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ON THE STRUCTURE OF FREE FINITE STATE MACHINES

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1. Introduction

As explained by Birkhoff and Lipson in [1], a finite state machine M (without outputs) can be considered as an algebra with two "phyla":

$$S = \text{set of states}, \quad I = \text{input alphabet}$$

and a single operator: $T : S \times I \to S$, the transition function of M.

Given $M = (S,I)$ and a pair of integers (m,n) there is an associated machine $U_{m,n}(M)$ freely generated as an algebra by states $t_1,\ldots,t_m$ and input symbols $e_1,\ldots,e_n$ subject to the relations which hold within M. Explicitly $U_{m,n}(M) = (\mathcal{S},\mathcal{I})$ where

$$\mathcal{I} = \{e_1,\ldots,e_n\}$$

and each state in $\mathcal{S}$ consists of an equivalence class of expressions of the form

$$t_i w(e_1,\ldots,e_n) \text{ where } 1 \leq i \leq m,\ w \in \mathcal{I}^*$$

and $t_i w(e_1,\ldots,e_n)$ and $t_j v(e_1,\ldots,e_n)$ are equivalent if for all pairs of maps $\{t_1,\ldots,t_m\} \xrightarrow{f} S$ and $\{e_1,\ldots,e_n\} \xrightarrow{g} I$ the relation:

$$f(t_i)\ w(g(e_1),\ldots,g(e_n)) = f(t_j)\ v(g(e_1),\ldots,g(e_n))$$

holds in M. The transition function then maps $(t_i w(e_1,\ldots,e_n),e_i)$ to $t_i(w(e_1,\ldots,e_n)e_i)$.

Definition Using the notation introduced above, it will be convenient to refer to a pair of maps $f : \{t_1,\ldots,t_m\} \to S$ and $g : \{e_1,\ldots,e_n\} \to I$ as a phyla-preserving mapping from $\{t_1,\ldots,t_m,e_1,\ldots,e_n\}$ to M or an interpretation of $\{t_1,\ldots,t_m,e_1,\ldots,e_n\}$ in M.

The proof of the following theorem is to be found in [1].

<u>Theorem</u> (i) $U_{m,n}(M)$ is a finite state machine.

(ii) $U_{m,n}(M)$ is generated by the m states $t_1,\ldots,t_m$ and n input symbols $e_1,\ldots,e_n$.

(iii) If π denotes the canonical phyla-preserving map from the set $\{t_1,\ldots,t_m,e_1,\ldots,e_n\}$ to $U_{m,n}(M)$, and θ is any phyla-preserving map from $\{t_1,\ldots,t_m,e_1,\ldots,e_n\}$ to M, then there is an unique algebra homomorphism $\phi : U_{m,n}(M) \to M$ such that $\theta = \phi\pi$.

(iv) $U_{m,n}(M)$ is an epimorphic image of any other finite state machine having property (iii).

<u>Definition</u> If $t_i\ w(e_1,\ldots,e_n)$ and $t_j\ v(e_1,\ldots,e_n)$ are equivalent in $U_{m,n}(M)$, then

$$t_i\ w(e_1,\ldots,e_n) = t_j\ v(e_1,\ldots,e_n)$$

is a <u>universal relation</u> in M.

2. <u>The case $m \geqslant 1$</u>

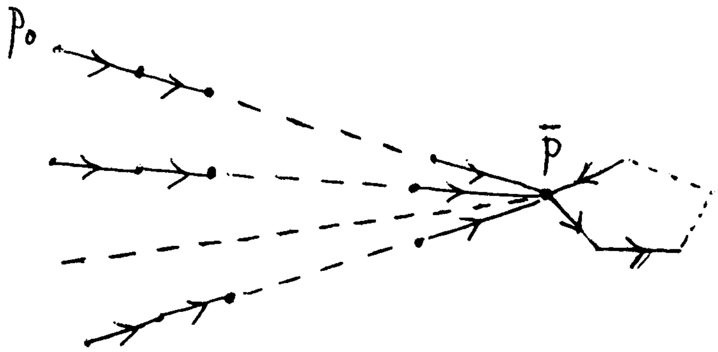
<u>Theorem 1</u>: For a machine $M = (S,I)$ to have a universal relation of the form

$$t_i\ w(e_1,\ldots,e_n) = t_j\ v(e_1,\ldots,e_n) \text{ with } i \neq j$$

it is necessary and sufficient that for each input α in I there should exist a state $t(\alpha)$ such that (a) $t(\alpha).\alpha = t(\alpha)$

and (b) for each s in S there is a non-negative integer $r(s)$ such that $s.\alpha^{r(s)} = t(\alpha)$

<u>Proof:</u> Suppose that M has a universal relation U of the form $t_i\ w(e_1,\ldots,e_n) = t_j\ v(e_1,\ldots,e_n)$ for $i \neq j$. If $\alpha \in I$, there is a submachine $M_\alpha = (S,\alpha^*)$ of M, which is a disjoint union of k machines of the following type:



Since U holds under all interpretations (f,g) for which $g(e_i) = \alpha$ for $1 \leqslant i \leqslant n$, it is clear that $k = 1$. Moreover, taking interpretations (f,g) such that $f(t_i) = p_o$, $f(t_j) = p_o\,\alpha^c$ for some non-negative integer c and $g(e_i) = \alpha$ for $1 \leqslant i \leqslant n$, it follows that

$$p_o\alpha^{\ell(w)} = p_o\alpha^{\ell(v)+c}$$

in M for $c = 0, 1, 2, \ldots$. This establishes that $\overline{p}\alpha = \overline{p}$, so that conditions (a) and (b) are satisfied with $t(\alpha) = \overline{p}$.

For the converse, suppose that given input α in I, there is a $t(\alpha)$ for which conditions (a) and (b) hold. Then let $r(\alpha) = \max_{s\epsilon S} r(s)$, and $r = \max_{\alpha\epsilon I} r(\alpha)$. It is clear that the relation $s\alpha^r = t\alpha^r$ holds for all s,t in S and all α in I; that is, the relation $t_1\, x_1^{\ r} = t_2\, x_1^{\ r}$ holds universally in M.

<u>Corollary to Theorem 1:</u>

Unless a relation of the form $t_1\, x_1^{\ r} = t_2\, x_1^{\ r}$ holds universally in the machine M, the finite state machine $U_{m,n}(M)$ is (up to isomorphism) m disjoint copies of $U_{1,n}(M)$.

3. <u>Structure of $U_{1,n}(M)$</u>

<u>Definition</u> Let K be a finite monoid generated by elements $x_1,\ldots,x_n$. The machine $\mathfrak{M}(K,X)$ <u>associated with the monoid K generated by X</u> has a set of states K, input alphabet $X = \{x_1,\ldots,x_n\}$ and transition function $K \times X \to K$ defined by multiplication in K. The machine $\mathfrak{M}(K,X)$ will be called a <u>monoid machine</u>. If K is a group, then $\mathfrak{M}(K,X)$ is a <u>group machine</u> or <u>Cayley diagram</u>.

Theorem 2: (i) If M is a finite state machine, then, for $n \geq 1$, $U_{1,n}(M)$ is isomorphic to the monoid machine $\mathcal{M}(K,X)$, where K is the monoid freely generated by $X = \{x_1,\ldots,x_n\}$ subject to the relations:

$$w(x_1,\ldots,x_n) = v(x_1,\ldots,x_n)$$

where $tw(e_1,\ldots,e_n) = tv(e_1,\ldots,e_n)$

is a universal relation in M.

(ii) Let K be a finite monoid generated by $X = \{x_1,\ldots,x_n\}$

For $\mathcal{M}(K,X)$ to be isomorphic with $U_{1,n}(M)$ for some finite state machine M, it is necessary and sufficient that for each relation $w(x_1,\ldots,x_n) = v(x_1,\ldots,x_n)$ in K and each map $f : \{1,2,\ldots,n\} \circlearrowleft$, the relation $w(x_{f(1)},\ldots,x_{f(n)}) = v(x_{f(1)},\ldots,x_{f(n)})$ also holds in K. If this condition is satisfied then $U_{1,n}(\mathcal{M}(K,X)) \simeq \mathcal{M}(K,X)$.

(iii) For $U_{1,n}(M)$ to be a group machine $(n \geq 1)$ it is necessary and sufficient that for some non-trivial w in $\mathcal{G}^*$, a relation of the form:

$$tw(e_1,\ldots,e_n) = t$$

holds universally in M.

Definition When the necessary and sufficient conditions (stated in (ii) above) for $\mathcal{M}(K,X)$ to be isomorphic with $U_{1,n}(M)$ for a finite state machine M are satisfied, X is said to generate K universally or to generate a universal presentation of K.

Proof: (i) The elements of $U_{1,n}$ are equivalence classes of expressions of the form:

$$tw(e_1,\ldots,e_n)$$

where $tw(e_1,\ldots,e_n)$ and $tv(e_1,\ldots,e_n)$ are equivalent if $tw(e_1,\ldots,e_n) = tv(e_1,\ldots,e_n)$ is a universal relation in M, with transition function defined by

$$(tw(e_1,\ldots,e_n),e_i) \mapsto t(w(e_1,\ldots,e_n)e_i)$$

The map $tw(e_1,\dots,e_n) \mapsto w(x_1,\dots,x_n)$ then clearly induces an isomorphism $U_{1,n}(M) \simeq K$.

(ii) Suppose $\mathfrak{M}(K,X) \simeq U_{1,n}(M)$. Then if the relation $w(x_1,\dots,x_n) = v(x_1,\dots,x_n)$ holds in K then $tw(e_1,\dots,e_n) = tv(e_1,\dots,e_n)$ is a universal relation in M. Thus given any map $f : \{1,2,\dots,n\}^{\circlearrowleft}$, the relation

$$tw(e_{f(1)},\dots,e_{f(n)}) = tv(e_{f(1)},\dots,e_{f(n)})$$

holds universally in M, whence $w(x_{f(1)},\dots,x_{f(n)}) = v(x_{f(1)},\dots,x_{f(n)})$ in K.

Conversely, suppose that if $w(x_1,\dots,x_n) = v(x_1,\dots,x_n)$ in K and f is a map $\{1,2,\dots,n\}^{\circlearrowleft}$, then $w(x_{f(1)},\dots,x_{f(n)}) = v(x_{f(1)},\dots,x_{f(n)})$. It follows that the relation $tw(e_1,\dots,e_n) = tv(e_1,\dots,e_n)$ holds universally in $\mathfrak{M}(K,X)$. Conversely if $tw(e_1,\dots,e_n) = tv(e_1,\dots,e_n)$ is a universal relation in $\mathfrak{M}(K,X)$ then certainly $w(x_1,\dots,x_n) = v(x_1,\dots,x_n)$ in K (interpreting t as 1, and e_i as x_i for $i = 1,2,\dots,n$). The isomorphism

$$U_{1,n}(\mathfrak{M}(K,X)) \simeq \mathfrak{M}(K,X)$$

follows from (i).

(iii) Let $x_1,\dots,x_n$ generate $U_{1,n}(M)$ freely subject to the relations:

$$w(x_1,\dots,x_n) = v(x_1,\dots,x_n)$$

where $tw(e_1,\dots,e_n) = tv(e_1,\dots,e_n)$

is a universal relation in M. Since $x_1{}^r = 1$ for some $r \geqslant 1$, the relation $te_i{}^r = t$ must hold universally in M for some r.

Conversely, suppose $tw(e_1,\dots,e_n) = t$ holds universally in M, with w non-trivial. Then given $f : \{1,2,\dots,n\}^{\circlearrowleft}$ the relation $w(x_{f(1)},\dots,x_{f(n)}) = 1$ holds in $U_{1,n}(M)$. In particular, $w(x_i,\dots,x_i) = 1$ for each i, which proves the existence of $x_i{}^{-1}$ for each i, as w is non-trivial.

Definition Let $M = (S,I)$ be a finite state machine, and let $F(S)$ denote the semigroup of mappings $S \to S$ under composition. For each α in I, let $T(\alpha)$ be the map $S \to S$ in $F(S)$. The map T extends naturally to a semigroup homomorphism $I^* \to F(S)$. The image of this homomorphism is the syntactic monoid $\mathcal{S}(M)$ of M.

Lemma: For each $n > 1$, $U_{1,n}(M)$ and $U_{1,n}(\mathcal{M}(\mathcal{S}(M), T(I)))$ are isomorphic.

Proof: Suppose that $sw(\alpha_1,\ldots,\alpha_n) = sv(\alpha_1,\ldots,\alpha_n)$ for all s in S and all α_i in I. Then $w(T(\alpha_1),\ldots,T(\alpha_n))$ and $v(T(\alpha_1),\ldots,T(\alpha_n))$ represent the same element of $\mathcal{S}(M)$, so that $fw(T(\alpha_1),\ldots,T(\alpha_n)) = fv(T(\alpha_1),\ldots,T(\alpha_n))$ for all f in $\mathcal{S}(M)$ and all α_i in I.

Conversely, if $fw(T(\alpha_1),\ldots,T(\alpha_n)) = fv(T(\alpha_1),\ldots,T\alpha_n))$ for all f in $\mathcal{S}(M)$ and all α_i in I, then $w(T(\alpha_1),\ldots,T(\alpha_n)) = v(T(\alpha_1),\ldots,T(\alpha_n))$ in $\mathcal{S}(M)$. Thus $sw(\alpha_1,\ldots,\alpha_n) = sv(\alpha_1,\ldots,\alpha_n)$ in M for all s in S and all α_i in I.

This proves the required isomorphism.

It is evident that a universal relation of the form $tw(e_1,\ldots,e_m) = tv(e_1,\ldots,e_m)$ holds in a monoid machine $\mathcal{M}(K,X)$ if and only if $w(x_1,\ldots,x_m) = v(x_1,\ldots,x_m)$ for all x_i in X. This result will be used in the proof of the next theorem, which describes a simple method for constructing $U_{1,m}(M)$ when M is a monoid machine.

Theorem 3: Let K be a finite monoid generated by $X = \{x_1,\ldots,x_m\}$. Let X^* be the set of rows of the n by m^n array whose columns are the elements of X^n. Then X^* has n elements $X_1,\ldots,X_n$, which generate a submonoid K^* of K^{m^n}, and $U_{1,n}(\mathcal{M}(K,X))$ and $\mathcal{M}(K^*,X^*)$ are isomorphic.

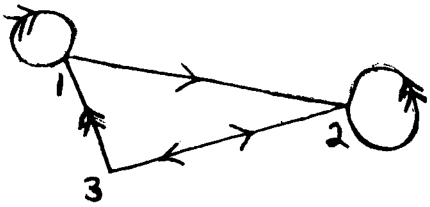
Proof: Suppose that $w(y_1,\ldots,y_n) = v(y_1,\ldots,y_n)$ in K for all interpretations of $y_1,\ldots,y_n$ in X. Then the identity $w(y_1,\ldots,y_n) = v(y_1,\ldots,y_n)$ necessarily holds in K^*.

Conversely $w(X_1,\ldots,X_n) = v(X_1,\ldots,X_n)$ in K^* entails $w(y_1,\ldots,y_n) = v(y_1,\ldots,y_n)$ for all interpretations of $y_1,\ldots,y_n$ in X, each interpretation corresponding to a projection of the identity $w(X_1,\ldots,X_n) = v(X_1,\ldots,X_n)$ onto a single component.

4. Illustrative Examples

Example 1:

Let M be the machine having three states, and input alphabet $\{a,b\}$, as indicated below:



(This machine is considered by Birkhoff and Lipson in [1]).

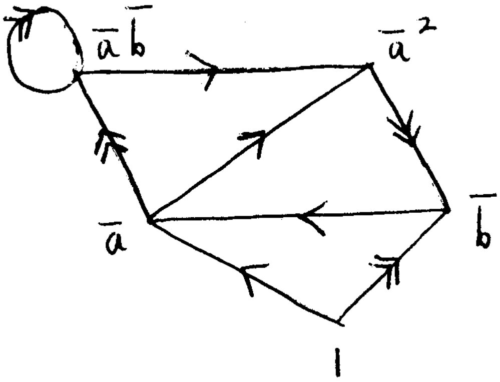
For this machine M, $\mathcal{S}(M)$ is the subsemigroup of maps $\{1,2,3\}^{\circlearrowleft}$ generated by a, b where

$\bar{a}(1) = 2, \quad \bar{a}(2) = 3, \quad \bar{a}(3) = 2$

and $\bar{b}(1) = 1, \quad \bar{b}(2) = 2, \quad \bar{b}(3) = 1$

The syntactic monoid then consists of five maps viz. $1, \bar{a}, \bar{b}, \bar{a}^2, \bar{a}\bar{b}$, and the additional relations $\bar{a}^3 = \bar{a}$, $\bar{b}^2 = \bar{b}$, $\bar{b}.\bar{a} = \bar{a}$, $\bar{a}^2\bar{b} = \bar{b}$, $\bar{a}\bar{b}\bar{a} = \bar{a}^2$ hold.

The machine $\mathcal{M} = \mathcal{M}(\mathcal{S}(M), \{\bar{a},\bar{b}\})$ is:



The free machine $U_{1,2}(\mathfrak{m}) \simeq U_{1,2}(M)$ is now the semigroup machine associated with the subsemigroup of $\mathcal{S}(M)^4$ generated by

$$A = (\bar{a},\bar{a},\bar{b},\bar{b}) \text{ and } B = (\bar{a},\bar{b},\bar{a},\bar{b})$$

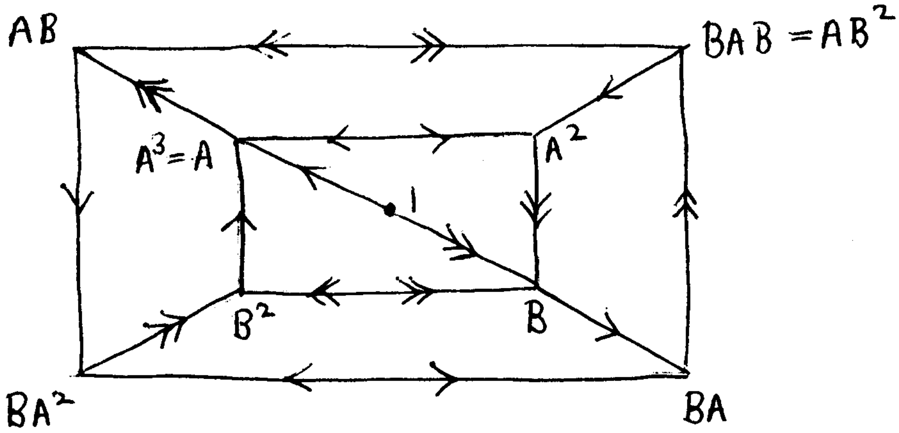
It has 9 elements viz.:

$$1,\ A,\ B,\ A^2 = (\bar{a}^2,\bar{a}^2,\bar{b},\bar{b}),\ B^2=(\bar{a}^2,\bar{b},\bar{a}^2,\bar{b}),$$

$$AB = (\bar{a}^2,\overline{ab},\bar{a},\bar{b}),\ BAB = (\bar{a},\overline{ab},\bar{a}^2,\bar{b})$$

$$BA = (\bar{a}^2,\bar{a},\overline{ab},\bar{b}) \text{ and } BA^2 = (\bar{a},\bar{a}^2,\overline{ab},\bar{b})$$

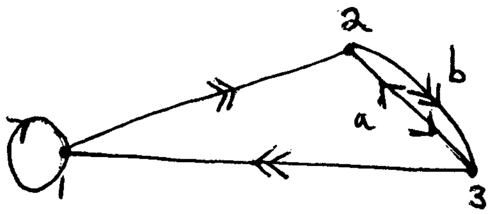
The resulting semigroup machine is:



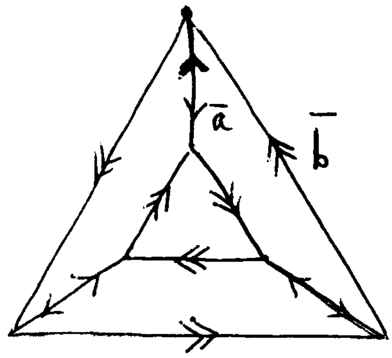
(Note that there is an error in the representation of $U_{1,n}(M)$ given by Birkhoff and Lipson in [1], and that a similar error occurs in [2]. The relation $AB^2 = BA^2$ does not hold universally in M as the diagrams in [1] and [2] suggest).

Example 2:

Let N be the machine with 3 states, and input alphabet {a, b}, as represented below:



In this case, $\mathcal{S}(N)$ is the subgroup of the semigroup of maps $\{1,2,3\}$ consisting of all permutations, with generators $\bar{a} = (23)$, $\bar{b} = (123)$. The machine $\mathcal{M}(\mathcal{S}(N), \{a,b\})$ is then a Cayley diagram for the symmetric group S_3 viz:



By the previous results, $U_{1,2}(M) \simeq U_{1,2}(\mathcal{M})$ is the group machine associated with the subgroup of $S_3^{\,4}$ generated by $A = (\bar{a},\bar{a},\bar{b},\bar{b},)$ and $B = (\bar{a},\bar{b},\bar{a},\bar{b})$ generated by $A = (\bar{a},\bar{a},\bar{b},\bar{b})$ and $B = (\bar{a},\bar{b},\bar{a},\bar{b})$. Since $(AB)^2 = (1,1,1,\bar{b})$, and it can be shown that $(\bar{a},\bar{a},\bar{b})$ and $(\bar{a},\bar{b},\bar{a})$ generate the subgroup of $C_1 \times S_3^{\,2}$ consisting of triples (c,p,q) such that pq and c are permutations of the same parity, it follows that $U_{1,2}(M)$ is (up to isomorphism) the group machine $\mathcal{M}(G,X)$ where $G = S_3 \times S_3 \times C_3$ and $X = \{(\bar{a},\bar{b},\bar{b}), (\bar{b},\bar{a},\bar{b})\}$. This result would be difficult to obtain by the direct method for computing $U_{1,2}(M)$ described in [1].

5. Monoids with a universal presentation

It has been shown in the previous section that every free finite state machine is of the form $\mathcal{M}(S,X)$ where S is a finite monoid, and X $\{x_1,\ldots,x_n\}$ is a set of generators for S. In this section, necessary conditions on S and X for $\mathcal{M}(S,X)$ to be a free finite state machine are described. For instance, it is evident that if X generates S universally then the set of relations between $x_1,\ldots,x_n$ holding in S is invariant under any permutation of $\{1,2,\ldots,n\}$. This fact is relevant for the interpretation of the results and proofs of this section.

Lemma 3: If the elements $x_1,\dots,x_n$ generate the finite monoid S universally, then either $x_1 = x_2 = \dots = x_n$ or $x_1,x_2,\dots,x_n$ are pairwise distinct and generate S irredundantly.

Proof: Suppose that $x_1 = w(x_2,\dots,x_n)$. Let f be the map $\{1,2,\dots,n\}$ such that $f(1) = 2$ and $f(i) = i$ for $i > 2$. Then $x_{f(1)} = x_2 = w(x_{f(2)},\dots,x_{f(n)}) = w(x_2,\dots,x_n) = x_1$, whence $x_1 = x_2 = \dots = x_n$.

Notation: Let P be a partition of $\{1,2,\dots,n\}$. The rank of P (the number of blocks in the partition P) will be denoted by $\rho(P)$.

Theorem 4: Let S be a finite monoid generated universally by distinct generators $x_1,\dots,x_n$. For each partition P let E(P) be the smallest congruence on S such that x_i and x_j are congruent for all (i,j) in P Then

(i) the map E is a join-preserving bijection from the lattice of partitions of $\{1,2,\dots,n\}$ (ordered by $P < Q$ if P is a refinement of Q), to the congruence lattice of S.

(ii) the quotient S/E(P) is isomorphic with the subsemigroup $S_{\rho(P)}$ of S generated by $x_1,\dots,x_{\rho(P)}$.

Proof: (ii) Let F be the monoid freely generated by $e_1,\dots,e_n$. Define $p : \{1,2,\dots,n\}$ by setting $p(i)$ = smallest integer in the block of P which contains i. There is a unique monoid homomorphism $\phi : F \to G$ such that $\phi(e_i) = x_{p(i)}$ for $i = 1,2,\dots,n$. Since $w(x_1,\dots,x_n) = v(x_1,\dots,x_n)$ implies $w(x_{p(1)},\dots,x_{p(n)}) = v(x_{p(1)},\dots,x_{p(n)})$ there is a monoid homomorphism $\phi' : G \to G$, induced by ϕ, such that $\phi'(w(x_1,\dots,x_n)) = w(x_{p(1)},\dots,x_{p(n)})$. Consider the equivalence relation E on S defined by $x \equiv y$ if and only if there exist w and v in F such that $w(x_1,\dots,x_n) = x$, $v(x_1,\dots,x_n) = y$ and $w(x_{p(1)},\dots,x_{p(n)}) = v(x_{p(1)},\dots,x_{p(n)})$. Clearly $(i,j) \in P$ implies $(x_i,x_j) \in E$, whilst it is easy to show that $\operatorname{Ker} \phi' = E \subseteq E(P)$. Since E(P) is the smallest congruence in which x_i and x_j are equivalent whenever $(i,j) \in P$, it follows that Ker $\phi' = E(P)$. Thus $S/E(P)$ is isomorphic to Im ϕ', the submonoid of S generated by $\{x_{p(1)},\ldots,x_{p(n)}\}$, and this set comprises $\rho(P)$ distinct elements.

(i) If P is a refinement of Q, then certainly $E(P) \subseteq E(Q)$. Moreover $P < Q$ ensures $\rho(P) > \rho(Q)$, so $S/E(P)$ and $S/E(Q)$ are non-isomorphic by (ii) and the previous lemma. Thus $P < Q$ entails $E(P) \underset{\neq}{\subset} E(Q)$.

Now $E(P \vee Q)$ is the congruence generated by the relations $x_i = x_j$ for $(i,j) \in P \vee Q$. Since $P \vee Q$ is the smallest equivalence relation which contains both P and Q, it follows that $E(P \vee Q) = E(P \cup Q) = E(P) \vee E(Q)$, showing that E is a join-preserving map.

Suppose that $E(P) = E(Q)$. Then $E(P) = E(P) \vee E(Q) = E(P \vee Q)$. Since $P \vee Q \geqslant P$, this implies $Q \leqslant P$. Similarly $P \leqslant Q$, so that E is a bijective map.

Note: The map E is not in general a lattice homomorphism.

Let S be universally generated by $x_1,\ldots,x_n$, and suppose that x_1 (and thus each generator) has stem of length c and period t.

Suppose that $w(e_1,\ldots,e_n)$ and $v(e_1,\ldots,e_n)$ are elements of length $\ell(w)$ and $\ell(v)$ respectively in F, the monoid freely generated by $e_1,\ldots,e_n$. If $w(x_1,\ldots,x_n) = v(x_1,\ldots,x_n)$ in S, then

$$x_1^{\ell(w)} = w(x_1,\ldots,x_1) = v(x_1,\ldots,x_1) = x_1^{\ell(v)}$$

whence either

(i) $\ell(w) = \ell(v) < c$

or (ii) $\min(\ell(w), \ell(v)) \geqslant c$

and $\ell(w) \equiv \ell(v) \pmod t$

Given an element x in S, it is then consistent to define the length of x as the unique number $\ell(x)$ such that if $w(x_1,\ldots,x_n) = x$ then $\ell(x) \equiv \ell(w) \pmod t$ and $\ell(x) < c + t$.

Corollary: Let U be the partition of $\{1,2,\ldots,n\}$ consisting of a single block. Then $(x,y) \in E(U)$ if and only if $\ell(x) = \ell(y)$.

Proof: Let $w(x_1,\ldots,x_n) = x$ and $v(x_1,\ldots,x_n) = y$. Then $\ell(x) = \ell(y)$ if and only if $w(x_1,\ldots,x_1) = v(x_1,\ldots,x_1)$, and this is equivalent to $(x,y) \in E(U)$ as observed in the proof of part (ii) of the theorem.

6. Algebras with a universal presentation

Necessary conditions for a finite monoid to possess a universal presentation have already been described. In this section, stronger conditions are derived for special varieties of monoid.

Theorem 5: Let S be an upper semilattice with least element 0 (i.e. a monoid $(S,\vee)$ in which the binary operation $\vee$ is commutative and idempotent and 0 is the identity element).

The generators $x_1,\ldots,x_n$ of S generate a universal presentation of S if and only if either $x_1 = x_2 = \ldots = x_n$ or $x_1,\ldots,x_n$ freely generate S as an upper semilattice with zero element.

Proof: The sufficiency of the stated conditions is clear. Accordingly, it suffices to show that if a relation of the form

$$(U) \quad \bigvee_{i \in A} x_i = \bigvee_{i \in B} x_i \qquad A,B \subseteq \{1,2,\ldots,n\}$$

holds in S then either $x_1 = x_2 = \ldots = x_n$ or $A = B$.

Assume without loss of generality that $A \neq \phi$. Then if $B = \phi$ the relation (U) is of the form

$$\bigvee_{i \in A} x_i = 0$$

whence $x_i = 0$ for all i in A, and $x_1 = x_2 = \ldots = x_n = 0$.

Suppose A, B both non-empty, and let $I = A \cap B$, $\overline{A} = A \setminus I$ and $\overline{B} = B \setminus I$. If $\overline{A} = \overline{B} = \phi$ then $A = B$. Otherwise assume without loss of generality that $\overline{A} \neq \phi$ and let $f : \{1,2,\ldots,n\}^{\circlearrowleft}$ be such that

$$f(i) = \begin{cases} 1 & \text{if } i \in \overline{A} \\ 2 & \text{otherwise} \end{cases}$$

Since (U) is a universal relation,

$\bigvee_{i \in A} x_{f(i)} = \bigvee_{i \in B} x_{f(i)}$ also holds in S. If $I = \phi$, this entails $x_1 = x_2$, whence $x_1 = x_2 = \ldots = x_n$. If $I \neq \phi$, then $x_2 \vee x_1 = x_2$ whence (by universality and commutativity)

$$x_1 = x_1 \vee x_2 = x_2 \vee x_1 = x_2 .$$

<u>Theorem 6</u>: Let G be a finite Abelian group. The elements $g_1,\ldots,g_n$ of G are the generators of a universal presentation if and only if for some t and some d dividing t, the group G is freely generated by $g_1,\ldots,g_n$ subject to the relations:

$$\left.\begin{array}{ll} \forall i & g_i^{\,t} = 1 \\ \forall i,j & g_i g_j = g_j g_i \\ \forall i,j & g_i^{\,d} = g_j^{\,d} \end{array}\right\} \quad (*)$$

<u>Proof</u>: The group G with free presentation on generators $g_1,\ldots,g_n$ subject to the relations (*) is universally presented on generators $g_1,\ldots,g_n$, since the set of relations (*) is closed under the application of any function $f : \{1,2,\ldots,n\}^{\circlearrowleft}$ to the indices of the g_i's.

Conversely, suppose that $g_1,\ldots,g_n$ universally generate G and have common order t. Then G has a free presentation on $g_1,\ldots,g_n$ with relations

$$\begin{array}{ll} \forall i & g_i^{\,t} = 1 \\ \forall i,j & g_i g_j = g_j g_i \end{array}$$

and other relations of the form $\prod_{i=1}^{n} g_i^{\,r_i} = 1$. Since $g_i^{\,t} = g_j^{\,t} = 1$, there is a least number d such that $g_i^{\,d} = g_j^{\,d}$ (for some, whence all pairs of indices (i,j)). As $g_i^{\,t} = g_j^{\,t}$ and $g_i^{\,d} = g_j^{\,d}$ ensure $g_i^{\,HCF(d,t)} = g_j^{\,HCF(d,t)}$, it must be that d divides t. If the relation $\prod_{i=1}^{n} g_i^{\,r_i} = 1$ holds in G, then $\prod_{i=1}^{n} g_{f(i)}^{\,r_i} = 1$ for all maps $f : \{1,2,\dots,n\}$. In particular, if $r = \sum_{i=1}^{n} r_i$ then $g_1^{\,r} = 1$, whence t divides r. Moreover, $g_1^{\,r_1} g_2^{\,r-r_1} = 1$, showing that $g_1^{\,r_1} = g_2^{\,r_1}$ and thus that d divides r_1. By symmetry, d divides r_i for each i, so that the relation $\prod_{i=1}^{n} g_i^{\,r_i} = 1$ is a consequence of the set of relations $g_i^{\,d} = g_j^{\,d}$ for all i and j.

Cor.1:

G is an Abelian group universally generated by elements $g_1,\dots,g_n$ of order t if and only if G and $C_t \times C_d^{\,n-1} \equiv \langle\alpha\rangle \times \langle\beta\rangle^{n-1}$ are isomorphic via the mapping ϕ such that $\phi(g_1) = (\alpha,1,\dots,1)$ and $\phi(g_i) = (\alpha,1,\dots,1,\beta,1,\dots,1)$ (ith component) for $i = 2,3,\dots,n$.

Proof: It is not difficult to show that the group freely generated by $g_1,\dots,g_n$ subject to the relations (*) is indeed isomorphic to $C_t \times C_d^{\,n-1}$ via the mapping ϕ.

7. Groups with a universal presentation

Necessary and sufficient conditions for a finite Abelian group to have a universal presentation are given in Theorem 6. The results and examples in this section relate to the harder (and unresolved) problem of determining which finite non-Abelian groups admit a universal presentation.

The following result is a corollary to Theorem 6:

Cor 2 to Thecrem 6:

Suppose that $g_1,\ldots,g_n$ generate a universal presentation for the finite group G. Let G' be the commutator subgroup of G. The images $\bar{g}_1,\ldots,\bar{g}_n$ of $g_1,\ldots,g_n$ generate a universal presentation of G/G'. In particular, G/G' is isomorphic with $C_t \times C_d^{\,n-1}$ for some positive integers k and d where d divides k.

Proof: The elements of G' are products of commutators. Thus if $w(g_1,\ldots,g_n) \in G'$ and $f : \{1,2,\ldots,n\}$ is any map, then $w(g_{f(1)},\ldots,g_{f(n)}) \in G'$. That is, the relations imposed upon $g_1,\ldots,g_n$ by taking the quotient by G' hold universally in G/G'. By Theorem 6, G/G' (being a finite Abelian group) is isomorphic with some $C_t \times C_d^{\,n-1}$.

The next result is the analogue for groups of Theorem 4.

Theorem 7: Let G be a finite group generated universally by distinct generators $g_1,\ldots,g_n$. For each partition P, let N(P) be the normal subgroup of G generated by all elements of the form $g_i g_j^{-1}$ such that $(i,j) \in P$.

Then

(i) the map N is a join-preserving bijection from the lattice of partitions of $\{1,2,\ldots,n\}$ (ordered by refinement) to the lattice of normal subgroups of G.

(ii) the quotient G/N(P) is isomorphic with the subgroup $G_{\rho(P)}$ of G generated by $g_1,\ldots,g_{\rho(P)}$, and G is isomorphic to a semi-direct product of $G_{\rho(P)}$ and N(P).

Proof: It suffices to show that $G \cong G_{\rho(P)} * N(P)$; the other results are interpretations of Theorem 4.

For $i = 1,2,\ldots,n$ let $q(i)$ be the least integer such that $(i,q(i)) \in P$. Let $\theta : G \to G$ be the group homomorphism such that $\theta(g_i) = g_{q(i)}$ (c.f. proof of Theorem 4 (ii)). Then $\mathrm{Ker}\theta = N(P)$, and $\mathrm{Im}\theta = \langle g_{q(1)},\ldots,g_{q(n)}\rangle$. Note that $g_i = g_{q(i)}(g_{q(i)}^{-1} g_i) \in \mathrm{Im}\theta.\mathrm{Ker}\theta$

so that $G = Im\theta . Ker\ \theta$. Moreover, if $g \in Ker\theta \cap Im\theta$, then $\theta(g) = g = 1$. Thus $G = Im\ \theta * Ker\ \theta \simeq G_{\rho(P)} * N(P)$

Corollary 1: If G has a universal presentation by generators $g_1,\dots,g_n$ of common order t then the elements of G of length 0 form a normal subgroup N of G, and $G \simeq C_t * N$.

Proof: See the corollary to Theorem 4, and apply Theorem 7 (ii).

Corollary 2: If $k \geqslant 4$, then the symmetric group S_k has no universal presentation.

Proof: Suppose $g_1,\dots,g_n$ are permutations generating S_k universally, and let $g_1,\dots,g_n$ have common order t. Since $S_k \simeq C_t * N$ for some normal subgroup N, it must be that $N = A_k$ and $t = 2$. On the other hand, in view of Theorem 7 (i), $n \leqslant 2$. But, if a group is generated by two elements of order 2 it is dihedral (see [4] p.49 Ex.1).

8. Examples of groups with universal presentations

Example 1:

As suggested by the proof of the previous corollary, the dihedral group D_n of order 2n has a universal presentation by two generators of order 2, viz. $< x,y \mid x^2 = y^2 = (xy)^n = 1 >$. In particular $S_3 (\simeq D_3)$ is universally generated by a pair of transpositions.

Example 2:

Every finite Burnside group B(t,n) (which is generated by n elements $x_1,\dots,x_n$ subject to relations $g^t = 1$ for every g in B(t,n))is universally generated by its canonical generating set.

The Burnside group B(3,3) of order 2187 illustrates that the map N in Theorem 7 (and likewise the map E in Theorem 4) is not in general a lattice homomorphism. As described in [3], evey element of B(3,3) has a unique representation of the form:

$$x_1^{a_1} x_2^{a_2} x_3^{a_3} (x_1,x_2)^{b_3} (x_1,x_3)^{b_2} (x_2,x_3)^{b_1} (x_1,x_2,x_3)^{c}$$

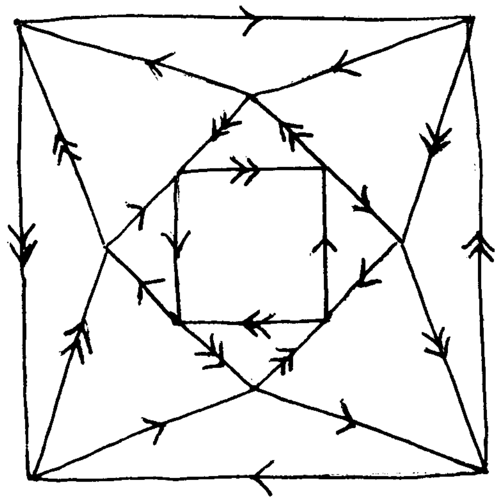
where $0 \leq a_i, b_i, c \leq 2$. Let P be the partition (12)(3) and Q the partition (1)(23). The partition $P \wedge Q$ is (1)(2)(3) whence $N(P \wedge Q) = \{1\}$ But $(x_1,x_2)^2 (x_1,x_3) (x_2,x_3)^2 \in N(P) \cap N(Q)$ (it reduces to 1 under adjunction of the relation $x_1 = x_2$ or $x_2 = x_3$) whence $N(P) \cap N(Q) \neq N(P \wedge Q)$.

Example 3:

The group A_4 is universally generated by x and y subject to the relations:

$$x^3 = y^3 = (xy)^3 = (xy^2)^2 = 1$$

The Cayley diagram associated with this presentation is:



Example 4:

The group G of order 56 generated by x and y subject to the relations

$$x^2y\ xy^3 = y^2xyx^3 = 1$$

is universally generated by x and y, which are elements of order 7 (see [4] p.60). The semi-direct product decomposition of G referred to in Corollary 2 to Theorem 7 exhibits G as $C_7 * C_2^3$.

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