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The computational complexity of the satisfiability of modal Horn clauses for modal propositional logics

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Abstract

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This paper presents complexity results about the satisfiability of modal Horn clauses for several modal propositional logics. Almost all these results are negative in the sense that restricting the input formula to modal Horn clauses does not decrease the inherent complexity of the satisfiability problem. We first show that, when restricted to modal Horn clauses, the satisfiability problem for any modal logic between K and S4 or between K and B is PSPACE-hard. As a result, the satisfiability of modal Horn clauses as well as the satisfiability of unrestricted formulas for any of K, T, B and S4 is PSPACE-complete. This result refutes the expectation (Fariñas del Cerro and Penttonen 1987) of getting a polynomial-time algorithm for the satisfiability of modal Horn clauses for these logics as long as $P \neq$ PSPACE. Next, we consider S4.3 and extensions of K5 including K5, KD5, K45, KD45 and S5, the satisfiability problem for each of which in general is known to be NP-complete, and show that for each extension of K5, a polynomial-time algorithm for the satisfiability of modal Horn clauses can be obtained; but for S4.3, together with some linear tense logics closely related to S4.3 like CL, SL and PL, the satisfiability of modal Horn clauses still remains NP-complete.

1. Introduction

Since the invention of Prolog, a number of languages based on nonclassical logics have been developed as extensions of Prolog. Some of these adopted nonclassical

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logics include modal logic [8], intuitionistic logic [11, 17], temporal logic [1, 10, 20], etc. The success of a programming language based on nonclassical logics usually lies in the definition of Horn clauses and the SLD-resolution-like inference rule. For modal logic these definitions are available and the programming language Molog has been developed based on the definition of modal Horn clauses and modal resolution [8, 2, 7]. It is therefore theoretically interesting to investigate the inherent complexity of the satisfiability problem of modal Horn clauses for various modal logics. It is well known, however, that the satisfiability problem of first-order modal Horn clauses is undecidable for its nonmodal part alone is already undecidable. For this reason we focus our attention on modal propositional logics.

For the classical propositional logic, we know that if we restrict the input formula to Horn clauses, the satisfiability problem can be solved in linear time [6], while the same problem in general is NP-complete [5]. We thus gain the benefit of saving much computation time for solving this problem by the restriction of the input formula to Horn clauses. But when considering modal logic, can we also obtain the same benefit by restricting the input formula to modal Horn clauses? For S5 the answer is yes: by the result of Ladner [15], the satisfiability problem for S5 is NP-complete, while by the result of Fariñas del Cerro and Penttonen [9], the same problem restricted to modal Horn clauses can be solved in polynomial time. But is it also true for other modal logics like K, T and S4? In [9] Fariñas del Cerro and Penttonen have given an algorithm for solving the satisfiability problem of modal Horn clauses for several normal logics based on the modal resolution principle, and an upper bound is induced accordingly. The upper bound, however, is exponential for modal logics like K, T and S4. Thus, the problem that whether the complexity of the satisfiability problem for modal logics like K, T, B, K5, K45, S4 and S4.3 can be reduced to polynomial time by restricting the input formula to modal Horn clauses still remains open.

In this paper we solve this problem for several normal modal logics and give negative answers for nearly all these logics. We show that the satisfiability of modal Horn clauses for any modal logic between K and S4 is PSPACE-hard. In particular, since the modal logics K, T and S4 have been shown by Ladner [15] to be PSPACE-complete, the satisfiability problem of modal Horn clauses for each of K, T and S4 is PSPACE-complete. Similarly, we can show that the satisfiability of modal Horn clauses for any modal logic between K and B is PSPACE-hard. Since the logics KB and B are also known to be PSPACE-complete [4], the satisfiability of modal Horn clauses for KB and B is thus PSPACE-complete too. We next consider S4.3 and some extensions of K5 including K5, KD5, K45, KD45 and S5; the satisfiability problem for each of these logics is NP-complete [15, 12, 18, 4]. We then show that for the extensions of K5, the satisfiability of modal Horn clauses can be decided in polynomial time, but for S4.3, together with some linear tense logics like CL, SL and PL that are closely related to S4.3, the satisfiability problem still remains NP-complete even if the input formula is restricted to modal Horn clauses.

The rest of the paper is organized as follows. In Section 2 we review various normal modal logics briefly and introduce modal Horn clauses. In Section 3 we prove that the satisfiability of modal Horn clauses for any modal logic between K and S4 or between K and B is PSPACE-hard. In Section 4 we show that the satisfiability of modal Horn clauses for S4.3 is NP-hard. In Section 5 we first introduce a simpler form of modal Horn clauses for all extensions of K5 and then show that the satisfiability of modal Horn clauses for each of K5, K45, KD5, KD45 and S5 is solvable in polynomial time by giving a polynomial-time algorithm. The final section concludes this paper.

2. Modal logic

2.1. Syntax

All modal logics considered in this paper share a common language, whose alphabet Σ includes

variable construction letters: \$, 0, 1,
logical connectives: $\neg$, $\wedge$, $\Box$,
parentheses: (,).

Each member of $VAR = \$\{0,1\}^+\$$ is called a propositional variable. The set of modal formulas MF is defined to be the least set of words over Σ including VAR such that if A and B are modal formulas, then so are $(A \wedge B)$, $\neg A$ and $\Box A$.

We regard other usual connectives such as $\vee$, $\supset$ and $\Diamond$ as defined operators so that $(A \vee B)$, $(A \supset B)$ and $\Diamond A$ are treated as if they are abbreviations of $\neg(\neg A \wedge \neg B)$, $\neg(A \wedge \neg B)$ and $\neg\Box\neg A$, respectively.

If S is a modal formula or a set of modal formulas, we use $var(S)$ to denote the set of propositional variables appearing in S. To avoid unnecessary parentheses, we assume the following order of precedence for the operators: $\neg, \Box, \Diamond > \wedge > \vee > \supset$; any parentheses may be dropped from formulas if there is no worry of confusion. Finally, the modal degree of a modal formula is defined to be the maximum depth of nested occurrences of modal operators appearing in the formula; a classical propositional formula is a modal formula whose modal degree is 0.

2.2. Axiomatics

Modal logics are extensions of classical propositional logic; they thus should contain all axioms of classical propositional logic. Now let PC be some complete set of axiom schemas of classical propositional logic with modus ponens as the inference rule. We define a modal logic $\mathscr{L}$ as a set of axiom schemas. For each modal logic $\mathscr{L}$, the provability relation $\vdash_{\mathscr{L}}$ is defined to be the least set of modal formulas closed under the following rules:

- $\vdash_{\mathscr{L}} A$ if A is an instance of any axiom schema of $\mathscr{L}$;
- $\vdash_{\mathscr{L}} A$ if $\vdash_{\mathscr{L}} B$ and $\vdash_{\mathscr{L}} B \supset A$ (modus ponens);
- $\vdash_{\mathscr{L}} \Box A$ if $\vdash A$ (rule of necessitation).

If $\vdash_{\mathscr{L}} A$, we say A is *$\mathscr{L}$-provable* (or say A is a *theorem* of $\mathscr{L}$).

We are interested in modal logics consisting of combinations of the following axiom schemas:

$$\mathrm{K} = \Box(A \supset B) \supset (\Box A \supset \Box B),$$

$$\mathrm{D} = \Box A \supset \Diamond A,$$

$$\mathrm{T} = \Box A \supset A,$$

$$\mathrm{B} = A \supset \Box \Diamond A,$$

$$4 = \Box A \supset \Box\Box A,$$

$$5 = \Diamond A \supset \Box \Diamond A,$$

$$\mathrm{H} = (\Diamond A \wedge \Diamond B) \supset \Diamond(A \wedge \Diamond B) \vee \Diamond(A \wedge B) \vee \Diamond(B \wedge \Diamond A).$$

We use the word $Kr_1 \ldots r_n$ to refer to the logic containing the set of axiom schemas $PC \cup \{K, r_1, \ldots, r_n\}$. According to the above nomenclature, the modal logics conventionally named T, B, S4, S5 and S4.3 are equal to KT, KTB, KT4, KT45 and KTH4, respectively. In the sequel when referring to these logics, we prefer using their conventional names.

For any modal logic L_1 and L_2, we say L_2 is an *extension* of L_1 if every theorem of L_1 is also a theorem of L_2. If L_3 is an extension of L_2 and L_2 is an extension of L_1, then we say L_2 is a logic between L_1 and L_3. It is easy to see that D, T, B and S4 are all extensions of K, and T is between K and S4.

2.3. Semantics

The semantics of normal modal logics discussed here can be defined by using *Kripke models* [14]. A (Kripke) model M is a triple $\langle W, R, h \rangle$ consisting of the following elements:

- W is a nonempty set (of *worlds*),
- R is a binary relation on W called the *accessibility relation*; if $(w, w') \in R$, we say w' is *accessible* from w. The pair (W, R) is called the *frame* of M.
- $h \in W \to 2^{VAR}$ is the *meaning function*, which assigns to each world w in W a subset $h(w)$ of VAR with the intention that p is true at world w iff $p \in h(w)$.

Given any Kripke model $M = \langle W, R, h \rangle$, a world $w \in W$ and a formula $A \in MF$, the truth of A at w of M, denoted $M, w \models A$, is defined inductively as follows:

- $M, w \models p$ where $p \in VAR$ iff $p \in h(w)$;
- $M, w \models \neg A$ iff $M(w) \not\models A$;
- $M, w \models A \wedge B$ iff $M, w \models A$ and $M, w \models B$;
- $M, w \models \Box A$ iff, for every $w' \in W$ accessible from w (i.e. wRw'), $M, w' \models A$.

We say A is *M-satisfiable* if there is a world w in W such that $M, w \models A$, and say A is *M-valid*, denoted $M \models A$, if $M, w \models A$ for every world w in W.

We are particularly interested in Kripke models whose accessibility relations R satisfy any of the following conditions:

serial(D): for any $w \in W$, there is a w' in W such that wRw'.

reflexive(T): for any $w \in W$, wRw.

symmetric(B): for any $w, w' \in W$, if wRw' then $w'Rw$.

transitive(4): for any $w, w', w'' \in W$, if wRw' and $w'Rw''$ then wRw''.

euclidean(5): for any $w, w', w'' \in W$, if wRw' and wRw'' then $w'Rw''$.

connected(H): for any $w, w', w'' \in W$, if wRw' and wRw'' then $w'Rw''$ or $w' = w''$ or $w''Rw'$.

The symbol enclosed in parentheses at the end of each head item listed above is a short-hand for the corresponding condition. To establish correspondence between axiom schema and class of models, we deliberately use the same symbol to stand for an axiom schema as well as the condition every member of its corresponding class of models satisfies; moreover, we also use the word $Kr_1 \ldots r_n$ ($n \geqslant 0$) to denote the class of all Kripke models whose accessibility relations satisfy the condition denoted by each r_i. So, for example, a K model is any Kripke model and a KT4 model is any Kripke model whose accessibility relation is reflexive and transitive. Finally, we regard T, B, S4, S5 and S4.3, respectively, as aliases of KT, KTB, KT4, KT45, and KTH4. So, when we say a model is an S4 model, we mean it is a KT4 model.

Let $\mathscr{L}$ be any class of models. We say a formula A is *$\mathscr{L}$-satisfiable* if there exists an $\mathscr{L}$-model M and a world w among the set of worlds of M such that $M, w \models A$, and say A is *$\mathscr{L}$-valid*, denoted $\models_{\mathscr{L}} A$, if, for every $\mathscr{L}$-model M, A is M-valid.

By treating every (finite) set of formulas as an abbreviation of the conjunction of all its members, we extend the definitions of previously defined notions like satisfiability, validity, etc., to sets of formulas in the obvious way. So, for example, $M, w \models S$ iff $M, w \models A$ for every $A \in S$.

The following well-known proposition establishes the equivalence of the semantical validity relation and the syntactic provability relation for each logic given here.

Proposition 2.1 (Chellas[1] [3]). *Let $\mathscr{L} = Kr_1 \ldots r_n$, where $n \geqslant 0$ and each $r_i \in \{\mathrm{D}, \mathrm{T}, \mathrm{B}, 4, 5, \mathrm{H}\}$, be any logic. Then any modal formula A is $\mathscr{L}$-provable iff it is $\mathscr{L}$-valid.*

For more details about modal logic, the readers are referred to [13, 3].

2.4. Modal Horn clauses

As the notion of clauses has been defined on the classical logic, it was also introduced to modal logic. We say a modal formula A is a *modal clause* if it is a formula of the form

$$L_1 \vee \cdots \vee L_t \vee \Box D_1 \vee \cdots \vee \Box D_u \vee \Diamond E_1 \vee \cdots \vee \Diamond E_v,$$

[1] Chellas indeed did not discuss logics containing the axiom schema H in [3]; it is very easy, however, to add it into the proof by following the approach he used for other axiom schemas.

where $t, u, v \geqslant 0$, each L_i is a propositional literal, each D_j is a modal clause and each E_i is a conjunction of modal clauses but is not a disjunctive modal formula.

In the above clause, each L_i, $\Box D_j$ or $\Diamond E_k$ is called a *modal literal*; every formula of the form p (or $\neg p$), where $p \in VAR$, is called a *positive* (*or negative*) *literal*; p and $\neg p$ are *complementary* to each other. Furthermore, if a modal clause contains at most one occurrence of positive literals and each D_i as well as each E_j is (inductively) a single modal Horn clause, we then say it is a *modal Horn clause*. It should be noted that we admit the use of the empty clause $\perp$, which is interpreted as "false".

Example 2.2. In

(1) $p \vee \Box(p \wedge q)$,
(2) $\Box(\neg p \vee \neg q) \vee \Diamond(\neg p \vee q)$,
(3) $\Box(\neg p \vee \neg q) \vee \Diamond(p \wedge \neg q)$,
(4) $\Box(p \vee \neg q) \vee \Diamond\Box(\neg p \vee \Box\neg q)$,
(5) $\Box(\neg p \vee \neg q) \vee \Diamond\Box(\neg p \vee \neg q)$,

neither (1) nor (2) is a modal clause: (1) is not a modal clause because $p \wedge q$ is not a modal clause; (2) is not a modal clause because $\neg p \vee q$ is a disjunctive formula. Formulas (3)–(5) are modal clauses and (4) and (5) are modal Horn clauses. Formula (3) is not a modal Horn clause because $p \wedge \neg q$ is not a single modal Horn clause.

Instead of writing a modal Horn clause A in disjunctive form

$$H \vee B_1 \vee \cdots \vee B_n \quad (n \geqslant 0),$$

where H is either empty in case A contains no positive literal or is the disjunct of A containing the only positive literal of A, we usually write it in the *rule* form

$$B_1' \wedge \cdots \wedge B_n' \supset H',$$

where H' is either H in case H contains no negative literals or the rule form of H in case H contains negative literals, and each B_i' is the normal form of $\neg B_i$ by performing the following negation-in rewrite rules:

(1) $\neg(\neg A) \rightarrow A$,
(2) $\neg(A \wedge B) \rightarrow \neg A \vee \neg B$,
(3) $\neg(A \vee B) \rightarrow \neg A \wedge \neg B$,
(4) $\neg\Box A \rightarrow \Diamond\neg A$,
(5) $\neg\Diamond A \rightarrow \Box\neg A$.

Example 2.3. In

(1) $A = p \vee \neg q \vee \Box(\neg p \vee \neg q) \vee \Diamond(\neg p)$,
(2) $B = \neg p \vee (\Box\neg q) \vee \Diamond\Box(q \vee \Box\neg q)$,

A has only one positive modal literal p, and the negations of other modal literals are equivalent to q, $\Diamond(p \wedge q)$ and $\Box p$, respectively. Therefore, A has the rule form $q \wedge \Diamond(p \wedge q) \wedge \Box p \supset p$. Similarly, B has the rule form $p \wedge \Diamond q \supset \Diamond\Box(\Diamond q \supset q)$.

It should be noted that in the literature there is no unified definition of modal or temporal Horn clauses [2, 9, 10], and our definition of modal Horn clauses is taken from Fariñas and Penttonen [9], which is syntactically the simplest among known definitions.

3. The complexity of modal Horn clauses for logics between K and S4 or between K and B

This section is devoted to the proof of the PSPACE-hardness of the satisfiability of modal Horn clauses for any modal logic between K and S4 or between K and B.

Theorem 3.1. (1) *Let $\mathscr{L}$ be any modal logic between* K *and* S4. *Then the $\mathscr{L}$-satisfiability problem of modal Horn clauses is* PSPACE-*hard with respect to log-space reducibility.*

(2) *Let $\mathscr{L}$ be any modal logic between* K *and* B. *Then the $\mathscr{L}$-satisfiability problem of modal Horn clauses is* PSPACE-*hard with respect to log-space reducibility.*

Since the satisfiability problems for K, T, S4, KB and B have been shown to be PSPACE-complete [15, 4], we thus have the following result.

Corollary 3.2. *The satisfiability problem of modal Horn clauses for each of* K, T, S4, KB *and* B *is* PSPACE-*complete with respect to log-space reducibility.*

The method that we will use to prove Theorem 3.1 is to find a problem log-space-complete for PSPACE and then show that the problem is log-space-reducible to the satisfiability problem of modal Horn clauses for any modal logic between K and S4 and between K and B. The problem that we selected is the QBF problem [19], which is the canonical one among many problems log-space-complete for PSPACE.

We say that a formula is a *quantified Boolean formula* (*QBF* formula for short) if it has the form $Q_1X_1 \dots Q_mX_mA(X_1, \dots, X_m)$, where ($m>0$) each Q_i ($1 \leqslant i \leqslant m$) is either $\forall$ or $\exists$, and $A(X_1, \dots, X_m)$ is a propositional formula with all variables occurring in $\{X_1, \dots, X_m\}$. The set of all quantified Boolean formulas is denoted as QBF. Assume that all variables in a QBF formula range over the domain $\{1(\text{true}), 0(\text{false})\}$ and the meaning of all connectives (including the quantifiers and two constants 1 and 0) is as usual. Then the QBF problem is to determine whether the truth value of a given QBF formula is equal to 1. We use $B \equiv 1$ (0) to mean that the truth value of B is 1 (0).

3.1. *Reducing the QBF problem to the satisfiability of modal Horn clauses for logics between* K *and* S4

We now show that the QBF problem can be reduced to the satisfiability problem of modal Horn clauses for any modal logic between K and S4.

Lemma 3.3. *There exists a log-space transformation function MH from QBF formulas to sets of modal Horn clauses such that for any $B \in QBF$,*
(1) *if $B \equiv 1$, then $MH(B)$ is* K*-unsatisfiable and*
(2) *if $B \equiv 0$, then $MH(B)$ is* S4*-satisfiable.*

As a consequence of Lemma 3.3, we have the following corollary.

Corollary 3.4. *Let $\mathscr{L}$ be any modal logic between* K *and* S4 *that has a sound and complete semantics. Then the QBF problem is log-space-reducible to the $\mathscr{L}$-satisfiability problem of modal Horn clauses.*

Proof. For any given QBF formula $B = Q_1 X_1 \ldots Q_m X_m A(X_1, \ldots, X_m)$, let $\bar{B}$ be the complement of B, i.e. $\bar{B} = \bar{Q}_1 X_1 \ldots \bar{Q}_m X_m \neg A(X_1, \ldots, X_m)$ where $\bar{Q}_i$ is $\forall$ (resp. $\exists$) if Q_i is $\exists$ (resp. $\forall$) for $1 \leqslant i \leqslant m$. It is clear that the truth value of B is 1 iff the truth value of $\bar{B}$ is 0. It is also easy to see that if MH is log-space-computable, then so is the function $f(B) = MH(\bar{B})$. Now we show that for any logic $\mathscr{L}$ between K and S4, $B \equiv 1$ iff $f(B)$ is $\mathscr{L}$-satisfiable, thus having proved the corollary.

If $B \equiv 1$, then $\bar{B} \equiv 0$, and by Lemma 3.3, $MH(\bar{B})$ is S4-satisfiable. Hence $\neg(\bigwedge MH(\bar{B}))$ is not S4-valid. So, by the soundness of S4, $\neg(\bigwedge MH(\bar{B}))$ is not S4-provable and thus is not $\mathscr{L}$-provable. However, since we assume $\mathscr{L}$ has a complete semantics, $\neg(\bigwedge MH(\bar{B}))$ is not $\mathscr{L}$-provable implies $\neg(\bigwedge MH(\bar{B}))$ is not $\mathscr{L}$-valid and $MH(\bar{B})$ thus is $\mathscr{L}$-satisfiable.

On the other hand, if $B \equiv 0$, then $\bar{B} \equiv 1$, and by Lemma 3.3, $MH(\bar{B})$ is K-unsatisfiable. Hence $\neg(\bigwedge MH(\bar{B}))$ is K-valid. By the completeness of K, $\neg(\bigwedge MH(\bar{B}))$ is K-provable and thus is $\mathscr{L}$-provable. However, we assume $\mathscr{L}$ has a sound semantics, so $\neg(\bigwedge MH(\bar{B}))$ is $\mathscr{L}$-valid in the underlying semantics and hence $MH(\bar{B})$ is not $\mathscr{L}$-satisfiable. $\square$

The first part of Theorem 3.1 is a direct consequence of Corollary 3.4. We now begin to define the function MH satisfying Lemma 3.3. Before proceeding, we need some more definitions.

Definition 3.5. For any given QBF formula $B = Q_1 X_1 \ldots Q_m X_m A(X_1, \ldots, X_m)$ $(m > 0)$, let $W_B = \{x \mid x \in \{1, 0\}^* \text{ and } |x| \leqslant m\}$, $R_B = \{(x, x \cdot a) \in W_B^2 \mid x \in W_B \text{ and } a \text{ is either 1 or 0}\}$. For any $x \in W_B$, we use $|x|$ to denote the length of x and use x_i $(i \leqslant |x|)$ to denote the ith bit of x.

We view the frame $T_B = (W_B, R_B)$ as a complete binary tree whose root is ε and every node x of length $< m$ has two children $x \cdot 1$ and $x \cdot 0$.

For any $x \in W_B$ of length i, we use $B(x)$ to stand for the QBF formula $Q_{i+1} X_{i+1} \ldots Q_m X_m A(x_1, \ldots, x_i, X_{i+1}, \ldots, X_m)$. It is thus easy to see that $B = B(\varepsilon)$ and the truth value of every $B(x)$ $(|x| < m)$ is uniquely determined by the truth values of $B(x \cdot 1)$, $B(x \cdot 0)$ and the quantifier $Q_{|x|+1}$.

The truth value of B can now be evaluated by the following procedure:

(1) Construct the tree T_B top-down.

(2) Determine the truth value of $B(x)$ $(=A(x))$ for every leaf x.

(3) Determine the truth value of $B(x)$ for each internal node x bottom-up according to the quantifier $Q_{|x|+1}$ and the truth values of $B(x\cdot 1)$ and $B(x\cdot 0)$.

Finally $B\equiv 1$ iff the truth value of $B(\varepsilon)$ is 1.

3.1.1. The transformation function MH

The function MH is essentially a description of the above procedure by using modal formulas.

Let $B=Q_1X_1\dots Q_mX_mA(X_1,\dots,X_m)$ $(m>0)$ be any QBF formula, n be the number of subformulas of A, and $A_1,\dots,A_n$ be any enumeration of all occurrences of subformulas of A with $A_p=A$ for some $1\leqslant p\leqslant n$. We shall then define two sets of modal Horn clauses, $MH'(B)$ and $MH(B)=MH'(B)\cup\{\neg Y_0\}$. $MH'(B)$, basically, is a description of the procedure for evaluating B. Before defining $MH'(B)$, we first state the intended usage of the variables appearing in $MH'(B)$.

The set of variables $var(MH'(B))$ includes the following elements.

- $X_1,\dots,X_m$ and $\bar{X}_1,\dots,\bar{X}_m$: Each X_i plays the same role as it is in B and $\bar{X}_i$ is intended to stand for the complement of X_i. The key property is that for every leaf x in the tree T_B, if $x_i=1$ (0), then X_i $(\bar{X}_i)$ should be true at x. As a result, each leaf x of T_B uniquely determines an interpretation I_x for $\{X_1\dots X_m\}$ with the convention that $I_x(X_i)=1$ (0) if X_i $(\bar{X}_i)\in x$. On the other hand, every interpretation for variables appearing in B must belong to the set of all interpretations determined by leaves of T_B.
- $L_0,\dots,L_m$: L_i is intended to represent the level (or length) of each node x in the tree such that L_i is true at x if and only if x is at level i.
- $U_1,\dots,U_m$ and $\bar{U}_1,\dots,\bar{U}_m$: Each U_i $(\bar{U}_i)$ is used as a shorthand for $X_i\wedge L_i$ $(\bar{X}_i\wedge L_i)$.
- $C_1,\dots,C_n$ and $\bar{C}_1,\dots,\bar{C}_n$: C_i $(\bar{C}_i)$ is used to represent the truth value of the subformula A_i under the interpretations determined by leaves of T_B. The convention is that C_i $(\bar{C}_i)\in x$ iff $I_x(A_i)=1$ (0), where I_x is the interpretation determined by x. C_i and $\bar{C}_i$ have no effects at internal nodes.
- $Y_0,\dots,Y_m$ and $\bar{Y}_0,\dots,\bar{Y}_m$: Y_i and $\bar{Y}_i$ are used to represent the truth value of $B(x)$ for any node x at level i. It is possible that both Y_i and $\bar{Y}_i$ are true at a node x if $|x|<i$.

Now we define $MH'(B)=\bigcup_{0\leqslant i\leqslant 5}T_i$, where each T_i is given as follows.

(1) $T_0=\{L_0\}$. T_0 states that the root node is at level 0.

(2) $T_1=\bigcup_{0\leqslant i<m}\{\Box^i(L_i\supset\Diamond U_{i+1}),\Box^i(L_i\supset\Diamond\bar{U}_{i+1})\}$.

(3) $T_2=\bigcup_{1\leqslant i\leqslant m}\{\Box^i(U_i\supset X_i),\Box^i(U_i\supset L_i),\Box^i(\bar{U}_i\supset\bar{X}_i),\Box^i(\bar{U}_i\supset L_i)\}$. T_1 and T_2 state that every node x in the tree at level $i<m$ should contain two children at level $i+1$ such that X_i is true at one of them and $\bar{X}_i$ is true at the other. T_1 and T_2 correspond to the top-down expansion of the tree T_B.

(4) $T_3=\bigcup_{1\leqslant i<m,\,i\leqslant j<m}\{\Box^j(X_i\supset\Box X_i),\Box^j(\bar{X}_i\supset\Box\bar{X}_i)\}$. T_3 states that the truth of X_i (or $\bar{X}_i$) at ancestor nodes should be propagated to all descendant nodes.

Therefore, the set of X-type and $\bar{X}$-type variables true at each leaf x constitutes the interpretation I_x for A, and the truth value of every subformula of A at the leaf x can be evaluated. T_0–T_3 correspond to the first step of the procedure for evaluating B.

(5) $T_4 = \bigcup_{0 \leqslant i < m} \{ \Box^i \varphi \mid \varphi \in \mathscr{E}_i \}$, where

$$\mathscr{E}_i = \begin{cases} \{ (\Diamond(\bar{Y}_{i+1} \wedge U_{i+1}) \supset \bar{Y}_i), (\Diamond(\bar{Y}_{i+1} \wedge \bar{U}_{i+1}) \supset \bar{Y}_i), \\ (\Diamond(Y_{i+1} \wedge U_{i+1}) \wedge \Diamond(Y_{i+1} \wedge \bar{U}_{i+1}) \supset \bar{Y}_i) \}, & \text{if } Q_{i+1} = \forall, \\ \{ (\Diamond(Y_{i+1} \wedge U_{i+1}) \supset Y_i), (\Diamond(Y_{i+1} \wedge \bar{U}_{i+1}) \supset Y_i), \\ (\Diamond(\bar{Y}_{i+1} \wedge U_{i+1}) \wedge \Diamond(\bar{Y}_{i+1} \wedge \bar{U}_{i+1}) \supset \bar{Y}_i) \} & \text{if } Q_{i+1} = \exists. \end{cases}$$

T_4 is used to describe how the truth values of Y_i and $\bar{Y}_i$ at each internal node at level i are determined by its children, obeying the meaning of quantification. This corresponds to the third step of the evaluation of B.

(6) $T_5 = \{ \Box^m \varphi \mid \varphi \in \mathscr{C} \}$, where $\mathscr{C} = \bigcup_{0 \leqslant i \leqslant n} \mathscr{C}_i$, and each $\mathscr{C}_i$ is defined, depending on subformula A_i of A, as follows.

- For any $1 \leqslant i \leqslant n$,
  - (a) if $A_i = X_j$ is a propositional variable, then $\mathscr{C}_i = \{ X_j \wedge L_m \supset C_i,\ \bar{X}_j \wedge L_m \supset \bar{C}_i \}$,
  - (b) if $A_i = \neg A_j$, then $\mathscr{C}_i = \{ C_j \wedge L_m \supset \bar{C}_i,\ \bar{C}_j \wedge L_m \supset C_i \}$,
  - (c) if $A_i = A_j \wedge A_k$, then $\mathscr{C}_i = \{ C_j \wedge C_k \wedge L_m \supset C_i,\ \bar{C}_j \wedge L_m \supset \bar{C}_i, \bar{C}_k \wedge L_m \supset \bar{C}_i \}$.
- $\mathscr{C}_0 = \{ C_p \wedge L_m \supset Y_m,\ \bar{C}_p \wedge L_m \supset \bar{Y}_m \}$.

T_5 encodes the boolean evaluation rules which can be used to evaluate the truth value of $I_x(A_i)$ for each subformula A_i of A at each leaf x of T_B. After the truth value of every subformula of A has been determined, we use the truth of Y_m (or $\bar{Y}_m$) at x to represent the fact that A is true (false) at x. Note that the L_m's used in each clause is to ensure that it has effect only at leaf nodes. T_5 corresponds to the second step of the evaluation of B.

(7) For technical reasons, we assume $MH'(B)_k = \{ \varphi \mid \Box^k \varphi \in MH'(B) \}$ for $0 \leqslant k \leqslant m$.

In addition to what is implied by T_0–T_5, $\neg Y_0 \in MH(B)$ means Y_0 cannot be true at the root node. As a result, if B is false, the evaluation tree T_B for B gives us a model in which $MH(B)$ holds at the root node. On the other hand, if B is true, the root node must contain Y_0 according to the rules specified by $MH'(B)$. We thus reach a contradiction and $MH(B)$ is hence not satisfiable.

Before formally proving all assertions described among the definitions, we note that $MH(B)$ can indeed be computed from B in log-space and leave the details to the reader.

3.1.2. *Correctness of the transformation function MH*

We now show that the function MH does satisfy Lemma 3.3.

Let x be any string in W_B of length i. Define $I(x) = \{ Z_1, \ldots, Z_i \}$, where Z_j $(1 \leqslant j \leqslant i)$ is X_j if $x_j = 1$ and Z_j is $\bar{X}_j$ if $x_j = 0$. In particular, define $I(\varepsilon) = \phi$. We also define

$U(x)=\{L_i, U_i\}$ if $x_i=1$ and $U(x)=\{L_i, \bar{U_i}\}$ if $x_i=0$. In particular, define $U(\varepsilon)=\{L_0\}$. The following lemma establishes the relation between W_B and any K-model satisfying $MH'(B)$.

Lemma 3.6. *Let* $B=Q_1X_1\ldots Q_mX_mA(X_1,\ldots,X_m)$ *be any QBF formula,* $M=\langle W,R,h\rangle$ *be any* K*-model and* w_0 *any world in* W *such that* $M,w_0\models MH'(B)$. *Then there exists a mapping* τ *from* W_B *to* W *satisfying the following properties:*

(1) $\tau(\varepsilon)=w_0$;

(2) *for any* $x\in W_B$ *of length* i, $M,\tau(x)\models I(x)\cup U(x)$;

(3) *for any* $x\in W_B$ *of length* i, $w_0R^i\tau(x)$;

(4) *for any* $x\in W_B$ *of length* m,

- *if* $B(x)\equiv 1$, *then* $M,\tau(x)\models C_p$, *and*
- *if* $B(x)\equiv 0$, *then* $M,\tau(x)\models \bar{C}_p$;

(5) *for any* $x\in W_B$ *of length* i, *if* $B(x)\equiv 1$, M, $\tau(x)\models Y_i$, *and if* $B(x)\equiv 0$, M, $\tau(x)\models \bar{Y}_i$.

Proof. See the appendix.

The following lemma is a direct consequence of Lemma 3.6.

Lemma 3.7. *If* $B\equiv 1$, *then* $MH(B)$ *is* K*-unsatisfiable.*

Proof. Assume that there exists a K-model $M=\langle W,R,h\rangle$ and a world $w_0\in W$ such that $M,w_0\models MH(B)$. Because $MH'(B)\subseteq MH(B)$, by property (5) of Lemma 3.6, if $B\equiv 1$ $(=B(\varepsilon))$, $M,w_0\models Y_0$. However, since $\neg Y_0$ is contained in $MH(B)$, we also have $M,w\not\models Y_0$. So M does not exist and $MH(B)$ is K-unsatisfiable. □

We now show the satisfiability of $MH(B)$ in case B is false.

Lemma 3.8. *If* $B\equiv 0$, $MH(B)$ *is* S4*-satisfiable.*

Proof. By construction. Let $M=\langle W_B,R,h\rangle$, where

- $R=R_B^*$, i.e. the reflexive and transitive closure of R_B, and
- h is any function from W_B to 2^{VAR} satisfying the following conditions. Let x be any world in W_B. Then:
  (1) For any $Z\in\{X_1,\ldots,X_m,\bar{X}_1,\ldots,\bar{X}_m, L_0,\ldots,L_m, U_1,\ldots,U_m, \bar{U}_1,\ldots,\bar{U}_m\}$, $Z\in h(x)$ iff $Z\in I(x)\cup U(x)$.
  (2) For $1\leqslant i\leqslant n$,
  $C_i\in h(x)$ iff $|x|=m$ and $A_i(x)\equiv 1$,
  $\bar{C}_i\in h(x)$ iff $|x|=m$ and $A_i(x)\equiv 0$.
  (3) For $0\leqslant i\leqslant m$,
  $Y_i\in h(x)$ iff either ($|x|=i$ and $B(x)\equiv 1$) or $i>|x|$, and
  $\bar{Y}_i\in h(x)$ iff either ($|x|=i$ and $B(x)\equiv 0$) or $i>|x|$.

To show that $M, \varepsilon \models MH(B)$, we note that clauses (1)–(5) given by

(1) $X_i \supset \Box X_i$, $\bar{X}_i \supset \Box \bar{X}_i$ for any $0 < i \leqslant m$,

(2) $L_i \supset \Diamond U_{i+1}$ and $L_i \supset \Diamond \bar{U}_{i+1}$ for $0 \leqslant i < m$,

(3) $U_i \supset X_i$, $U_i \supset L_i$ and $\bar{U}_i \supset \bar{X}_i$, $\bar{U}_i \supset L_i$ for $0 < i \leqslant m$,

(4) any clause in $\bigcup_{0 \leqslant i < m} \mathscr{E}_i$ and

(5) any clause in $\mathscr{C}$

are all valid in M because, for any world $w \in W_B$ and for each clause $l \supset r$ listed above, either $M, \not\models l$ or $M, w \models r$, as the reader can easily verify.

Therefore, for any world $w \in W_B$, for any clause A listed above and for any $i \geqslant 0$, we have $M, w \models \Box^i A$. As a result, $M, \varepsilon \models \bigcup_{1 \leqslant i \leqslant 5} T_i$. Furthermore, we have $M, \varepsilon \models \neg Y_0 \wedge L_0$, so $M, \varepsilon \models MH(B)$. $\Box$

Now we have proven Lemma 3.3, which is a direct consequence of Lemmas 3.7 and 3.8 and the fact that $MH(B)$ can be computed from B in log-space.

Remark. Since $\Box A \equiv \Box\Box A$ is valid for S4, if we are merely concerned with S4-satisfiability of modal Horn clauses, it is possible to get a set of Horn clauses simpler than $MH(B)$. We can replace the sequence of modal operators $\Box^i$ $(i \geqslant 1)$ appearing at the front of each clause of $MH(B)$ by a single $\Box$. The simplified $MH(B)$ has modal degree 2 and can also be used to prove the PSPACE-hardness of the satisfiability of modal Horn clauses for S4. As a result, the satisfiability of modal Horn clauses for S4 is PSPACE-hard even if the modal degree of the input set of Horn clauses is restricted to not greater than 2 and thus is unlikely to be solvable in polynomial time. This suggests that the claim of Fariñas del Cerro and Penttonen [9] that the satisfiability of modal Horn clauses can be solved in polynomial time if the modal degree of the Horn clauses is limited to a constant is incorrect for S4.

3.2. *Reducing the QBF problem to the satisfiability of modal Horn clauses for any modal logic between* K *and* B

We now begin to show the second part of Theorem 3.1. The proof strategy is analogous to the proof of the first part. We first show the analog of Lemma 3.3.

Lemma 3.9. *There exists a log-space transformation function MB from QBF formulas to sets of modal Horn clauses such that for any $B \in QBF$,*

(1) *if $B \equiv 1$, then $MB(B)$ is* K*-unsatisfiable,*

(2) *if $B \equiv 0$, then $MB(B)$ is* B*-satisfiable.*

The proof of the following corollary is analogous to that of Corollary 3.4.

Corollary 3.10. *Let $\mathscr{L}$ be any modal logic between* K *and* B *which has a sound and complete semantics. Then the QBF problem is log-space-reducible to the $\mathscr{L}$-satisfiability problem of modal Horn clauses.*

The second part of Theorem 3.1 is a direct consequence of Corollary 3.10.

3.2.1. The transformation function MB

By slightly modifying the set of modal Horn clauses $MH(B)$ used in Section 3.1, we can obtain the $MB(B)$ needed for Lemma 3.9. Formally, let

$$MB(B)=\{L_0,\neg Y_0\}\cup T_1\cup T_4\cup T_5\cup S,$$

where T_1, T_4 and T_5 are the same as those given in the definition of $MH(B)$ and

$$S=\bigcup_{0<i\leqslant m}\{\Box^i(U_i\supset\Box^{m-i}X_i),\Box^i(U_i\supset L_i),\Box^i(\bar{U}_i\supset\Box^{m-i}\bar{X}_i),\Box^i(\bar{U}_i\supset L_i)\}.$$

Before describing the informal meaning of S, we note that, due to the reflexive and symmetric nature of the accessibility relation for the modal logic B, we can no longer use T_2 and T_3 defined in Section 3.1 to propagate X_i (and $\bar{X}_i$) to descendant leaves without resulting in inconsistency. Consider the case that $m=3$. In order to propagate X_1 and $\bar{X}_1$ to leaves, we have in T_3 the rules $\Box(X_1\supset\Box X_1)$, $\Box^2(X_1\supset\Box X_1)$, $\Box(\bar{X}_1\supset\Box\bar{X}_1)$ and $\Box^2(\bar{X}_1\supset\Box\bar{X}_1)$. Let R be the reflexive and symmetric closure of R_B. Then since $1\,R^2\,0$ and $0\,R^2\,1$, the node 1 (resp. node 0) containing X_1 (resp. $\bar{X}_1$) must enforce the node 0 (1) to contain X_1 (resp. $\bar{X}_1$) in order to obey the T_3 rules. As a result, both node 0 and node 1 will contain X_1 and $\bar{X}_1$, which again will enforce all leaves to contain X_1 and $\bar{X}_i$. So we no longer will be able to use all X_i and $\bar{X}_i$ at each leaf to determine a unique interpretation.

S is essentially another way of propagating X_i and $\bar{X}_i$ to descendant leaves suitable for reflexive and symmetric accessibility relations. It says that if a node x contains U_i ($\bar{U}_i$), which by T_1 means that x is at level i and $x_i=1$ (0), then every node x' with $xR^{m-i}x'$ must contain X_i ($\bar{X}_i$). As a result, all descendant leaves of x will contain X_i ($\bar{X}_i$). Since every leaf has exactly one ancestor at each level i, which by T_1 must contain U_i or $\bar{U}_i$ but not both, all the X_j's and $\bar{X}_j$'s that each leaf x contains thus constitute the interpretation I_x for the variables appearing in the formula B. It should be noted, however, that by S it is possible that some node contains both X_i and $\bar{X}_i$, but it may happen only when it is an internal node.

We claim that $MB(B)$ does satisfy Lemma 3.9; the proof resembles that used in the proof of Lemma 3.3. To avoid unnecessary duplication, however, we do not present the proof here. The interested readers can follow the same line as we did for Lemma 3.3 to obtain it.

4. The complexity of modal Horn clauses for S4.3

In this section we will show that the satisfiability problem for S4.3 with the input restricted to modal Horn clauses is NP-complete. Since the satisfiability problem for S4.3 in general has been shown to be NP-complete [18], we thus only have to show its hardness part.

Theorem 4.1. *The satisfiability problem for* S4.3 *with the input restricted to modal Horn clauses is* NP-*hard.*

Proof. We will show that the satisfiability problem for classical propositional clauses can be reduced to the S4.3-satisfiability of modal Horn clauses in polynomial time. The problem is thus NP-hard.

The polynomial-time transformation function MT is defined as follows. Assume the input $\mathscr{C}$ is a set of propositional clauses

$$\{C_1, C_2, \dots, C_m\},$$

where $m \geqslant 1$ and each C_i $(1 \leqslant i \leqslant m) = \{L_{i1}, \dots, L_{ip_i}\}$ is a set of literals that does not contain complementary literals.

Let $var(\mathscr{C}) = \{X_1, \dots, X_n\}$ be the set of propositional variables appearing in $\mathscr{C}$. The set of propositional variables used in $MT(\mathscr{C})$ includes not only $var(\mathscr{C})$ but also the set $\{\bar{X} \mid X \in var(\mathscr{C})\}$, each element $\bar{X}_i$ of which is a new variable not occurring in $\mathscr{C}$ and is intended to represent the complement of X_i.

Define $MT(\mathscr{C}) = \bigcup_{0 \leqslant i \leqslant 3} S_i$, where each S_i is given as follows:

(1) $S_0 = \bigcup_{1 \leqslant i \leqslant n} \{\Box(\neg X_i \vee \neg \bar{X}_i)\}$;

(2) $S_1 = \bigcup_{1 \leqslant i \leqslant n} \{\Diamond X_i, \Diamond \bar{X}_i\}$;

(3) for $1 \leqslant i \leqslant n$, let $g_i = \Box(\neg \bar{X}_i \vee \Box \neg X_i)$, $\bar{g}_i = \Box(\neg X_i \vee \Box \neg \bar{X}_i)$, and let $S_2 = \bigcup_{1 \leqslant i \leqslant n} \{g_i \vee \bar{g}_i\}$;

(4) for each $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant p_i$, let

$$l_{ij} = \begin{cases} \bar{g}_k & \text{if } L_{ij} = \neg X_k, \\ g_k & \text{if } L_{ij} = X_k, \end{cases}$$

and let

$$S_3 = \bigcup_{1 \leqslant i \leqslant m} \{l_{i1} \vee \cdots \vee l_{ip_i}\}.$$

The formula set $MT(\mathscr{C})$ defined above is clearly a set of modal Horn clauses; it is also easy to see that $MT(\mathscr{C})$ can be constructed from $\mathscr{C}$ in time polynomial in the size of $\mathscr{C}$. Moreover, Lemma 4.3 states that MT is satisfiability-preserving, and so we have proved the theorem. $\Box$

The intuition behind the construction of S_0–S_2 can be best explained by the proof of the following key lemma.

Lemma 4.2. *Let* $M = \langle W, R, h \rangle$ *be any* S4.3-*model and* w *any world in* W *such that* $M, w \models S_0 \cup S_1 \cup S_2$. *Then for any* $1 \leqslant i \leqslant n$, $M, w \models g_i$ *iff* $M, w \not\models \bar{g}_i$.

Proof. By S_1, $M, w \models \Diamond X_i \wedge \Diamond \bar{X}_i$. So there must exist $w', w'' \in W$ accessible from w such that $M, w' \models X_i$ and $M, w'' \models \bar{X}_i$.

Since R is connected, we have $w'Rw''$ or $w''Rw'$ or $w'=w''$. But the last case is impossible, for if $w'=w''$ we would have $M,w \models \Diamond(X_i \wedge \bar{X}_i)$, which is contradictory to S_0.

In case $w'Rw''$, $M,w' \models X_i \wedge \Diamond\bar{X}_i$. So $M,w \models \Diamond(X_i \wedge \Diamond\bar{X}_i)\ (=\neg\bar{g}_i)$ and hence $M,w \not\models \bar{g}_i$. Similarly, if $w''Rw'$, we have $M,w \not\models g_i$. Therefore, either $M,w \not\models g_i$ or $M,w \not\models \bar{g}_i$.

But, by S_2, at least one of g_i and $\bar{g}_i$ must be true at w; therefore, $M,w \models g_i$ iff $M,w \not\models \bar{g}_i$. □

Now by Lemma 4.2 every model–world pair (M,w) satisfying S_0–S_2 uniquely determines an interpretation for $\mathscr{C}$ which interprets X_i as true (resp. false) if g_i ($\bar{g}_i$) holds at w. Moreover, every interpretation for $\mathscr{C}$ must be identical to some interpretation determined in this way. Hence we can use g_i and $\bar{g}_i$, respectively, to simulate X_i and $\neg X_i$; S_3 then is just a substitution of g_i and $\bar{g}_i$, respectively, for each X_i and $\neg X_i$ occurring in $\mathscr{C}$. Now we show that MT is satisfiability-preserving.

Lemma 4.3. *$\mathscr{C}$ is satisfiable iff $MT(\mathscr{C})$ is* S4.3-*satisfiable.*

Proof. $\Rightarrow$: Since $\mathscr{C}$ is satisfiable, there exists a literal set $E=\{L_1,\dots,L_n\}$, where L_i is either X_i or $\neg X_i$ such that, for each $C_i \in \mathscr{C}$, $E \cap C_i \neq \phi$.

Now let $M=\langle W,R,h\rangle$, where

- W is the set of rational numbers Q,
- R is the "less than or equal to" relation $\leqslant$ on Q,
- $h(1)=\{X_i \mid X_i \in E\} \cup \{\bar{X}_i \mid \neg X_i \in E\}$,
- $h(2)=\{\bar{X}_i \mid X_i \in E\} \cup \{X_i \mid \neg X_i \in E\}$ and
- $h(n)=\phi$ for any $n \in Q\setminus\{1,2\}$.

Note that M is indeed an S4.3-model. It is easy to verify that $M,0 \models A$ for every $A \in S_0 \cup S_1 \cup S_2$; it is also easy to verify that if X_i ($\neg X_i$)$\in E$ then $M,0 \models g_i(\bar{g}_i)$. Since for each $1 \leqslant i \leqslant m$, there exists a literal $L_{ir_i} \in E \cap C_i$, we thus have $M,0 \models l_{ir_i}$ and hence $M,0 \models l_{i1} \vee \cdots \vee l_{ip_i}$. As a result, $M,0 \models S_3$ as well.

$\Leftarrow$: Let $M=\langle W,R,h\rangle$ be any S4.3-model such that $M,w_0 \models MT(\mathscr{C})$ for some $w_0 \in W$. From M and w_0, we construct a literal set $E=\{L_1,\dots,L_n\}$, where

$$L_i=\begin{cases} X_i & \text{if } M,w_0 \models g_i,\\ \neg X_i & \text{if } M,w_0 \models \bar{g}_i.\end{cases}$$

By Lemma 4.2, for $1 \leqslant i \leqslant n$, exactly one of X_i and $\neg X_i$ belongs to E. Now we show that $E \cap C \neq \phi$ for each clause $C \in \mathscr{C}$, so $\mathscr{C}$ is satisfiable. Let $C=\{X_{u_1},\dots,X_{u_\alpha}, \neg X_{v_1},\dots,\neg X_{v_\beta}\}$ be any clause in $\mathscr{C}$ where $\alpha \geqslant 0$, $\beta \geqslant 0$. By S_3, we have

$$M,w_0 \models \left(\bigvee_{1\leqslant i\leqslant\alpha} g_{u_i} \vee \bigwedge_{1\leqslant j\leqslant\beta} \bar{g}_{v_j}\right).$$

So either $M,w_0 \models g_{u_i}$ for some $1 \leqslant i \leqslant \alpha$, or $M,w_0 \models \bar{g}_{v_j}$ for some $1 \leqslant j \leqslant \beta$. Accordingly, either $X_{u_i} \in E$ or $\neg X_{v_j} \in E$, and $E \cap C \neq \phi$. □

Remark. It should be noticed that our proof about the complexity of modal Horn clauses for S4.3 can also be used without change to show that the satisfiability of modal Horn clauses for some linear tense logics like CL, SL and PL (see [16] for an introduction to these logics) is NP-complete. Like S4.3, the general satisfiability problem for all these logics is known to be NP-complete [18].

5. The complexity of modal Horn clauses for extensions of K5

5.1. K5 *Horn clause*

The modal clause and modal Horn clause defined in Section 2 are general for all normal logics; for specific modal logics more specialized definitions are possible. For example, since for S5 every set of modal clauses can be translated into an equivalent set of modal clauses of modal degree at most 1, the S5 modal clause is defined in [7] to be of the form

$$C \vee \Box D_1 \vee \cdots \vee \Box D_m \vee \Diamond E_1 \vee \cdots \vee \Diamond E_n,$$

where $C, D_1, \ldots, D_m$ are classical clauses and $E_1, \ldots, E_n$ are sets of classical clauses.

Indeed, we can show for all extensions, not only of S5 but also of K5, that every set of modal clauses can be translated into an equivalent set of modal clauses of modal degree at most 2. We can thus obtain a simpler form of the modal Horn clause for all extensions of K5. The possibility of such a translation is based on the following proposition.

Proposition 5.1 (Chen and Lin [4]). *Let* $\odot$, $\odot_1$ *and* $\odot_2$ *be any modal operators. Then the formulas*

(1) $\Box(A \vee \odot B) \equiv \Box A \vee \Box \odot B$,

(2) $\Diamond(A \wedge \odot B) \equiv \Diamond A \wedge \Diamond \odot B$,

(3) $\odot_1 \odot \odot_2 A \equiv \odot_1 \odot_2 A$

are valid for K5 (*and hence valid for* KD5, KD45, K45 *and* S5 *as well*).

According to Proposition 5.1, every set of modal Horn clauses can be rewritten to an equivalent set of modal Horn clauses of modal degree 2 or less in polynomial time by the following rewrite rules:

(1) $\Box(D_1 \vee \Box D_2) \rightarrow \Box D_1 \vee \Box\Box D_2$;

(2) $\Box(D_1 \vee \Diamond D_2) \rightarrow \Box D_1 \vee \Box\Diamond D_2$;

(3) $\Diamond(D_1 \vee D_2) \rightarrow \Diamond D_1 \vee \Diamond D_2$;

(4) $\odot_1 \odot \odot_2 D_1 \equiv \odot_1 \odot_1 D_1$.

In the above rules D_1 and D_2 are any modal Horn clauses, $\odot$, $\odot_1$ and $\odot_2$ are any modal operators, i.e. $\Box$ or $\Diamond$.

The first three rules are used to distribute the principal modal operator of any modal clause to each disjunct of the clause; the last rule is used to eliminate intermediate modality. Note that both sides of the third rule are equivalent for K.

Now we may assume without loss of generality that each clause of the set of modal Horn clauses under consideration has the form

$$M_1C_1 \vee M_2C_2 \vee \cdots \vee M_nC_n,$$

where $n \geqslant 0$, and each M_iC_i is of the form $L, \odot\Diamond L$ or $\odot\Box C$, where L is any propositional literal, C is any propositional Horn clause and $\odot$ is $\Box$, $\Diamond$ or *empty*.

We shall call any modal Horn clause of the above form a *K5 Horn clause* and each M_iC_i is called a *K5-literal*; in particular, the K5-literal which contains a positive propositional literal is called a *positive* K5-literal and other K5-literals containing no positive propositional literals are called *negative* K5-literals. Moreover, if a positive K5-literal contains no negative literals, it is called a *K5-atom*. Finally, any occurrence of a modal operator in a K5 Horn clause is called a *level-1* (occurrence of) modal operator if it is not in the scope of any modal operators; otherwise it is in the scope of another modal operator and we shall call it a *level-2* modal operator.

Example 5.2. Let $S=\{A,B\}$ be a set of modal Horn clauses where

(1) $A=\Box(\Diamond p \vee \Diamond\Box(\Diamond\neg q \vee \neg r) \vee \Diamond\neg p)$ and

(2) $B=\Box\Diamond\Box(\neg p \vee q)$.

By applying the rewrite rules, we get

$A \xrightarrow{*} \Box\Diamond p \vee \Box\Diamond\Box(\Diamond\neg q \vee \neg r) \vee \Box\Diamond\neg p$,

$\xrightarrow{*} \Box\Diamond p \vee \Box\Diamond\Box\Diamond\neg q \vee \Box\Diamond\Box\neg r \vee \Box\Diamond\neg p$,

$\xrightarrow{*} \Box\Diamond p \vee \Box\Diamond\neg q \vee \Box\Box\neg r \vee \Box\Diamond\neg p\,(=A')$,

$B \rightarrow \Box\Box(\neg p \vee q)\ (=B')$.

In A', there is one K5-atom $\Box\Diamond p$ in which the $\Diamond$ is a level-2 occurrence, and all the other K5-literals are negative; in B' there is one positive K5-literal $\Box\Box(\neg p \vee q)$.

5.2. *A modal Herbrand theorem for* KD5

Before demonstrating our algorithm, we need to establish a modal version of the Herbrand theorem for KD5. The classical Herbrand theorem says that to determine satisfiability of a given set of classical clauses, you only have to consider all structures whose domain is the Herbrand universe. Likewise, our modal Herbrand theorem tells us that, to test the KD5-satisfiability of a given set S of K5 Horn clauses, we only have to consider all KD5-models with a common fixed frame determined by the skolemization of S.

The first step of the theorem is to skolemize the given set of K5 Horn clauses, by which we simply associate each occurrence of $\Diamond$ with a unique number. The goal of the skolemization is to determine the frame common to all KD5-models that need to be considered.

Definition 5.3. Let S be a set of K5 Horn clauses and N the set of nonzero natural numbers. A skolemization of S is a 1–1 mapping *sk* from the set of all occurrences of

$\Diamond$ in S to N. If sk is a skolemization of S, we use $I'_1(sk)$ (resp. $I_2(sk)$) to denote the image of all level-1 (resp. level-2) occurrences of $\Diamond$ in S. We define $I_1(sk)=I'_1(sk)$ if $I'_1(sk)\neq\phi$; otherwise define $I_1(sk)=\{0\}$. Note that 0 never occurs in $I_2(sk)$. Finally, define $I(sk)=I_1(sk)\cup I_2(sk)$. $I(sk)$ is called the index set of sk and $I_1(I_2)$ is called the level-1 (level-2) index set of sk. Since all occurrences of $\Diamond$ in S are typed the same, to help distinguish among different occurrences of $\Diamond$, we use $\Diamond$ subscripted with an index i to refer to the occurrence skolemized by i, i.e. $sk(\Diamond_i)=i$.

Definition 5.4. Let S be any set of K5 Horn clauses and sk any skolemization of S. Then we call any KD5-model $M_{sk}=\langle W_{sk}, R_{sk}, h\rangle$ a Herbrand KD5-model based on S and sk, where

- $W_{sk}=\{\varepsilon\}\cup I(sk)$,
- $R_{sk}=\{(\varepsilon,i)\mid i\in I_1(sk)\}\cup\{(i,j)\mid i,j\in I(sk)\}$ and
- h is any function from W_{sk} to 2^{VAR}.

In other words, every Herbrand KD5-model based on S and sk has the indexed set together with a distinguished initial world "ε" as the set of worlds, and has an accessibility relation in which every index is accessible from every index and every level-1 index is accessible from the initial world.

Moreover, we also want every skolemized $\Diamond_i$ to be interpreted as the world i instead of as an existentially quantified world variable. Therefore, besides the standard satisfaction relation $\models$ common to all Kripke models, we also need another satisfaction relation $\models_{sk}$ for Herbrand KD5-models, whose definition is basically the same as that of the standard satisfaction relation except that if a formula of the form $\Diamond_i A$ is to be interpreted as true, it means that A is true at the world i and i is accessible from the current world. In other words, we have

$$M_{sk}, w \models_{sk} \Diamond_i A \text{ iff } wR_{sk}i \quad\text{and}\quad M_{sk}, i \models_{sk} A \text{ for any } i\in I(sk)$$

in the definition of $\models_{sk}$; the definitions for other connectives such as $\neg$, $\vee$ and $\Box$ are all the same as those defined for the standard $\models$.

It should be noticed that it is with respect to the relation $\models_{sk}$ instead of the standard satisfaction relation that our algorithm determines KD5-satisfiability of modal Horn clauses. Our modal Herbrand theorem states, however, that these two relations determine the same satisfiable sets of modal Horn clauses.

Lemma 5.5. *Let S be any set of* K5 *Horn clauses and sk any skolemization of S. Then*

(1) *for any Herbrand* KD5*-model $M_{sk}=\langle W_{sk}, R_{sk}, h\rangle$ based on S and sk, $M_{sk}, w\models_{sk} A$ implies $M_{sk}, w\models A$, where w is any world in W_{sk} and A is any occurrence of a subformula of S;*

(2) *(modal Herbrand theorem for* KD45*) S is* KD5*-satisfiable if and only if there is a Herbrand* KD5*-model $M_{sk}=\langle W_{sk}, R_{sk}, h\rangle$ based on S and sk such that $M_{sk},\varepsilon\models_{sk} S$.*

Proof. See the appendix.

5.3. *An algorithm for testing* KD5*-satisfiability of* K5 *Horn clauses*

According to Lemma 5.5, we can now present a polynomial-time algorithm for testing the KD5-satisfiability of a set of K5 Horn clauses. The algorithm is given as follows.

Algorithm KD5-SAT(S);; The input S is a set of K5 Horn clauses.

(1) Skolemize S by labelling each occurrence of $\Diamond$ in S with a unique number. Let *sk* be the resulting skolemization and I (resp. I_1 and I_2) the index set (resp. level-1 and level-2 index sets) of *sk*.
(2) $Atom$ = the set of all K5-atoms of S.
(3) $Cmp = S \backslash Atom$.
(4) **Repeat**
  for each K5 Horn clause C in Cmp **do**
  (4.1) Do one of the following depending on the format of C (if no case matches, do nothing):
  Case 1: $C = D \vee MC'$, where C' is a negative classical clause (i.e. no modality). Then if *inconsistent-with-Atom*(MC') then replace C in Cmp by D.
  Case 2: $C = Mp$ is a K5-atom. Then remove C from Cmp and add it to $Atom$.
  Case 3: $C = M(p \vee \neg p_1 \vee \cdots \vee \neg p_n)$ $(n \geqslant 1)$ is a positive K5-literal. Then $Cmp = Cmp \backslash \{C\} \cup \{\Box \Diamond_i p \vee \Box \Diamond_i \neg p_1 \vee \cdots \vee \Box \Diamond_i \neg p_n \mid i \in val(M_1) \cap I_2\} \cup \{\Diamond_i p \vee \Diamond_i \neg p_1 \vee \cdots \vee \Diamond_i \neg p_n \mid i \in val(M_1) \cap I_1\}$.
  end for
  until either the empty clause $\bot \in Cmp$ or Cmp is not changed in the last for-loop.
(5) If $\bot \in Cmp$ return ("unsatisfiable") else return ("satisfiable").
(6) **end**.

In algorithm KD5-SAT, some terms require an explanation.

(1) The set variable *Atom* is used to collect the set of K5-atoms that must be true at the initial world ε, and *Cmp* contains the remaining K5 Horn clauses.

(2) The function $val(M)$ is used to return the set of worlds that the modality M denotes in W_{sk} and hence is defined as follows:

- if M is empty then $val(M) = \{\varepsilon\}$;
- if $M = \Diamond_i$ or $\Box \Diamond_i$ or $\Diamond_j \Diamond_i$ then $val(M) = \{i\}$;
- if $M = \Box$ then $val(M) = I_1$;
- if $M = \Diamond_i \Box$ or $\Box\Box$ then $val(M) = I$.

(3) The predicate *inconsistent-with-Atom*(MC), where $C = (\neg p_1 \vee \cdots \vee \neg p_n)$ is a negative clause, is used to check if MC is inconsistent with *Atom*. In other words, if there is a world $i \in val(M)$ such that, according to *Atom*, all p_k''s $(1 \leqslant k \leqslant n)$ are true at world i, then *inconsistent-with-Atom*(MC) returns true; otherwise it returns false. To implement this predicate, we can maintain for each propositional variable p a set variable $twd(p)$ recording all worlds in which p must be true according to the current

value of *Atom*. So, for example, if *Atom* contains $\Box p$, $\Box \Diamond_i p$ and p about p, then $twd(p) = I_1 \cup \{i, \varepsilon\}$. With such a data structure available, it is easy to implement a quadratic-time algorithm for this predicate.

We now analyse the time complexity of this algorithm. That all steps in the algorithm but the **repeat** loop can be completed in polynomial time is easy to see; the critical part of the algorithm is the **repeat** loop, which requires time $O(k \cdot \alpha)$, where k is the number of times step 4.1 is executed and α is the maximum number of steps required to execute any one of the three cases inside step 4.1.

For case 1 of step 4.1, the most expensive operation is the test *inconsistent-with-atom*(MC), which requires time $O(|S|^2)$. For case 2, $O(|S|)$ time is sufficient, while for case 3, the split of C results in the generation of at most $|W_{sk}|$ instances of C to be added to *Cmp*, thus requiring time $O(|S|^2)$ at most. To sum up, $\alpha = O(|S|^2)$.

Now we see at most how many times step 4.1 would be executed before termination. The strategy is to define a well-founded ordering on sets of K5 Horn clauses and show that the order of *Cmp* decreases for every execution of the for-loop at step 4.

The ordering is defined inductively as follows:

(1) For each clause $C = D \vee MC'$ where C' is a negative Horn clause, define $C\# = D\# + 1$. This corresponds to case 1 of step 4.1.

(2) For each clause $C = Mp$ being a K5-atom, define $C\# = 1$. This corresponds to case 2 of step 4.1.

(3) For each positive clause $C = M(p \vee p_1 \vee \cdots \vee p_n)$ $(n > 0)$ define $C\# = 1 + |W_{sk}| \times (1 + n)$. This corresponds to case 3 of step 4.1.

Finally, for a set of K5-clauses S, define $S\# = \Sigma_{C \in S} C\#$.

It is now easy to see that the value of $Cmp\#$ decreases at least by 1 after each execution of any case of step 4.1. But the **until** condition of the **repeat** statement requires that at least one case be executed for any but the last iteration of the repeat loop; the number k is thus bounded by the initial value of $Cmp\#$, which has order $O(|W_{sk}| \times |S|) = O(|S|^2)$, times the maximum possible cardinality of *Cmp*, which has order $O(|S|^2)$ as well. To sum up, $k = O(|S|^4)$. As a result, KD5-SAT takes time $O(|S|^6)$ totally; we thus have the following lemma.

Lemma 5.6. KD5-SAT(S) *always terminates in time polynomial in the size of* S.

Before proving the correctness of KD5-SAT, we first give an example to show how KD5-SAT works.

Example 5.7. Let the set of K5 Horn clauses $S = \{\Diamond p \vee \Box\Box(\neg q \vee \neg r), \Box\Diamond q, \Diamond\Diamond\neg q \vee \Diamond q, \Box\Box r, \Box\neg p\}$.

After skolemization, we might get

$$S = \{\Diamond_1 p \vee \Box\Box(\neg q \vee \neg r), \Box\Diamond_2 q, \Diamond_3\Diamond_4\neg q \vee \Diamond_5 q, \Box\Box r, \Box\neg p\},$$

with

$$I_1=\{1,3,5\}, \qquad I_2=\{2,4\} \quad \text{and} \quad W_{sk}=\{\varepsilon,1,2,3,4,5\}.$$

After step 3, we get

$$Atom=\{\Box\Box r, \Box\Diamond_2 q\},$$

$$twd(r)=\{1,2,3,4,5\}, \qquad twd(q)=\{2\}$$

and

$$Cmp=\{\Diamond_1 p \vee \Box\Box(\neg q \vee \neg r), \Diamond_3\Diamond_4\neg q \vee \Diamond_5 q, \Box\neg p\}.$$

The first iteration of the repeat loop will remove the K5-literal $\Box\Box(\neg q \vee \neg r)$ from the clause $\Diamond_1 p \vee \Box\Box(\neg q \vee \neg r)$ because $\Box\Box(\neg p \vee \neg q)$ is inconsistent with *Atom*: $\Box\Box(\neg q \vee \neg r)$ implies one of q and r must be false at each world, but by *Atom*, both q and r must be true at the world "2".

After the second iteration of the repeat loop, $\Diamond_1 p$ will be moved from *Cmp* to *Atom* and "1" will be added to $twd(p)$. And after the third iteration, the program will terminate with "unsatisfiable" returned for the empty clause will be generated at this iteration by virtue of the inconsistence of $\Box\neg p$ and $\Diamond_1 p$.

5.4. *Correctness of* KD5-SAT

We now prove the correctness of KD5-SAT.

Lemma 5.8. *Assume* KD5-SAT(S) *terminates after the kth execution of step* 4.1. *Let* $Atom^0$ *and* Cmp^0 *be the values of Atom and Cmp, respectively, immediately before the first execution of step* 4.1, *and* $Atom^i$ *and* Cmp^i *be the values of Atom and Cmp, respectively, immediately after the ith execution of step* 4.1 *for* $1\leqslant i\leqslant k$. *Then for any Herbrand* KD5-*models* M_{sk} *based on S and sk determined at step* 1, *we have*

$$M_{sk},\varepsilon\models_{sk} Atom^i\cup Cmp^i \text{ iff } M_{sk},\varepsilon\models_{sk} Atom^{i+1}\cup Cmp^{i+1}(S)$$

for any $0\leqslant i<k$.

Proof. Let C be the K5 Horn clause selected from Cmp^i for the $(i+1)$th execution of step 4.1. Then there are 4 conditions to be considered depending on which case of step 4.1 is executed.

(1) Case 1 is executed. Then $C=D \vee MC'$, MC' is a negative K5-literal and is inconsistent with $Atom^i$. So $Cmp^{i+1}(S)=Cmp^i\setminus\{C\}\cup\{D\}$, and $Atom^i=Atom^{i+1}$. Since D subsumes C, $M_{sk},\varepsilon\models_{sk} D$ implies $M_{sk},\varepsilon\models_{sk} C$. On the other hand, since $Atom^i$ and MC' are inconsistent with each other, $M_{sk},\varepsilon\models_{sk} Atom^i\cup\{D\vee MC'\}$ implies $M_{sk},\varepsilon\models_{sk} Atom^i\cup\{D\}$. Therefore, $M_{sk},\varepsilon\models_{sk} Cmp^i\cup Atom^i$ implies $M_{sk},\varepsilon\models_{sk} Cmp^{i+1}\cup Atom^{i+1}$.

(2) Case 2 is executed. Then $Cmp^{i+1}\cup Atom^{i+1}=Cmp^i\cup Atom^i$; the lemma obviously holds.

(3) Case 3 is executed. Then the C at Cmp is replaced by the set T of all instances of C with respect to W_{sk}. The lemma thus holds because $M_{sk}, \varepsilon \models_{sk} C$ iff $M_{sk}, \varepsilon \models_{sk} T$.

(4) No cases match. Since $Atom$ and Cmp are not changed, the lemma obviously holds. $\square$

Lemma 5.9. (1) *If* KD5-SAT(S) *returns "unsatisfiable", S is not* KD5-*satisfiable.*

(2) *If* KD5-SAT(S) *returns "satisfiable", S is* KD5-*satisfiable.*

Proof. Assume KD5-SAT(S) terminates after the kth execution of step 4.1.

(1) Since KD5-SAT(S) returns "unsatisfiable", the empty clause $\bot \in Cmp^k$. $Atom^k \cup Cmp^k$, which by Lemma 5.8 is equivalent to S, thus is unsatisfiable (with respect to $\models_{sk}$). So, by the modal Herbrand theorem, S is not KD5-satisfiable.

(2) Let $M_{sk} = \langle W_{sk}, R_{sk}, h \rangle$ where $h(i) = \{p \mid Mp \in Atom^k$ and $i \in val(M)\}$. Namely, $h(i)$ contains only those propositional variables that must be true at i for each $i \in W_{sk}$.

It is obvious that $M_{sk}, \varepsilon \models_{sk} Atom^k$. To see that $M_{sk}, \varepsilon \models_{sk} Cmp^k$ as well, we note that since Cmp was not changed in the last repeat loop, Cmp^k must be either an empty set, which is vacuumly true, or a set of K5 Horn clauses of the form $C = D \vee MC'$, where $C' = \neg p_1 \vee \cdots \vee \neg p_\alpha$ is a negative propositional clause, such that MC is consistent with $Atom^k$, i.e. for each i in $val(M)$, there exists a p_j $(1 \leqslant j \leqslant \alpha)$ such that $i \notin twd(p_j)$. Accordingly, $M_{sk}, \varepsilon \models_{sk} MC'$ and hence $M_{sk}, \varepsilon \models_{sk} C$ for each C in Cmp^k. Finally, by Lemmas 5.5 and 5.8, S is KD5-satisfiable. $\square$

5.5. *Complexity results*

Now we have shown the first result of this section.

Theorem 5.10. *The satisfiability problem for* KD5, KD45 *and* S5 *with the input restricted to modal Horn clauses can be solved in polynomial time.*

Proof. The case for KD5 is a direct consequence of Lemmas 5.6 and 5.9. To avoid unnecessary duplication, we do not provide algorithms for KD45 and S5 here. But in fact they are almost the same as KD5-SAT except that the definition of *twd*, *val* and *inconsistent-with-Atom* should be slightly modified to reflect the differences among the corresponding models, and are indeed simpler than KD5-SAT. $\square$

After we have shown that the KD5/KD45-satisfiability of modal Horn clauses can be solved in polynomial time, it is easy to obtain the same result for K5 and K45 as well since, according to the following proposition, to test the K5/K45-satisfiability of any formula it suffices to test whether it is KD5/KD45-satisfiable or satisfiable in a model with one world only.

Proposition 5.11. *Let A be any modal formula. Then A is* K5-*satisfiable (resp.* K45-*satisfiable) iff either A is* KD5-*satisfiable (resp.* KD45-*satisfiable) or A is satisfiable in a model with one world only.*

Proof. The if part is trivial since every single-world model and every KD5-model (resp. KD45) are K5-models (resp. K45-models).

For the proof of the only-if part, let $M = \langle W, R, h \rangle$ be any K5-model such that $M, w_0 \models A$ for some $w_0 \in W$.

If there is no world in W accessible from w_0, then let $M' = \langle \{w_0\}, \phi, h' \rangle$ and $h'(w_0) = h(w_0)$. It is easy to verify that $M', w_0 \models A$.

On the other hand, if w_0 is not an ending world, i.e. there is a world accessible from w_0, then let $M' = \langle W', R', h' \rangle$, where $W' = \{w' \in W \mid w_0 R^* w', R^*$ is the reflexive and transitive closure of $R\}$ and R' and h' are R and h restricted to W', respectively.

It is easy to verify that $M', w_0 \models A$ and M' is a KD5-model. As a result, A is KD5-satisfiable. The K45–KD45 case is similar. $\square$

But to test the satisfiability of a given set S of modal Horn clauses in single-world models is very easy: we simply replace every subformula of the form $\square A$ in S with *true* and replace every subformula of the form $\diamond A$ in S with *false*. It is easy to show that S is satisfiable in a single-world model if and only if the resulting set of classical Horn clauses is satisfiable for classical propositional logic, which is well known to be solvable in linear time [6]. We thus have the following theorem.

Theorem 5.12. *The satisfiability problem for* K5 *and* K45 *with the input restricted to modal Horn clauses can be solved in polynomial time.*

6. Conclusion

We have shown in this paper that the satisfiability problem for any modal propositional logic between K and S4 still remains PSPACE-hard even if we restrict the input formula to modal Horn clauses. This result refutes the expectation of getting a polynomial-time algorithm for these logics as long as $\mathrm{P} \neq \mathrm{PSPACE}$. Likewise, we have shown that the same problem for any modal logic between K and B is PSPACE-hard as well. Accordingly, the satisfiability problem for K, T, KB, B and S4 is PSPACE-complete whether the formula is restricted to modal Horn clauses or not. We also showed that the satisfiability of modal Horn clauses for S4.3 and for some linear tense logics like CL, SL and PL is NP-complete. Again, each have the same complexity as the unrestricted case. All the above results are negative in the sense that restricting the formula to modal Horn clauses does not decrease the inherent difficulty of the satisfiability problem. Fortunately, we did find some extensions of K5 including K5, KD5, K45, KD45 and S5, for which the satisfiability problem in general is NP-complete, but when restricted to modal Horn clauses, the problem can be solved in polynomial time.

Appendix

A.1. Proof of Lemma 3.6

In order to prove Lemma 3.6, we need some facts about $MH'(B)$ and any K-model satisfying it.

Proposition A.1. *Let* $M=\langle W,R,h\rangle$ *be any* K-*model such that* $M,w_0\models MH'(B)$ *for some* $w_0\in W$. *Then, for any world* $w\in W$ *with* w_0R^kw *where* $0\leqslant k\leqslant m$,

$$M,w\ \models\ MH'(B)_k.$$

In particular, for $0\leqslant k<m$, *the formulas*
(1) $L_k\supset\Diamond U_{k+1}, L_k\supset\Diamond\bar{U}_{k+1}$, *and*
(2) *any clause in* $\mathscr{E}_k$ *are satisfied at* w; *for* $0<k\leqslant m$, *the formulas*
(3) $U_k\supset X_k, U_k\supset L_k, \bar{U}_k\supset\bar{X}_k, \bar{U}_k\supset L_k$,
(4) $X_i\supset\Box X_i, \bar{X}_i\supset\Box\bar{X}_i$, *where* $1\leqslant i\leqslant k$, *are satisfied at* w; *for* $k=m$,
(5) *any clause in* $\mathscr{C}$ *is satisfied at* w.

Proof. Simple inductive proof. $\Box$

The proof of Lemma 3.6 is now shown below.

Proof of Lemma 3.6. (1), (2), (3): The mapping τ is defined by induction on the ordering (W_B, R_B); in the meantime, the first three properties are proved simultaneously.

Basic case: $x=\varepsilon$. Let $\tau(x)=w_0$. Obviously, the τ so constructed satisfies properties (1), (2) and (3) as far as x is concerned.

Induction case: Assume that the value of τ for every element of W_B of length $\leqslant k$ has been defined. Now consider any $x\in W_B$ of length $k+1\leqslant m$.

Case 1: $x=z\cdot 1$. Since $M,w_0\models MH'(B)$ and, by induction hypothesis of (3), $w_0R^k\tau(z)$, according to Proposition A.1, we have $M,\tau(z)\models MH'(B)_k$. In particular, $M,\tau(z)\models L_k\supset\Diamond U_{k+1}$. But, by induction hypothesis of (2), $M,\tau(z)\models L_k$ $(\in U(z))$, therefore $M,\tau(z)\models\ \Diamond U_{k+1}$, and there must exist a $w\in W$ accessible from $\tau(z)$ such that $M,w\models U_{k+1}$. So let $\tau(x)=w$. Now we verify that w satisfies properties (2) and (3).

Since $M,\tau(z)\models I(z)\cup MH'(B)_k$ and, for any $Z\in I(z)$, $Z\supset\Box Z\in MH'(B)_k$, we have $M,\tau(z)\models\Box Z$ for any $Z\in I(z)$. Hence $M,w\models I(z)$. Moreover, $M,w\models\{U_{k+1}\supset L_{k+1}, U_{k+1}\supset X_{k+1}\}$ $(\subset MH'(B)_{k+1})$ and $M,w\models U_{k+1}$, so $M,w\models L_{k+1}\wedge U_{k+1}\wedge X_{k+1}$. Hence $M,w\models I(x)\cup U(x)$ and property (2) was verified.

That w satisfies property (3) is easy to see since, by $\tau(z)R\tau(x)$ and by the hypothesis $w_0R^k\tau(z)$, we have $w_0R^{k+1}\tau(x)$.

Case 2: $x=z\cdot 0$. Similar to case 1. Omitted here.

(4) To prove property (4) by induction on the structure of (subformulas of) A, we need a stronger version of property (4), namely,

(4′) *For any* $x = x_1 \dots x_m \in W_B$ *of length* m *and for any subformula* A_i *of* A,

- *if* $A_i(x_1, \dots, x_m) \equiv 1$, *then* $M, \tau(x) \models C_i$, *and*
- *if* $A_i(x_1, \dots, x_m) \equiv 0$, *then* $M, \tau(x) \models \bar{C}_i$.

The proof of (4′) is as follows. Assume $\tau(x) = w$. By properties (2) and (3) of Lemma 3.6, and by property (5) of Proposition A.1, we have $M, w \models I(x) \cup \{L_m\} \cup \mathscr{C}$.

Now consider any subformula A_i of A.

Basic case: $A_i = X_j$ for some propositional variable X_j.

If $x_j = 1$ (hence $A_i(x) \equiv 1$), then $X_j \in I(x)$. Since $X_j \wedge L_m \supset C_i \in \mathscr{C}$, $M, w \models C_i$;

If $x_j = 0$ (hence $A_i(x) \equiv 0$), then $\bar{X}_j \in I(x)$. However, $\bar{X}_j \wedge L_m \supset \bar{C}_i$ is also contained in $\mathscr{C}$, so $M, w \models \bar{C}_i$.

Induction case

Case 1: $A_i = \neg A_j$. If $A_i(x) \equiv 1$, then $A_j(x) \equiv 0$. Thus, by induction hypothesis, $M, w \models \bar{C}_j$. Moreover, $M, w \models \bar{C}_j \wedge L_m \supset C_i \in \mathscr{C}$, so $M, w \models C_i$.

The case that $A_i(x) \equiv 0$ is similar to the above; we omit it here.

Case 2: $A_i = A_j \wedge A_k$. If $A_i(x) \equiv 1$, then $A_j(x) \equiv 1$ and $A_k(x) \equiv 1$. Hence, from $C_j \wedge C_k \wedge L_m \supset C_i \in \mathscr{C}$ and $M, w \models C_j \wedge C_k$ obtained by induction hypothesis, we have $M, w \models C_i$. On the other hand, if $A_i(x) \equiv 0$, then either $A_j(x) \equiv 0$ or $A_k(x) \equiv 0$. So, by induction hypothesis, either $M, w \models \bar{C}_j$ or $M, w \models \bar{C}_k$. However, since $M, w \models \{\bar{C}_j \wedge L_m \supset \bar{C}_i, \bar{C}_k \wedge L_m \supset \bar{C}_i\} \subseteq \mathscr{C}$, both cases imply $M, w \models \bar{C}_i$.

(5) Property (5) is proved by induction on the ordering (W_B, R_B^{-1}), where R_B^{-1} is the reverse of R_B.

Basic case: $|x| = m$. Since $B(x) = A(x)$ $(= A_p(x))$, $M, \tau(x) \models L_m$, and $M, \tau(x) \models \{C_p \wedge L_m \supset Y_m, \bar{C}_p \wedge L_m \supset \bar{Y}_m\} \subseteq \mathscr{C}$, according to property (4), if $B(x) \equiv 1$, then $M, \tau(x) \models C_p$ and, consequently, $M, \tau(x) \models Y_m$; on the other hand, if $B(x) \equiv 0$, then $M, \tau(x) \models \bar{C}_p$ and, consequently, $M, \tau(x) \models \bar{Y}_m$.

Induction case: $0 \leqslant |x| = i < m$.

Case 1: $Q_{i+1} = \forall$. If $B(x) \equiv 1$, then $B(x \cdot 1) \equiv 1$ and $B(x \cdot 0) \equiv 0$. By induction hypothesis and property (2), $M, \tau(x \cdot 1) \models U_{i+1} \wedge Y_{i+1}$ and $M, \tau(x \cdot 0) \models \bar{U}_{i+1} \wedge Y_{i+1}$. Moreover, we have $M, \tau(x) \models \Diamond(U_{i+1} \wedge Y_{i+1}) \wedge \Diamond(\bar{U}_{i+1} \wedge Y_{i+1}) \supset Y_i$ $(\in MH'(B)_i)$, $\tau(x) R \tau(x \cdot 1)$, and $\tau(x) R \tau(x \cdot 0)$. Hence $M, \tau(x) \models Y_i$. For the converse, if $B(x) \equiv 0$, then either $B(x \cdot 1) \equiv 0$ or $B(x \cdot 0) \equiv 0$. By induction hypothesis and property (2), we have either $M, \tau(x \cdot T) \models U_{i+1} \wedge \bar{Y}_{i+1}$ or $M, \tau(x \cdot F) \models \bar{U}_{i+1} \wedge \bar{Y}_{i+1}$. Moreover, we also have $M, \tau(x) \models \{\Diamond(U_{i+1} \wedge \bar{Y}_{k+1}) \supset \bar{Y}_i, \Diamond(\bar{U}_{i+1} \wedge \bar{Y}_{k+1}) \supset \bar{Y}_i\}$ $(\subseteq MH'(B)_i)$, $\tau(x) R \tau(x \cdot 1)$ and $\tau(x) R \tau(x \cdot 0)$. Hence $M, \tau(x) \models \bar{Y}_i$.

Case 2: $Q_{i+1} = \exists$. Dual to case 1. Omitted here. □

A.2. Proof of Lemma 5.5

Proof. The proof is by induction on subformulas of S.

The cases that $A = p$, $A = \neg p$, $A = B \vee C$, $A = B \wedge C$ or $A = \Box p$ are obvious since then $\models_{sk}$ and $\models$ have the same definitions. So we only have to consider the case that A has the form $\Diamond_i C$. By definition, $M_{sk}, w \models_{sk} \Diamond_i C$ iff $w R_{sk} i$ and $M_{sk}, i \models_{sk} C$, which, by induction hypothesis, implies $M_{sk}, i \models C$. Hence $M_{sk}, w \models \Diamond C$.

(2) The if part is a direct consequence of (1), since $M_{sk}, \varepsilon \models S$ implies S is KD5-satisfiable.

For the proof of the only-if part, let $M = \langle W, R, h \rangle$ be any KD5-model such that $M, w_0 \models S$ for some $w_0 \in W$. Our goal is to construct a Herbrand KD5-model $M_{sk} = \langle W_{sk}, R_{sk}, h' \rangle$ from M, w_0 and sk such that $M_{sk}, \varepsilon \models_{sk} S$. The contents of W_{sk} and R_{sk} depend on sk only and have been defined in Definition 5.4. To determine h', we first define a mapping τ from W_{sk} to W as follows.

(1) $\tau(\varepsilon) = w_0$.

(2) For each $i \in I'_1(sk)$, let $\Diamond_i C$ be the K5-literal in S with $\Diamond_i$ as principal operator. If $M, w_0 \models \Diamond C$, then arbitrarily choose any world w' accessible from w_0 with $M, w' \models C$ and let $\tau(i) = w'$; otherwise, arbitrarily choose any world w' accessible from w_0 and let $\tau(i) = w'$. In case $I'_1(sk) = \phi$, let $\tau(0)$ be any world accessible from w_0.

(3) For each $i \in I_2(sk)$, let $\odot \Diamond_i C$ be the K5-literal in S in which $\Diamond_i$ occurs. If $M, w_0 \models \odot \Diamond C$, then arbitrarily choose any $w'' \in W$ such that $w_0 R^2 w''$ and $M, w'' \models C$ and let $\tau(i) = w''$; otherwise, choose any $w'' \in W$ such that $w_0 R^2 w''$ and let $\tau(i) = w''$.

For any $i \in W_{sk}$, define $h'(i) = h(\tau(i))$.

We now prove that $M_{sk}, \varepsilon \models_{sk} C$ $(= M_1 C_1 \vee \cdots \vee M_n C_n)$ for each K5 Horn clause C in S. Hence $M_{sk}, \varepsilon \models_{sk} S$.

By definition, $M, w_0 \models M_1 C_1 \vee \cdots \vee M_n C_n$ iff $M, w_0 \models M_i C_i$ for some $1 \leqslant i \leqslant n$.

There are five cases we have to consider depending on M_i.

Case 1: M_i is empty. Since the truth of C_i at w_0 depends on $h(w_0) = h'(\varepsilon)$ only, we thus have $M_{sk}, \varepsilon \models_{sk} M_i C_i$ $(= C_i)$.

Case 2: $M_i = \Diamond_j$ (and $j \in I_1(sk)$). Since $M, w_0 \models \Diamond C_i$, by definition of τ, $M, \tau(j) \models C_i$. But C_i contains no modality; we thus have $M_{sk}, j \models_{sk} C_i$. Hence $M_{sk}, \varepsilon \models_{sk} \Diamond_j C_i$.

Case 3: $M_i = \Box$, since $M, w_0 \models M_i C_i$, for every world w' accessible from w_0, $M, w' \models C_i$. In particular, $M, \tau(j) \models C_i$ for every $j \in I_1(sk)$, which implies $M_{sk}, j \models_{sk} C_i$ for every $j \in I_1(sk)$. Hence $M_{sk}, \varepsilon \models M_i C_i$.

Case 4: $M_i = \odot \Diamond_j$, where $\odot$ is either $\Box$ or $\Diamond$ (and $j \in I_2(sk)$). Similar to case 2.

Case 5: $M_i = \odot \Box$, where $\odot$ is either $\Box$ or $\Diamond$. Similar to case 3. $\Box$

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