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Process versus unfolding semantics for Place/Transition Petri nets

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Abstract

In the last few years, the semantics of Petri nets has been investigated in several different ways. Apart from the classical "token game", one can model the behaviour of Petri nets via nonsequential processes, via unfolding constructions, which provide formal relationships between nets and domains, and via algebraic models, which view Petri nets as essentially algebraic theories whose models are monoidal categories.

In this paper we show that these three points of view can be reconciled. In our formal development a relevant role is played by DecOcc, a category of occurrence nets appropriately decorated to take into account the history of tokens. The structure of decorated occurrence nets at the same time provides natural unfoldings for Place/Transition (PT) nets and suggests a new notion of processes, the decorated processes, which induce on Petri nets the same semantics as that of unfolding. In addition, we prove that the decorated processes of a net can be axiomatized as the arrows of a symmetric monoidal category which, therefore, provides the aforesaid unification.

0. Introduction

Petri nets, introduced by Petri in [22] (see also [23, 25, 26]), are a widely used model of concurrency. This model is attractive from a theoretical point of view because of its

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simplicity and because of its intrinsically concurrent nature, and has often been used as a semantic basis on which to interpret concurrent languages (see e.g. [33, 21, 9, 5]). Concerning Petri nets themselves, several different semantics have been proposed in the literature. Most of them can be coarsely classified as *process-oriented* semantics, *unfolding* semantics, or *algebraic* semantics, though the latter is not as clearly delimited and not as widely known as the former two classes. Of course, such classes are not at all disjoint, as this paper aims to support. We further discuss these approaches below.

To account for computations involving many different transitions and for the *causal connections* between the "events" which constitute them, the basic notion of computation of Petri nets has been formalized using various notions of *process* [24, 10, 2]. The main criticism raised against process models is that they do not provide a semantics for a net as a whole, but specify only the meaning of single, deterministic computations, while the accurate description of the fine interplay between concurrency and nondeterminism is one of the most valuable features of nets.

Other semantic investigations have capitalized on the *algebraic structure* of Place/Transition (PT) nets, first noticed by Reisig [25] and later exploited by Winskel [35]. The clear advantage of these approaches resides in the fact that they tend to clarify both the structure of the single PT net, so giving insights about their essential properties, and the global structure of the class of all nets; providing, for example, useful combinators able to describe operations such as parallel and nondeterministic composition of nets [34, 35, 14, 3, 4, 13, 16, 18].

The formal framework which has proved superior for this kind of investigations is *category theory*. The discovery of categories, occurred in the context of algebraic topology in the early forties, emphasized the by now well-established conviction that mathematical entities are to be studied in terms of their structure, i.e. in terms of the abstract properties that they enjoy, rather than in terms of their actual elements. Indeed, the theory of categories builds on such conceptual guidelines introducing a new idea: the entities we intend to investigate can be equipped with a notion of *morphism* by means of which all their relevant structural properties can be expressed. (Of course, the actual meanings of "morphism" and "structure" depend on the specific nature of the subject one is considering.) This paradigm is clearly well suited for the study of models of computation, where the entities one considers, i.e. system or behaviour descriptions of some kind, come naturally with an associated notion of "morphism", e.g. simulations, bisimulations, or similar behaviour-based relationships, which encapsulates their real essence. This is in fact also the case of Petri nets whose very structure suggests a notion of morphism which captures the intuitive idea of simulation and, therefore, the idea of behaviour itself. Then, with this understanding of the role of category theory, founding an algebraic theory of Petri nets on categories simply means considering an abstract framework in which *behaviour* is a "first-class citizen". One of the first direct benefits of the use of a categorical framework is that, as a generalization of universal algebra, it provides *universal constructions* which can give fully satisfactory justifications to otherwise ad hoc defined combinators. For example, the parallel and nondeterministic compositions of nets discussed above can be understood, respectively, as products and coproducts in the category of nets.

An original interpretation of the algebraic structure of PT nets has been proposed in [14], where the theory of *monoidal categories* is exploited to the purpose. Unlike the preceding approaches, [14] yields an algebraic theory of Petri nets in which notions such as firing sequence, case graph, relationships between net descriptions at different levels of abstraction, duality, and invariants find adequate algebraic/categorical (universal) formulations. Alternative interesting categorical approaches are [3, 4].

In addition to that, since from the formal viewpoint categories are simply *algebraic graphs*, and in particular graphs whose arcs are closed under an operation of sequential composition, it is often the case that the computations of a single behavioural entity, say a Petri net, can be modelled themselves as a category, yielding in this way an axiomatization of its *space of computations*. One may call this use of categories "in the small", as opposed to their use "in the large" to study the global properties of the entire class of nets as illustrated above. This idea has been exploited in [6], where it is shown that the *commutative processes* [2] of a net N are isomorphic to the arrows of a symmetric monoidal category $\mathcal{T}[N]$. Moreover, [6] introduced the *concatenable processes* of N – a slight variation of Goltz–Reisig processes [10] – and structured them as the arrows of the symmetric monoidal category $\mathcal{P}[N]$. In particular, the distributivity of tensor product and arrow composition in monoidal categories is shown to capture the basic identifications of net computations, thus providing a *model of computation* for Petri nets.

Roughly speaking, the *unfolding semantics* consists, as the name indicates, in "unfolding" a net to simple denotational structures such that the identity of every event in their computations is unambiguous. However, *not* every assignment of denotations yields an appropriate semantics for nets. In other words, when defining an unfolding semantics, an integral part of the work is to provide some justification of adequacy of the obtained semantics. Exploiting the categorical framework, it is possible to achieve such a justification implicitly and more satisfactorily than appealing to mere intuition. The idea is to ensure that the denotation assigned to each net enjoys a certain universal property whose role is exactly to guarantee that, for the given target category, the assignment is, informally speaking, "as good as possible". The theory of categories provides the right notion to express this: the notion of *adjunction*. Thus, one would like to identify an *adjoint functor* assigning a denotation to each PT net and preserving certain compositional properties in the assignment. This is exactly what the present authors – building on Winskel's work on safe nets [34] – have done in [15, 16] for PT nets (see [8, 11] for related approaches).

In Winskel's work – which in turn builds on the previous work [19] – the denotation of a *safe* net is a coherent finitary prime algebraic *Scott domain* [32], or *dI-domain* [1]. Winskel shows that there exists a coreflection – a particularly nice form of adjunction – between the category $\underline{\mathsf{Dom}}$ of (coherent) *finitary prime algebraic domains* and the category $\underline{\mathsf{Safe}}$ of *safe Petri nets*. This coreflection factorizes through the chain of coreflections

$$\begin{tikzcd} \underline{\mathsf{Safe}} \arrow[r, shift left, "\mathcal{U}[\text{-}]"] & \underline{\mathsf{Occ}} \arrow[l, shift left, hook'] \arrow[r, shift left, "\mathcal{E}[\text{-}]"] & \underline{\mathsf{PES}} \arrow[l, shift left, "\mathcal{N}[\text{-}]"] \arrow[r, shift left, "\mathcal{L}[\text{-}]"] & \underline{\mathsf{Dom}} \arrow[l, shift left, "\mathcal{P}r[\text{-}]"] \end{tikzcd}$$

where $\underline{\mathsf{PES}}$ is the category of *prime event structures* (with binary conflict relation), which is equivalent to $\underline{\mathsf{Dom}}$, $\underline{\mathsf{Occ}}$ is the category of *occurrence nets* [34] and $\hookrightarrow$ is the inclusion functor. In [15, 16], such a chain has been extended to a quite general category $\underline{\mathsf{PTNets}}$ of PT nets by defining the *unfoldings* of PT nets and relating them by means of an adjunction to occurrence nets and therefore – exploiting the already existing adjunctions – to prime event structures and finitary prime algebraic domains. Namely, the adjunction between $\underline{\mathsf{Dom}}$ and $\underline{\mathsf{PTNets}}$ is the composition of the chain of adjunctions

$$\begin{tikzcd} \underline{\mathsf{PTNets}} \arrow[r, shift left, "\mathcal{U}[\text{-}]"] & \underline{\mathsf{DecOcc}} \arrow[l, shift left, "(\text{-})^{+}"] \arrow[d, shift left, "\mathcal{F}[\text{-}]"] & & \\ & \underline{\mathsf{Occ}} \arrow[u, shift left, "\mathcal{D}[\text{-}]"] \arrow[r, shift left, "\mathcal{E}[\text{-}]"] & \underline{\mathsf{PES}} \arrow[l, shift left, "\mathcal{N}[\text{-}]"] \arrow[r, shift left, "\mathcal{L}[\text{-}]"] & \underline{\mathsf{Dom}} \arrow[l, shift left, "\mathcal{P}r[\text{-}]"] \end{tikzcd}$$

where $\underline{\mathsf{DecOcc}}$ is the "key" category of *decorated occurrence nets*. These are occurrence nets in which places belonging to the post-set of the same transition are partitioned into *families*. In this way, since families are used to relate places corresponding in the unfolding to multiple instances of the same place in the original net, they naturally represent the unfoldings of PT nets and can account for the multiplicities of places in transitions. It is worth mentioning that, although the adjunction $\langle(\mathcal{D}[\_])^{+}, \mathcal{F}\mathcal{U}[\_]\rangle : \underline{\mathsf{Occ}} \rightharpoonup \underline{\mathsf{PTNets}}$ is *not* a coreflection, a fact which would guarantee the ideal situation, it is a quite natural construction; moreover, it does restrict to Winskel's coreflection from $\underline{\mathsf{Occ}}$ to $\underline{\mathsf{Safe}}$, and, therefore, all the right adjoints with source category $\underline{\mathsf{PTNets}}$ in the chain above are proper "conservative" extensions of the corresponding functors with source $\underline{\mathsf{Safe}}$ in Winskel's chain.

We have already mentioned that these three views of net semantics are not mutually exclusive and, in fact, we have discussed how [6] provides a unification of the process-oriented and algebraic views via the categories $\mathcal{T}[N]$ and $\mathcal{P}[N]$ modelling, respectively, *commutative* and *concatenable processes*. Concerning the relationships between process and unfolding semantics, in the case of safe nets the question is easily answered by exploiting the existence of a coreflection of $\underline{\mathsf{Occ}}$ into $\underline{\mathsf{Safe}}$, which directly implies the existence of an isomorphism between the processes of N and the deterministic finite subnets of $\mathcal{U}[N]$, i.e. the finite configurations of $\mathcal{E}\mathcal{U}[N]$. (More details about such correspondence will be given in Section 3.) Thus, in this case, the process and unfolding semantics coincide, although it should not be forgotten that the latter has the great merit of collecting together all the processes of N as a *whole*, thus accounting at the same time for concurrency and nondeterminism.

In this paper we study the relationships between the algebraic paradigm, the process semantics described above, and the unfolding semantics for PT nets given in [15, 16]. We find that, in the context of general PT nets, the latter two notions do not coincide. In particular, the unfolding of a net N contains information strictly more concrete than the collection of the processes of N. However, we show that the difference between the two semantics can be axiomatized quite neatly and simply. In particular, we introduce a new notion of processes, whose definition is suggested by the idea of families in decorated occurrence nets, and which are therefore called *decorated processes*, and we show that they capture the unfolding semantics, in the precise sense that there is a one-to-one translation between decorated processes of N and finite configurations of $\mathcal{E}\mathcal{F}\mathcal{U}[N]$. Then, following the approach proposed in [6] for the case of nonsequential processes, we introduce the notion of *decorated concatenable process* and we axiomatize it in terms of monoidal categories. More precisely, we define an abstract symmetric monoidal category $\mathcal{D}\mathcal{P}[N]$ and we show that its arrows represent precisely the decorated concatenable processes of N. Clearly, decorated concatenable processes are structures strictly more concrete than concatenable processes; remarkably, such a difference can be captured in our algebraic/categorical setting by the weakening of a single axiom.

The natural environment for the development of a theory of net processes based on monoidal categories is, as illustrated in [6], a category $\underline{\mathsf{Petri}}$ of *unmarked* nets, i.e. nets without initial markings, whose transitions have finite pre- and post-sets. However, since the unfolding of a net is considered with respect to an initial marking, $\underline{\mathsf{PTNets}}$ and all the categories of nets considered in [15] (and in related works) are categories of *marked* nets whose transitions, because of technical reasons, are forced to have possibly infinite pre- and post-sets and nonempty pre-sets. In order to solve this discrepancy, we simply restrict our attention to the subcategory of $\underline{\mathsf{PTNets}}$, say $\underline{\mathsf{MPetri}}^*$, consisting of the nets whose transitions have finite pre- and post-sets, i.e. the nets with nonempty pre-sets in $\underline{\mathsf{Petri}}$ equipped with an additional initial marking. Therefore, summing up, our result is that the following diagram commutes up to isomorphism

$$\begin{tikzcd}
\underline{\mathsf{MPetri}}^* \arrow[rrrr, "\mathcal{D}\mathcal{P}^*[\text{-}]"] \arrow[d, hook] & & & & \underline{\mathsf{MSMonCat}} \arrow[dr, "\langle\text{-}\downarrow\text{-}\rangle"] & \\
& & & & & \underline{\mathsf{Cat}} \\
\underline{\mathsf{PTNets}} \arrow[r, "\mathcal{U}[\text{-}]"'] & \underline{\mathsf{DecOcc}} \arrow[r, "\mathcal{F}[\text{-}]"'] & \underline{\mathsf{Occ}} \arrow[rr, "\mathcal{E}[\text{-}]"'] & & \underline{\mathsf{PES}} \arrow[ur, "\mathcal{L}_F[\text{-}]"'] &
\end{tikzcd}$$

where $\hookrightarrow$ is the inclusion of $\underline{\mathsf{MPetri}}^*$ in $\underline{\mathsf{PTNets}}$, $\underline{\mathsf{MSMonCat}}$ is the category of the "marked" symmetric strict monoidal categories, i.e. symmetric strict monoidal categories $\underline{\mathsf{C}}$ with a distinguished object $c \in \underline{\mathsf{C}}$, $\mathcal{D}\mathcal{P}^*[\_]$ maps the marked net (N, u_N) to $(u_N, \mathcal{D}\mathcal{P}[N])$, $\underline{\mathsf{Cat}}$ is the category of the categories, $\langle\_\downarrow\_\rangle$ is the comma category functor $(c, \underline{\mathsf{C}}) \mapsto \langle c\downarrow\underline{\mathsf{C}}\rangle$ (see Definition 3.15), and $\mathcal{L}_F$ returns the finite configurations of prime event structures ordered by inclusion. We remark that a similar approach has been followed in [20] in the case of elementary net systems – a particular class of safe nets without self-looping transitions – for unfoldings and nonsequential processes.

It should be stressed that our concern here is at the level of a *single* net, which means that the diagram above is defined only at the object level, i.e. the correspondence we establish is not functorial; more precisely, $\mathcal{DP}[\_]$ – as well as the closely related $\mathcal{P}[\_]$ – fails to be a functor. Observe, however, that since the lower edge of the diagram is clearly a functor, it would be immediately possible to extend also to a functor the upper edge and, therefore, to obtain a functorial correspondence. Nevertheless, we prefer to avoid this approach because on the one hand it would not give any further real insight into the subject, whilst, on the other hand, it would still leave unresolved the key issue of functoriality for $\mathcal{DP}[\_]$ (and $\mathcal{P}[\_]$). (Further research is currently ongoing on these open questions, e.g. [27, 29, 30].) Although $\mathcal{DP}[\_]$ is defined only at the object level, we think that the paper presents interesting results, providing a natural and unified account of the *algebraic*, the *process-oriented*, and the *denotational* views of net semantics. It is worth remarking once again that the notion underlying such a unification is that of decorated occurrence net which, therefore, appears to be of some interest on its own.

Concluding this discussion, we would like to mention that the correspondence of semantics presented here can be lifted smoothly to *infinite computations*. In [31], the present authors show that the symmetric monoidal category $\mathcal{P}[N]^{\omega}$ obtained as the completion of $\mathcal{P}[N]$ by colimits of ω-diagrams can be understood as the category of possibly *infinite* concatenable processes of N. Working analogously, one can see that the arrows of the symmetric strict monoidal category $\mathcal{DP}[N]^{\omega}$ are possibly *infinite* decorated concatenable processes. Then, one can prove the commutativity (up to equivalence) of a diagram analogous to the one above involving all the configurations of $\mathcal{EFU}[N]$ and the comma category $\langle u_N \downarrow \mathcal{DP}[N]^{\omega} \rangle$. However, we shall not say more about this extension here; the details of the construction can be found in [27].

Concerning the organization of the paper, in Section 1 we recall the basic facts about the algebraic approach to Petri nets as given in [14] and [6]. Then, in Section 2 we give a brief overview of the formal development concerning the unfolding semantics introduced in [15]. In Section 3 we introduce the decorated processes and we illustrate their relationships with the unfolding semantics. Finally, we study the decorated concatenable processes of N and their axiomatization as the arrows of the symmetric monoidal category $\mathcal{DP}[N]$.

The following exposition assumes that the reader is acquainted with a few very basic notions of category theory, namely, category, functor and adjunction; an excellent introductory textbook is [12]. Some of the results presented here appear also in [27]. A short version of the paper appears as [17].

Notation: We denote indifferently by juxtaposition (from right to left) and by $\_\circ\_$ the composition of functors, while the composition of arrows is always written as $\_\circ\_$, except in the categories – such as those of net processes – in which we want to emphasize the computational interpretation of composition as sequentialization. In these cases we write it as $\_;\_$ and we use the (left to right) diagrammatic order.

1. Petri nets and their computations

In this section we briefly recall some of the basic definitions about Petri nets [22, 25]. In particular, we recall their algebraic description as introduced in [14] and their processes [24, 10, 2, 6, 7].

Given a set S and a function μ from S to the set of natural numbers ω, we write $[\![\mu]\!]$ to indicate the support of μ that is the subset of S consisting of those elements s such that $\mu(s) > 0$. Moreover, we denote by $S^{\oplus}$ the set of *finite multisets* of S, i.e. the set of all functions from S to ω with finite support. Of course, any function $g : S_0 \to S_1$ can be "freely" extended to a function $g^{\oplus} : S_0^{\oplus} \to S_1^{\oplus}$ defined by

$$g^{\oplus}(\mu)(s') = \sum_{s \in g^{-1}(s')} \mu(s).$$

Notation: We shall represent a finite multiset $\mu \in S^{\oplus}$ as a formal sum $\bigoplus_{s\in S} \mu(s) \cdot s$. Moreover, we shall often denote $\mu \in S^{\oplus}$ by $\bigoplus_{i\in I} n_i s_i$ where $\{s_i \mid i \in I\} = [\![\mu]\!]$ and $n_i = \mu(s_i)$, i.e. as a sum whose summands are all nonzero. For instance, the multiset which contains the unique element s with multiplicity one is written as $1 \cdot s$, or simply s. Moreover, given $S' \subseteq S$, we will write $\bigoplus S'$ for $\bigoplus_{s\in S'} 1 \cdot s = \bigoplus_{s \in S'} s$.

Definition 1.1 (*Petri nets*). A *Place/Transition* (*PT*) *Petri net* is a structure $N = (\partial_N^0, \partial_N^1 : T_N \to S_N^{\oplus})$, where T_N is a set of *transitions*, S is a set of *places*, and ∂_N^0 and ∂_N^1 are functions such that $\partial_N^0(t) \neq 0$.

This describes a Petri net precisely as a graph whose set of nodes is a free commutative monoid, i.e. the set of *finite multisets* on a given set of *places*. The source and target of an arc, here called a *transition*, are meant to represent, respectively, the *marking* consumed by the transition, i.e. the minimum multiset of tokens which allows the transition to fire, and the marking produced by the firing of the transition. The restriction to nets in which $\partial_N^0(t) \neq 0$ for each transition t is due to the fact that such transitions are highly degenerated. In particular, the firing of any number of parallel instances of them is enabled at any marking, and this represents a serious problem for the unfolding semantics.

It is rather common to consider the nets we just defined as closer to *system schemes* than to *systems*, since they lack an initial state from which to start computing and, of course, different initial markings can give rise to very different behaviours for the same net. Although this distinction is clearly reasonable, we shall not put much emphasis on it, since in the categorical framework this is not always necessary. We shall for instance define processes and computations of unmarked nets, so obtaining the collection of the computations for any possible initial marking, the point being that it is always possible to recover all the relevant information about the behaviour for a given initial marking via canonical constructions such as comma categories [12] (see also Definition 3.15).

Definition 1.2 (*Marked Petri nets*). A *marked PT net* is a pair (N, u_N), where N is a PT net and $u_N \in S_N^{\oplus}$ is the *initial marking*.

The formalization of nets as graphs with additional algebraic structure on the set of nodes suggests considering graph morphisms which respect such a structure as morphisms of nets; alternative definitions have been investigated in e.g. [20, 3, 4, 18].

Definition 1.3 (*PT nets morphisms*). A *morphism* of PT nets f from N_0 to N_1 consists of a pair of functions $\langle f_t, f_p \rangle$, where $f_t : T_{N_0} \to T_{N_1}$ and $f_p : S^{\oplus}_{N_0} \to S^{\oplus}_{N_1}$ is a *monoid homomorphism*, such that $\langle f_t, f_p \rangle$ respects source and target, i.e. it makes the two diagrams below commute:

$$\begin{tikzcd} T_{N_0} \arrow[r, "\partial^0_{N_0}"] \arrow[d, "f_t"'] & S^{\oplus}_{N_0} \arrow[d, "f_p"] \\ T_{N_1} \arrow[r, "\partial^0_{N_1}"'] & S^{\oplus}_{N_1} \end{tikzcd} \qquad \begin{tikzcd} T_{N_0} \arrow[r, "\partial^1_{N_0}"] \arrow[d, "f_t"'] & S^{\oplus}_{N_0} \arrow[d, "f_p"] \\ T_{N_1} \arrow[r, "\partial^1_{N_1}"'] & S^{\oplus}_{N_1} \end{tikzcd}$$

A *morphism* of marked PT nets from N_0 to N_1 is a PT net morphism $f : N_0 \to N_1$ which preserves the initial marking, i.e. such that $f(u_{N_0}) = u_{N_1}$.

Notation: To simplify notation we shall almost always omit the subscripts t and p which distinguish the components of a morphism f. In these cases, the type of the argument will identify which component we are referring to. Observe further that by the very definition of free algebras, an $(\_)^{\oplus}$-homomorphism $f_p : S^{\oplus}_{N_0} \to S^{\oplus}_{N_1}$ is completely determined by its behaviour on S_{N_0}, the generators of the free algebra $S^{\oplus}_{N_0}$. Therefore, we will often define morphisms between nets by giving their transition components f_t and a map $f_p : S_{N_0} \to S^{\oplus}_{N_1}$ for their place components: it is implicit that they have to be thought of as lifted to the corresponding $(\_)^{\oplus}$-homomorphisms.

Transitions are the basic units of computation in a PT net. A transition t with $\partial^0_N(t) = u$ and $\partial^1_N(t) = v$ – usually written $t : u \to v$ – performs a computation *consuming* the tokens in u and *producing* the tokens in v. A finite number of transitions can be composed in parallel to form a *step*, which, therefore, is a finite multiset of transitions. We write $u[\alpha\rangle v$ to denote a step α with source u and target v. The set $\mathcal{S}[N]$ of steps of N is generated by the rules:

$$\frac{t : u \to v \text{ in } N \text{ and } w \text{ in } S^{\oplus}}{(u \oplus w)[t\rangle(v \oplus w) \text{ in } \mathcal{S}[N]} \qquad \frac{u[\alpha\rangle v \text{ and } u'[\beta\rangle v' \text{ in } \mathcal{S}[N]}{(u \oplus u')[\alpha \oplus \beta\rangle(v \oplus v') \text{ in } \mathcal{S}[N]}.$$

A finite number of steps of N can be sequentially composed, thus yielding a *step sequence*. The set of step sequences, denoted by $\mathcal{SS}[N]$, is given by the rules:

$$\frac{u \text{ in } S^{\oplus}}{u[\emptyset\rangle u \text{ in } \mathcal{SS}[N]} \qquad \frac{u_0[\alpha_0\rangle \cdots [\alpha_{n-1}\rangle u_n \text{ in } \mathcal{SS}[N] \text{ and } u_n[\alpha_n\rangle u_{n+1} \text{ in } \mathcal{S}[N]}{u_0[\alpha_0\rangle \cdots [\alpha_{n-1}\rangle[\alpha_n\rangle u_{n+1} \text{ in } \mathcal{SS}[N]}.$$

Given a PT net N and a marking $u \in S^{\oplus}_N$, the set $\mathcal{R}_u[N]$ of markings of N *reachable from* u is the set of markings which are target of some step sequence leaving from u, i.e. $\mathcal{R}_u[N] = \{v \mid \exists (u[\alpha_0\rangle \cdots [\alpha_n\rangle v) \text{ in } \mathcal{SS}[N]\}$.

A seriously restricted class of nets, which however plays a relevant role in the literature, is the class of *safe nets*. These are nets which, in their dynamic behaviour, never have multiple instances of tokens.

Definition 1.4 (*Safe nets*). A marked PT net N is *safe* if and only if
(i) for all $v \in \mathscr{R}_{u_N}[N]$, the multiset v is actually a set;
(ii) for all $t \in T_N$, the multisets $\partial_N^i(t)$, for $i = 0, 1$, are actually sets.

Unlike step sequences, processes provide a causal explanation of net behaviours, which is achieved by decorating the step sequences with explicit information about the *causal links* which ruled the firing of the transitions in the sequence. Usually one assumes that such links can be expressed faithfully as a partial order of transitions, the ordering being considered a cause/effect relationship. Thus, roughly speaking, a process of a net N consists of a partial order built on a multiset of transitions of N. The formalization of this gives the following notion of deterministic occurrence net.

Notation: In the following, we shall use the standard notation ${}^\bullet a$, for $a \in S_N$, to mean the *pre-set* of a, that is ${}^\bullet a = \{t \in T_N \mid a \in [\![\partial_N^1(t)]\!]\}$. Symmetrically, $a^\bullet$ indicates $\{t \in T_N \mid a \in [\![\partial_N^0(t)]\!]\}$, the *post-set* of a. These notations are extended in the obvious way to the case of sets of places. Recall that the terminology pre- and post-set is used also for transitions to indicate, respectively, ${}^\bullet t = [\![\partial_N^0(t)]\!]$ and $t^\bullet = [\![\partial_N^1(t)]\!]$. As usual, $|\_|$ indicates the cardinality of sets.

Definition 1.5 (*Occurrence and process nets*). An *occurrence net* is a PT net Θ such that
(i) for all $t \in T_\Theta$, for all $a \in S_\Theta$ one has $\partial_\Theta^0(t)(a) \leqslant 1$ and $\partial_\Theta^1(t)(a) \leqslant 1$;
(ii) for all $a \in S_\Theta$, $|{}^\bullet a| \leqslant 1$;
(iii) $\prec$ is irreflexive, where $\prec$ is the transitive closure of the relation

$$\prec^1 = \{(a,t) \mid a \in S_\Theta, t \in a^\bullet\} \cup \{(t,a) \mid a \in S_\Theta, t \in {}^\bullet a\};$$

moreover, $\forall t \in T_\Theta$, $\{t' \in T_\Theta \mid t' \prec t\}$ is finite;
(iv) the binary "conflict" relation $\#$ on $T_\Theta \cup S_\Theta$ is irreflexive, where

$$\forall t_1, t_2 \in T_\Theta,\ t_1 \#_m t_2 \Leftrightarrow [\![\partial_\Theta^0(t_1)]\!] \cap [\![\partial_\Theta^0(t_2)]\!] \neq \emptyset \text{ and } t_1 \neq t_2,$$

$$\forall x, y \in T_\Theta \cup S_\Theta,\ x \# y \Leftrightarrow \exists t_1, t_2 \in T_\Theta:\ t_1 \#_m t_2 \text{ and } t_1 \preccurlyeq x \text{ and } t_2 \preccurlyeq y,$$

where $\preccurlyeq$ is the reflexive closure of $\prec$.

Given $x, y \in T_\Theta \cup S_\Theta$, we say that x and y are concurrent, in symbols $x \; co \; y$, if it is not the case that ($x \prec y$ or $y \prec x$ or $x \# y$). A set $X \subseteq T_\Theta \cup S_\Theta$ is concurrent, in symbols $Co(X)$, if $\forall x, y \in X$, $x \; co \; y$ and $|\{t \in T_\Theta \mid \exists x \in X,\ t \preccurlyeq x\}| \in \omega$. We say that an occurrence net Θ is *deterministic* if for all $a \in S_\Theta$, $|a^\bullet| \leqslant 1$. Observe that, in this case, we have $\# = \emptyset$. We shall refer to deterministic occurrence nets also as *process nets*.

Thus, in an occurrence net each place belongs at most to one post-set and, if the net is a process net, at most to one pre-set. This makes the "flow" relation $\preccurlyeq$ be a preorder. Thus, requiring $\prec$ to be irreflexive, which is equivalent to requiring that the net be acyclic, identifies a partial order on the transitions. The constraint about the cardinality of the set of predecessors of a transition is then the fairly intuitive requirement that each transition be *finitely caused.* (See [34] for a discussion in terms of event structures of this issue.)

We stipulate that occurrence nets are to be considered also as *marked* nets whose minimal (w.r.t. $\prec$) places constitute the initial marking. Observe that this matches exactly with the standard definition, according to which occurrence nets can be marked only by assigning a single token to each of its minimal places. In the following, therefore, we shall use occurrence nets both in contexts in which marked nets are expected and in contexts in which unmarked nets are. Observe that, by virtue of (i) and (ii) in Definition 1.5, (marked) occurrence nets are safe.

Thanks to their nicely stratified structure, it is possible to define the notion of *depth* of an element of an occurrence net.

Definition 1.6 (*Depth*). Let Θ be an occurrence net. The *depth* of $x \in T_\Theta \cup S_\Theta$ is inductively defined by:

- $\mathrm{depth}(x) = 0$ if $x \in S_\Theta$ and ${}^\bullet x = \emptyset$;
- $\mathrm{depth}(x) = \max\{\mathrm{depth}(b) \mid b \prec x\} + 1$ if $x \in T_\Theta$;
- $\mathrm{depth}(x) = \mathrm{depth}(t)$ if $x \in S_\Theta$ and ${}^\bullet x = \{t\}$.

Given an occurrence net Θ its *subnet* of depth n is the net $\Theta^{(n)}$ consisting of the elements of Θ whose depth is not greater than n.

Definition 1.7 (*Nonsequential processes* [10]). Given a net N, a *process* of N is a PT net morphism $\pi : \Theta \to N$ which maps places to places (as opposed to morphisms which map places to markings), where Θ is a finite process net.

Similarly, a *process* of a marked net N is a morphism $\pi : \Theta \to N$ of marked PT nets which maps places to places, for a finite process net Θ.

For the purpose of defining processes at the right level of abstraction, we need to make some identifications among process nets. Of course, we shall consider as identical process nets which are isomorphic and, consequently, we shall make no distinction between two processes $\pi : \Theta \to N$ and $\pi' : \Theta' \to N$ for which there exists an isomorphism $\varphi : \Theta \to \Theta'$ such that $\pi' \circ \varphi = \pi$. Observe that the particular form of π is relevant, since we certainly want process morphisms to be total and to map a single component of the process net to a single component of N. Otherwise said, process morphisms are nothing but *labellings* of Θ with an appropriate element of N. Moreover, as usual, in the case of marked nets, we want to consider only processes whose source is the initial marking.

Inspired by the current trends in the development of the theory of computation, one would certainly like to describe the processes of a net N as an algebra whose operations model a minimal set of combinators on processes which capture the essence of concurrency. Clearly, in the present case the core of such an algebra must consist of the operations of *sequential* and *parallel* composition of processes. The problem which arises immediately is that nonsequential processes cannot be concatenated when multiplicities are present: in order to support such an operation one must *disambiguate* the identity of all the tokens in the multisets source and target of processes. In other words, one must recognize that process concatenation has to do with tokens rather than with places. This is the approach followed in [6], which led to the introduction of the *concatenable processes* of N. These are, as already sketched above, nonsequential processes enriched by total orderings of the minimal and maximal places carrying the same label. Then, exploiting the additional information, it is easy to define an operation of concatenation of such processes, and thus to organize them as the arrows of a category $\mathcal{CP}[N]$. In particular, since concatenable processes also admit an operation of parallel composition, $\mathcal{CP}[N]$ is a symmetric monoidal category. In addition, [6] shows that $\mathcal{CP}[N]$ can be axiomatized by means of an abstract symmetric monoidal category $\mathcal{P}[N]$. Next, we briefly recall this construction. The axiomatization of $\mathcal{P}[N]$ presented here has been proved to be equivalent to the original formulation in [28].

Recall that a *symmetric strict monoidal category* (see [12] for a thorough elementary introduction) is a category $\underline{\mathsf{C}}$ together with a functor $\otimes : \underline{\mathsf{C}} \times \underline{\mathsf{C}} \to \underline{\mathsf{C}}$, called the *tensor product*, and a selected object $e \in \underline{\mathsf{C}}$, the *unit object*, such that $\otimes$, when viewed as a pair of operations respectively on objects and arrows of $\underline{\mathsf{C}}$, forms two monoids whose units are e and id_e, and together with a family of arrows $\gamma_{x,y} : x \otimes y \to y \otimes x$, for x and y objects of $\underline{\mathsf{C}}$, such that, for each $f : x \to x'$ and $g : y \to y'$ in $\underline{\mathsf{C}}$,

$$
\begin{aligned}
(id_y \otimes \gamma_{x,z}) \circ (\gamma_{x,y} \otimes id_z) &= \gamma_{x,y\otimes z},\\
(g \otimes f) \circ \gamma_{x,y} &= \gamma_{x',y'} \circ (f \otimes g),\\
\gamma_{y,x} \circ \gamma_{x,y} &= id_{x\otimes y}.
\end{aligned} \tag{1}
$$

Notice that the equations above mean, respectively, that γ satisfies the relevant Kelly–MacLane [12] coherence axiom, that $\gamma = \{\gamma_{x,y}\}_{x,y\in\underline{\mathsf{C}}}$ is a natural transformation $\otimes \xrightarrow{\cdot} \otimes \circ \Delta$, where Δ is the endofunctor on $\underline{\mathsf{C}} \times \underline{\mathsf{C}}$ which "swaps" its arguments, and that $\gamma_{x,y}$ is an isomorphism with inverse $\gamma_{y,x}$. A *symmetry* in a symmetric monoidal category is any arrow obtained as composition and tensor of components of γ and identities. We shall write $Sym_{\underline{\mathsf{C}}}$ to denote the subcategory of a symmetric monoidal category $\underline{\mathsf{C}}$ whose objects are those of $\underline{\mathsf{C}}$ and whose arrows are the symmetries of $\underline{\mathsf{C}}$. It is important to stress that, in our context, i.e. from the point of view of the semantics of concurrency, symmetries provide a precise and elegant way to account for *causality streams* in computations. This will be clear shortly. A *symmetric strict monoidal functor* from $(\underline{\mathsf{C}}, \otimes, e, \gamma)$ to $(\underline{\mathsf{D}}, \otimes', e', \gamma')$ is a functor $\mathsf{F} : \underline{\mathsf{C}} \to \underline{\mathsf{D}}$ such that

$$
\begin{aligned}
\mathsf{F}(e) &= e',\\
\mathsf{F}(x \otimes y) &= \mathsf{F}(x) \otimes' \mathsf{F}(y),\\
\mathsf{F}(\gamma_{x,y}) &= \gamma'_{\mathsf{F}x,\mathsf{F}y}.
\end{aligned} \tag{2}
$$

Given a symmetric monoidal category $\underline{\mathsf{C}}$ and a set of equations $\mathscr{E}$ on parallel arrows, i.e. on arrows with the same domain and codomain, the *monoidal quotient* of $\underline{\mathsf{C}}$ modulo $\mathscr{E}$ is the category $\underline{\mathsf{C}}/\mathscr{E}$ whose objects are those of $\underline{\mathsf{C}}$ and whose arrows are the equivalence classes of the arrows of $\underline{\mathsf{C}}$ modulo the *least* equivalence closed with respect to composition and tensor which contains $\mathscr{E}$. In the language of categories, the quotient of $\underline{\mathsf{C}}$ is characterized by a universal property which identifies it uniquely up to isomorphism.

Proposition 1.8 (Quotient monoidal categories). *Given the symmetric monoidal categories $\underline{\mathsf{C}}$ and $\underline{\mathsf{D}}$ and a set of equations $\mathscr{E}$ on parallel arrows of $\underline{\mathsf{C}}$, suppose that there exists a symmetric strict monoidal functor $\mathsf{Q}:\underline{\mathsf{C}} \to \underline{\mathsf{D}}$ such that*

(i) *if $f\mathscr{E}g$ then $\mathsf{Q}(f) = \mathsf{Q}(g)$;*

(ii) *for each symmetric strict monoidal functor $\mathsf{H}:\underline{\mathsf{C}} \to \underline{\mathsf{C}}'$ such that $f\mathscr{E}g$ implies $\mathsf{H}(f) = \mathsf{H}(g)$ there exists a unique functor $\mathsf{K}:\underline{\mathsf{D}} \to \underline{\mathsf{C}}'$, which is necessarily symmetric strict monoidal, such that the following diagram commutes:*

$$\begin{tikzcd} \underline{\mathsf{C}} \arrow[r, "\mathsf{Q}"] \arrow[dr, "\mathsf{H}"'] & \underline{\mathsf{D}} \arrow[d, "\mathsf{K}"] \\ & \underline{\mathsf{C}}' \end{tikzcd}$$

Then $\underline{\mathsf{D}}$ is isomorphic to $\underline{\mathsf{C}}/\mathscr{E}$. On the contrary, if $\underline{\mathsf{D}}$ is isomorphic to $\underline{\mathsf{C}}/\mathscr{E}$, then there exists $\mathsf{Q}:\underline{\mathsf{C}} \to \underline{\mathsf{D}}$ satisfying (i) *and* (ii) *above.*

Proof. Let $\mathsf{Q}_{\mathscr{E}}:\underline{\mathsf{C}} \to \underline{\mathsf{C}}/\mathscr{E}$ be the "projection" functor which is the identity on the objects and which maps each arrow to its equivalence class in $\underline{\mathsf{C}}/\mathscr{E}$. The category $\underline{\mathsf{C}}/\mathscr{E}$ and the functor $\mathsf{Q}_{\mathscr{E}}$ certainly satisfy the above conditions, as can be easily checked exploiting the definitions. Now consider $\underline{\mathsf{D}}$ and Q as in the hypothesis. By the above consideration we conclude that there exists $K:\underline{\mathsf{C}}/\mathscr{E} \to \underline{\mathsf{D}}$ such that $\mathsf{Q} = \mathsf{K} \circ \mathsf{Q}_{\mathscr{E}}$. Moreover, since $\underline{\mathsf{D}}$ and Q satisfy (ii), there exists a functor $\mathsf{K}':\underline{\mathsf{D}} \to \underline{\mathsf{C}}/\mathscr{E}$ such that $\mathsf{Q}_{\mathscr{E}} = \mathsf{K}' \circ \mathsf{Q}$. Then, exploiting the uniqueness condition in (ii), one proves as usual that $\mathbf{1} = \mathsf{K}' \circ \mathsf{K}$ and $\mathbf{1} = \mathsf{K} \circ \mathsf{K}'$, i.e. $\underline{\mathsf{D}} \cong \underline{\mathsf{C}}/\mathscr{E}$.

Suppose now that $\underline{\mathsf{C}}/\mathscr{E}$ and $\underline{\mathsf{D}}$ are isomorphic via the symmetric strict monoidal functor $\mathsf{F}:\underline{\mathsf{C}}/\mathscr{E} \to \underline{\mathsf{D}}$ and let Q be $\mathsf{F} \circ \mathsf{Q}_{\mathscr{E}}$. Clearly, Q satisfies (i). Moreover, for any $\mathsf{H}:\underline{\mathsf{C}} \to \underline{\mathsf{C}}'$, let $\mathsf{K}:\underline{\mathsf{C}}/\mathscr{E} \to \underline{\mathsf{C}}'$ be the unique functor such that $\mathsf{K} \circ \mathsf{Q}_{\mathscr{E}} = \mathsf{H}$. Then, it is immediate to see that $\mathsf{K} \circ \mathsf{F}^{-1}$ is the functor required by (ii). $\square$

We can now give the definition of $\mathscr{P}[N]$.

Definition 1.9 (*The category $\mathscr{P}[N]$*). Let $N = (\partial_N^0, \partial_N^1 : T_N \to S_N^{\oplus})$ be a PT net. Then $\mathscr{P}[N]$ is the monoidal quotient of the *free* symmetric strict monoidal category on N modulo the axioms

$$\gamma_{a,b} = id_{a\oplus b} \quad \text{if } a,b \in S_N \text{ and } a \neq b,$$

$$s;t;s' = t \quad \text{if } t \in T_N \text{ and } s,s' \text{ are symmetries.}$$

The intended interpretation of the data above is as follows. As usual, a single transition $t_0 : u_0 \to v$ consumes the tokens in u_0 and produces those in v. Of course, given $t_0' : v \to w_0$, in the composition $t_0; t_0'$ we say that t_0' causally depends on t_0. Consider now $t_1 : u_1 \to v$ and $t_1' : v \to w_1$. Then, in accordance with the fact that $(t_0 \otimes t_1); (t_0' \otimes t_1') = (t_0; t_0') \otimes (t_1; t_1')$, we may stipulate that in $(t_0 \otimes t_1); (t_0' \otimes t_1') : u_0 \oplus u_1 \to w_0 \oplus w_1$ the transition t_0' depends on t_0 and the transition t_1' depends on t_1, while in $(t_0 \otimes t_1); (t_1' \otimes t_0')$ it is t_0 that causes t_1' and t_1 that causes t_0'. Of course, both of those scenarios are possible since in $\mathscr{P}[N]$ we have that $(t_0' \otimes t_1') \neq (t_1' \otimes t_0')$. Now, since

$$(t_0 \otimes t_1); \gamma_{v,v}; (t_0' \otimes t_1') = (t_0 \otimes t_1); (t_1' \otimes t_0'),$$

symmetries may be viewed as formal operations that "exchange causes", by exchanging the tokens produced by parallel transitions. Observe that this interpretation is also well supported by the particular form that the symmetry takes on disjoint pairs u and v. Then, $\gamma_{u,v}$ is the identity, corresponding to the fact that in this case no ambiguity is possible concerning what transition produced what token in $u \oplus v$ and, therefore, $(t_0 \otimes t_1); (t_0' \otimes t_1')$ and $(t_0 \otimes t_1); (t_1' \otimes t_0')$ have in this case to be considered as the same process. Now, the meaning of the "naturality" of γ is apparent. The same applies to the axiom $s; t; s' = t$, called axiom (Ψ) in [6], since exchanging two tokens consumed by or produced by a single t does not influence the causal behaviour.

As mentioned earlier, this nice interpretation of the arrows of $\mathscr{P}[N]$ may be pursued further by relating them to a slight refinement of the classical notion of process: the concatenable processes of N. In order to introduce them, we need the following definition.

Definition 1.10 (*f-indexed orderings*). Given sets A and B together with a function $f : A \to B$, an *f-indexed ordering* of A is a family $\{\ell_b \mid b \in B\}$ of bijections $\ell_b : f^{-1}(b) \to \{1, \ldots, |f^{-1}(b)|\}$, $f^{-1}(b)$ being as usual the set $\{a \in A \mid f(a) = b\}$.

Therefore, an f-indexed ordering of A is a family of total orderings, one for each of the partitions of A induced by f. By abuse of language, we shall keep calling an f-indexed ordering of $C \subseteq A$ any ordering obtained by restricting f to C. In the following, given a process net Θ, let $\min(\Theta)$ and $\max(\Theta)$ denote, respectively, its minimal and maximal elements, which must be places.

Definition 1.11 (*Concatenable processes*). A *concatenable process* of N is a triple $CP = (\pi, \ell, L)$ where
- $\pi : \Theta \to N$ is a process of N;
- ℓ is a π-indexed ordering of $\min(\Theta)$;
- L is a π-indexed ordering of $\max(\Theta)$.

Two concatenable processes CP and CP' are isomorphic if their underlying processes are isomorphic via an isomorphism φ which respects the ordering, i.e. such that $\ell'_{\pi'(\varphi(a))}(\varphi(a)) = \ell_{\pi(a)}(a)$ and $L'_{\pi'(\varphi(b))}(\varphi(b)) = L_{\pi(b)}(b)$ for all $a \in \min(\Theta)$ and

Fig. 1. A marked net and one of its concatenable processes.

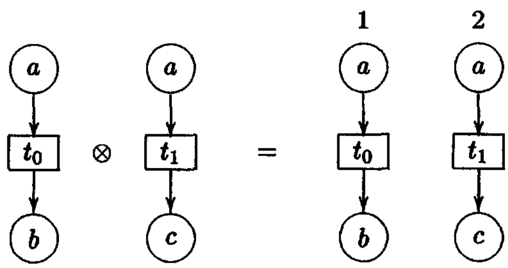

Fig. 2. The process of Fig. 1 as a tensor of two simpler processes.

$b \in \max(\Theta)$. As in the case of processes, we identify isomorphic concatenable processes.

Concatenable processes can be represented by drawing the underlying process nets and labelling their elements according to π, ℓ and L. When $|\pi^{-1}(a)| = 1$ for some place a, we omit the trivial labelling. Fig. 1 shows a simple example. We use the standard graphical representation of nets in which circles are places, boxes are transitions, and sources and targets are directed arcs whose weights represent multiplicities, unitary weights being omitted. The initial marking is given by the number of "tokens" in the places.

It is clearly possible to define an operation of concatenation of concatenable processes, whence their name. We can associate a source and a target in $S_N^{\oplus}$ to any concatenable process CP, namely by taking the image through π of, respectively, $\min(\Theta)$ and $\max(\Theta)$, where Θ is the underlying process net of CP. Then, the concatenation of $(\pi_0, \ell_0, L_0): u \to v$ and $(\pi_1, \ell_1, L_1): v \to w$ is defined in the obvious way exploiting the information given by the labellings in order to merge the underlying process nets. Under this operation the concatenable processes of N form a *category* $\mathcal{CP}[N]$ with objects the finite multisets on S_N and identities those processes consisting only of places, which therefore are both minimal and maximal, and such that $\ell = L$.

Concatenable processes admit also a tensor operation $\otimes$ which represents the parallel composition of processes. In particular, $CP_0 \otimes CP_1$ is the concatenable process which may be graphically represented by putting side by side, from left to right, the graphical representations of CP_0 and CP_1 and reorganizing the labellings appropriately as shown in Fig. 2. It is easy to see that the concatenable processes consisting only of places are the symmetries which make $\mathscr{CP}[N]$ into a *symmetric strict monoidal category*. Then, since the transitions t of N are faithfully represented in the obvious way by concatenable processes with a unique transition which is in the post-set of any minimal place and in the pre-set of any maximal place, minimal and maximal places being in one-to-one correspondence, respectively, with $\partial_N^0(t)$ and $\partial_N^1(t)$, it is possible to show the following.

Theorem 1.12 ([6, Theorem 20, p. 184]). *$\mathscr{CP}[N]$ and $\mathscr{P}[N]$ are isomorphic.*

2. Unfolding Place / Transition nets

In this section we sketch the basic notions concerning the unfolding of PT Petri nets as defined in [15, 16]. In order to keep the exposition of the background material as short as possible, we limit ourselves to the definitions of the object components of the functors $\mathscr{U}[\_]$, $\mathscr{F}[\_]$, $\mathscr{E}[\_]$ and $\mathscr{L}[\_]$. In particular, we shall not introduce explicitly the categories involved. The reader interested in the details is referred to [15, 34]. A complete survey of the topic is also given in [27].

As a first step, we define *decorated occurrence nets*, a type of occurrence net in which places are grouped into families. They allow a convenient treatment of multiplicity issues in the unfolding of PT nets. We shall use $[n]$ to denote the segment $\{1,\ldots,n\}$ of ω.

Definition 2.1 (*Decorated occurrence nets* [15]). A *decorated occurrence net* is an occurrence net Θ such that:

(i) S_Θ is of the form $\bigcup_{a\in A_\Theta}\{a\}\times[n_a]$, for some set A_Θ, where the set $\{a\}\times[n_a]$ is called the family of a. We will use a^F to denote the family of a regarded as a multiset;

(ii) $\forall a\in A_\Theta,\ \forall x,y\in\{a\}\times[n_a],\ {}^\bullet x={}^\bullet y$.

A family is thus a collection of finitely many places with the same pre-set, and a decorated occurrence net is an occurrence net where each place belongs to exactly one family. Families, and therefore decorated occurrence nets, are capable of describing *relationships* between places by grouping them together. We will use families to relate places which are *instances* of the same place obtained in a process of unfolding.

Notation: Since decorated occurrence nets are in particular occurrence nets, in the following we shall use concepts such as causal dependence ($\prec$), conflict (#), depth, ..., for decorated occurrence nets referring to the corresponding notions for the underlying occurrence nets.

Next, we define an unfolding procedure which maps marked PT nets to decorated occurrence nets.

Definition 2.2 (*PT nets unfoldings: $\mathscr{U}[\_]$* [15]). Let $N=(\partial_N^0,\partial_N^1:T_N\to S_N^\oplus,u_N)$ be a marked net. We define the decorated occurrence net $\mathscr{U}[N]$ to be $(\partial^0,\partial^1:T\to S^\oplus)$, where T, S and ∂^0 are generated inductively by the following inference rules:

$$\frac{u_N(b) = n}{\{(\emptyset, b)\} \times [n] \subseteq S}$$

$$\frac{B = \{((\varepsilon_j, b_j), i_j) \mid j \in J\} \subseteq S,\ Co(B),\ t \in T_N,\ \partial^0_N(t) = \bigoplus_{j \in J} b_j}{(B, t) \in T \quad \text{and} \quad \partial^0(B, t) = \bigoplus B}$$

$$\frac{x = (B, t) \in T,\ \partial^1_N(t)(b) = n}{\{(\{x\}, b)\} \times [n] \subseteq S}$$

and for $x \in T$, $\partial^1(x) = \bigoplus_{b,i}((\{x\}, b), i)$.

Informally speaking, the definition above can be explained as follows, where we use $\mathscr{U}[N]^{(n)}$, $n \in \omega$, to denote the nth approximation of $\mathscr{U}[N]$, i.e. the subnet of $\mathscr{U}[N]$ consisting of the elements at depth not greater than n. The net $\mathscr{U}[N]^{(0)}$ is obtained by exploding in families the initial marking of N, and $\mathscr{U}[N]^{(n+1)}$ is obtained, inductively, by generating a new transition for each possible subset of concurrent places of $\mathscr{U}[N]^{(n)}$ whose corresponding multiset of places of N constitutes the source of some transition t of N; the target of t is also exploded in families which are added to $\mathscr{U}[N]^{(n+1)}$. As a consequence, the transitions of the nth approximant net are instances of transitions of N, in the precise sense that each of them corresponds to a unique occurrence of a transition of N in one of its step sequences of length at most n.

There is an obvious *forgetful* functor from decorated occurrence nets to occurrence nets which forgets about the structure of families. It allows us to drop the additional structure of decorated occurrence nets and to bring the unfolding of PT nets into <u>Occ</u>. Moreover, exploiting Winskel's coreflections in [34], we obtain an explanation of the causal behaviour of nets in <u>PES</u> and in <u>Dom</u> as already explained in the introduction.

Definition 2.3 ($\mathscr{F}[\_]$*: from* <u>DecOcc</u> *to* <u>Occ</u>). Given a decorated occurrence net Θ define $\mathscr{F}[\Theta]$ to be the occurrence net underlying Θ.

Fig. 3 shows a simple example of unfolding of PT nets. To make explicit the nature of the elements of $\mathscr{U}[N]$ and $\mathscr{F}\mathscr{U}[N]$, in the picture we label them with the corresponding element a, b, ..., t_3 of N. In particular, the places of the unfolding labelled by a and b are respectively $(\emptyset, a)$ and $(\emptyset, b)$, the transitions labelled by t_0 and t_2 are $\bar{t}_0 = (\{(\emptyset, a)\}, t_0)$ and $\bar{t}_2 = (\{(\emptyset, b)\}, t_2)$, and thus the three instances of c are $((\{\bar{t}_0\}, c), 1)$, $((\{\bar{t}_0\}, c), 2)$ and $((\{\bar{t}_2\}, c), 1)$. A family is represented by enclosing its elements into an oval. The numbers which label the outgoing arcs from $\bar{t}_0$ take into account the ordering of the elements in the family $(\{\bar{t}_0\}, c)^F$; since $\mathscr{U}[N]$ is an occurrence net, no confusion is possible with arc multiplicities. Families of cardinality one are not explicitly indicated. We call $\mathscr{U}[N]$ and $\mathscr{F}\mathscr{U}[N]$ respectively the unfolding of N in <u>DecOcc</u> and in <u>Occ</u>. However, in the following we shall avoid explicit reference to <u>DecOcc</u> and <u>Occ</u>.

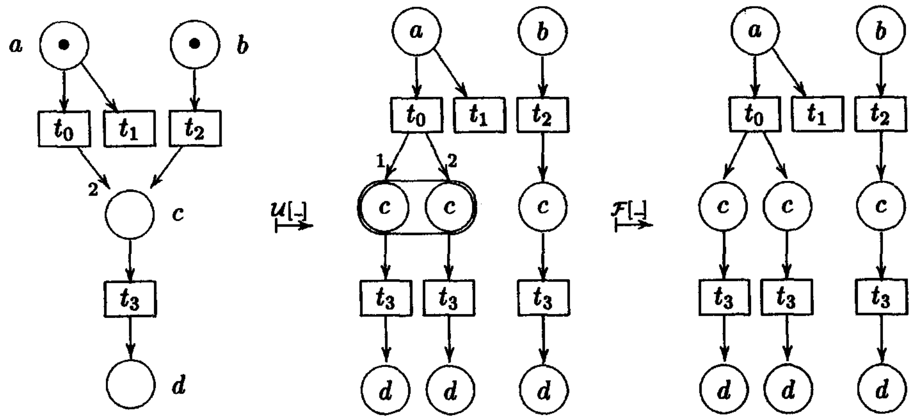

Fig. 3. A net N, its unfolding $\mathcal{U}[N]$, and $\mathcal{F}\mathcal{U}[N]$.

The correspondence between elements of the unfolding and elements of the original net should be clear from Definition 2.2, since elements of $\mathcal{U}[N]$ carry explicitly the "name" of the element of N they correspond to. Such a notion can be formalized via the following definition of folding morphism.

Proposition 2.4 (Folding morphism). *Consider the map* $\varepsilon_N = \langle \varepsilon_t, \varepsilon_p \rangle : \mathcal{F}\mathcal{U}[N] \to N$ *defined by*

- $\varepsilon_t(B,t) = t$;
- $\varepsilon_p\big(\bigoplus_i (x_i, y_i)\big) = \bigoplus_i y_i$.

Then, ε_N *is a morphism of marked nets, called the* folding *of* $\mathcal{F}\mathcal{U}[N]$ *into* N.

Prime event structures [19, 34] are the simplest event based model of concurrency. They consist of a set of events, intended as indivisible *quanta* of computation, which are related to each other by two binary relations: *causality*, modelled by a partial order relation $\leqslant$, and *conflict*, modelled by an irreflexive, symmetric, and hereditary relation #.

Definition 2.5 (*Prime Event Structures*). A *prime event structure* is a structure $\mathrm{E} = (E, \#, \leqslant)$ consisting of a set of *events* E partially ordered by $\leqslant$, and a symmetric, irreflexive relation $\# \subseteq E \times E$, the *conflict* relation, such that

$$\{e' \in E \mid e' \leqslant e\} \text{ is finite for each } e \in E,$$

$$e \mathbin{\#} e' \leqslant e'' \text{ implies } e \mathbin{\#} e'' \text{ for each } e, e', e'' \in E.$$

The computational intuition behind event structures is really simple: an event e can occur when all its *causes* have occurred and no event that is in conflict with the given event has already occurred. This is formalized by the following notion of *configuration*.

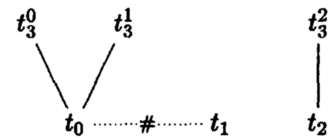

Fig. 4. The event structure $\mathscr{E}\mathscr{F}\mathscr{U}[N]$ for the net in Fig. 3.

Definition 2.6 (*Configurations*). Given a prime event structure $(E, \#, \leqslant)$, define its *configurations* to be those subsets $x \subseteq E$ which are:

Conflict free: $\forall e_1, e_2 \in x,\ \text{not}(e_1 \# e_2)$.

Left closed: $\forall e \in x\ \forall e' \leqslant e,\ e' \in x$.

Let $\mathscr{L}(\mathrm{E})$ denote the set of configurations of the prime event structure E and $\mathscr{L}_{\mathrm{F}}(\mathrm{E})$ the set of *finite* configurations of E.

The following definition recalls how to translate occurrence nets into prime event structures. An example of this translation is shown in Fig. 4, where, using the standard graphical representation of event structures, $\leqslant$ is indicated by (bottom-up) solid lines and # by a dotted line; we use superscripts to distinguish between the three instances of t_3 in $\mathscr{F}\mathscr{U}[N]$.

Definition 2.7 ($\mathscr{E}[\_]$: *from* $\underline{\mathsf{Occ}}$ *to* $\underline{\mathsf{PES}}$ [34]). Let Θ be an occurrence net. Then, $\mathscr{E}[\Theta]$ is the event structure $(T_\Theta, \preccurlyeq, \#)$, where $\preccurlyeq$ and # are the restrictions to the set of transitions of Θ of, respectively, the flow ordering and the conflict relation implicitly defined by Θ.

Finitary prime algebraic domains or dI-domains – introduced by Berry while studying sequentiality of functions [1] – are particular Scott's domains which are distributive and in which each finite element is preceded only by a finite number of elements of the domain. Here we are interested in their *"coherent"* version, i.e. in the version in which the underlying partial order is pairwise complete.

Definition 2.8 (*Finitary* (*coherent*) *prime algebraic domains*). Let $(D, \sqsubseteq)$ be a partial order. Recall that a set $X \subseteq D$ is *directed* if all the pairs $x, y \in X$ have an upper bound in X, is *compatible* if there exists $d \in D$ such that $x \sqsubseteq d$ for all $x \in X$ and is *pairwise compatible* if $\{x, y\}$ is compatible for all $x, y \in X$. We say that D is a (coherent) *domain* if it is *pairwise complete*, i.e. if for all pairwise compatible $X \subseteq D$ the least upper bound $\bigsqcup X$ of X exists.

A *complete prime* of D is an element $p \in D$ such that, for any compatible $X \subseteq D$, if $p \sqsubseteq \bigsqcup X$, then there exists $x \in X$ such that $p \sqsubseteq x$. We say that a domain D is *prime algebraic* if for all $d \in D$ we have $d = \bigsqcup\{p \sqsubseteq d \mid p \text{ is a complete prime}\}$.

Moreover, an element $e \in D$ is *finite* if for any directed $S \subseteq D$, if $e \sqsubseteq \bigsqcup S$, then there exists $s \in S$ such that $e \sqsubseteq s$. We say that D is *finitary* if for all finite elements $e \in D$, $|\{d \in D \mid d \sqsubseteq e\}| \in \omega$.

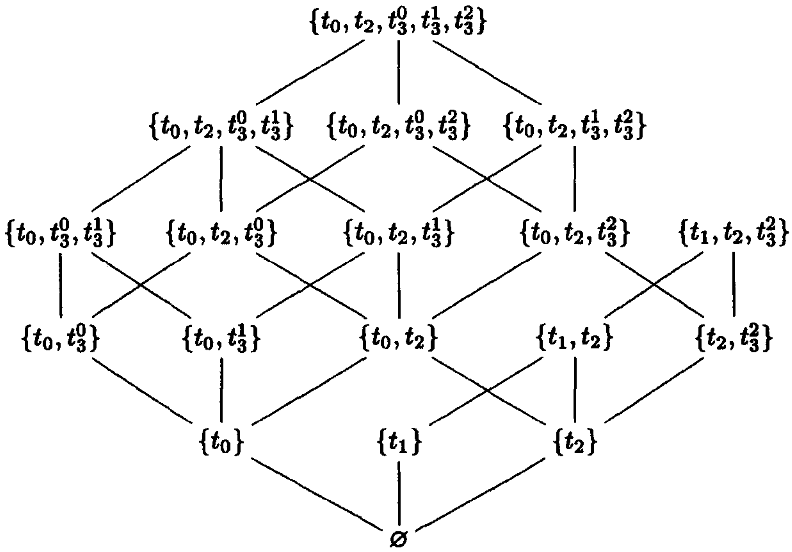

Fig. 5. The Hasse diagram of the domain $\mathcal{L}\mathcal{E}\mathcal{F}\mathcal{U}[N]$ for the net in Fig. 3.

Finitary prime algebraic domains can be equipped with a notion of morphism in such a way that the category $\underline{\mathsf{Dom}}$ so obtained is *equivalent* to $\underline{\mathsf{PES}}$ (see [34]). We conclude this section by recalling the object component of the equivalence functor $\mathcal{L}[\_] : \underline{\mathsf{PES}} \to \underline{\mathsf{Dom}}$. An example is provided by Fig. 5.

Proposition 2.9 ($\mathcal{L}[\_]$: from $\underline{\mathsf{PES}}$ to $\underline{\mathsf{Dom}}$ [34]). *Let* E *be a prime event structure. Then,* $\mathcal{L}(\mathrm{E}) = (\mathcal{L}(\mathrm{E}), \subseteq)$, *i.e. the set of configurations of* E *ordered by inclusion is a finitary (coherent) prime algebraic domain.*

3. Process vs. unfolding semantics for nets

The semantics obtained via the unfolding yields an explanation of the behaviour of nets in terms of event structures, that is, in terms of domains. Domains can be unambiguously thought of as partial orderings of computations, where a computation is represented by a configuration, which, in our context, is a "downward" closed, conflict free set of occurrences of transitions. On the other hand, processes are by definition left closed and conflict free (multi)sets of transitions. Moreover, the processes from a given initial marking are naturally organized in a preorder-like fashion via a comma category construction which formalizes the usual notion of prefix ordering of processes. The question which therefore arises spontaneously concerns the relationships between these two notions; this is the question addressed in this section.

It is worth noticing that in the case of safe nets the question is readily answered exploiting Winskel's coreflection $\langle \hookrightarrow, \mathcal{U}[\_] \rangle : \underline{\mathsf{Occ}} \rightharpoonup \underline{\mathsf{Safe}}$. In fact, by definition an adjunction $\langle \mathsf{F}, \mathsf{G} \rangle : \underline{\mathsf{C}} \rightharpoonup \underline{\mathsf{D}}$ determines an isomorphism between arrows of the kind

$\mathsf{F}(c) \to d$ in $\underline{\mathsf{D}}$ and arrows of the kind $c \to \mathsf{G}(d)$ in $\underline{\mathsf{C}}$. Then, in the case of safe nets, we have a one-to-one correspondence

$$\pi : \Theta \to N \quad \Longleftrightarrow \quad \pi' : \Theta \to \mathcal{U}[N]$$

for each safe net N and each occurrence net Θ. Therefore, since such correspondence is easily seen to map processes to processes, in this special case, the correspondence between process and unfolding semantics of N is very tidy: they are the same notion in the precise sense that there is an isomorphism between the processes of N and the processes of $\mathcal{U}[N]$, i.e. the deterministic finite subnets of the unfolding of N, i.e. the finite configurations of $\mathcal{E}\mathcal{U}[N]$.

In our context, however, we have that the unfolding of N is strictly more concrete than the processes of N. For example, consider again the net N and its unfolding $\mathcal{F}\mathcal{U}[N]$ shown in Fig. 3. Clearly, there is a unique process of N in which t_0, t_2 and a single instance of t_3 caused by t_0 has occurred. Nevertheless, there are two deterministic subnets of $\mathcal{F}\mathcal{U}[N]$ which correspond to such a process, namely those obtained by choosing respectively the left and the right instance of t_3 below t_0. It is worth noticing that such subnets are isomorphic and that this is not a fortunate case, since it is easy to show that two finite deterministic subnets of $\mathcal{F}\mathcal{U}[N]$ correspond to the same process of N if and only if they are isomorphic via an isomorphism which sends instances of an element of N to instances of the same element. More interestingly, the results of this section will prove that this is the exact relationship between the two semantics of N: the unfolding contains several copies of the same process which, as illustrated in [15, 16], are needed to provide a fully *causal* explanation of the behaviour of N, i.e. to obtain an occurrence net whose transitions represent exactly the instances of the transitions of N in all the possible causal contexts and which can therefore account for concurrent multiple instances of the same element of N, that is for *autoconcurrency*. More precisely, we shall see that the finite deterministic subnets of the unfolding of N can be characterized by appropriately *decorating* the processes of N, which directly shows that the difference between the process and the unfolding semantics of N is only due to the replication of data needed in the latter. Of course, as we have already mentioned, the appropriate decoration of processes is immediately suggested by the notion of family in decorated occurrence nets: a *decorated process* is simply a process whose underlying process net is a decorated occurrence net.

Summing up the above discussion, this result is twofold: it yields both a *process-oriented* account of the unfolding construction (in terms of decorated processes) and an explanation of the lack of coincidence of such a construction with the standard notion of nonsequential process.

In addition, we shall give an abstract representation of the decorated *concatenable* processes of N by providing, in the style of [6, 28], an axiomatic construction of a *symmetric strict monoidal* category $\mathcal{D}\mathcal{P}[N]$ whose arrows are in one-to-one correspondence with such processes. Therefore, building on top of the previous argument, we can conclude that $\mathcal{D}\mathcal{P}[N]$ provides both the *algebraic* and the *process-oriented* account of the unfolding construction. In particular, as already stated in the introduction, for each marked PT net (N, u_N) we have

$$\begin{tikzcd}[column sep=small] & u_N, \mathscr{DP}[N]) \arrow[dr] & \\ (N, u_N) \arrow[ur] \arrow[dr] & & \mathscr{L}_F\mathscr{EFU}[(N,u)] \cong \langle u_N \downarrow \mathscr{DP}[N] \rangle \\ & \mathscr{EFU}[(N, u_N)] \arrow[ur] & \end{tikzcd} \tag{3}$$

where the role of the comma category construction is to consider only the decorated concatenable processes from the initial marking u_N.

Finally, the axiomatization of the decorated concatenable processes of N in abstract terms via the category $\mathscr{DP}[N]$ will also "axiomatize" the essential difference between occurrence nets and decorated occurrence nets, and therefore between (concatenable) processes and decorated (concatenable) processes. In fact, it will show that the latter is captured by a single axiom, namely the part $t;s = t$ of axiom (Ψ) of Definition 1.9. This completes our study of the relationships between the various semantics characterizing *formally* the relative concreteness of decorated (concatenable) processes, and thus of the unfolding semantics, with respect to standard (concatenable) processes.

It is worth observing that decorated (*deterministic*) occurrence nets which at first seem to be just a convenient technical solution to establish the adjunction from PT nets to occurrence nets, provide useful insights, being the notion underlying *both* the process and the algebraic counterpart of the unfolding semantics. It is also easy to realize that they are the minimal refinement of Goltz–Reisig processes which guarantees the identity of all tokens in processes. In fact, in order to achieve this, it is necessary to disambiguate both the tokens in the same place of the initial marking and the tokens which are multiple instances of the same place, and, therefore, to introduce the notion of *families*. All this seems to indicate that decorated process nets and their algebraic formalization $\mathscr{DP}[\_]$ may be structures of interest on their own.

Getting to the task, we start by showing an easy fact that we already mentioned, namely that the processes of an occurrence net Θ coincide with the finite configurations of $\mathscr{E}[\Theta]$. Clearly, since $\mathscr{FU}[N]$ is an occurrence net, we also obtain that the processes of $\mathscr{FU}[N]$ coincide with the finite configurations of $\mathscr{EFU}[N]$. We shall need the following lemmas which state three easy properties of morphisms between occurrence nets, namely that they preserve the depth of elements (Lemma 3.1), that they reflect causal links (Lemma 3.2), and that they preserve concurrency, i.e. that they reflect the relation $(\# \cup =) \subseteq (T_\Theta \cup S_\Theta) \times (T_\Theta \cup S_\Theta)$ (Lemma 3.3).

Lemma 3.1. *Let Θ_0 and Θ_1 be occurrence nets and let $f : \Theta_0 \to \Theta_1$ be a morphism of marked PT nets which maps places to places. Then, for all $x \in T_{\Theta_0} \cup S_{\Theta_0}$ we have* $\mathrm{depth}(f(x)) = \mathrm{depth}(x)$.

Proof. By induction on the depth of x. Since marked PT net morphisms map initial markings to initial markings, the thesis holds in the base case, i.e. if $\text{depth}(x) = 0$.

Inductive step: Let n be the depth of x and suppose that x is a transition. Then, by definition of depth, we have that $\text{depth}(y) \leqslant n-1$ for all $y \in {}^\bullet x$ and that there exists $z \in {}^\bullet x$ such that $\text{depth}(z) = n-1$. Then, since $f({}^\bullet x) = {}^\bullet f(x)$, the thesis follows immediately by induction. If, instead, x is a place we have that $\text{depth}(t) = n$, where t is the unique element in ${}^\bullet x$. Then, as we just proved, $\text{depth}(f(t)) = n$ and since $f(x) \in f(t)^\bullet$ the proof is concluded. $\square$

Lemma 3.2. *Let Θ_0 and Θ_1 be occurrence nets and let $f : \Theta_0 \to \Theta_1$ be a morphism of marked PT nets which maps places to places. Consider $x \in T_{\Theta_0} \cup S_{\Theta_0}$ and suppose that $y \preccurlyeq f(x)$ for some $y \in T_{\Theta_1} \cup S_{\Theta_1}$. Then, there exists $\bar{y} \preccurlyeq x$ such that $f(\bar{y}) = y$.*

Proof. In order to show the thesis, it is enough to consider the following two cases.

(i) Suppose that $a \in t^\bullet$ and $f(\bar{a}) = a$. Since a does not belong to the initial marking of Θ_1, then $\bar{a}$ cannot belong to the initial marking of Θ_0. Therefore, there exists a unique $\bar{t} \in {}^\bullet\bar{a}$ and, necessarily, $f(\bar{t}) = t$.

(ii) Suppose that $a \in {}^\bullet t$ and that $f(\bar{t}) = t$. Then, since $f({}^\bullet\bar{t}) = {}^\bullet t$ and since f maps places to places, there exists $\bar{a} \in S_{\Theta_0}$ such that $f(\bar{a}) = a$. $\square$

Lemma 3.3. *Let Θ_0 and Θ_1 be occurrence nets, let $f : \Theta_0 \to \Theta_1$ be a morphism of marked PT nets which maps places to places, and consider elements x and y in $T_{\Theta_0} \cup S_{\Theta_0}$. Then, if $f(x) = f(y)$ or $f(x) \# f(y)$, we have $x = y$ or $x \# y$.*

Proof. We proceed by induction on the least of the depths of x and y.

Base case: If $\text{depth}(x) = \text{depth}(y) = 0$, then $f(x) = f(y)$. In fact, in this case x and y belong to the initial marking of Θ_0 and thus, by definition of marked morphism, $f(x)$ and $f(y)$ are in the initial marking of Θ_1. It follows that they cannot be in conflict, since $\# \cap [\![u_\Theta]\!] \times [\![u_\Theta]\!] = \emptyset$. Now, if $x \neq y$, we have $f(u_{\Theta_0}) = f(x \oplus y \oplus u) = f(x) \oplus f(y) \oplus f(u) = 2f(x) \oplus f(u)$. But this is impossible, since $f(u_{\Theta_0}) = u_{\Theta_1}$ and each token in u_{Θ_1} has multiplicity one.

Inductive step: Let $n \geqslant 1$ be the least of the depths of x and y. Without loss of generality, assume $\text{depth}(x) = n$. First suppose that $f(x) = f(y)$. Then, there exist $z \in {}^\bullet x$ and $z' \in {}^\bullet y$ such that $f(z) = f(z')$. Then, if x is a transition, $\text{depth}(z) < n$ and therefore, by induction, $f(z) = f(z')$ or $f(z) \# f(z')$, whence it follows that $f(x) = f(y)$ or $f(x) \# f(y)$. If, instead, x is a place, then z is a transition at depth n and the induction is maintained exploiting the proof given above for such a case.

Suppose instead that $f(x) \# f(y)$. By definition, this means that there exist t_0 and t_1 in T_{Θ_1} such that $t_0 \#_m t_1$, $t_0 \preccurlyeq f(x)$ and $t_1 \preccurlyeq f(y)$. Then, by Lemma 3.2, there exist $\bar{t}_0 \preccurlyeq x$ and $\bar{t}_1 \preccurlyeq y$ in T_{Θ_0} such that $f(\bar{t}_0) = t_0$ and $f(\bar{t}_1) = t_1$. This concludes the proof since it follows easily that $\bar{t}_0 \# \bar{t}_1$, which implies $x \# y$. $\square$

It is easy to observe that the restriction to morphisms which map places to places is not necessary to show that morphisms of occurrence nets preserve the depth of elements and reflect $\preccurlyeq$-chains and the conflict relation. However, the formulations above suffices for application in what follows.

Proposition 3.4. *Let Θ_0 be ar. occurrence net. There is an isomorphism between the set of finite configurations of $\mathscr{E}[\Theta_0]$ and the set of processes of Θ_0.*

Proof. Let ϕ be the function which maps a process $\pi : \Theta \to \Theta_0$ to the set of transitions $\pi(T_\Theta)$. Recall that π is a marked net morphism between occurrence nets which maps places to places. Then, by Lemma 3.3, we have that π maps concurrent transitions to concurrent transitions. Since Θ is a process net, and thus deterministic, $\pi(T_\Theta)$ is conflict free. Consider now $t \in \pi(T_\Theta)$ and let $t' \in T_{\Theta_0}$ be such that $t' \preccurlyeq t$. Then, by Lemma 3.2, there exists $x \in T_\Theta$ such that $\pi(x) = t'$, i.e. $\pi(T_\Theta)$ is downwards closed and, thus, a finite configuration of $\mathscr{E}[\Theta_0]$.

On the contrary, let X be a finite configuration of $\mathscr{E}[\Theta_0]$. By depth of an element x of X we mean the length of the shortest chain in X whose maximal element is x; the depth of X is the greatest of the depths of its elements. We show by induction on the depth of X that there exists a unique (up to isomorphism) process $\pi : \Theta \to \Theta_0$ such that $\pi(T_\Theta) = X$.

Base case: If $X = \emptyset$, let Θ be the subnet of depth zero of Θ_0, i.e. the net consisting of the minimal places of Θ_0, and let π be the inclusion $\Theta \hookrightarrow \Theta_0$. Clearly, π is the unique (marked) process of Θ_0 such that $\phi(\pi) = \emptyset$.

Inductive step: Suppose that the depth of X is $n + 1$. Let Z be the set of elements of X at depth $n + 1$. Since the elements of Z are necessarily maximal in X, the set $Y = X \setminus Z$ is a configuration of $\mathscr{E}[\Theta_0]$. Moreover, the depth of Z is n. Then, by induction, there exists a unique $\pi : \Theta \to \Theta_0$ such that $\pi(T_\Theta) = Y$. Let $t \in Z$ and consider $a \in \partial^0_{\Theta_0}(t)$. We show that there exists a unique place $x_a \in S_\Theta$, which in addition is maximal, such that $\pi(x_a) = a$. The following two cases are possible.

(i) ${}^\bullet a = \emptyset$. Then, a belongs to the initial marking of Θ_0 and thus, by definition of marked net morphism, there exists a unique $x_a \in u_\Theta$ such that $\pi(x_a) = a$. Moreover, since by Lemma 3.1 π preserves the depth of elements, there is no other $x \in S_\Theta$ such that $\pi(x) = a$.

(ii) ${}^\bullet a = \{t'\}$. Then, $t' \prec t$ and thus, since X is downwards closed, there exists $x \in T_\Theta$ such that $\pi(x) = t'$. It follows that we can find a unique $x_a \in x^\bullet$ such that $\pi(x_a) = a$. Now, since by Lemma 3.3 π maps concurrent transitions to concurrent transitions, x is the unique transition of Θ mapped to t'. Therefore, x_a is the unique place of Θ mapped to a.

Observe that x_a must be maximal in Θ. In fact, if there were $x \in x_a^\bullet$, there would be $\pi(x) \in X$ with $\pi(x) \# t$, which is impossible since X is a configuration.

Now, it is easy to see that π can be extended to a process π' such that $\phi(\pi') = X$ in essentially a unique way. To this purpose, consider the net Θ' obtained by adding to Θ, for each $t \in Z$, a new transition x_t and a new place $\bar{a}$ for each $a \in \partial^1_{\Theta_0}(t)$ with

$$\partial^0_{\Theta'}(x_t) = \bigoplus\{x_a \mid a \in \partial^0_{\Theta_0}(t)\} \quad \text{and} \quad \partial^1_{\Theta'}(x_t) = \bigoplus\{\bar{a} \mid a \in \partial^1_{\Theta_0}(t)\}.$$

Since Θ_0 is an occurrence net, we have that $\partial^1_{\Theta_0}(t_0) \cap \partial^1_{\Theta_0}(t_1) = \emptyset$, for $t_0 \neq t_1$ in Z, and therefore, by definition, Θ' is an occurrence net. Moreover, since Z is a set of concurrent transitions, we also have $\partial^0_{\Theta_0}(t_0) \cap \partial^0_{\Theta_0}(t_1) = \emptyset$. Then, considering also that each x_a is maximal in Θ, we conclude that Θ' is deterministic. Therefore, π' defined as

$$\pi'(x) = \begin{cases} \pi(x) & \text{if } x \in T_\Theta \cup S_\Theta \\ t & \text{if } x = x_t \text{ for } t \in Z \\ a & \text{if } x = \bar{a} \text{ for } a \in \partial^1_{\Theta_0}(t) \text{ and } t \in Z \end{cases}$$

is a process of Θ_0 such that $\phi(\pi') = \pi'(T_{\Theta'}) = X$. Observe that, given the uniqueness of x_a, the only possible variation in the construction of π' is in the choice of "names" for the transitions and the places added to Θ. Then, since π is by inductive hypothesis the unique process such that $\pi(T_\Theta) = Y$, we conclude that π' is (up to isomorphism) the unique process such that $\pi'(T_{\Theta'}) = X$.

Therefore, ϕ is an isomorphism. $\square$

In particular, we have that there exists an isomorphism between the processes of $\mathcal{FU}[N]$ and the finite configurations of $\mathcal{EFU}[N]$. Our next task will be to characterize the processes of $\mathcal{FU}[N]$ in terms of processes of N. We shall do it by means of the following notion of *decorated process*.

Definition 3.5 (*Decorated processes*). A *decorated process* of a marked net N is a triple $DP = (\pi, \ell, \tau)$ where

- $\pi : \Theta \to N$ is a (marked) process of N;
- ℓ is a π-indexed ordering of $\min(\Theta)$;
- τ is a family $\{\tau(t)\}$ indexed by the transitions t of Θ, where each $\tau(t)$ is a π-indexed ordering of the post-set of t in Θ.

The decorated processes $(\pi : \Theta \to N, \ell, \tau)$ and $(\pi' : \Theta' \to N, \ell', \tau')$ are isomorphic, and then identified, if their underlying processes are isomorphic via an isomorphism φ which respects all the orderings, i.e. $\ell'_{\pi'(\varphi(a))}(\varphi(a)) = \ell_{\pi(a)}(a)$ for all $a \in \min(\Theta)$, and $\tau'(\varphi(t))_{\pi'(\varphi(a))}(\varphi(a)) = \tau(t)_{\pi(a)}(a)$ for all $t \in T_\Theta$ and $a \in t^\bullet$.

Fig. 6 shows the two decorated processes of the net N in Fig. 3 corresponding to the (unique) process of N in which t_0, t_2 and an instance of t_3 caused by t_0 have occurred. In the pictures, we represent a process $\pi : \Theta \to N$ by drawing Θ and labelling its element x by $\pi(x)$. Observe that Fig. 6 also gives a hint about the announced correspondence between processes of $\mathcal{FU}[N]$ and decorated processes of N.

We say that $(\pi : \Theta \to N, \ell, \tau) \leqslant (\pi' : \Theta' \to N, \ell', \tau')$ if there exists $\varphi : \Theta \to \Theta'$ which preserves all the orderings and such that $\pi = \pi' \circ \varphi$. The set of decorated processes

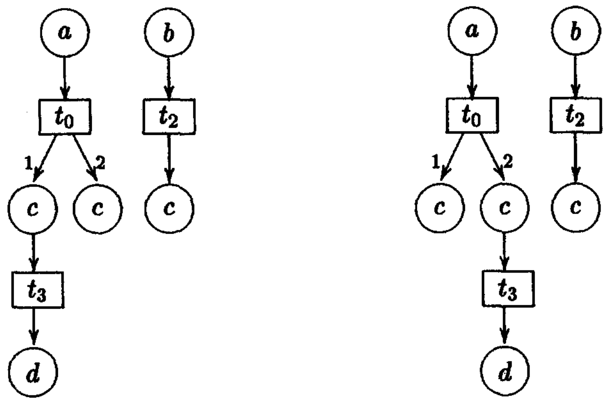

Fig. 6. Two decorated processes of the net in Fig. 3.

of N is clearly preordered by $\leqslant$. Let us write $DP[N]$ to indicate such preordering. The next proposition shows that actually $\leqslant$ is a partial order.

Proposition 3.6. *$DP[N]$ is a partial order.*

Proof. Consider $DP = (\pi : \Theta \to N, \ell, \tau)$ and $DP' = (\pi' : \Theta' \to N, \ell', \tau')$, and suppose that $DP \leqslant DP'$ and $DP' \leqslant DP$. Then, by definition, there exist $\varphi : \Theta \to \Theta'$ and $\varphi' : \Theta' \to \Theta$ which respect all the orderings and such that $\pi = \pi' \circ \varphi$ and $\pi' = \pi \circ \varphi'$. Since we identify isomorphic decorated processes, to conclude the proof it is enough to show that φ is an isomorphism. Observe however that, since π and π' map places to places and since $\pi = \pi' \circ \varphi$, it follows that φ has to map places to places. The same of course holds for φ'. Then, we show the thesis by showing the following more general fact: whenever the process nets Θ and Θ' are linked by marked PT net morphisms $\varphi : \Theta \to \Theta'$ and $\varphi' : \Theta' \to \Theta$ which map places to places, then φ (φ') is an isomorphism. Observe that, because of the aforesaid property of its place component, in order to show that φ (φ') is an isomorphism it is enough to show that it is injective and surjective on both places and transitions.

Injectivity: Since Θ is deterministic, it follows immediately by virtue of Lemmas 3.1 and 3.3 that φ is injective. Of course, for the same reason, also φ' is injective.

Surjectivity: By Lemma 3.1, we know that, for each $n \geqslant 1$ ($n \geqslant 0$), φ and φ' restrict to functions between the sets of transitions (places) at depth n of Θ and Θ'. Moreover, by definition of process nets, we have that such sets are finite. Then, the surjectivity of φ follows immediately from the injectivity of φ and φ' and from the following general fact which is readily shown by cardinality arguments: if $f : A \to B$ is an injective function between the *finite* sets A and B, and if there exists an injective function $g : B \to A$, then f (g) is surjective. $\square$

We are now ready to prove the correspondence between the decorated processes and the unfolding of N. To this purpose, recall that the folding morphism $\varepsilon_N : \mathcal{F}\mathcal{U}[N] \to N$

given in Proposition 2.4 is the marked net morphism such that

$$((x,a),i) \mapsto a \quad \text{and} \quad (B,t) \mapsto t.$$

The folding ε_N provides an obvious way to map a process $\pi : \Theta \to \mathscr{F}\mathscr{U}[N]$ to a process of N, namely $\varepsilon_N \circ \pi : \Theta \to N$. Moreover, we also have the following natural way of finding ℓ and τ which decorate this process and make it be a decorated process $P(\pi) = (\varepsilon_N \circ \pi, \ell, \tau)$ of N.

- Let b be in $\min(\Theta)$ and suppose that $\pi(b) = ((\emptyset, a), i)$. Then, defining $\ell_a(b) = i$ clearly gives a $(\varepsilon_N \circ \pi)$-indexed ordering of $\min(\Theta)$.
- Let t be a transition of Θ, and consider $b \in t^\bullet$. Since π is a process morphism, its image through π must be a place in the post-set of $\pi(t)$, i.e. a component of some family in $\pi(t)^\bullet$, say $\pi(b) = ((\pi(t), a), j)$. Then, taking $\tau(t)_a(b) = j$ clearly gives a $(\varepsilon_N \circ \pi)$-indexed ordering of $t^\bullet$.

In the opposite direction, we define a mapping F as follows. Let (π, ℓ, τ) be a decorated process of N with $\pi : \Theta \to N$. Then, $F(\pi, \ell, \tau)$ is $f : \Theta \to \mathscr{F}\mathscr{U}[N]$ defined inductively as follows:

depth 0: For $b \in \min(\Theta)$, consider $f_p(b) = ((\emptyset, a), i)$ with $a = \pi(b)$ and $i = \ell_a(b)$, while, of course, for $t \in T_\Theta$, $f_t(t)$ is $([\![f_p({}^\bullet t)]\!], \pi(t))$.

depth n+1: If t is a transition of Θ of depth $n+1$, then once again $f_t(t)$ is $([\![f_p({}^\bullet t)]\!], \pi(t))$, whilst if $b \in t^\bullet$ we take $f_p(b) = ((\{f_t(t)\}, \pi(b)), i)$ for $i = \tau(t)_a(b)$.

Informally, the behaviour of P and F may be explained by saying that P and F just move the information about families, respectively, in ℓ and τ from π and back in π from ℓ and τ. Of course, we have that $FP(\pi) = \pi$ and it shows clearly in the construction of $F(\pi, \ell, \tau)$ that $PF(\pi, \ell, \tau)$ is (up to isomorphism) again (π, ℓ, τ). Therefore, we have shown the following proposition.

Proposition 3.7. *The set of decorated processes of N is isomorphic to the set of (marked) processes of $\mathscr{F}\mathscr{U}[N]$ via the maps F and P given above.*

We complete the study of the relationship between process and unfolding semantics by showing that the correspondence we established above is easily lifted to a correspondence between the partial order of the decorated processes of N and the partial order of the finite configurations of $\mathscr{E}\mathscr{F}\mathscr{U}[N]$.

Proposition 3.8. *$DP[N]$ is isomorphic to $\mathscr{L}_F\mathscr{E}\mathscr{F}\mathscr{U}[N]$.*

Proof. To prove the claim we only need to show that, given the decorated processes $DP = (\pi : \Theta \to N, \ell, \tau)$ and $DP' = (\pi' : \Theta' \to N, \ell', \tau')$, we have $DP \leqslant DP'$ if and only if $\phi F(DP) \subseteq \phi F(DP')$, where ϕF gives the configuration corresponding to a marked decorated process as described by Propositions 3.4 and 3.7.

If $DP \leqslant DP'$, then there exists $\varphi : \Theta \to \Theta'$ which preserves the labellings and such that $\pi = \pi' \circ \varphi$. It follows immediately that φ is a morphism between the process nets underlying $F(DP)$ and $F(DP')$, and therefore $\phi F(DP) \subseteq \phi F(DP')$. The other implication comes along the same lines: if $\phi F(DP) \subseteq \phi F(DP')$, then there is a morphism φ from the process net underlying $F(DP)$, i.e. Θ, to the process net underlying $F(DP')$, i.e., Θ', such that $F(DP) = F(DP') \circ \varphi$. Clearly, φ is the marked net morphism which maps the element x of Θ to the unique element of Θ' in $\mathsf{F}(DP')^{-1}(F(DP)(x))$. Then, φ is a morphism from Θ to Θ' which preserves the labellings ℓ and τ and such that $\pi = \pi' \circ \varphi$, i.e. φ shows that $DP \leqslant DP'$. □

As already mentioned, the results established above on the one hand show that the unfolding construction can be reconciled with a process-oriented view, whilst, on the other hand, they illustrate precisely the differences between it and the standard notion of process. The question which then arises is whether decorated processes can be understood in more abstract terms. In the following we shall prove that this is the case by developing a theory which parallels that of concatenable processes. This will provide an algebraic account of the unfolding which will characterize it yet more neatly.

The same conceptual step which led from nonsequential processes to concatenable processes now suggests the following definition.

Definition 3.9 (*Decorated concatenable processes*). A *decorated concatenable process* of the (unmarked) net N, is a quadruple (π, ℓ, τ, L) where (π, ℓ, L) is a concatenable process of N and τ is a family $\{\tau(t)\}$ indexed by the transitions t of Θ, where each $\tau(t)$ is a π-indexed ordering of the post-set of t in Θ.

An isomorphism of decorated concatenable processes is an isomorphism of the underlying concatenable processes which, in addition, preserves all the orderings given by τ, i.e. $\tau'(\varphi(t))_{\pi'(\varphi(a))}(\varphi(a)) = \tau(t)_{\pi(a)}(a)$ for all $t \in T_\Theta$ and $a \in t^\bullet$.

So, a decorated concatenable process is a concatenable process where the post-sets of the transitions are π-indexed ordered. Such a definition makes the difference between concatenable and decorated concatenable processes immediate to grasp. The difference between decorated and decorated concatenable processes is also clear, being analogous to that between nonsequential and concatenable processes.

Since decorated concatenable processes are concatenable processes, they can be given a source and a target, namely those of the underlying concatenable process. Moreover, the concatenation of concatenable processes can be lifted to an operation on decorated concatenable processes. The concatenation of $(\pi_0, \ell_0, \tau_0, L_0): u \to v$ and $(\pi_1, \ell_1, \tau_1, L_1): v \to w$ is the decorated concatenable process $(\pi, \ell, \tau, L): u \to w$ defined as follows (see also Fig. 7, where $\tau(t)$ is depicted by decorating the arcs outgoing from t). In order to simplify notation, we assume that the process nets corresponding to π_0 and π_1, say Θ_0 and Θ_1, are disjoint.

- Let A be the set of pairs (y,x) such that $x \in \max(\Theta_0)$, $y \in \min(\Theta_1)$, $\pi_0(x) = \pi_1(y)$ and $(\ell_1)_{\pi_1(y)}(y) = (L_0)_{\pi_0(x)}(x)$. By the definitions of decorated concatenable processes and of their sources and targets, A determines an isomorphism $A: \min(\Theta_1) \to$

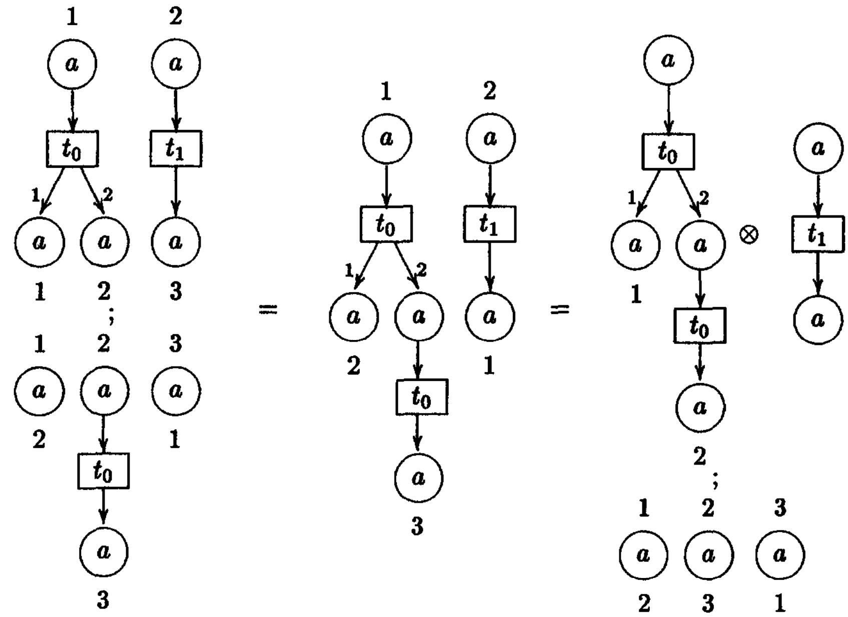

Fig. 7. An example of the algebra of decorated concatenable processes.

$\max(\Theta_0)$. Consider $S_1 = S_{\Theta_1} \setminus \min(\Theta_1)$, and let $in : S_{\Theta_1} \to S_{\Theta_0} \cup S_1$ be the function which is the identity on S_1 and maps $y \in \min(\Theta_1)$ to $A(y)$. Then,

$$\Theta = (\partial^0_\Theta, \partial^1_\Theta : T_{\Theta_0} \cup T_{\Theta_1} \to (S_{\Theta_0} \cup S_1)^\oplus),$$

where

- $\partial^0_\Theta(t) = \partial^0_{\Theta_0}(t)$ if $t \in T_{\Theta_0}$ and $\partial^0_\Theta(t) = in^\oplus(\partial^0_{\Theta_1}(t))$ if $t \in T_{\Theta_1}$;
- $\partial^1_\Theta(t) = \partial^1_{\Theta_i}(t)$ if $t \in T_{\Theta_i}$.

Then, $\pi : \Theta \to N$ coincides with π_0 on $S_{\Theta_0} \cup T_{\Theta_0}$ and with π_1 on $S_1 \cup T_{\Theta_1}$.

- $\ell = \ell_0$.
- $\tau(t) = \tau_i(t)$ if $t \in T_{\Theta_i}$.
- $L_a(y) = (L_1)_a(y)$ if $y \in S_1$, $L_a(x) = (L_1)_a(A^{-1}(x))$ if $x \in \max(\Theta_0)$.

Therefore, we can consider the category $\mathscr{DCP}[N]$ whose objects are the finite multisets on S_N and whose arrows are the decorated concatenable processes.

Proposition 3.10. *Under the above-defined operation of sequential composition, $\mathscr{DCP}[N]$ is a category with identities those decorated concatenable processes consisting only of places, which therefore are both minimal and maximal, and such that $\ell = L$.*

Decorated concatenable processes admit also a tensor operation $\otimes$ such that, given $DCP_0 = (\pi_0, \ell_0, \tau_0, L_0) : u \to v$ and $DCP_1 = (\pi_1, \ell_1, \tau_1, L_1) : u' \to v'$, $DCP_0 \otimes DCP_1$ is the decorated concatenable process $(\pi, \ell, \tau, L) : u \oplus u' \to v \oplus v'$ given below (see also

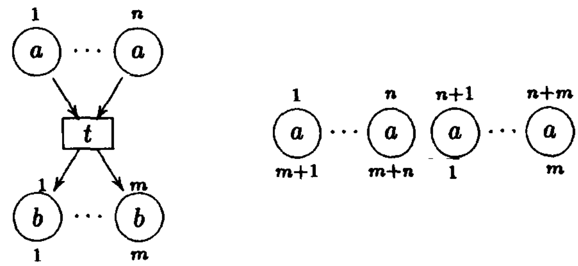

Fig. 8. A transition $t : n \cdot a \to m \cdot b$ and the symmetry $\gamma_{n \cdot a, m \cdot a}$ in $\mathscr{DCP}[N]$.

Figure 7), where again we suppose that Θ_0 and Θ_1, the underlying process nets, are disjoint.

- $\Theta = (\partial^0_\Theta, \partial^1_\Theta : T_{\Theta_0} \cup T_{\Theta_1} \to (S_{\Theta_0} \cup S_{\Theta_1})^{\oplus})$,
  where
  $-\partial^0_\Theta(t) = \partial^0_{\Theta_i}(t)$ if $t \in T_{\Theta_i}$;
  $-\partial^1_\Theta(t) = \partial^1_{\Theta_i}(t)$ if $t \in T_{\Theta_i}$.
  Then, $\pi : \Theta \to N$ is obviously given by $\pi(x) = \pi_i(x)$ for $x \in T_{\Theta_i} \cup S_{\Theta_i}$.
- $\ell_a(x) = (\ell_0)_a(x)$ if $x \in S_{\Theta_0}$, and $\ell_a(x) = |\pi_0^{-1}(a) \cap \min(\Theta_0)| + (\ell_1)_a(x)$ otherwise.
- $\tau(t) = \tau_i(t)$ if $t \in T_{\Theta_i}$.
- $L_a(x) = (L_0)_a(x)$ if $x \in S_{\Theta_0}$, and $L_a(x) = |\pi_1^{-1}(a) \cap \max(\Theta_1)| + (L_1)_a(x)$ otherwise.

It is easy to see that $\otimes$ is a functor from $\mathscr{DCP}[N] \times \mathscr{DCP}[N] \to \mathscr{DCP}[N]$. Moreover, as in the case of concatenable processes, we have that the decorated concatenable processes consisting only of places play the role of the symmetries of monoidal categories. In particular, for any $u = n_1 a_1 \oplus \cdots \oplus n_k a_k$ and $v = m_1 b_1 \oplus \cdots \oplus m_h b_h$, the concatenable process having as many places as elements in the multiset $u \oplus v$ mapped by π to the corresponding places of N and such that $L_{a_i}(x) = v(a_i) + \ell_{a_i}(x)$ and $\ell_{b_i}(x) = L_{b_i}(x) - u(b_i)$ (see also Fig. 8) is the symmetry coherence isomorphism $\gamma_{u,v}$ with respect to which $\mathscr{DCP}[N]$ is a *symmetric monoidal category*, i.e. Eqs (1) hold in $\mathscr{DCP}[N]$ for the given family of $\gamma_{u,v}$. Therefore, we have the following.

Proposition 3.11. *$\mathscr{DCP}[N]$ is a* symmetric strict monoidal category *with the symmetry isomorphism $\{\gamma_{u,v}\}_{u,v \in S_N^{\oplus}}$ given above.*

Observe that, since the decorated concatenable processes consisting only of places are just concatenable processes, in fact the subcategory $Sym_{\mathscr{DCP}[N]}$ of symmetries of $\mathscr{DCP}[N]$ coincides with the corresponding one of $\mathscr{CP}[N]$. Such observation will be useful later on. Observe also that the transitions t of N are represented by decorated concatenable processes with a unique transition and two layers of places: the minimal, in one-to-one correspondence with $\partial^0_N(t)$, and the maximal, in one-to-one correspondence with $\partial^1_N(t)$ (see also Fig. 8). The decoration, of course, consists in taking $\tau(t) = L$.

Recalling that the concatenable processes of N correspond to the arrows of $\mathscr{P}[N]$, and observing that the π-indexed orderings of the post-sets of the transitions of decorated concatenable processes are manifestly linked to the $t;s = t$ part of axioms (Ψ) in Definition 1.9, we are led to the following definition of the symmetric monoidal category $\mathscr{DP}[N]$ which captures the *algebraic essence* of decorated (concatenable) processes, and thus of the unfolding construction, simply by dropping that axiom in the definition of $\mathscr{P}[N]$.

Definition 3.12 (*The category* $\mathscr{DP}[N]$). Let N be a PT net. Then $\mathscr{DP}[N]$ is the monoidal quotient of the free symmetric strict monoidal category on N modulo the axioms

$$\begin{aligned} \gamma_{a,b} &= id_{a\oplus b} \quad \text{if } a,b \in S_N \text{ and } a \neq b,\\ s;t &= t \qquad\quad \text{if } t \in T_N \text{ and } s \text{ is a symmetry.} \end{aligned} \tag{4}$$

Explicitly, $\mathscr{DP}[N]$ is the category whose objects are the elements of $S_N^\oplus$ and whose arrows are generated by the inference rules

$$\frac{u \in S_N^\oplus}{id_u : u \to u \text{ in } \mathscr{DP}[N]} \qquad \frac{u, v \text{ in } S_N^\oplus}{c_{u,v} : u \oplus v \to u \oplus v \text{ in } \mathscr{DP}[N]} \qquad \frac{t : u \to v \text{ in } T_N}{t : u \to v \text{ in } \mathscr{DP}[N]}$$

$$\frac{\alpha : u \to v \text{ and } \beta : u' \to v' \text{ in } \mathscr{DP}[N]}{\alpha \otimes \beta : u \oplus u' \to v \oplus v' \text{ in } \mathscr{DP}[N]} \qquad \frac{\alpha : u \to v \text{ and } \beta : v \to w \text{ in } \mathscr{DP}[N]}{\alpha;\beta : u \to w \text{ in } \mathscr{DP}[N]}$$

modulo the axioms expressing that $\mathscr{DP}[N]$ is a strict monoidal category, namely,

$$\begin{gathered} \alpha; id_v = \alpha = id_u; \alpha \text{ and } (\alpha;\beta);\delta = \alpha;(\beta;\delta),\\ (\alpha\otimes\beta)\otimes\delta = \alpha\otimes(\beta\otimes\delta) \text{ and } id_0 \otimes \alpha = \alpha = \alpha \otimes id_0,\\ id_u \otimes id_v = id_{u\oplus v} \text{ and } (\alpha\otimes\alpha');(\beta\otimes\beta') = (\alpha;\beta)\otimes(\alpha';\beta'), \end{gathered} \tag{5}$$

the latter whenever the right-hand term is defined, the following axioms corresponding to axioms (1) expressing that $\mathscr{DP}[N]$ is symmetric with symmetry isomorphism c

$$\begin{aligned} c_{u,v\oplus w} &= (c_{u,v} \otimes id_w);(id_v \otimes c_{u,w}),\\ c_{u,u'};(\beta\otimes\alpha) &= (\alpha\otimes\beta);c_{v,v'} \quad \text{for } \alpha : u \to v,\ \beta : u' \to v',\\ c_{u,v};c_{v,u} &= id_{u\oplus v}, \end{aligned} \tag{6}$$

and the following axioms corresponding to axioms (4)

$$\begin{aligned} c_{a,b} &= id_{a\oplus b} \quad \text{if } a,b \in S_N \text{ and } a \neq b,\\ (id_u \otimes c_{a,a} \otimes id_v);t &= t \quad \text{if } t \in T. \end{aligned} \tag{7}$$

It is worthwhile to remark that in the definition above, axioms (5) and (6) define $\mathscr{F}(N)$, the free symmetric strict monoidal category on N [27, 28]. Observe that, exploiting the coherence axiom, i.e. the first of (6), a symmetry in $\mathscr{F}(N)$ can always be written as a composition of symmetries of the kind $(id_u \otimes c_{a,b} \otimes id_v)$ for $a, b \in S_N$. Then, since we have $c_{a,b} = id_{a\oplus b}$ if $a \neq b$, the second of (4) takes the particular form stated in (7).

Our next task is to show that $\mathcal{DP}[N]$ and $\mathcal{DCP}[N]$ are isomorphic categories. We need the following fundamental lemma about symmetries in monoidal categories. In the following, $\underline{C}^n$ denotes the nth power of $\underline{C}$, i.e. the cartesian product of n copies of $\underline{C}$. Moreover, for $n \geqslant 2$, we use $\otimes^n : \underline{C}^n \to \underline{C}$ to indicate $\otimes \circ (\mathbf{1}_{\underline{C}} \times \otimes) \circ \cdots \circ (\mathbf{1}_{\underline{C}^{n-2}} \times \otimes)$.

Lemma 3.13. *Let $\underline{C}$ be a symmetric strict monoidal category. For each permutation σ of n elements, $n \geqslant 2$, let $\mathsf{F}_\sigma : \underline{C}^n \to \underline{C}^n$ be the functor which "swaps" its arguments according to σ, i.e.*

$$\begin{tikzcd}
\underline{C}^n \arrow[rr, "F_\sigma"] & & \underline{C}^n \\
(x_1, \ldots, x_n) \arrow[rr, mapsto] \arrow[d, "{(f_1, \ldots, f_n)}"'] & & (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \arrow[d, "{(f_{\sigma(1)}, \ldots, f_{\sigma(n)})}"] \\
(y_1, \ldots, y_n) \arrow[rr, mapsto] & & (y_{\sigma(1)}, \ldots, y_{\sigma(n)})
\end{tikzcd}$$

Then, there exists a natural isomorphism $\gamma_\sigma : \otimes^n \stackrel{\bullet}{\to} \otimes^n \circ \mathsf{F}_\sigma$. We shall call γ_σ the "σ-interchange" *symmetry.*

Proof. Recall that a permutation of n elements is an *isomorphism* of the segment $\{1, \ldots, n\}$ of the first n positive natural numbers with itself. It is well known that each permutation of n elements can be written as a composition of *transpositions*, where, for $i = 1, \ldots, n-1$, the transposition τ_i is the permutation which leaves fixed all the elements but i and $i+1$, which are (of course) exchanged. This formalizes the intuitive fact that a permutation can always be achieved by performing a sequence of "*swappings*" of adjacent integers. Then, assume that σ is $\tau_{i_k} \circ \cdots \circ \tau_{i_1}$. We show the thesis by induction on k.

Base case: If $k = 0$ then $\sigma = id$, and thus $\mathbf{1}_{\otimes^n}$ is the isomorphism looked for.

Inductive step: Let σ' be $\tau_{i_{k-1}} \circ \cdots \circ \tau_{i_1}$. Then, by inductive hypothesis, we have a σ'-interchange symmetry $\gamma_{\sigma'} : \otimes^n \to \otimes^n \circ \mathsf{F}_{\sigma'}$. Now, let i_k be $\sigma'(i)$ and consider the natural isomorphism

$$\bar{\tau} = id_{x_{\sigma'(1)}} \otimes \cdots \otimes id_{x_{\sigma'(i-1)}} \otimes \gamma_{x_{\sigma'(i)}, x_{\sigma'(i+1)}} \otimes id_{x_{\sigma'(i+2)}} \otimes \cdots \otimes id_{x_{\sigma'(n)}}$$

from $\otimes^n \circ \mathsf{F}_{\sigma'}$ to $\otimes^n \circ \mathsf{F}_{\sigma' \circ \tau_{i_1}}$. Of course, since $\tau_{i_k} \circ \sigma' = \sigma$, we have that γ_σ is the (vertical) composition $\bar{\tau} \cdot \gamma_{\sigma'} : \otimes^n \stackrel{\bullet}{\to} \otimes^n \circ \mathsf{F}_\sigma$.

Observe that, since σ admits several factorizations in terms of transpositions, in principle many different γ_σ may exist. However, it is worth noticing that this is not the case. In particular, there exists a unique σ-interchange symmetry, as follows from the Kelly–MacLane coherence theorem (see [12]) which, informally speaking, states that, given any pair of functors built up from identity functors and $\otimes$, there is at most one natural transformation built up from identities and components of the symmetry γ between them. $\square$

The following announced result matches Theorem 1.12 in the context of decorated concatenable processes. Although some of the ideas of the proof of Theorem 1.12 have parallels in this case, our result cannot follow from it. It requires a separate proof that we give below.

Proposition 3.14. *$\mathscr{DCP}[N]$ and $\mathscr{DP}[N]$ are isomorphic.*

Proof. Let $\mathscr{F}(N)$ be the free symmetric strict monoidal category on N (see the remark following Definition 3.12). Corresponding to the inclusion morphism $N \hookrightarrow \mathscr{DCP}[N]$, i.e. to the PT net morphism whose place component is the identity and whose transition component sends $t \in T_N$ to the corresponding decorated concatenable process (see Fig. 8), there is a symmetric strict monoidal functor $\mathsf{H} : \mathscr{F}(N) \to \mathscr{DCP}[N]$. Observe that $\mathscr{DCP}[N]$ satisfies axioms (4), the symmetries and the transitions being as explained before (see Fig. 8). In fact, if $a, b \in S_N$ and $a \neq b$, by definition of f-indexed ordering, we have $\gamma_{a,b} = id_{a \oplus b}$. Moreover, for S a symmetry and T a transition in $\mathscr{DCP}[N]$, it follows easily from the definition of $\_;\_$ that $S;T$ is (isomorphic to) T. Therefore, we have $\mathsf{H}(c_{a,b}) = \gamma_{a,b} = id_{a \oplus b}$ when $a \neq b \in S_N$, and since any symmetric monoidal functor sends symmetries to symmetries, we have $\mathsf{H}(s;t) = \mathsf{H}(s);\mathsf{H}(t) = S;\mathsf{H}(t) = \mathsf{H}(t)$, i.e., taking (4) as our set $\mathscr{E}$ of equations, H satisfies condition (ii) of Proposition 1.8. Therefore, denoting by Q the quotient functor from $\mathscr{F}(N)$ to $\mathscr{DP}[N]$ induced by Eqs. (4), by Proposition 1.8, there exists a (unique) symmetric strict monoidal functor $\mathsf{K} : \mathscr{DP}[N] \to \mathscr{DCP}[N]$ such that the diagram below commutes:

$$\begin{tikzcd}
\mathscr{F}(N) \arrow[r, "\mathsf{Q}"] \arrow[dr, "\mathsf{H}"'] & \mathscr{DP}[N] \arrow[d, "\mathsf{K}"] \\
& \mathscr{DCP}[N]
\end{tikzcd}$$

In the following we shall prove that K is an isomorphism. Observe that, by definition, for any $u \in S_N^{\oplus}$, we have $\mathsf{K}(u) = \mathsf{K}(\mathsf{Q}(u)) = \mathsf{H}(u) = u$, i.e. K is the identity on the objects. Moreover, we can easily conclude that it is an isomorphism on the symmetries. In fact, as already remarked, the decorated concatenable processes of depth zero, i.e. the symmetries of $\mathscr{DCP}[N]$, are exactly the concatenable processes of depth zero, i.e. the symmetries of $\mathscr{CP}[N]$. Therefore, we have $Sym_{\mathscr{DCP}[N]} = Sym_{\mathscr{CP}[N]}$. Now observe that, by definition, $\mathscr{P}[N]$ is the monoidal quotient of $\mathscr{DP}[N]$ modulo the axiom $t;s = t$. Since none of the axioms of $\mathscr{DP}[N]$ can discharge transitions from terms, axiom $t;s = t$ can never be used in a proof of equality of symmetries, i.e. it does not induce any equality on the symmetries. Therefore, we have that $Sym_{\mathscr{P}[N]} = Sym_{\mathscr{DP}[N]}$. Moreover, Proposition 1.12 shows that $Sym_{\mathscr{P}[N]}$ and $Sym_{\mathscr{CP}[N]}$ are isomorphic via a functor whose object component is the identity (see also [7, 28]). Now observe that, once the object component is fixed, there can be at most one symmetric strict monoidal functor F between two categories of symmetries. In fact, on the one hand we have that, by definition, the symmetries of a symmetric strict monoidal category are generated by the identities and the components of the isomorphism γ, while, on the other hand, it must necessarily be $\mathsf{F}(id_u) = id_{\mathsf{F}(u)}$ and $\mathsf{F}(\gamma_{u,v}) = \gamma_{\mathsf{F}(u),\mathsf{F}(v)}$ (see axioms (2)). Then, since K is a symmetric strict monoidal functor whose object component is the identity, its restriction to $Sym_{\mathcal{DP}[N]}$ is an isomorphism $Sym_{\mathcal{DP}[N]} = Sym_{\mathcal{P}[N]} \cong Sym_{\mathcal{CP}[N]} = Sym_{\mathcal{DCP}[N]}$. We proceed now to show that K is full and faithful.

Fullness: It is completely obvious that any decorated concatenable process DCP may be obtained as a concatenation $DCP_0; \ldots ; DCP_n$ of decorated concatenable processes DCP_i of depth one. Now, each of these DCP_i may be split into the concatenation of a symmetry S_0^i, the tensor of the (processes representing the) transitions which appear in DCP_i plus some identities, say $id_{u_i} \otimes \bigotimes_j \mathsf{K}(t_j^i)$, and finally another symmetry S_1^i. The intuition about this factorization is as follows. We take the tensor of the transitions which appear in DCP_i in any order and multiply the result by an identity concatenable process in order to get the correct source and target. Then, in general, we need a pre- and a post-concatenation with a symmetry in order to get the right indexing of minimal and maximal places and of the post-sets of each $\mathsf{K}(t_j^i)$. Thus, we finally have

$$DCP = S_0^0; (id_{u_1} \otimes \bigotimes_j \mathsf{K}(t_j^1)); (S_1^0; S_0^1); \ldots ; (S_1^{n-1}; S_0^n); (id_{u_n} \otimes \bigotimes_j \mathsf{K}(t_j^n)); S_1^n$$

which shows that every decorated concatenable process is in the image of K.

Faithfulness: The arrows of $\mathcal{DP}[N]$ are equivalence classes modulo the axioms stated in Definition 3.12 of terms built by applying tensor and sequentialization to the identities id_u, the symmetries $c_{u,v}$, and the transitions t. We have to show that, given two such terms α and β, whenever $\mathsf{K}(\alpha) = \mathsf{K}(\beta)$ we have $\alpha =_{\mathscr{E}} \beta$, where $=_{\mathscr{E}}$ is the equivalence induced by (5), (6) and (7).

First of all, observe that if $\mathsf{K}(\alpha)$ is a decorated process DCP of depth n, then α can be proved equal to a term

$$\alpha' = s_0; (id_{u_1} \otimes \bigotimes_j t_j^1); s_1; \ldots ; s_{n-1}; (id_{u_n} \otimes \bigotimes_j t_j^n); s_n$$

where, for $1 \leqslant i \leqslant n$, the transitions t_j^i, for $1 \leqslant j \leqslant n_i$, are exactly the transitions of DCP at depth i and where $s_0, \ldots , s_n$ are symmetries. Moreover, we can assume that in the ith tensor product $\bigotimes_j t_j^i$ the transitions are indexed according to a global ordering $\leqslant$ of T_N assumed for the purpose of this proof, i.e. $t_1^i \leqslant \cdots \leqslant t_{n_i}^i$, for $1 \leqslant i \leqslant n$. Let us prove our claim. It is easily shown by induction on the structure of terms that using axioms (5) α can be rewritten as $\alpha_1; \ldots ; \alpha_h$, where $\alpha_i = \bigotimes_k \xi_k^i$ and ξ_k^i is either a transition or a symmetry. Now, observe that by functoriality of $\otimes$, for any $\alpha' : u' \to v'$, $\alpha'' : u'' \to v''$ and $s : u \to u$, we have $\alpha' \otimes s \otimes \alpha'' = (id_{u'} \otimes s \otimes id_{u''}); (\alpha' \otimes id_u \otimes \alpha'')$, and thus, by repeated applications of (5), we can prove that α is equivalent to $\bar{s}_0; \bar{\alpha}_1; \bar{s}_1 \ldots ; \bar{s}_{h-1}; \bar{\alpha}_h$, where $\bar{s}_0, \ldots , \bar{s}_{h-1}$ are symmetries and each $\bar{\alpha}_i$ is a tensor $\bigotimes_k \bar{\xi}_k^i$ of transitions and identities. The fact that the transitions at depth i can be brought to the ith tensor product follows intuitively from the fact that they are "disjointly enabled", i.e. concurrent to each other, and that they depend causally on some transition at depth $i-1$. In particular, the sources of the transitions of depth 1 can be target only of symmetries. Therefore, reasoning formally as above, they can be pushed up to $\bar{\alpha}_1$ exploiting axioms (5). Then, the same happens for the transitions of depth 2, which can be brought to $\bar{\alpha}_2$. Proceeding in this way, eventually we show that α is equivalent to the composition $\bar{\bar{s}}_0; \bar{\bar{\alpha}}_1; \bar{\bar{s}}_1 \ldots ; \bar{\bar{s}}_{n-1}; \bar{\bar{\alpha}}_n; \bar{\bar{s}}_n$

of the symmetries $\bar{\bar{s}}_0, \ldots, \bar{\bar{s}}_n$ and the products $\bar{\bar{\alpha}}_i = \bigotimes_k \bar{\bar{\xi}}_k^i$ of transitions at depth i and identities. Finally, exploiting Lemma 3.13, the order of the $\bar{\bar{\xi}}_k^i$ can be permuted in the way required by $\leqslant$. This is achieved by pre- and post-composing each product by appropriate σ-interchange symmetries. More precisely, let σ be a permutation such that $\bigotimes_k \bar{\bar{\xi}}_{\sigma(k)}^i$ coincides with $id_{u_i} \otimes \bigotimes_j t_j^i$, suppose that $\bar{\bar{\xi}}_k^i : u_k^i \to v_k^i$, for $1 \leqslant k \leqslant k_i$, and let γ_σ be the σ-interchange symmetry guaranteed by Lemma 3.13 in $\mathscr{DP}[N]$. Then, since γ_σ is a natural transformation, we have that

$$\gamma_{\sigma\, u_1^i, \ldots, u_{k_i}^i} ; (\bigotimes_k \bar{\bar{\xi}}_{\sigma(k)}^i) = (\bigotimes_k \bar{\bar{\xi}}_k^i); \gamma_{\sigma\, v_1^i, \ldots, v_{k_i}^i},$$

and then, since γ_σ is an isomorphism, we have that

$$(id_{u_i} \otimes \bigotimes_j t_j^i) = \gamma^{-1}_{\sigma\, u_1^i, \ldots, u_{k_i}^i} ; (\bigotimes_k \bar{\bar{\xi}}_k^i); \gamma_{\sigma\, v_1^i, \ldots, v_{k_i}^i}.$$

Now, applying the same argument to β, one proves that it is equivalent to a term $\beta' = p_0; \beta_0; p_1; \ldots p_{n-1}; \beta_n; p_n$, where $p_0, \ldots, p_n$ are symmetries and β_i is the product of the transitions at depth i in $\mathsf{K}(\beta)$ and of identities. Then, since $\mathsf{K}(\alpha) = \mathsf{K}(\beta)$, and since the transitions occurring in β_i are indexed in a predetermined way, we conclude that $\beta_i = (id_{u_i} \otimes \bigotimes_j t_j^i)$, i.e.

$$\begin{aligned} \alpha' &= s_0; (id_{u_1} \otimes \bigotimes_j t_j^1); s_1; \ldots; s_{n-1}; (id_{u_n} \otimes \bigotimes_j t_j^n); s_n, \\ \beta' &= p_0; (id_{u_1} \otimes \bigotimes_j t_j^1); p_1; \ldots; p_{n-1}; (id_{u_n} \otimes \bigotimes_j t_j^n); p_n. \end{aligned} \tag{8}$$

In other words, the only possible differences between α' and β' are the symmetries. Observe now that the steps which led from α to α' and from β to β' have been performed by using the axioms which define $\mathscr{DP}[N]$, and since such axioms hold in $\mathscr{DCP}[N]$ as well and K preserves them, we have that $\mathsf{K}(\alpha') = \mathsf{K}(\alpha) = \mathsf{K}(\beta) = \mathsf{K}(\beta')$. Thus, we conclude the proof by showing that, if α and β are terms of the form given in (8) which differ only by the intermediate symmetries and if $\mathsf{K}(\alpha) = \mathsf{K}(\beta)$, then α and β are equal in $\mathscr{DP}[N]$.

We proceed by induction on n. Observe that if n is zero then there is nothing to show: since we know that K is an isomorphism on the symmetries, s_0 and p_0, and thus α and β, must coincide. To provide a correct basis for the induction, we need to prove the thesis also for $n = 1$.

depth 1: In this case, we have

$$\begin{aligned} \alpha &= s_0; (id_u \otimes \bigotimes_j t_j); s_1, \\ \beta &= p_0; (id_u \otimes \bigotimes_j t_j); p_1. \end{aligned}$$

Without loss of generality, we may assume that p_0 and p_1 are identities. In fact, we can multiply both terms by p_0^{-1} on the left and by p_1^{-1} on the right and obtain a pair of terms whose images through K still coincide and whose equality implies the equality in $\mathscr{DP}[N]$ of the original α and β.

Let $(\pi:\Theta \to N,\ell,\tau,L)$ be the decorated concatenable process $\mathsf{K}(id_u \otimes \bigotimes_j t_j)$. Of course, we can assume that $\mathsf{K}(s_0)$ and $\mathsf{K}(s_1)$ are respectively $(\pi_0:\Theta_0 \to N,\ell',\emptyset,\ell)$ and $(\pi_1:\Theta_1 \to N,L,\emptyset,L')$, where Θ_0 is $\min(\Theta)$, Θ_1 is $\max(\Theta)$, π_0 and π_1 are the corresponding restrictions of π, and ℓ' and L' are π-indexed orderings respectively of the minimal and the maximal places of Θ.

Then, we have that $\mathsf{K}(s_0;(id_u\otimes\bigotimes_j t_j);s_1)$ is $(\pi:\Theta \to N,\ell',\tau,L')$, and by hypothesis there is an isomorphism $\varphi:\Theta \to \Theta$ such that $\pi\circ\varphi = \pi$ and which respects all the orderings, i.e. $\ell'_{\pi(\varphi(a))}(\varphi(a)) = \ell_{\pi(a)}(a)$ and $L'_{\pi(\varphi(b))}(\varphi(b)) = L_{\pi(b)}(b)$, for all $a \in \Theta_0$ and $b \in \Theta_1$, and $\tau(\varphi(t))_{\pi\varphi(a)}(\varphi(a)) = \tau(t)_{\pi(a)}(a)$ for all $t \in \Theta$ and $a \in t^\bullet$. Let us write $id_u\otimes\bigotimes_j t_j$ as $\bigotimes_k \xi_k$, where ξ_k is either a transition t_j or the identity of a place in u. Moreover, let $\xi_k:u_k \to v_k$, for $1\leqslant k\leqslant k_i$. Clearly, φ induces a permutation of the symbols ξ_k, namely the permutation σ such that $\xi_{\sigma(k)} = \varphi(\xi_k)$. Then, in order to be a morphism of nets, φ must map the (places corresponding to the) pre-set, respectively post-set, of t_j to the (the places corresponding to the) pre-set, respectively post-set, of $t_{\sigma(j)}$. Observe now that this identifies φ uniquely on the maximal places of Θ, which implies that $\mathsf{K}(s_1)$ is completely determined. In fact, if a maximal place x is also minimal, then the corresponding ξ_k is the identity id_{u_k} and thus x must be mapped to the object for which $\xi_{\sigma(k)}$ is the identity. If, instead, x is in the post-set of t_j then x must be mapped to the post-set of $t_{\sigma(j)}$ in the unique way compatible with the family of π-indexed orderings τ. In other words, $\mathsf{K}(s_1)$ is the component at $(v_1,\dots,v_{k_i})$ of the σ-interchange symmetry. Then, since K is an isomorphism between $Sym_{\mathscr{DP}[N]}$ and $Sym_{\mathscr{DCP}[N]}$, s_1 must necessarily be the corresponding component of the σ-interchange symmetry in $\mathscr{DP}[N]$.

Concerning $\mathsf{K}(s_0)$, we cannot be so precise. However, since we know that the pre-sets of transitions are mapped by φ according to σ, reasoning as above we can conclude that $(\pi_0,\ell,\emptyset,\ell')$, which is $\mathsf{K}(s_0)^{-1}$, must be a symmetry obtained by concatenating the component at $(u_1,\dots,u_{k_i})$ of the σ-interchange symmetry and some product $\bigotimes_j S_j$ of symmetries, one for each t occurring in α, whose role is to reorganize the tokens in the pre-sets of each transitions. It follows that s_0 is $\gamma^{-1}_{\sigma\, u_1,\dots,u_{k_i}};(id_u\otimes\bigotimes_j s_j)$, where s_j is a symmetry on the source of t_j.

Then, by distributing the tensor of symmetries on the transitions and using the second of (7), we show that $\alpha = \gamma^{-1}_{\sigma\, u_1,\dots,u_{k_i}};(id_u\otimes\bigotimes_j t_j);\gamma_{\sigma\, v_1,\dots,v_{k_i}}$, which, by definition of σ-interchange symmetry, is $(id_u\otimes\bigotimes_j t_j)$. Thus, we have $\alpha =_{\mathscr{E}} \beta$ as required.

Inductive step: Suppose that $n > 1$ and let $\alpha = \alpha';\alpha''$ and $\beta = \beta';\beta''$, where

$$\begin{aligned} \alpha' &= s_0;(id_{u_1}\otimes\textstyle\bigotimes_j t_j^1);s_1;\dots;s_{n-1} \quad\text{and } \alpha'' = (id_{u_n}\otimes\textstyle\bigotimes_j t_j^n);s_n,\\ \beta' &= p_0;(id_{u_1}\otimes\textstyle\bigotimes_j t_j^1);p_1;\dots;p_{n-1} \text{ and } \beta'' = (id_{u_n}\otimes\textstyle\bigotimes_j t_j^n);p_n. \end{aligned}$$

We show that there exists a symmetry s in $\mathscr{DP}[N]$ such that $\mathsf{K}(\alpha';s) = \mathsf{K}(\beta')$ and $\mathsf{K}(s^{-1};\alpha'') = \mathsf{K}(\beta'')$. Then, by the induction hypothesis, we have $(\alpha';s) =_{\mathscr{E}} \beta'$ and $(s^{-1};\alpha'') =_{\mathscr{E}} \beta''$. Therefore, we conclude that $(\alpha';s;s^{-1};\alpha'') =_{\mathscr{E}} (\beta';\beta'')$, i.e. that $\alpha = \beta$ in $\mathscr{DP}[N]$.

Let $(\pi:\Theta \to N,\ell,\tau,L)$ be the decorated concatenable process $\mathsf{K}(\alpha) = \mathsf{K}(\beta)$. Without loss of generality we may assume that the decorated occurrence nets $\mathsf{K}(\alpha')$ and $\mathsf{K}(\beta')$

are, respectively, $(\pi' : \Theta' \to N, \ell', \tau', L^{\alpha'})$ and $(\pi' : \Theta' \to N, \ell', \tau', L^{\beta'})$, where Θ' is the subnet of depth $n-1$ of Θ, ℓ' and τ' are the appropriate restrictions of ℓ and τ and finally $L^{\alpha'}$ and $L^{\beta'}$ are π-indexed orderings of the places at depth $n-1$ of Θ. Consider the symmetry $S = (\bar{\pi}, \bar{\ell}, \emptyset, \bar{L})$ in $\mathscr{DCP}[N]$, where

- $\bar{\Theta}$ is the process net consisting of the maximal places of Θ';
- $\bar{\pi} : \bar{\Theta} \to N$ is the restriction of π to $\bar{\Theta}$;
- $\bar{\ell} = L^{\alpha'}$;
- $\bar{L} = L^{\beta'}$.

Then, by definition, we have $\mathsf{K}(\alpha'); S = \mathsf{K}(\beta')$. Let us consider now α'' and β''. Clearly, we can assume that $\mathsf{K}(\alpha'')$ and $\mathsf{K}(\beta'')$ are $(\pi'' : \Theta'' \to N, \ell^{\alpha''}, \tau'', L'')$ and $(\pi'' : \Theta'' \to N, \ell^{\beta''}, \tau'', L'')$, where Θ'' is the process net obtained by removing from Θ the subnet Θ', τ'' and L'' are respectively the restrictions of τ and L to Θ'', and $\ell^{\alpha''}$ and $\ell^{\beta''}$ are π-indexed orderings of the places at depth $n-1$ of Θ. Now, in our hypothesis, it must be $L^{\alpha'} = \ell^{\alpha''}$ and $L^{\beta'} = \ell^{\beta''}$, which shows directly that $S^{-1}; \mathsf{K}(\alpha'') = \mathsf{K}(\beta'')$. Then, $s = \mathsf{K}^{-1}(S)$ is the required symmetry of $\mathscr{DP}[N]$.

Then, since K is full and faithful and is an isomorphism on the objects, it is an isomorphism and the proof is concluded. $\square$

We conclude the paper by proving the commutativity (up to equivalence) of diagram (3). We first recall the following simple notion from category theory.

Definition 3.15 (*Comma categories*). Let $\underline{\mathsf{C}}$ be a category and c an object of $\underline{\mathsf{C}}$. Then, the *comma category* $\langle c{\downarrow}\underline{\mathsf{C}}\rangle$, also called the *category of elements under* c, is the category whose objects are the arrows $f : c \to c'$ of $\underline{\mathsf{C}}$ and whose arrows $h : (f : c \to c') \to (g : c \to c'')$ are commutative diagrams of the form

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ c' \arrow[rr, "h"'] & & c'' \end{tikzcd}$$

Identities and arrows composition are inherited in the obvious way from $\underline{\mathsf{C}}$.

The first step to achieve the result is the following easy observation about the structure of the comma category $\langle u_N{\downarrow}\mathscr{DP}[N]\rangle$, which shows that the edge $\langle \_ \downarrow \_\rangle \circ \mathscr{DP}^*[\_]$ of the diagram discussed in the introduction actually maps $\underline{\mathsf{MPetri}}^*$ to $\underline{\mathsf{PreOrd}}$, the category of preordered sets.

Proposition 3.16. *The category* $\langle u_N{\downarrow}\mathscr{DP}[N]\rangle$ *is a preorder.*

Proof. We have to show that in $\langle u_N{\downarrow}\mathscr{DP}[N]\rangle$ there is at most one arrow between any pair of objects $\alpha : u_N \to v$ and $\alpha' : u_N \to w$. Exploiting the characterization of arrows of $\mathscr{DP}[N]$ in terms of decorated concatenable processes established by Proposition 3.14, the thesis can be reformulated as follows: for each pair of concatenable decorated processes $DCP_0 : u_N \to v$ and $DCP_1 : u_N \to w$ there exists at most one decorated concatenable process $DCP : v \to w$ such that $DCP_0; DCP = DCP_1$.

In order to show the claim, suppose that there exist DCP and DCP' from v to w such that $DCP_0; DCP = DCP_1 = DCP_0; DCP'$. Let $\bar{\pi} : \bar{\Theta} \to N$ and $\bar{\pi}' : \bar{\Theta}' \to N$ be the (plain) processes underlying, respectively, $DCP_0; DCP$ and $DCP_1; DCP'$. Without loss of generality, we can assume that $\bar{\Theta}$, respectively $\bar{\Theta}'$, is formed by joining Θ_0, the process net underlying DCP_0, with Θ, the process net underlying DCP, respectively Θ', the process net underlying DCP'. Then, since $DCP_0; DCP = DCP_0; DCP'$, there exists an isomorphism $\varphi : \bar{\Theta} \to \bar{\Theta}'$ which respects all the orderings and such that $\bar{\pi} = \bar{\pi}' \circ \varphi$. Since we can assume that φ restricts to the identity of Θ_0 (as a subnet of $\bar{\Theta}$ and $\bar{\Theta}'$), it follows that it restricts to an isomorphism $\varphi' : \Theta \to \Theta'$ which shows $DCP = DCP'$. □

The next proposition establishes the commutativity of diagram (3) essentially by showing that the canonical partial order associated to the preorder $\langle u_N \downarrow N \rangle$ is $DP[(N, u_N)]$, and concludes our exposition. As for the previous proposition, the proof follows easily from Proposition 3.14, and the intuition behind it can be grasped from Fig. 9, where the self-looping arrows represent the nonidentity symmetries. We warn the reader that not all the symmetries are shown in the picture; this is the meaning of the double arrows which stand for several of them.

Proposition 3.17. *For any marked PT net* (N, u_N),

$$\langle u_N \downarrow \mathscr{DP}[N] \rangle \cong DP[(N, u_N)] \cong \mathscr{L}_F \mathscr{E} \mathscr{F} \mathscr{U}[(N, u_N)].$$

Proof. Consider the mapping from the objects of $\langle u_N \downarrow \mathscr{DP}[N] \rangle$ to the elements of $DP[(N, u_N)]$ given by $(\pi, \ell, \tau, L) \mapsto (\pi, \ell, \tau)$. Now, observe that there is a morphism from $DCP = (\pi : \Theta \to N, \ell, \tau, L)$ to $DCP' = (\pi' : \Theta' \to N, \ell', \tau', L')$ in $\langle u_N \downarrow \mathscr{DP}[N] \rangle$ if and only if there exists a decorated concatenable process DCP'' such that $DCP; DCP'' = DCP'$ if and only if there exists $\varphi : \Theta \to \Theta'$ such that $\pi = \pi' \circ \varphi$ and which preserves all orderings, i.e. if and only if $(\pi, \ell, \tau) \leqslant (\pi', \ell', \tau')$ in $DP[(N, u_N)]$. Thus, since from Proposition 3.16 we know that $\langle u_N \downarrow \mathscr{DP}[N] \rangle$ is a preorder, the mapping above is clearly a full and faithful functor. Moreover, since such a mapping is surjective on the objects, it is an equivalence of categories.

Observe that the second equivalence is actually an isomorphism, as shown by Proposition 3.8. □

4. Conclusions

In this paper we have shown how the unfolding semantics given in [15, 16] can be reconciled with a process-oriented semantics based on the new notion of decorated process. Moreover, we have seen that the algebraic structure of the decorated processes of a net can be faithfully expressed by a symmetric monoidal category. The key of this

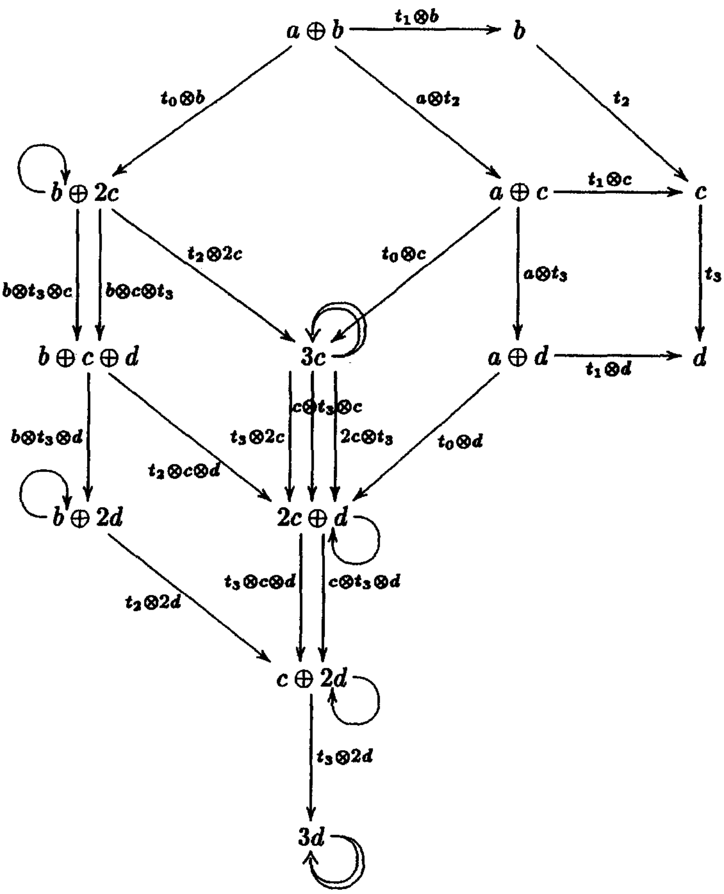

Fig. 9. Some of the arrows with source $a \oplus b$ in $\mathscr{DP}[N]$ for the net of Fig. 3.

formal achievement is the notion of *decorated occurrence nets*. Although the category DecOcc arose from the need of factorizing the involved adjunction from PTNets to Occ, and, thus, decorated occurrence nets were at first just a convenient technical solution, we have shown that there are in fact some insights on the semantics of nets given by the unfolding construction and the associated notion of decorated occurrence nets. In fact, decorated *deterministic* occurrence nets, suitably axiomatized as arrows of the symmetric monoidal category $\mathscr{DP}[N]$, provide both the process-oriented and the algebraic counterpart of the unfolding semantics. Moreover, they can be characterized as the minimal refinement of Goltz–Reisig processes which guarantees the identity of all tokens, i.e. as the minimal refinement of occurrence nets which guarantees the existence of an unfolding for PT nets.

A possible objection to decorated concatenable processes is that they are based on an undesired "*colouring*" of tokens. The categorical characterization of decorated concatenable processes given in Proposition 3.14 helps in clarifying this matter. First of all, since the source and target of a decorated concatenable process are plain markings, and *not* coloured entities, it is certainly not possible to classify the present approach as "coloured". It is nevertheless true that the identities of the tokens are somehow taken into account as "first-class" components of the internal structure of processes. What actually goes on becomes immediately clear looking at the axiomatization provided by $\mathscr{DP}[N]$, where a certain notion of identity of tokens is "built" into the categorical notion of symmetries. Then, it is important to stress that this is accomplished without manoeuvering tokens: it is the structure of the process itself that takes tokens into account. Moreover, it should be generally accepted that distinguishing tokens by structural means is the primary purpose of processes. Of course, this purpose can be dealt with by considering morphisms $\pi : \Theta \to N$ and also, as this paper indicates, by algebraic means. Summing up, we want to stress the idea that decorated concatenable processes are a reasonable (and intentionally concrete) version of the standard notion of process. The same argument, of course, applies to concatenable processes and the results in [6].

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