangle$  for  $k \geq \ell(p)$ ,  $\beta < \kappa_\omega^{++}$  and  $\beta \notin \bigcup_{\ell(p) \leq k < \omega} (\text{dom } p_{k1} \cup \text{dom } p_{k2})$ . Then there is a direct extension  $q$  of  $p$  such that  $\beta \in \bigcup_{k \geq \ell(q)} \text{dom } q_{k,1}$ , where  $q = \langle q_k \mid k < \omega \rangle$  and  $q_k = \langle q_{k0}, q_{k1}, q_{k2} \rangle$  for every  $k \geq \ell(q)$ .

**Proof:** Let us assume for simplicity that  $\ell(p) = 0$ . Set  $a = \bigcup_{k < \omega} \text{dom } p_{k1}$ .

**Case 1.**  $\beta \geq \bigcup a$ .

Then for every  $n < \omega$ , pick some  $\xi_n$   $\delta(p_n) < \xi_n < \kappa_n^{+n+2}$  which is  $n$ -good. It exists by Lemma 2.3. Extend  $p_{n0}$  to a condition  $q_{n0}$  obtained by adding  $\xi_n$  and some  $\xi$  which is above  $\xi_n$  and  $mc(p_n)$  to  $\text{supp}(p_{n0})$ . Set  $q_{n1} = p_{n1} \cup \{\langle \beta, \xi_n \rangle\}$ ,  $q_{n2} = p_{n2}$  and  $q_n = \langle q_{n0}, q_{n1}, q_{n2} \rangle$ . Then  $q = \langle q_n \mid n < \omega \rangle$  will be as desired.

**Case 2.**  $\beta < \bigcup a$ .

Then pick the least  $\alpha \in a$   $\alpha > \beta$ . By the definition of  $\mathcal{P}^*$ , namely (2) of 2.9,  $\alpha \in \text{dom } p_{n1}$  starting with some  $n^* < \omega$ . by 2.9(3) there is a nondecreasing converging to infinity sequence of natural numbers  $\langle k_m \mid \omega > m \geq n^* \rangle$  such that for every  $m \geq n^*$   $p_{m,1}(\alpha)$  is  $k_m$ -good. Let  $n^{**} \geq n^*$  be such that  $k_{n^{**}} > 0$ . For every  $n \geq n^{**}$  we like to extend  $p_n$  in order to include  $\beta$  into the extension. So, let  $n \geq n^{**}$ . Set  $\gamma = \bigcup \{p_{n2}(\delta) \mid \delta < \alpha\}$ . Since  $p_{n1}(\alpha)$  is good.  $cf p_{n1}(\alpha) > \kappa_n^{++}$  and hence  $\gamma < p_{n1}(\alpha)$ . by Lemma 2.5.1, there  $k_n - 1$ -good  $\delta$ ,  $\gamma < \delta < p_{n1}(\alpha)$ . Extend  $p_{n0}$  to some  $q_{n0}$  having  $\delta$  in support. Set  $q_{n1} = p_{n1} \cup \{\langle \beta, \delta \rangle\}$ ,  $q_{n2} = p_{n2}$  and  $q_n = \langle q_{n0}, q_{n1}, q_{n2} \rangle$ .

Now for every  $n \geq n^{**}$   $q_{n1}(\beta)$  will be  $k_n - 1$ -good. Clearly,  $\langle k_n - 1 \mid n \geq n^{**} \rangle$  is nondecreasing sequence converging to infinity. So  $q = \langle q_n \mid n < \omega \rangle$  is a condition in  $\mathcal{P}^*$  as desired.  $\square$

$\mathcal{P}^*$  still collapses  $\kappa_\omega^{++}$  to  $\kappa_\omega^+$ . The reason of this as those of Lemma 1.15.

**Lemma 2.16.** In  $V[G]$   $|(\kappa_\omega^{++})^\vee| = \kappa_\omega^+$ .

The following lemma will be the key lemma for defining the projection of  $\mathcal{P}^*$  satisfying  $\kappa_\omega^{++}$ -c.c. in the next section.

But first a definition.

**Definition 2.17.** Let  $p = \langle p_n \mid n < \omega \rangle$ ,  $q = \langle q_n \mid n < \omega \rangle$  be two conditions in  $\mathcal{P}^*$ . They are called similar iff

- (1)  $\ell(p) = \ell(q)$
- (2) for every  $n < \ell(p)$  the following holds
- (2a)  $p_{n0} = q_{n0}$

- (2b)  $\min(\text{dom } q_{n1} \setminus (\text{dom } q_{n1} \cap \text{dom } p_{n1})) > \bigcup_{n < \omega} \sup(\text{dom } p_{n1})$
- (2c) for every  $\beta \in \text{dom } p_{n1} \cap \text{dom } q_{n1}$   $p_{n1}(\beta) = q_{n1}(\beta)$
- (2d)  $|p_{n1}| = |q_{n1}|$  where  $p_n = \langle p_{n0}, p_{n1} \rangle$ ,  $q_n = \langle q_{n0}, q_{n1} \rangle$
- (3) for every  $n \geq \ell(p)$  the following holds
- (3a)  $p_{n0} = q_{n0}$   
for every  $j \in \{1, 2\}$
- (3b)  $\min(\text{dom } q_{nj} \setminus (\text{dom } q_{nj} \cap \text{dom } p_{nj})) > \bigcup_{n < \omega} \sup(\text{dom } p_{nj})$
- (3c) for every  $\beta \in \text{dom } p_{nj} \cap \text{dom } q_{nj}$   $p_{nj}(\beta) = q_{nj}(\beta)$
- (3d)  $|p_{nj}| = |q_{nj}|$  where  $p_n = \langle p_{n0}, p_{n1}, p_{n2} \rangle$  and  $q_n = \langle q_{n0}, q_{n1}, q_{n2} \rangle$ .

**Lemma 2.18.** *Suppose  $p$  and  $q$  are similar conditions. Then there are  $s \geq p$  and  $t \geq q$  such that*

- (1)  $\ell(s) = \ell(t)$  and  $s \restriction \ell(s) = t \restriction \ell(t)$
- (2) for every  $n \geq \ell(s)$  the following holds
- (2a)  $\text{dom } s_{n1} = \text{dom } t_{n1} = \text{dom } p_{n1} \cup \text{dom } q_{n1}$
- (2b)  $s_{n2} = t_{n2} = p_{n2} \cup q_{n2}$
- (2c) for every  $\beta \in \text{dom } s_{n1} = \text{dom } t_{n1}$   $mc(s_{n0})$  projects to  $s_{n1}(\beta)$  exactly in the same way as  $mc(t_{n0})$  projects to  $t_{n1}(\beta)$
- (3) there exists a nondecreasing converging to infinity sequence of natural numbers  $\langle k_n \mid n \geq \ell(s) \rangle$  with  $k_{\ell(s)} \geq 2$  such for every  $n \geq \ell(s)$  the  $\mathcal{L}_{n, k_n, \rho_n}$ -type realized by  $mc(s_n)$  and  $mc(t_n)$  are identical, where  $\rho_n$  the least upper bound of or the code of  $p''_{n1}(\text{dom } p_{n1} \cap \text{dom } q_{n1})$ .

Moreover, if in addition  $\min(\bigcup_{\ell(q) \leq n < \omega} \text{dom } q_{n1}) \setminus \bigcup_{\ell(q) \leq n < \omega} (\text{dom } p_{n1} \cap \text{dom } q_{n1})$  is in  $\text{dom } q_{\ell(q), 1}$ , then  $s \geq^* p$ ,  $t \geq^* q$ .

**Proof:** Let  $\beta$  be the least element of  $(\bigcup_{\ell(q) \leq n < \omega} \text{dom } q_{n1}) \setminus \bigcup_{\ell(q) \leq n < \omega} (\text{dom } p_{n1} \cap \text{dom } q_{n1})$ . Pick some  $n^*, \omega > n^* \geq \ell(q)$  such that  $\beta \in \text{dom } q_{n^*, 1}$  and for every  $n \geq n^*$   $q_{n, 1}(\beta)$  is at least 5-good. In order to obtain  $s$  and  $t$  we first extend  $p, q$  to  $p', q'$  by adding Prikry sequence up to level  $n^* - 1$  such that  $\ell(p') = \ell(q') = n^*$ ,  $p' \restriction n^* = q' \restriction n^*$  and  $p' \setminus n^* = p \setminus n^*$ ,  $q' \setminus n^* = q \setminus n^*$ . Then we apply Lemma 2.8.1. for every  $n, \omega > n \geq n^*$  to  $\beta, q'_n$  and  $p'_n$  to produce  $t_n$  and  $s_n$ . Finally,  $t = p' \restriction n^* \cap \langle t_n \mid \omega > n \geq n^* \rangle$  and  $s = p' \restriction n^* \cap \langle s_n \mid \omega > n \geq n^* \rangle$

will be as required. □

The standard  $\Delta$ -system argument gives the following

**Lemma 2.19.** *Among any  $\kappa_\omega^{++}$ -conditions in  $\mathcal{P}^*$  there are  $\kappa_\omega^{++}$  which are alike.*

### 3. The Projection

Our aim will be to project  $\mathcal{P}^*$  to a forcing notion satisfying  $\kappa_\omega^{++}$ -c.c. but still producing  $\kappa_\omega^{++}$ -Prikry sequences.

**Definition 3.0.** Let  $n < \omega$  and suppose  $\langle p, f \rangle, \langle q, g \rangle \in Q_n^*$  are such that  $f = g$  then we call them  $k$ -equivalent for every  $k \leq n$  and denote this by  $\longleftrightarrow_{n,k}$ .

**Definition 3.1.** Let  $2 \leq k \leq n < \omega$ . Suppose  $\langle p, a, f \rangle, \langle q, b, g \rangle \in Q_n^*$ . We call  $\langle p, a, f \rangle$  and  $\langle q, b, g \rangle$   $k$ -equivalent and denote this by  $\longleftrightarrow_{n,k}$  iff

- (0)  $f = g$
- (1)  $\text{dom } a = \text{dom } b$
- (2)  $mc(p)$  and  $mc(q)$  are realizing the same  $k$ -type
- (3)  $T(p) = T(q)$ , i.e. the sets of measure 1 are the same
- (4) for every  $\delta \in \text{dom } a = \text{dom } b$   $a(\delta)$  and  $b(\delta)$  are realizing the same  $k$ -type
- (5) for every  $\delta \in \text{dom } a = \text{dom } b$  and  $\ell \leq k$   $a(\delta)$  is  $\ell$ -good iff  $b(\delta)$  is  $\ell$ -good
- (6) for every  $\delta \in \text{dom } a = \text{dom } b$   $mc(p)$  projects to  $a(\delta)$  the same way as  $mc(q)$  projects to  $b(\delta)$ .

**Definition 3.2.** Let  $p = \langle p_n \mid n < \omega \rangle, q = \langle q_n \mid n < \omega \rangle \in \mathcal{P}^*$ . We call  $p$  and  $q$  equivalent and denote this by  $\longleftrightarrow$  iff

- (1)  $\ell(p) = \ell(q)$
- (2) for every  $n < \ell(p)$   $p_n \longleftrightarrow_{n,n} q_n$ , i.e.  $p_{n1} = q_{n1}$ , where  $p_n = \langle p_{n0}, p_{n1} \rangle$  and  $q_n = \langle q_{n0}, q_{n1} \rangle$ .

Notice that we require only the parts producing the function from  $\kappa_\omega^{++}$  to be equal. So, actually the finite portions of the Prikry type forcing become unessential.

- (3) there is a nondecreasing sequence  $\langle k_n \mid \ell(p) \leq n < \omega \rangle, \lim_{n \rightarrow \infty} k_n = \infty, k_0 \geq 2$  such that for every  $n, \ell(p) \leq n < \omega$   $p_n$  and  $q_n$  are  $k_n$ -equivalent.



It is easy to check that  $\longleftrightarrow$  is an equivalence relation.

Now paraphrasing Lemma 2.18 we obtain the following

**Lemma 3.3.** *Suppose that  $p$  and  $q$  are similar. Then there are equivalent  $s$  and  $t$  such that  $s \geq p$  and  $t \geq q$ .*

Note that for every  $n \geq \ell(s) = \ell(t)$   $mc(s_{n0})$ ,  $mc(t_{n0})$  are realizing the same  $\mathcal{L}_{n,k_n}$ -type for  $k_n \geq 2$ , where  $s, t$  are produced by Lemma 2.18. There are at most  $\kappa_n^{++}$  different measures over  $\kappa_n$ . So, the measures corresponding  $mc(s_{n0})$  and  $mc(t_{n0})$  are the same. Now we can shrink sets of measure one  $T(s_{n0})$  and  $T(t_{n0})$  to the same set in order to satisfy the condition (3) of Definition 3.1.

**Definition 3.4.** Let  $p, q \in \mathcal{P}^*$ . Then  $p \longrightarrow q$  iff there is a sequence of conditions  $\langle r_k \mid k < m < \omega \rangle$  so that

- (1)  $r_0 = p$
- (2)  $r_{m-1} = q$
- (3) for every  $k < m - 1$

$$r_k \leq r_{k+1} \quad \text{OR} \quad r_k \longleftrightarrow r_{k+1} .$$

See diagram:

$$\begin{array}{ccc}
 r_{m-2} & \longleftrightarrow & r_{m-1} = q \\
 \vee | & & \\
 r_{m-3} & \longleftrightarrow & r_{m-4} \\
 \dots & & \\
 r_4 & \longleftrightarrow & r_5 \\
 \vee | & & \\
 r_3 & \longleftrightarrow & r_2 \\
 & & \vee | \\
 p = r_0 & \longleftrightarrow & r_1
 \end{array}$$

Obviously,  $\longrightarrow$  is reflexive and transitive.

**Lemma 3.5.** *Suppose  $p, q, s \in \mathcal{P}^*$   $p \longleftrightarrow q$  and  $s \geq p$ . Then there are  $s' \geq s$  and  $t \geq q$  such that  $s' \longleftrightarrow t$ .*

**Proof:** Pick a nondecreasing sequence  $\langle k_n \mid \ell(p) = \ell(q) \leq n < \omega \rangle$ ,  $\lim_{n \rightarrow \infty} k_n = \infty$  such that  $p_n \longleftrightarrow_{n,k_n} q_n$  for every  $n \geq \ell(p)$ . For each  $n$ ,  $\ell(p) \leq n < \ell(s)$  we extend  $q_n = \langle q_{n0}, q_{n1}, q_{n2} \rangle$  to  $t_n = \langle t_{n0}, t_{n1} \rangle$  by putting  $s_{n0}^{mc(p_{n0})}$  over  $mc(q_{n0})$  projecting it over the

rest of the coordinates in  $\text{supp } q_{n0}$  and  $\text{rng } q_{n1}$  and setting  $t_{n1} = s_{n1}$ , where  $s_n = \langle s_{n0}, s_{n1} \rangle$ ,  $p_n = \langle p_{n0}, p_{n1}, p_{n2} \rangle$  and  $s_{n0}^{mc(p_{n0})}$  is the one element sequence standing over the maximal coordinate of  $p_{n0}$ . Notice that this is possible since  $T(p_{n0}) = T(q_{n0})$  and  $s_{n0}^{m(p_{n0})} \in T(p_{n0})$ . Then  $s_n$  and  $t_n$  will be  $n$ -equivalent. Set  $s'_n = s_n$ .

Suppose now that  $n \geq \ell(s)$ . Let  $s_n = \langle s_{n0}, s_{n1}, s_{n2} \rangle$ ,  $p_n = \langle p_{n0}, p_{n1}, p_{n2} \rangle$  and  $q_n = \langle q_{n0}, q_{n1}, q_{n2} \rangle$ .

**Case 1.**  $k_n > 2$ .

By Lemma 2.0, there is  $\delta$  realizing the same  $k_n - 1$ -type over  $mc(q_{n0})$  as  $mc(s_{n0})$  does over  $mc(p_{n0})$ . Now pick  $t_n = \langle t_{n0}, t_{n1}, t_{n2} \rangle$  to be a condition with  $mc(t_{n0}) = \delta$   $k_n - 1$ -equivalent to  $s_n$ . Set  $s'_n = s_n$ .

**Case 2.**  $k_n \leq 2$ .

We first extend  $s_n$  to a stronger condition  $s'_n = \langle s'_{n0}, s'_{n1} \rangle$ . Then we proceed as in the case  $\ell(p) \leq n < \ell(s)$ .

By the construction  $s' = \langle s'_n \mid n < \omega \rangle$  and  $t = \langle t_n \mid n < \omega \rangle$  will be stronger than  $s$  and  $q$  respectively. Also  $\ell(s') = \ell(t)$  and for every  $n < \ell(s)$   $s'_n \longleftrightarrow_{n,n} t_n$ . The sequence  $\langle k_n - 1 \mid \ell(s') \leq n < \omega \rangle$  will witness the condition (2) of Definition 3.2.  $\square$

Now let us define the projection.

**Definition 3.5.** Set

$$\mathcal{P}^{**} = \mathcal{P} / \longleftrightarrow .$$

For  $x, y \in \mathcal{P}^{**}$  let  $x \preceq y$  iff there are  $p \in x$  and  $q \in y$  such that  $p \longrightarrow q$ .

**Lemma 3.7.** A function  $\pi : \mathcal{P}^* \rightarrow \mathcal{P}^{**}$  defined by  $\pi(p) = p / \longleftrightarrow$  projects  $\langle \mathcal{P}^*, \leq \rangle$  nicely onto  $\langle \mathcal{P}^{**}, \preceq \rangle$ .

**Proof:** It is enough to show that for every  $p, q \in \mathcal{P}^*$  if  $p \rightarrow q$  then there is  $s \geq p$  such that  $q \rightarrow s$ . Suppose for simplicity that we have the following diagram witnessing  $p \rightarrow q$ .

In a general case the same argument should be applied inductively.

$$\begin{array}{ccc}
q & \longleftrightarrow & h \\
& & \vee | \\
f & \longleftrightarrow & g \\
\vee | & & \\
d & \longleftrightarrow & c \\
& & \vee | \\
a & \longleftrightarrow & b \\
\vee | & & \\
p & & 
\end{array}$$

Using Lemma 3.5 we find equivalent  $f' \geq f$  and  $h' \geq h$ . Then applying it to  $d, c, f'$  find equivalent  $f'' \geq f'$  and  $c'' \geq c$ . Finally, using Lemma 3.5 for  $c'', b, a$  we find equivalent  $a''' \geq a$  and  $c''' \geq c''$ . In the diagram it looks like:

$$\begin{array}{ccccccc}
& & & & h' & & \\
& & & & \vee | & & \\
& & q & \longleftrightarrow & h & & \\
& & & & \vee | & & \\
f'' & \geq & f' & \geq & f & \longleftrightarrow & g \\
& & & & \vee | & & \\
& & d & \longleftrightarrow & c & \leq & c'' \leq c''' \\
& & & & \vee | & & \\
a''' & \geq & a & \longleftrightarrow & b & & \\
& & & & \vee | & & \\
& & & & p & & 
\end{array}$$

We claim that  $a'''$  is as required, i.e.  $a''' \geq p$  and  $q \longrightarrow a'''$ . Clearly,  $a''' \geq p$ . In order to prove  $q \longrightarrow a'''$  we consider the following diagram:

$$\begin{array}{ccc}
a''' & \longleftrightarrow & c''' \\
& & \vee | \\
f'' & \longleftrightarrow & c'' \\
\vee | & & \\
f' & \longleftrightarrow & h' \\
& & \vee | \\
q & \longleftrightarrow & h
\end{array}$$

So the sequence  $\langle q, h, h', f', f'', c'', c''', a''' \rangle$  witnessing  $q \longrightarrow a'''$ . □

The next lemma follows from Lemma 3.3.

**Lemma 3.8.**  $\mathcal{P}^{**}$  satisfies  $\kappa_\omega^{++}$ -c.c.

Let  $G \subseteq \mathcal{P}^*$  be generic. We like to show that for every  $\beta < \kappa_\omega^{++}$   $G(\beta) \in V[\pi''(G)]$ .

The following will be sufficient.

**Lemma 3.9.** *Let  $p \longleftrightarrow q$ ,  $\beta < \kappa_\omega^{++}$ . Suppose that for some  $n < \ell(p)$   $\beta \in \text{dom } p_{n1}$  then  $\beta \in \text{dom } q_{n1}$  and  $p_{n1}(\beta) = q_{n1}(\beta)$ . Where  $p_n = \langle p_{n0}, p_{n1} \rangle$  and  $q_n = \langle q_{n0}, q_{n1} \rangle$ .*

**Proof:** By the definition of equivalence  $q_{n1} = p_{n1}$ . □

So using Lemma 2.14 we obtain the following

**Theorem 3.10.** *Let  $G$  be a generic subset of  $\mathcal{P}^*$ . Then  $V[\pi''(G)]$  is a cardinal preserving extension of  $V$  such that  $GCH$  holds below  $\kappa_\omega$  and  $2^{\kappa_\omega} = \kappa_\omega^{++}$ .*

#### 4. Down to $\aleph_\omega$

In this section we sketch an additional construction needed for moving  $\kappa_\omega$  to  $\aleph_\omega$ . The construction will be similar to those of [Git-Mag1].

Let  $G$  be a generic subset of the forcing  $\mathcal{P}^{**}$  of the previous section. Denote by  $\langle \rho_n \mid n < \omega \rangle$  a Prikry sequence corresponding to normal measures over  $\kappa_n$ 's. Then  $cf(\prod_{n < \omega} \rho_n^{+n+2} / \text{finite}) = \kappa_\omega^{++}$ . Just  $G(\beta)$ 's ( $\beta < \kappa_\omega^{++}$ ) which are Prikry sequences are witnessing this. The idea will be to collapse  $\rho_{n+1}$  to  $\kappa_n^{+n+2}$  and all the cardinals between  $\rho_{n+1}^{+n+4}$  and  $\kappa_{n+1}$  to  $\rho_{n+1}^{+n+4}$ . In order to perform this avoiding collapse of  $\kappa_\omega^{++}$ , we need modify  $\mathcal{P}^*$ . For collapsing cardinals between  $\rho_{n+1}^{+n+4}$  and  $\kappa_{n+1}$  the method used in [Git-Mag 1] applies directly since the length of the extender used over  $\kappa_{n+1}$  is only  $\kappa_{n+1}^{+(n+1)+2}$ . Hence let us describe only the way  $\rho_{n+1}$  will be collapsed to  $\kappa_n^{+n+2}$ .

Let us deal with a fixed  $n < \omega$  and drop the lower index  $n$  for a while. Fix a nonstationary set  $A \subseteq \kappa^{+n+2}$ . In Definition 1.2 we require in addition that  $\text{rng } p \cap A = \emptyset$  and  $\text{supp } p \cap A = \emptyset$ . In the definition of the order on  $Q$ , Definition 1.4 (2) for  $\gamma \in A$  we replace  $p^\gamma$  by  $\kappa$  only if  $p^\gamma \geq \kappa_{n+1}$ . Now, the definition of  $\mathcal{P}$ , Definition 1.8 is changed as follows:

**Definition 4.1.** A set of forcing conditions  $\mathcal{P}$  consists of all elements  $p$  of the form  $\langle p_n \mid n < \omega \rangle$  so that

- (1) for every  $n < \omega$   $p_n \in Q_n$
- (2) there exists  $\ell < \omega$  such that for every  $n \geq \ell$   $p_n \in Q_n^0$
- (3) if  $0 < n < \ell(p)$ , then for every  $\gamma \in A_{n-1} \cap \delta(p_{n-1,0})$   $p_{n-1,0}^\gamma < p_{n,0}^0$ , where  $p_n = \langle p_{n0}, p_{n1} \rangle$  and  $p_{n-1} = \langle p_{n-1,0}, p_{n-1,1} \rangle$ .

The meaning of the new condition (3) is that  $p_{n0}^0$  which is  $\rho_n$  is always above all the sequences mentioned in  $p_{n-1,0}$ . This will actually produce a cofinal function from  $A_n$  into  $\rho_n$ .

Finally, in order to keep it while going to the projection  $\mathcal{P}^{**}$ , we strengthen the notion of similarity. Thus, in Definition 2.17 we require in addition that for every  $\gamma \in a_n \cap \delta(p_{n0})$   $p_{n0}^\gamma = q_{n0}^\gamma$ . I.e. the values of the cofinal function  $A_n \mapsto \rho_n$  are never changed.

There is no problem in showing the Prikry condition, (i.e. Lemma 1.11) since passing from level  $n - 1$  to level  $n$  we will have a regressive function on a set of measure one for a normal measure over  $\kappa_n$ .

## 5. Loose Ends

We do not know if it is possible under the same initial assumption to make a gap between  $\kappa_\omega$  and  $2^{\kappa_\omega}$  wider. Our conjecture is that it is possible. Namely, it is possible to obtain countable gaps. Also we think that uncountable gaps are impossible.

## References

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