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Convergence, Continuity and Iteration in Mathematical Morphology

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This paper examines continuity properties of morphological operators, or more generally, operators mapping a complete lattice into itself. For any sequence of lattice elements one can define its $\limsup$ and $\liminf$, and in case both elements coincide one says that the sequence converges. Using this notion of convergence one can define $\downarrow$- and $\uparrow$-continuity of lattice operators. For the case where the lattice is the family of all subsets of the discrete space $\mathbb{Z}^d$, one can easily find explicit expressions for the $\limsup$ and $\liminf$, and give criteria for the $\downarrow$- and $\uparrow$-continuity of an operator. Special attention is given to well-known morphological operators such as dilations, erosions, closings, openings, and hit-or-miss operators. Order continuity properties of a morphological operator play a major role in settling the problem when iteration of such an operator yields an idempotent one, e.g. an opening or a closing, and other strong filters, such as the middle element. In the final section, an application to numerical functions is given.

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0. Introduction

For the classification, recognition, or analysis of an image, it is generally required to transform it into another image (or even a collection of images) which is better suited for this goal. For this reason there has been (and still is) a great deal of interest in image transformations or, as we shall call them, image operators. Among the image processing techniques which have been developed during the past three decades, mathematical morphology, after a period of relative obscureness, receives a fast increasing amount of attention. The reason for this is twofold. First of all, mathematical morphology, whose main strength lies in the quantitative description of geometrical structure and shape, has proved extremely useful in many image processing tasks. But secondly, it is a theory which also appeals to theoreticians since it allows a rigorous mathematical description and derives its tools from several mathematical disciplines such as algebra, topology, probability, and integral geometry, just to name a few.

As far as the algebraic features are concerned, this double aspect of mathematical morphology may be put together with its evolution. Initially designed by Matheron [14] and Serra [21] to formalize Euclidean set transformations (or operators as we will call them), in particular those which are translation-invariant, the method was progressively extended to numerical functions [16, 26,27], to planar graphs [10], and to context dependent cases such as geodesics [11]. The initial framework turned out too restrictive and was later replaced by Matheron and Serra by a more general one, namely the framework of complete lattices. An extensive account of this more general approach, along with a number of results on morphological filtering, is given in [22]. For a number of related results see [8,19].

This paper stems from the following reflections. One can sometimes obtain powerful morphological algorithms by iterating a certain operator until stability is reached. We mention the following examples: extraction of the connected components which contain a given marker, middle element, ultimate erosion, skeleton, watersheds, etc. (in this list the first two examples pertain to morphological filtering). However, iterative techniques do not always work, since they may give rise to oscillations. For example, repeated application of the median to the square chessboard pattern leads to period-two oscillations. However, as soon as the the operator ψ reduces or expands the object under study, we must necessarily have in practice (that is, in the case that the underlying support is finite)

$$\psi^{n+1} = \psi^n$$

for finite n. But here, convergence is due to the fact that we work on a finite grid. Figure 3 below exhibits an example (first reported in [5]) based on a very simple operator acting on the space of all subsets of $\mathbb{Z}$. Under iteration this operator shows the following behaviour:

$$\psi^\infty \neq \psi \circ \psi^\infty.$$

Here ψ^∞ denotes the operator obtained after (countably) infinitely many iterations of ψ (a precise definition of ψ^∞ will be given below). In the continuous case similar situations may be encountered [22, p.113].

A remedy against the occurence of such anomalies is to restrict oneselves to finite spaces. But this excludes the use of translation invariant operators which presuppose the underlying space to be infinite, and which are of great importance to the practicioner of mathematical morphology. Instead, we propose to look for conditions on the operator ψ, rather than on the underlying support space, which guarantee convergence of ψ^n under iteration. Again we adopt the general framework of complete lattices.

In Section 1 we will briefly recall some of the necessary terminology. In Section 2 we introduce the "lim sup" and "lim inf" for sequences of lattice elements. It is worth noting that no topological requirement about the lattices under consideration is assumed. In Section 3, sequential order convergence is used to define $\uparrow$- and $\downarrow$-continuity of an operator. In Section 4 we briefly state some known results about Matheron's hit-or-miss topology [14], and compare the resulting topological notion of convergence with its algebraic counterpart. We specialize to the Boolean lattice $\mathcal{P}(E)$, where $E = \mathbb{R}^d$ or $\mathbb{Z}^d$ in Sections 5 and 6. In this particular case the two notions of convergence coincide. In Section 5 we investigate the continuity properties of morphological operators on $\mathcal{P}(E)$ and in Section 6 we show that many of these operators are nothing but particular examples of so-called finite operators. There we also show that every finite operator is continuous. In Section 7, we deal with morphological filters, i.e., increasing morphological operators which are idempotent. In particular we show that continuity properties of operators can be exploited to construct filters by iteration. Finally, in Section 8 we illustrate some of the theoretical results by considering a particular class of mappings on the lattice of numerical functions on $\mathbb{R}^d$.

1. Some basic notions

In this section we briefly recall some basic vocabulary and some results that we shall use later on; for more details consult [14,21,22] and also [8,19]. One of the basic features of mathematical morphology is that the underlying space is a complete lattice; we shall denote this lattice by $\mathcal{L}$. This means that there is a partial order $\leq$ on $\mathcal{L}$, and that every subset $\mathcal{H}$ of $\mathcal{L}$ has a supremum and infimum respectively denoted by $\bigvee \mathcal{H}$ and $\bigwedge \mathcal{H}$. The greatest and least element of $\mathcal{L}$ are denoted by O and $\mathcal{I}$ respectively. As important examples we mention $\mathcal{P}(\mathbb{Z}^d)$ and $\mathcal{P}(\mathbb{R}^d)$, the space of all subsets of $\mathbb{Z}^d$ resp. $\mathbb{R}^d$, $\mathcal{F}(\mathbb{R}^d)$, the space of all closed subsets of $\mathbb{R}^d$, the space of all (closed) convex subsets of $\mathbb{R}^d$, and spaces of numerical functions on $\mathbb{R}^d$ (see Section 8). By an operator we shall mean a mapping of the lattice into itself. The identity operator which maps any element of $\mathcal{L}$ onto itself is denoted by id. We say that the operator ψ is

- *increasing* if $X \leq Y$ implies that $\psi(X) \leq \psi(Y)$;
- a *dilation* if ψ distributes over suprema, i.e., $\psi(\vee_{i\in I} X_i) = \vee_{i\in I}\psi(X_i)$;
- an *erosion* if ψ distibutes over infima, i.e.,$\psi(\wedge_{i\in I} X_i) = \wedge_{i\in I}\psi(X_i)$;
- *(anti-)extensive* if $\psi(X) \geq X$ ($\psi(X) \leq X$);
- *idempotent* if $\psi^2 = \psi$;
- an *opening* if ψ is anti-extensive and idempotent;
- a *closing* if ψ is extensive and idempotent;
- a *(morphological) filter* if ψ is increasing and idempotent.

Note that e.g. extensivity of ψ is also expressed by the inequality $\psi \geq \mathrm{id}$.

We denote by $\mathcal{O}(\mathcal{L})$ and $\mathcal{O}_+(\mathcal{L})$ the set of all operators respectively all increasing operators on $\mathcal{L}$. It is easily seen that both sets become a complete lattice under the ordering

$$\phi \leq \psi \iff \phi(X) \leq \psi(X) \text{ for any } X \in \mathcal{L}.$$

A useful principle for complete lattices and mappings on complete lattices is the so-called *duality principle*. This principle states that to any concept or statement concerning (mappings between) complete lattices there always corresponds an opposite concept or statement which is obtained by reversing the order of the lattices involved. For example, the opposite of a dilation is an erosion, and hence to every statement concerning dilations there corresponds the opposite statement concerning erosions. As a second illustration of the duality principle we mention the following example: the opposite of the statement "a *dilation followed by an erosion yields a closing*" is the statement "*an erosion followed by a dilation yields an opening*" (see below). Opposite statements are obtained by interchanging $\leq$ and $\geq$, infima and suprema, dilations and erosions, openings and closings, extensivity and anti-extensivity, etc.

An important relation between dilations and erosions is given by *adjunctions*: let ε and δ be two operators on $\mathcal{L}$. We call (ε, δ) an adjunction if for every $X, Y \in \mathcal{L}$, we have

$$\delta(X) \leq Y \quad \iff \quad X \leq \varepsilon(Y).$$

Adjunctions constitute a bijection between dilations and erosions, in other words:

- if (ε, δ) is an adjunction, then ε is an erosion and δ a dilation;
- for every dilation δ, there is a unique erosion ε such that (ε, δ) is an adjunction;
- for every erosion ε, there is a unique dilation δ such that (ε, δ) is an adjunction.

Adjunctions offer a key to access openings and closings. Namely, if (ε, δ) is an adjunction, then

$$\varepsilon\delta\varepsilon = \varepsilon \quad \text{and} \quad \delta\varepsilon\delta = \delta,$$

and

$$\varepsilon\delta \geq \mathsf{id} \quad \text{and} \quad \delta\varepsilon \leq \mathsf{id}.$$

In particular, this means that $\delta\varepsilon$ is an opening and $\varepsilon\delta$ is a closing.

The interest in these four basic mappings stems from the fact hat they generate a comprehensive class of useful operators by considering suprema, infima, compositions, and ... iterations.

If $\mathcal{L}$ is a Boolean lattice, e.g. the space $\mathcal{P}(E)$ consisting of all subsets of some space E, then every element of $\mathcal{L}$ has a unique complement which we denote by X^c. For such lattices one can introduce another duality concept, namely duality with respect to complementation. The dual operator of ψ is given by $\psi^*(X) = (\psi(X^c))^c$. It is clear that ψ^* is increasing if and only if ψ is.

Throughout the remainder of this section we consider translation-invariant operators. Let $E = \mathbb{R}^d$ or $\mathbb{Z}^d$ and take $\mathcal{L}$ to be the complete lattice $\mathcal{P}(E)$.We define the translate of the set X along the vector $h \in E$ by $X_h = \{x + h \mid x \in X\}$. The operator ψ is said to be translation-invariant if $\psi(X_h) = (\psi(X))_h$, for every X and h. Every translation-invariant dilation δ is a Minkowski addition with some fixed set A, called the structuring element. That is

$$\delta(X) = X \oplus A = \bigcup_{a \in A} X_a. \tag{1.1}$$

The adjoint erosion ε is the Minkowski subtraction with the same set, that is,

$$\varepsilon(X) = X \ominus A = \bigcap_{a \in A} X_{-a}. \tag{1.2}$$

Now the operators $X \mapsto X_A$ and $X \mapsto X^A$, where

$$X_A = (X \ominus A) \oplus A \tag{1.3}$$
$$X^A = (X \oplus A) \ominus A \tag{1.4}$$

define an opening respectively a closing.

The importance of erosions and dilations is best illustrated by Proposition 1.1. which is due to Matheron [14]. If ψ is an operator on $\mathcal{P}(E)$, then we define its kernel $\mathcal{V}(\psi)$ by

$$\mathcal{V}(\psi) = \{A \in \mathcal{P}(E) \mid 0 \in \psi(A)\}. \tag{1.5}$$

For a subset A of E the reflected set $\check{A}$ is defined as

$$\check{A} = \{-a \mid a \in A\}. \tag{1.6}$$

Proposition 1.1. *Let ψ be an increasing translation-invariant operator on $\mathcal{P}(E)$. Then ψ can be decomposed as a union of erosions, and, likewise as an intersection of dilations. More precisely*

$$\psi(X) = \bigcup_{A \in \mathcal{V}(\psi)} X \ominus A = \bigcap_{A \in \mathcal{V}(\psi^*)} X \oplus \check{A}. \tag{1.7}$$

In discrete morphology, a large class of operators (skeleton, skiz, pseudo-convex hull, etc., see [21]) derive from the *hit-or-miss operator* which is defined as

$$\begin{aligned} X \circledast (A, B) &= \{h \in E \mid A_h \subset X \text{ and } B_h \subset X^c\} \\ &= (X \ominus A) \cap (X^c \ominus B). \end{aligned} \tag{1.8}$$

Here A, B are structuring elements. The thickening operator is defined as

$$X \odot (A, B) = X \cup [X \circledast (A, B)], \tag{1.9}$$

and the thinning operator is defined as

$$\begin{aligned} X \bigcirc (A, B) &= X \setminus [X \circledast (A, B)] \\ &= X \cap [X \circledast (A, B)]^c. \end{aligned} \tag{1.10}$$

Recently Banon and Barrera [1] have extended Matheron's result for operators which are not necessarily increasing. Before we formulate their result we need some further definitions. For $A, B \subseteq E$ we define the operator

$$X \circledS (A, B) = \{h \in E \mid A_h \subseteq X \subseteq B_h\}. \tag{1.11}$$

Obviously, $X \circledS (A,B) = \emptyset$ if $A \not\subseteq B$. Furthermore, there exists the following relation with the hit-or-miss operator

$$X \circledS (A,B) = X \circledast (A, B^c). \tag{1.12}$$

For $A, B \subseteq E$ we define the "interval" $[A, B]$ as

$$[A,B] := \{X \in \mathcal{P}(E) \mid A \subseteq X \subseteq B\}.$$

Finally, we define the *bi-kernel* $\mathcal{W}(\psi)$ of the operator ψ as

$$\mathcal{W}(\psi) = \{(A,B) \in \mathcal{P}(E) \times \mathcal{P}(E) \mid [A,B] \subseteq \mathcal{V}(\psi)\}.$$

Proposition 1.2. *Let ψ be an arbitrary translation-invariant operator on $\mathcal{P}(E)$. Then*

$$\psi(X) = \bigcup_{(A,B) \in \mathcal{W}(\psi)} X \circledS (A,B). \tag{1.13}$$

PROOF.
"$\supseteq$": Let $h \in X \circledS (A,B)$ for some $(A,B) \in \mathcal{W}(\psi)$. Then $A_h \subseteq X \subseteq B_h$, hence $X_{-h} \in [A,B] \subseteq \mathcal{V}(\psi)$. Therefore $0 \in \psi(X_{-h})$, and by the translation-invariance of ψ, $h \in \psi(X)$.
"$\subseteq$": Let $h \in \psi(X)$, that is, $0 \in \psi(X_{-h})$. Then $(X_{-h}, X_{-h}) \in \mathcal{W}(\psi)$. It is obvious that $h \in X \circledS (X_{-h}, X_{-h})$, and thus $h \in \bigcup_{(A,B) \in \mathcal{W}(\psi)} X \circledS (A,B)$.

∎

Note that if the operator ψ is increasing and $A \in \mathcal{V}(\psi)$ then $(A, E) \in \mathcal{W}(\psi)$ and since $X \circledS (A,B)$ is decreasing with respect to B, and $X \circledS (A,E) = X \ominus A$, formula (1.13) reduces to $\psi(X) = \bigcup_{A \in \mathcal{V}(\psi)} X \ominus A$. So Proposition 1.2 indeed generalizes Proposition 1.1.

2. Order convergence on complete lattices

Throughout this section we assume that $\mathcal{L}$ is a complete lattice.

Definition 2.1. For a sequence X_n in $\mathcal{L}$ we define

$$\liminf X_n = \bigvee_{N \geq 1} \bigwedge_{n \geq N} X_n$$
$$\limsup X_n = \bigwedge_{N \geq 1} \bigvee_{n \geq N} X_n.$$

Obviously, for any sequence X_n,

$$\liminf X_n \leq \limsup X_n. \tag{2.1}$$

We say that $X_n \to X$ if $\liminf X_n = \limsup X_n = X$.

We point out that throughout this paper we restrict to *sequential* order convergence: c.f. [9, Chapter IV]. In [2], Birkhoff more generally defines the liminf and limsup for an arbitrary net X_α in $\mathcal{L}$, and says that X_α "order-converges" to X if $\limsup X_\alpha = \liminf X_\alpha = X$. This notion of convergence defines an intrinsic topology on $\mathcal{L}$ (i.e., a topology which is preserved under isomorphisms), called the *order topology* and coincides with our definition if one restricts to sequences. Notice however that as a drawback of our restriction the underlying topological space is not completely specified unless it has a countable base. Another interesting intrinsic topology on the complete lattice $\mathcal{L}$ is the so-called *interval topology*, defined by taking the closed intervals $[A, B]$, where $A, B \in \mathcal{L}$, as a subbasis of closed sets. In the complete atomic Boolean lattice $\mathcal{P}(\mathbb{Z}^d)$ the order and interval topology are equivalent ([2, p.252]). A *topological lattice* is a lattice with a specified convergence topology where $X_\alpha \to X$ and $Y_\beta \to Y$ imply

(i) $X_\alpha \wedge Y_\beta \to X \wedge Y$

(ii) $X_\alpha \vee Y_\beta \to X \vee Y$.

In [4], Gierz et al thoroughly investigate so-called continuous lattices. These are lattices with certain properties on which one can define different topologies. One of them is the so-called *Lawson topology*. It turns out that $\mathcal{F}(\mathbb{R}^d)$ with the opposite ordering (that is, $X \leq Y$ if and only if $Y \subseteq X$) is a continuous lattice, and that the Lawson topology coincides with the hit-or-miss topology.

In a recent and more synthetic study Matheron [15] classifies the various topologies for complete lattices and focusses on compactness questions.

It is clear that

$$\limsup(X_n \wedge Y_n) \leq \limsup X_n \wedge \limsup Y_n \tag{2.2}$$

$$\limsup(X_n \vee Y_n) \geq \limsup X_n \vee \limsup Y_n, \tag{2.3}$$

and the same relations hold for liminf. We point out that these relations more generally hold for an arbitrary collection of sequences.

Let $\mathcal{L}$ be a complete lattice. We say that the *infinite distributivity laws* hold if for any collection $A \in \mathcal{L}$, $X_i \in \mathcal{L}$ $(i \in I)$ we have

$$A \wedge \bigvee_{i \in I} X_i = \bigvee_{i \in I} (A \wedge X_i) \tag{2.4}$$

$$A \vee \bigwedge_{i \in I} X_i = \bigwedge_{i \in I} (A \vee X_i). \tag{2.5}$$

In particular these relations hold in any complete Boolean lattice as well as for lattices of numerical functions.

Proposition 2.2. *Let $\mathcal{L}$ be a complete lattice for which the infinite distributivity laws hold. Then for any two sequence X_n, Y_n in $\mathcal{L}$ we have*

$$\limsup(X_n \vee Y_n) = \limsup X_n \vee \limsup Y_n \tag{2.6}$$

$$\liminf(X_n \wedge Y_n) = \liminf X_n \wedge \liminf Y_n. \tag{2.7}$$

PROOF. Define $\overline{X}_N = \bigvee_{n \geq N} X_n$ and $\overline{Y}_N = \bigvee_{n \geq N} Y_n$. It is obvious that for every $N \geq 1$,

$$\bigwedge_{M \geq N} (\overline{X}_M \vee \overline{Y}_M) = \limsup(X_n \vee Y_n).$$

Therefore, since $\overline{X}_N, \overline{Y}_N$ are decreasing in N,

$$\begin{aligned}
\limsup(X_n \vee Y_n) &= \bigwedge_{N \geq 1} \bigwedge_{M \geq N} (\overline{X}_M \vee \overline{Y}_M) \leq \bigwedge_{N \geq 1} \bigwedge_{M \geq N} (\overline{X}_N \vee \overline{Y}_M) \\
&= \bigwedge_{N \geq 1} \bigwedge_{M \geq 1} (\overline{X}_N \vee \overline{Y}_M) = \bigwedge_{N \geq 1} (\overline{X}_N \vee \bigwedge_{M \geq 1} \overline{Y}_M) \\
&= (\bigwedge_{N \geq 1} \overline{X}_N) \vee (\bigwedge_{M \geq 1} \overline{Y}_M) = \limsup X_n \vee \limsup Y_n.
\end{aligned}$$

Here we have used (2.5) twice. The reverse inequality is given in (2.3).

∎

A sequence X_n in $\mathcal{L}$ is said to be decreasing if $\ldots \leq X_{n+1} \leq X_n \leq X_{n-1} \leq \ldots$. We write $X_n \downarrow X$ if X_n is a decreasing sequence and $X = \bigwedge_{n \geq 1} X_n$. Analogously, we write $X_n \uparrow X$ if X_n is an increasing sequence and $X = \bigvee_{n \geq 1} X_n$. The following result is trivial.

Proposition 2.3. *In any complete lattice*

(a) $X_n \downarrow X$ *implies that* $X_n \to X$.

(b) $X_n \uparrow X$ *implies that* $X_n \to X$.

Examples 2.4.

(a) On the lattice $\overline{\mathbb{R}}$ with the usual ordering order convergence is equivalent with Euclidean convergence as long as one restricts to finite limits.

(b) Let $\mathcal{L} = \mathcal{P}(\mathbb{R})$, and let $X_n = [0,1]$ if n is odd, and $X_n = [-1,0]$ if n is even. Then $\liminf X_n = \{0\}$, and $\limsup X_n = [-1,1]$. If furthermore $Y_n = [0,1]$ if n is even and $Y_n = [-1,0]$ if n is odd then $X_n \wedge Y_n = \{0\}$ hence $\limsup(X_n \wedge Y_n) = \{0\}$, but $\limsup X_n \wedge \limsup Y_n = [-1,1]$. This example shows that in general the inequality in (2.2) is strict.

(c) Let $\mathcal{L}$ be the complete lattice of all closed convex subsets, and $X_n = \{(-1)^n\}$. Then $\liminf X_n = \emptyset$ and $\limsup X_n = [-1,1]$.

3. Order continuity of lattice operators

Definition 3.1. Let $\mathcal{L}_1, \mathcal{L}_2$ be complete lattices and let $\psi : \mathcal{L}_1 \to \mathcal{L}_2$ be an arbitrary operator. We say that ψ is $\downarrow$-continuous if $X_n \to X$ implies that $\limsup \psi(X_n) \leq \psi(X)$, and that ψ is $\uparrow$-continuous if $X_n \to X$ implies that $\psi(X) \leq \liminf \psi(X_n)$. If ψ is both $\uparrow$- and $\downarrow$-continuous, that is, $X_n \to X$ implies that $\psi(X_n) \to \psi(X)$, then we say that ψ is (order) continuous.

The following result follows immediately from the relations (2.2)–(2.3) and Proposition 2.2.

Proposition 3.2. *The operations $\bigvee, \bigwedge : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ given by*

$$(X,Y) \mapsto X \vee Y$$
$$(X,Y) \mapsto X \wedge Y$$

are $\uparrow$-continuous respectively $\downarrow$-continuous.
If $\mathcal{L}$ satisfies the infinite distributivity laws (2.4)-(2.5) then both operations are continuous.

In fact this proposition is a somewhat weaker version of Theorem X.19 in [2] which says that any complete distributive lattice for which the infinite distributivity laws hold is a topological lattice: c.f. Section 2.

From now on we shall exclusively deal with operators mapping $\mathcal{L}$ into $\mathcal{L}$. Most of the results however, immediately carry over to the situation where the operators act between two different complete lattices.

Obviously, there exist a duality relation between $\uparrow$-continuous and $\downarrow$-continuous operators in the following way: if ψ is an $\uparrow$-continuous operator on the lattice $(\mathcal{L}, \leq)$, then ψ is a $\downarrow$-continuous operator on the complete lattice $(\mathcal{L}, \geq)$ with the opposite ordering. Therefore we may restrict in the sequel to $\downarrow$-continuous operators. For the special but important subclass of increasing operators we have the following characterizations.

Proposition 3.3. *The increasing operator ψ is $\downarrow$-continuous if and only if $X_n \downarrow X$ implies that $\psi(X_n) \downarrow \psi(X)$.*

PROOF. "only if": assume that ψ is $\downarrow$-continuous, and that $X_n \downarrow X$. Then, by Proposition 2.3, $X_n \to X$, hence $\limsup \psi(X_n) \leq \psi(X)$. This implies that

$$\bigwedge_{N\geq 1} \psi(X_N) = \bigwedge_{N\geq 1} \bigvee_{n\geq N} \psi(X_n) \leq \psi(X).$$

But the reverse inequality holds trivially, which proves that $\psi(X_n) \downarrow \psi(X)$.
"if": assume that $X_n \to X$. We must show that $\limsup \psi(X_n) \leq \psi(X)$. Define $Y_N := \vee_{n\geq N} X_n$. Then $Y_N \downarrow X$, hence $\psi(Y_N) \downarrow \psi(X)$. But $X_N \leq Y_N$, hence $\limsup \psi(X_N) \leq \limsup \psi(Y_N) = \psi(X)$.

∎

Proposition 3.4. *Let ψ be an increasing operator on $\mathcal{L}$. Then ψ is $\downarrow$-continuous if and only if*

$$\limsup \psi(X_n) \leq \psi(\limsup X_n)$$

for any sequence X_n in $\mathcal{L}$.

PROOF. The "if" part is trivial.
"only if": let the operator ψ be $\downarrow$-continuous and let X_n be an arbitrary sequence in $\mathcal{L}$. Define $Y_n := \vee_{k\geq n} X_k$, then $Y_n \downarrow \limsup X_n$. This implies that

$$\psi(\limsup X_n) = \bigwedge_{n\geq 1} \psi(Y_n) = \bigwedge_{n\geq 1} \psi(\bigvee_{k\geq n} X_k)$$
$$\geq \bigwedge_{n\geq 1} \bigvee_{k\geq n} \psi(X_k) = \limsup \psi(X_n).$$

This proves the result.

∎

Corollary 3.5.

(a) *Any erosion is* $\downarrow$*-continuous.*

(b) *Any dilation is* $\uparrow$*-continuous.*

(c) *Any automorphism is continuous.*

Proposition 3.6. *The infimum of any collection of* $\downarrow$*-continuous operators is* $\downarrow$*-continuous.*

PROOF. Let, for every i in the index set I, the operator ψ_i be $\downarrow$-continuous. Define $\psi = \bigwedge_{i\in I}\psi_i$. Let $X_n \to X$, we must show that $\limsup \psi(X_n) \leq \psi(X)$. For every $i \in I$ we have

$$\bigwedge_{N\geq 1}\bigvee_{n\geq N}\psi_i(X_n) \leq \psi_i(X),$$

which yields that for every i

$$\bigwedge_{N\geq 1}\bigvee_{n\geq N}\bigwedge_{i\in I}\psi_i(X_n) \leq \psi_i(X),$$

and hence that $\limsup \psi(X_n) \leq \psi(X)$.

∎

One can easily construct examples which show that a similar result for suprema does not hold in general.

Definition 3.7. An element X in the complete lattice $\mathcal{L}$ is called a *co-prime* (see [4]) if $X \leq X_1 \vee X_2 \vee \ldots \vee X_p$ for some $X_1, X_2, \ldots, X_p \in \mathcal{L}$ implies that $X \leq X_k$ for some k between 1 and p.

Every atom of $\mathcal{L}$ is also a co-prime, but not conversely. (Recall that a nonzero element $X \in \mathcal{L}$ is an atom if $Y \leq X$ implies that $Y = \mathcal{O}$ or $Y = X$.)

Proposition 3.8. *Let* $\mathcal{L}$ *be a complete lattice possessing a sup-generating family of co-primes (i.e., every element of* $\mathcal{L}$ *can be written as a supremum of co-primes). Then any finite supremum of* $\downarrow$*-continuous operators is* $\downarrow$*-continuous.*

PROOF. Let, for $i = 1, \ldots, p$ the operator ψ_i be $\downarrow$-continuous. We show that $\psi = \bigvee_{i=1}^{p}\psi_i$ is $\downarrow$-continuous. Let $X_n \to X$, we must show that

$$\bigwedge_{N\geq 1}\bigvee_{n\geq N}\psi(X_n) \leq \psi(X).$$

It is sufficient to prove that for any co-prime Y which satisfies $Y \leq \bigwedge_{N\geq 1}\bigvee_{n\geq N}\psi(X_n)$ one also gets that $Y \leq \psi(X)$. The first inequality implies that

$$Y \leq \bigvee_{n\geq N}\bigvee_{i=1}^{p}\psi_i(X_n) = \bigvee_{i=1}^{p}\bigvee_{n\geq N}\psi_i(X_n),$$

for every $N \geq 1$. Since Y is a co-prime we may conclude that for every $N \geq 1$ there exists an i_N between 1 and p such that

$$Y \leq \bigvee_{n\geq N}\psi_{i_N}(X_n).$$

Then at least one i must occur infinitely often among the i_N, $N = 1, 2, \ldots$. This implies that for this value of i,

$$Y \le \bigvee_{n \ge N} \psi_i(X_n),$$

for all $N \ge 1$. By the $\downarrow$-continuity of ψ_i we get that

$$Y \le \limsup \psi_i(X_n) \le \psi_i(X) \le \psi(X),$$

and the result is proved.

∎

As an example of a complete lattice which has a sup-generating family of co-primes (besides the atomic lattice $\mathcal{P}(E)$) we mention the lattice of functions from some space E to the extended real line: here the co-primes are the pulse functions $f_{x,t}$ which take the value t at the point x and which are $-\infty$ elsewhere.

One can easily show that the latter proposition more generally holds for the case where $\mathcal{L}$ is a complete sublattice of some complete lattice $\mathcal{L}_0$ which possesses a sup-generating family of co-primes.

In general, composition of two $\downarrow$-continuous operators does not yield a $\downarrow$-continuous operator as we show in Example 3.12 below. But one can establish the following result.

Proposition 3.9. *Let ψ_1, ψ_2 be arbitrary operators. Under either of the following assumptions the composition $\psi_2\psi_1$ is $\downarrow$-continuous:*
(a) ψ_1 is continuous and ψ_2 is $\downarrow$-continuous.
(b) ψ_1 is $\downarrow$-continuous and ψ_2 is increasing and $\downarrow$-continuous.

PROOF.
(a): trivial.
(b): let $X_n \to X$. Then $\limsup \psi_1(X_n) \le \psi_1(X)$. Using the increasingness of ψ_2 and Proposition 3.4 we get that

$$\limsup \psi_2\psi_1(X_n) \le \psi_2(\limsup \psi_1(X_n)) \le \psi_2\psi_1(X).$$

∎

Finally we state some easy results which hold in the case that the underlying lattice $\mathcal{L}$ is Boolean. Note that in this case we have

$$\limsup X_n^* = (\liminf X_n)^*$$
$$\liminf X_n^* = (\limsup X_n)^*.$$

Suppose that $\mathcal{L}$ is a complete lattice which is a sublattice of a Boolean lattice $\mathcal{L}_0$. Let the complement of an element $X \in \mathcal{L}$ be given by X^*. We define

$$\mathcal{L}^* = \{X^* \mid X \in \mathcal{L}\}.$$

Then $\mathcal{L}^*$ becomes a complete lattice under the order $\le^*$ given by

$$X^* \le^* Y^* \quad \text{iff} \quad X \ge Y,$$

with supremum $\vee^*$ and infimum $\wedge^*$ given by $\bigvee^*_{i \in I} X_i^* = (\bigwedge_{i \in I} X_i)^*$ and $\bigwedge^*_{i \in I} X_i^* = (\bigvee_{i \in I} X_i)^*$.

Proposition 3.10. *Let $\mathcal{L}_0$ be a complete Boolean lattice, let $\mathcal{L}$ be a complete sublattice, and let $\mathcal{L}^*$ be as above. Furthermore, let $\mathcal{K}$ be any other complete lattice. If the operator $\psi : \mathcal{K} \to \mathcal{L}$ is $\uparrow$-continuous, then the operator ψ' form $\mathcal{K}$ into $\mathcal{L}^*$ given by $\psi'(X) = (\psi(X))^*$ is $\downarrow$-continuous (and vice versa).*

The proof is straightforward.

Corollary 3.11. *Let $\mathcal{L}$ be a complete Boolean lattice.*

(a) The operator $X \mapsto X^*$ is continuous.

(b) If the operator ψ is $\downarrow$-continuous then the dual operator ψ^* is $\uparrow$-continuous and vice versa.

(c) If ψ is $\downarrow$-continuous then the operator $X \mapsto (\psi(X))^*$ is $\uparrow$-continuous and vice versa.

Example 3.12.

(a) Let $\mathcal{L}$ be the Boolean lattice $\mathcal{P}(E)$, where E is $\mathbb{R}^d$ or $\mathbb{Z}^d$. We first show that the operator given by the Minkowski addition with an infinite structuring element is **not** $\downarrow$-continuous (since it is a dilation it is $\uparrow$-continuous). Let $A \subseteq E$ with $|A| = \infty$: here $|A|$ denotes the number of elements in A. Let $a_n \in A$ all be different, and define $X_n = \{-a_n, -a_{n+1}, \ldots\}$. Then $X_n \downarrow \emptyset$, hence $X_n \to \emptyset$. But $0 \in X_n \oplus A$ for all n, hence $0 \in \limsup(X_n \oplus A)$. This proves that the operator $X \mapsto X \oplus A$ is not $\downarrow$-continuous: see also Figure 1.

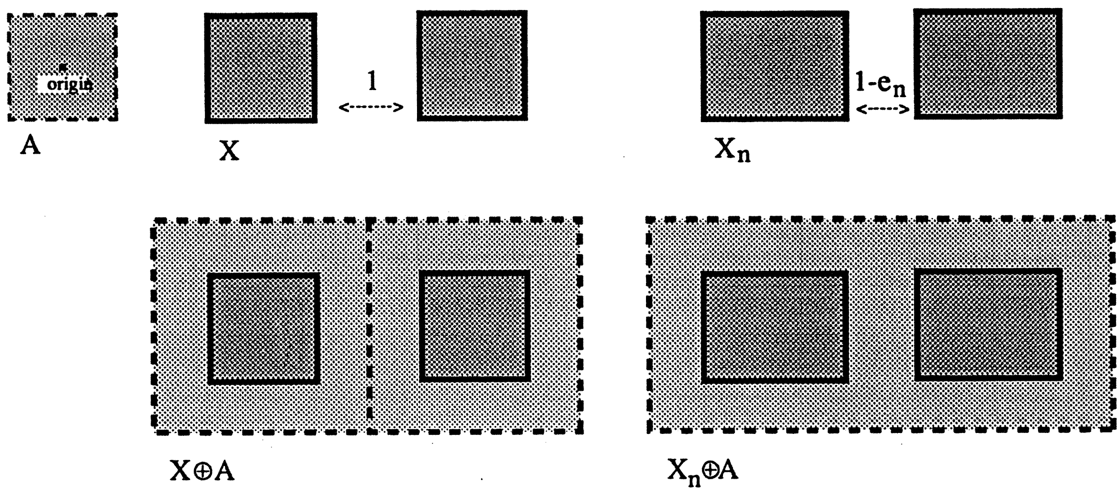

FIGURE 1. *Dilation with an infinite structuring element is not $\downarrow$-continuous. Let X_n consist of two rectangles at distance $1 - e_n$, where $e_n \downarrow 0$ as $n \to \infty$. Assume that the position of the utmost left and right border of X_n are independent of n. Then $X_n \downarrow X$, where X consists of two squares at distance 1. If the structuring element A is the open square with sides 1, then $\bigcap_{n \geq 1} X_n \oplus A$ is the rectangle with width 3 and height 1, whereas $X \oplus A$ is the smaller set obtained by deleting the vertical edge in the middle.*

(b) Using the previous example it is easy to find a $\downarrow$-continuous operator ψ_1 and a $\downarrow$-continuous operator ψ_2 such that their composition $\psi_2\psi_1$ is not $\downarrow$-continuous.
Let A be an infinite structuring element in E. From Proposition 3.9(a) it follows that the non-increasing operator $\psi_1 : X \mapsto X^c \ominus \check{A}$ (where X^c is the complement of X, and $\check{A}$ is the reflected structuring element) is $\downarrow$-continuous. By Corollary 3.11(a), the operator $\psi_2 : X \mapsto X^c$ is continuous. Using the identity

$$(X^c \ominus \check{A})^c = X \oplus A$$

we get that $\psi_2\psi_1(X) = X \oplus A$, which, by the above example, does not define a $\downarrow$-continuous operator.

Example 3.13. In this example we take $\mathcal{L} = \mathcal{F}(\mathbb{R}^d)$, the closed subsets of $\mathbb{R}^d$. Here the translation-invariant dilations and erosions are given by (c.f. [19, Subsection 4.2]):

$$X \overline{\oplus} A = \overline{X \oplus A} = \overline{\bigcup_{a \in A} X_a}$$
$$X \overline{\ominus} A = X \ominus A = \bigcap_{a \in A} X_{-a}.$$

From Corollary 3.5 we know that dilation (resp. erosion) defines a $\uparrow$-continuous (resp. $\downarrow$-continuous) operator. Furthermore, if A is compact, then $X \mapsto X \oplus A$ is also $\downarrow$-continuous, and hence continuous. Namely, let $X_n \downarrow X$, we must show that $X_n \oplus A \downarrow X \oplus A$. Here we have used that $X \overline{\oplus} A = X \oplus A$ if A is compact. We are done if we can prove that $\bigcap_{n \geq 1} X_n \oplus A \subseteq X \oplus A$. Let $y \in \bigcap_{n \geq 1} X_n \oplus A$. So for every $n \geq 1$ there is a $x_n \in X_n$ and $a_n \in A$ such that $y = x_n + a_n$. Since A is compact, $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$. Let $a_{n_k} \to a$ as $k \to \infty$. Then $x_{n_k} \to y - a$. Since for every p, $x_{n_k} \in X_p$ for k large enough and since X_p is closed we get that $y - a \in X_p$ and hence that $y - a \in X$. Then $y = (y-a) + a \in X \oplus A$. This proves the assertion.

On the other hand it is easy to find compact structuring elements A such that the erosion $X \mapsto X \ominus A$ is not $\uparrow$-continuous.

From Proposition 3.9 we now conclude that the closing $X \mapsto X^A$ and the opening $X \mapsto X_A$ are also $\downarrow$-continuous if A is compact.

4. Relations with the hit-or-miss topology

In this section we shall be concerned with convergence and continuity on the complete lattice $\mathcal{L} = \mathcal{F}(E)$, the closed subsets of E. Here E is a topological space which is Hausdorff, locally compact, and admits a countable base. On this space one can define a topology, named the *hit-or-miss topology* by Matheron in [14]. This topology is Hausdorff, compact, and has a countable base. The latter means that this topology is completely determined by specifying convergence of sequences. It turns out that convergence and (semi-) continuity with respect to the hit-or-miss topology is closely related to our notions of order convergence and continuity, though they are not completely equivalent: see in particular Propositions 4.2 and 4.7.

In this section we shall write $\mathcal{F}-\limsup$ (resp. $\mathcal{F}-\liminf$) for the $\limsup$ and $\liminf$ in the lattice $\mathcal{F}(\mathbb{R}^d)$ to emphasize the dependence on the underlying lattice $\mathcal{F}(E)$. Thus we have

$$\mathcal{F}-\limsup X_n = \bigcap_{N\geq 1} \overline{(\bigcup_{n\geq N} X_n)}$$
$$\mathcal{F}-\liminf X_n = \overline{\bigcup_{N\geq 1} (\bigcap_{n\geq N} X_n)}.$$

We first recall the definition and some properties of the hit-or-miss topology.

Throughout this section we assume that E is a topological space which is Hausdorff, locally compact, and admits a countable base. We denote by $\mathcal{F}(E), \mathcal{G}(E)$, and $\mathcal{K}(E)$ respectively the space of all closed, open, and compact subsets of E. The family of subsets of $\mathcal{F}(E)$ given by

$$\mathcal{F}^K_{G_1,G_2,\ldots,G_p} = \{F \in \mathcal{F}(E) \mid F \cap K = \emptyset \text{ and } F \cap G_i \neq \emptyset,\ i = 1, \ldots p\}$$

defines a base for a topology on $\mathcal{F}(E)$, called *the hit-or-miss topology*: see [14]. In the literature this topology is also known as the *Fell topology*: see [18]. The hit-or-miss topology on the space $\mathcal{F}(E)$ is compact, Hausdorff, and has a countable base. So in order to study continuity properties of operators on $\mathcal{F}(E)$ we may restrict ourselves to convergence of sequences. If a sequence X_n in $\mathcal{F}(E)$ converges to X with respect to the hit-or-miss topology the we write $X_n \xrightarrow{\tau} X$ to distinguish it from the order convergence denoted by $X_n \to X$.

It follows immediately that $X_n \xrightarrow{\tau} X$ if and only if

(A) for every open set G, $X \cap G \neq \emptyset \implies X_n \cap G \neq \emptyset$, for n large enough
(B) for every compact set K, $X \cap K = \emptyset \implies X_n \cap K = \emptyset$, for n large enough.

Following Matheron [14] we define $\underline{\lim}\ X_n$ as the largest closed set satisfying (A), and $\overline{\lim}\ X_n$ as the smallest closed set satisfying (B). The following results have been obtained by Matheron [14].

Proposition 4.1. *Let X_n be a sequence in $\mathcal{F}(E)$.*
(a) $\underline{\lim}\ X_n \subset \overline{\lim}\ X_n$
(b) $X_n \xrightarrow{\tau} X$ *if and only if* $\underline{\lim}\ X_n = \overline{\lim}\ X_n = X$.
(c) $\overline{\lim}\ X_n = \bigcap_{N\geq 1} \overline{\bigcup_{n\geq N} X_k}$.
(d) $x \in \underline{\lim}\ X_n$ *if and only if there is a sequence* $x_n \in X_n$ *such that* $x_n \to x$.
(e) $x \in \overline{\lim}\ X_n$ *if and only if there is a subsequence* $x_{n_k} \in X_{n_k}$ *such that* $x_{n_k} \to x$.

There is a close relation between Matheron's concepts $\underline{\lim}$ and $\overline{\lim}$ and our definition of $\mathcal{F}-\limsup$ and $\mathcal{F}-\liminf$, which is expressed by the following result.

Proposition 4.2. *Let X_n be an arbitrary sequence in $\mathcal{F}(E)$. Then*

$$\mathcal{F}-\limsup X_n = \overline{\lim}\ X_n \tag{4.1}$$

$$\mathcal{F}-\liminf X_n \subseteq \underline{\lim}\ X_n. \tag{4.2}$$

PROOF. The first relation is a reformulation of Proposition 4.1(c). To prove the second we show that $\mathcal{F}-\liminf X_n$ satisfies (A). Then from the definition of $\underline{\lim}\ X_n$, being the largest closed set satisfying (A), we may conclude that (4.2) holds. Let G be open and assume that

$$\mathcal{F}-\liminf X_n \cap G \neq \emptyset,$$

that is,

$$\overline{\bigcup_{N\geq 1}(\bigcap_{n\geq N} X_n)}\bigcap G \neq \emptyset.$$

But then also

$$\bigcup_{N\geq 1}(\bigcap_{n\geq N} X_n)\bigcap G \neq \emptyset,$$

or equivalently,

$$\bigcup_{N\geq 1}\bigcap_{n\geq N}(X_n \cap G) \neq \emptyset.$$

So there is an $N \geq 1$ such that $X_n \cap G \neq \emptyset$, for all $n \geq N$. This concludes the proof. ∎

The following is a straightforward consequence of relations (4.1) and (4.2).

Corollary 4.3. *If $X_n \to X$ then $X_n \xrightarrow{\mathcal{T}} X$.*

Example 4.4. Let $\mathcal{L} = \mathcal{F}(\mathbb{R}^2)$ and take X_n to be the circle with center $(0,0)$ and radius $1-\frac{1}{n}$, and X the circle with radius 1. Then $\mathcal{F}-\limsup X_n = \overline{\lim}\ X_n = X$, $\underline{\lim}\ X_n = X$ and $\mathcal{F}-\liminf X_n = \emptyset$: see also Figure 2. In particular, $X_n \xrightarrow{\mathcal{T}} X$ whereas X_n does not converge relative to the order topology. This example shows that the inclusion in (4.2) may be strict and that the converse of Corollary 4.3 needs not hold.

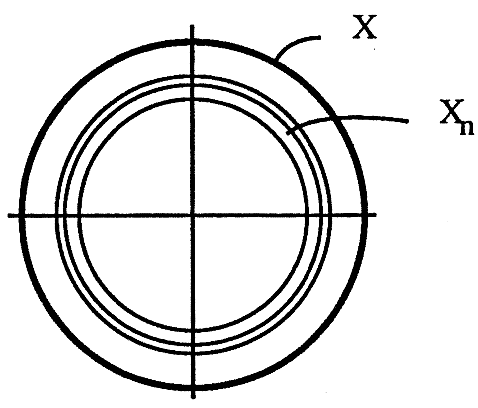

FIGURE 2. *For an explanation see Example 4.4.*

In Remark 4.8 below we consider the case that the topology on E is trivial, that is, every subset of E is open. In this case the inclusion in (4.2) also becomes an equality, and hence the topological and algebraic notion of convergence coincide.

Definition 4.5. An operator $\psi : \mathcal{F}(E) \to \mathcal{F}(E)$ is called *upper-semi-continuous* (u.s.c.) if $X_n \xrightarrow{\mathcal{T}} X$ implies that $\overline{\lim}\ \psi(X_n) \subseteq \psi(X)$. It is called *lower-semi-continuous* (l.s.c.) if $X_n \xrightarrow{\mathcal{T}} X$ implies that $\psi(X) \subseteq \underline{\lim}\ \psi(X_n)$.

This definition as well as the following result is due to Matheron [14].

Proposition 4.6. *An increasing operator ψ on $\mathcal{F}(E)$ is u.s.c. if and only if $X_n \downarrow X$ implies that $\psi(X_n) \downarrow \psi(X)$.*

Proposition 4.7. *Let ψ be an arbitrary operator on $\mathcal{F}(E)$.*

(a) If ψ is u.s.c. then ψ is $\downarrow$-continuous.

(b) If ψ is increasing and $\downarrow$-continuous then ψ is u.s.c.

(c) If ψ is increasing and l.s.c. then ψ is $\uparrow$-continuous.

PROOF.
(a): follows easily with Corollary 4.3 and relation (4.1).
(b): use Proposition 3.3 and Proposition 4.6.
(c): we use the analogue of Proposition 3.3 for $\uparrow$-continuous operators. Assume $X_n \uparrow X$, we must show that $\psi(X_n) \uparrow \psi(X)$. Since the sequence $\psi(X_n)$ is increasing we get that

$$\underline{\lim}\ \psi(X_n) = \overline{\bigcup_{n\geq 1} \psi(X_n)}.$$

From the fact that $X_n \xrightarrow{\mathcal{T}} X$ we may conclude that

$$\psi(X) \subseteq \underline{\lim}\ \psi(X_n) = \overline{\bigcup_{n\geq 1} \psi(X_n)}.$$

Since $X_n \subseteq X$ and ψ is increasing also the reverse inequality holds, and therefore

$$\psi(X) = \overline{\bigcup_{n\geq 1} \psi(X_n)}.$$

This yields that $\psi(X_n) \uparrow \psi(X)$.

∎

Remark 4.8. We can specialize the results of this section to the case where E is an arbitrary countable space and has the trivial topology, that is, every subset of E is open, and hence closed. Obviously, $\mathcal{K}(E)$ now consists of all finite subsets of E. Furthermore, for any sequence x_n in E we have $x_n \to x$ if and only if $x_n = x$ eventually. We show that under the given assumptions, the topological notions $\overline{\lim}$ and $\underline{\lim}$ respectively coincide with their algebraic counterparts $\mathcal{F} - \limsup$ and $\mathcal{F} - \liminf$. This means in particular that $X_n \to X$ if and only if $X_n \xrightarrow{\mathcal{T}} X$, for any sequence X_n in $\mathcal{P}(E)$. From Proposition 4.2 it is clear that we have only to show that $\underline{\lim}\ X_n \subseteq \liminf X_n$. Let $x \in \underline{\lim}\ X_n$. From Proposition 4.1(d) we know that there is a sequence $x_n \in X_n$ such that $x_n \to x$, that is, $x_n = x$ eventually. This implies that $x \in X_n$ eventually. But then $x \in \bigwedge_{n \geq N} X_n$ for N large enough, whence it follows that $x \in \liminf X_n$. This proves the assertion.

Note also that finite intersection which turns out to be continuous for the sequential order convergence, at least if the conditions of Proposition 3.8 are fulfilled, is only upper-semi-continuous on $\mathcal{F}(E)$ supplied with the hit-or-miss topology.

5. Convergence and continuity on the Boolean lattice $\mathcal{P}(E)$

Let E be an arbitrary set and consider the complete Boolean atomic lattice $\mathcal{P}(E)$. Note that the only co–primes are the atoms (singletons of the form $\{x\}$) and that the assumption of Proposition 3.8 is satisfied. This means that every finite supremum of $\downarrow$-continuous operators is $\downarrow$-continuous, and dually, that every finite infimum of $\uparrow$-continuous operators is $\uparrow$-continuous.

We have the following expressions for $\limsup X_n$ and $\liminf X_n$. We point out that in [9] one can find similar expressions.

$$\limsup X_n = \{x \in E \mid x \in X_{n_k} \text{ for some subsequence } n_k \to \infty\} \tag{5.1}$$

$$\liminf X_n = \{x \in E \mid x \in X_n \text{ eventually}\}. \tag{5.2}$$

Throughout the remainder of this section we shall restrict to the case where $E = \mathbb{R}^d$ or $\mathbb{Z}^d$. We are particularly interested in continuity properties of the basic operations of mathematical morphology on the space $\mathcal{P}(E)$. We have already seen in Corollary 3.5 that every dilation is $\uparrow$-continuous, and that every erosion is $\downarrow$-continuous. For finite structuring elements we can prove a lot more.

Proposition 5.1. *Let A be a finite structuring element. Then the dilation $X \mapsto X \oplus A$, the erosion $X \mapsto X \ominus A$, the closing $X \mapsto X^A$, and the opening $X \mapsto X_A$ are all continuous.*

PROOF. Since every translation operator $X \mapsto X_h$ on $\mathcal{P}(E)$ is obviously an automorphism, and therefore continuous, we may conclude from Proposition 3.8 that any dilation and erosion with a finite structuring element is $\downarrow$-continuous respectively $\uparrow$-continuous. The results for the closing and the opening follow now immediately from Proposition 3.9.

∎

Let $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ be an arbitrary operator. For $h \in E$ we define the kernel of ψ at h by $\mathcal{V}_h(\psi) = \{A \in \mathcal{P}(E) \mid h \in \psi(A)\}$. Obviously, $\psi(X) = \{h \in E \mid X \in \mathcal{V}_h(\psi)\}$. If ψ is translation-invariant (e.g. $E = \mathbb{R}^d$ or $\mathbb{Z}^d$) then $\mathcal{V}_h(\psi) = (\mathcal{V}_0(\psi))_h$. We have the following result (compare [14, Proposition 8-2-1]).

Proposition 5.2. *Let ψ be an arbitrary operator. Then ψ is $\downarrow$-continuous if and only if for every $h \in E$ the kernel $\mathcal{V}_h(\psi)$ is closed under* $\limsup$ *(i.e., if $X_n \in \mathcal{V}_h(\psi)$ for all $n \geq 1$, then $\limsup X_n \in \mathcal{V}_h(\psi)$).*

PROOF.
"only if": easy.
"if": let $X_n \to X$. We must show that $\limsup \psi(X_n) \subseteq \psi(X)$. Let $h \in \limsup \psi(X_n)$, hence $h \in \psi(X_{n_k})$ for some subsequence n_k going to infinity. So $X_{n_k} \in \mathcal{V}_h(\psi)$, and since $\mathcal{V}_h(\psi)$ is closed under lim sup we get that $X = \limsup X_{n_k} \in \mathcal{V}_h(\psi)$ which is equivalent to $h \in \psi(X)$.

∎

Example 5.3. In Example 3.12(a) we have seen that the condition that A is finite is necessary for the dilation $X \mapsto X \oplus A$ to be continuous. It is easy to build examples which show that the finiteness of A is required to prove $\uparrow$-continuity of the opening $X \mapsto X_A$. It turned out less trivial to find an example (with A infinite) such that this opening is not $\downarrow$-continuous. For completeness we present such an example. Let

$$A = \{0, 1, 3, 7, 15, \ldots\} = \{\sum_{k=0}^{n-1} 2^k \mid n \geq 0\},$$

and let for $n \geq 0$ the set X_n be given by

$$X_n = \{\ldots, -2, -1, 0, 2^n, 2 \cdot 2^n, 3 \cdot 2^n, \ldots\}.$$

It is easy to check that $0 \in (X_n)_A$ for every $n \geq 0$. Therefore $0 \in \limsup X_n$. However

$$X_n \to X = \{\ldots, -2, -1, 0\},$$

and $X_A = \emptyset$. Thus the opening $X \mapsto X_A$ is not $\downarrow$-continuous.

Proposition 5.4. *Let $A, B \subset E$.*

(a) The hit-or-miss operator $X \mapsto X \circledast (A, B)$ is $\downarrow$-continuous. If both A and B are finite then it is continuous.

(b) The thickening operator $X \mapsto X \odot (A, B)$ is $\downarrow$-continuous. If both A and B are finite then it is continuous.

(c) The thinning operator $X \mapsto X \circ (A, B)$ is $\uparrow$-continuous. If both A and B are finite then it is continuous.

PROOF.
(a): since $X \mapsto X \ominus A$ is an erosion it is $\downarrow$-continuous. From Example 3.12(b) we know that $X \mapsto X^c \ominus B$ is $\downarrow$-continuous as well, and from Proposition 3.6 we conclude that

$$X \mapsto X \circledast (A, B) = (X \ominus A) \cap (X^c \ominus B)$$

is $\downarrow$-continuous. The continuity for finite A, B can be proved similarly if one uses Proposition 5.1.
(b): is now trivial.
(c): use (a) and (b) above together with Corollary 3.11.

∎

From this result and relation (1.12) it follows that $X \circledcirc (A, B)$ is $\downarrow$-continuous; it is continuous if both A and B^c are finite.

6. Finite operators

Like in the previous section, we shall deal exclusively with operators on the Boolean lattice $\mathcal{P}(E)$, where $E = \mathbb{R}^d$ or $\mathbb{Z}^d$.

Definition 6.1. Let $M : E \to \mathcal{P}(E)$. We say that the operator $\psi : \mathcal{P}(E) \to \mathcal{P}(E)$ is *finite with mask* M if for every $h \in E$ there exists a finite subset $M(h) \subset E$ such that for every $X \in \mathcal{P}(E)$ and for every $N \supset M(h)$ we have

$$h \in \psi(X) \Leftrightarrow h \in \psi(X \cap N).$$

It should be remarked that for increasing operators this definition coincides with the definition given in [19]. If the operator ψ is translation invariant then one may assume that $M(h) = M_h$, i.e., the translate of some fixed mask $M \subset E$.

The result to be given next can be considered the main result of this section.

Proposition 6.2. *Every finite operator on $\mathcal{P}(E)$ is continuous.*

The proof of this result requires the following lemma.

Lemma 6.3. *Let C be a subset of E. Then $X_n \to X$ implies that $X_n \cap C \to X \cap C$. If, furthermore, C is finite, then $X_n \cap C = X \cap C$ eventually.*

PROOF. Let X_n be a sequence in $\mathcal{P}(E)$. Then

$$\begin{aligned}\limsup(X_n \cap C) &= \bigcap_{N\geq 1}\bigcup_{n\geq N}(X_n \cap C) = \bigcap_{N\geq 1}\Big(\bigcup_{n\geq N} X_n\Big)\cap C \\ &= (\limsup X_n) \cap C,\end{aligned}$$

and the same argument applies to liminf. This proves the first assertion.

The second assertion follows easily if one uses expressions (5.1) and (5.2). This completes the proof.

∎

PROOF OF PROPOSITION 6.2: let ψ be finite. We prove that only that ψ is $\downarrow$-continuous: the proof of the $\uparrow$-continuity follows similarly.
Let $X_n \to X$: we show that $\limsup \psi(X_n) \leq \psi(X)$. Let $h \in \bigwedge_{N\geq 1}\bigvee_{n\geq N}\psi(X_n)$. We show that $h \in \psi(X)$. We first observe that $h \in \psi(X_{n_k})$ for some subsequence n_k tending to ∞. But ψ is finite which means that $h \in \psi(X_{n_k} \cap M(h))$. By the previous lemma, $X_{n_k} \cap M(h) = X \cap M(h)$ eventually, which yields that $h \in \psi(X \cap M(h))$. Then, since ψ is finite, it follows that $h \in \psi(X)$.

∎

The remainder of this section will be mainly concerned with an investigation of properties of finite operators.

Proposition 6.4. *The operator ψ is finite if and only if its dual operator ψ^* is finite.*

PROOF. Let ψ be finite with mask M. We show that ψ^* is finite also with mask M. Let $X \in \mathcal{P}(E)$ and $N \supset M(h)$. We must show that

$$h \in \psi^*(X) \Leftrightarrow h \in \psi^*(X \cap N).$$

We know that

$$h \in \psi(X) \Leftrightarrow h \in \psi(X \cap N).$$

From this, it follows easily that

$$\begin{aligned} h \in \psi^*(X) &\Leftrightarrow \notin \psi(X^c) \\ &\Leftrightarrow h \notin \psi(X^c \cap N) \\ &\Leftrightarrow h \notin \psi((X \cup N^c)^c) \\ &\Leftrightarrow h \in \psi(X \cup N^c). \end{aligned}$$

From the fact that $X \cup N^c = (X \cap N) \cup N^c$ and the previous relation we find that

$$h \in \psi^*(X) \Leftrightarrow h \in \psi^*((X \cap N) \cup N^c) \Leftrightarrow h \in \psi^*(X \cap N).$$

This completes the proof.

∎

The following result is easy and therefore stated without proof.

Proposition 6.5. *Let for every i in the index set I, the operator ψ_i be finite with mask M_i. If $M(h) := \bigcup_{i \in I} M_i(h)$ is finite for every $h \in E$, then $\bigcap_{i \in I} \psi_i$ and $\bigcup_{i \in I} \psi_i$ are finite, both with mask M.*

An immediate consequence of this result is that *finite* unions and intersections of finite operators are finite. The next result states that a similar remark applies to finite compositions.

Proposition 6.6. *Let ψ_1, ψ_2 be finite operators with respective masks M_1 and M_2. Then $\psi_2\psi_1$ is a finite operator with mask M given by $M(h) = \bigcup_{k \in M_2(h)} M_1(k)$.*

PROOF. Let M be as defined, take $h \in E$ and assume that $M(h) \subseteq N$. We first show that for every set X,

$$\psi_1(X) \cap M_2(h) = \psi_1(X \cap N) \cap M_2(h). \tag{$*$}$$

Namely, take $k \in M_2(h)$. Then, since $M(h) \subseteq N$, it follows that $M_1(k) \subseteq N$. Therefore $k \in \psi_1(X)$ iff $k \in \psi_1(X \cap N)$, or equivalently, $\psi_1(X) \cap \{k\} = \psi_1(X \cap N) \cap \{k\}$. Taking on both sides the union over all $k \in M_2(h)$, we find (*). To complete the proof, observe that

$$\begin{aligned} h \in \psi_2(\psi_1(X)) &\Leftrightarrow h \in \psi_2(\psi_1(X) \cap M_2(h)) \\ &\Leftrightarrow h \in \psi_2(\psi_1(X \cap N) \cap M_2(h)) \\ &\Leftrightarrow h \in \psi_2(\psi_1(X \cap N)). \end{aligned}$$

∎

Remark 6.7. Our definition of finite operator much resembles Serra's so-called *local knowledge principle*: see [21, Section I.B.3, p.11]. This becomes even more apparent if we use the following equivalent characterization of a finite operator. The operator ψ is finite if and only if for every finite subset U of E there is finite subset M_U such that for all $N \supseteq M_U$ we have

$$\psi(X) \cap U = \psi(X \cap N) \cap U,$$

for every $X \in \mathcal{P}(E)$. It is clear that every operator with this property is finite: just choose $U = \{h\}$. To prove the converse, consider a finite operator ψ and let U be finite. Define $M_U := \bigcup_{h \in U} M(h)$. Then one can easily show that the property mentioned above is satisfied

Obviously, erosions and dilations with finite structuring elements are finite operators. Therefore Proposition 6.2 provides an alternative proof of Proposition 5.1 and the statements of Proposition 5.4 concerning finite structuring elements. We conclude this section by mentioning an easy but useful result stated in [5].

Proposition 6.8. *Every increasing, translation-invariant, finite operator can be written as a finite union of finite erosions, or, alternatively, as a finite intersection of finite dilations.*

PROOF. We only prove the first statement. Let ψ be an increasing, translation-invariant, finite operator, and let M be the mask belonging to ψ. Since ψ is translation-invariant we may assume that $M(h) = M_h$, with M finite. Let $\mathcal{V}$ be the kernel of ψ and define $\mathcal{V}_0 := \{A \cap M \mid A \in \mathcal{V}\}$. Since $0 \in \psi(A)$ iff $0 \in \psi(A \cap M)$, we get that $\mathcal{V}_0 \subset \mathcal{V}$. It is obvious that

$$\psi(X) = \bigcup_{A \in \mathcal{V}} X \ominus A = \bigcup_{A_0 \in \mathcal{V}_0} X \ominus A_0.$$

Since M is finite, every $A_0 \in \mathcal{V}_0$ is also finite, and the proof is completed.

∎

In the terminology of Maragos this result says that any increasing, translation-invariant, finite operator has a finite basis: see [12].

7. Iteration and morphological filtering

Throughout this section we let $\mathcal{L}$ be an arbitrary complete lattice. Recall that $\mathcal{O}_+(\mathcal{L})$ denotes the complete lattice of all increasing operators on $\mathcal{L}$.

Definition 7.1. Let ψ be an increasing operator on $\mathcal{L}$. We say that ψ is

(a) a *(morphological) filter* if $\psi^2 = \psi$
(b) an *underfilter* if $\psi^2 \leq \psi$
(c) an *overfilter* if $\psi^2 \geq \psi$
(d) an *inf-overfilter* if $\psi = \psi(\mathrm{id} \wedge \psi)$
(e) a *sup-underfilter* if $\psi = \psi(\mathrm{id} \vee \psi)$
(f) a *strong filter* if ψ is both an inf-overfilter and a sup-underfilter.

For an extensive account on the theory of morphological filtering we refer to [22], in particular Chapter 6 written by G. Matheron: see also [24]. Note that every inf-overfilter (sup-underfilter) is an overfilter (underfilter). Important examples of strong filters are the openings and closings. The class of underfilters as well as that of sup-underfilters is closed under infima. On the other hand, both the class of overfilters and that of inf-overfilters is closed under suprema. All four classes are closed under selfcomposition, that is, if ψ belongs to one of those classes then so does ψ^n for $n \geq 2$. We define $\hat{\psi}$ to be the smallest closing $\geq \psi$. Note that $\hat{\psi}$ is uniquely determined by its domain of invariance

$$\mathsf{Inv}(\hat{\psi}) = \mathsf{Inv}(\mathsf{id} \vee \psi) = \{X \mid \psi(X) \leq X\}$$

which is inf-closed (i.e., closed under arbitrary infima). It follows that

$$\psi\hat{\psi} \leq \hat{\psi}. \tag{7.1}$$

Similarly we define $\check{\psi}$ as the largest opening $\leq \psi$. Then

$$\mathsf{Inv}(\check{\psi}) = \mathsf{Inv}(\mathsf{id} \wedge \psi) = \{X \mid \psi(X) \geq X\},$$

and

$$\psi\check{\psi} \geq \check{\psi}. \tag{7.2}$$

The mappings given by $\psi \mapsto \hat{\psi}$ and $\psi \mapsto \check{\psi}$ define a closing resp. an opening on the complete lattice $\mathcal{O}_+(\mathcal{L})$. We recall some facts from [22, Chapter 6]. The lattice $\mathcal{L}$ is *modular* if for any $A, B, X \in \mathcal{L}$,

$$A \leq B \implies B \wedge (A \vee X) = A \vee (B \wedge X).$$

Any distributive lattice is modular, but the converse is not true in general.

Proposition 7.2. *Let ψ be an increasing operator on $\mathcal{L}$.*

(a) $\hat{\psi}\psi$ is an underfilter. It is the smallest underfilter $\geq \psi$.
(b) $\check{\psi}\psi$ is an overfilter. It is the largest overfilter $\leq \psi$.
(c) If ψ is an overfilter then $\hat{\psi}\psi$ is a filter.
(d) If ψ is an underfilter then $\check{\psi}\psi$ is a filter.
(e) $\psi\hat{\psi}$ is a sup-underfilter. It is the smallest sup-underfilter $\geq \psi$.
(f) $\psi\check{\psi}$ is an inf-overfilter. It is the largest inf-overfilter $\leq \psi$.

Assume furthermore that $\mathcal{L}$ is modular.

(g) If ψ is an inf-overfilter then $\psi\hat{\psi}$ is a strong filter.
(h) If ψ is a sup-underfilter then $\psi\check{\psi}$ is a strong filter.

PROOF. We prove (a),(c),(e),(g). The other results follow by duality.
(a): from (7.1) we get that $\hat{\psi}\psi\hat{\psi}\psi \leq \hat{\psi}^2\psi = \hat{\psi}\psi$, so $\hat{\psi}\psi$ is an underfilter. Let ϕ be an underfilter $\geq \psi$. Then $\phi^2 \leq \phi$, hence $\phi(X) \in \mathsf{Inv}(\hat{\phi}) = \{X \mid \phi(X) \leq X\}$, for any $X \in \mathcal{L}$. So $\phi = \hat{\phi}\phi \geq \hat{\psi}\psi$.
(c): If ψ is an overfilter then $\hat{\psi}\psi\hat{\psi}\psi \geq \hat{\psi}\psi^2 \geq \hat{\psi}\psi$. Combining this with (a) the result follows.
(e): From (7.1) we get that

$$\psi\hat{\psi}(\mathsf{id} \vee \psi\hat{\psi}) \leq \psi\hat{\psi}(\mathsf{id} \vee \hat{\psi}) = \psi\hat{\psi}^2 = \psi\hat{\psi},$$

hence $\psi\hat{\psi}$ is a sup-underfilter. Let ϕ be any sup-underfilter. Then $(\mathrm{id} \vee \phi)(\mathrm{id} \vee \phi) = \mathrm{id} \vee \phi$, hence $\mathrm{id} \vee \phi$ is a closing, the smallest closing $\geq \phi$. Therefore $\hat{\phi} = \mathrm{id} \vee \phi$, and $\phi\hat{\phi} = \phi(\mathrm{id} \vee \phi) = \phi$. If, in addition, $\phi \geq \psi$ then $\phi = \phi\hat{\phi} \geq \psi\hat{\psi}$, hence $\psi\hat{\psi}$ is the smallest sup-underfilter $\geq \psi$.
(g): we first show that $\phi = \theta(\mathrm{id} \wedge \theta)$ is a sup-underfilter if θ is a sup-underfilter. Thereto we use the modularity of $\mathcal{L}$. We must show that $\phi(\mathrm{id} \vee \phi) \leq \phi$. Note first that, since $\phi \leq \theta$,

$$\theta \leq \theta(\mathrm{id} \vee \phi) \leq \theta(\mathrm{id} \vee \theta) = \theta,$$

hence $\theta = \theta(\mathrm{id} \vee \phi)$. Then

$$\begin{aligned}
\phi(\mathrm{id} \vee \phi) &= \theta(\mathrm{id} \wedge \theta)(\mathrm{id} \vee \phi) = \theta\big[\mathrm{id} \vee \phi \wedge \theta(\mathrm{id} \vee \phi)\big] \\
&= \theta\big[(\mathrm{id} \vee \phi) \wedge \theta\big] = \theta\big[(\mathrm{id} \wedge \theta) \vee \phi\big] \\
&= \theta\big[(\mathrm{id} \wedge \theta) \vee \theta(\mathrm{id} \wedge \theta)\big] = \theta(\mathrm{id} \vee \theta)(\mathrm{id} \wedge \theta) \\
&= \theta(\mathrm{id} \wedge \theta) = \phi.
\end{aligned}$$

Suppose that ψ is an inf-overfilter, then

$$\psi = \psi(\mathrm{id} \wedge \psi) \leq \psi\hat{\psi}(\mathrm{id} \wedge \psi\hat{\psi}) \leq \psi\hat{\psi}.$$

From (e) we know already that $\psi\hat{\psi}$ is a sup-underfilter, and using the result above we find that $\psi\hat{\psi}(\mathrm{id} \wedge \psi\hat{\psi})$ is a sup-underfilter. Since it majorates ψ we derive from (e) that

$$\psi\hat{\psi}(\mathrm{id} \wedge \psi\hat{\psi}) \geq \psi\hat{\psi}$$

whence it follows that $\psi\hat{\psi}$ is an inf-overfilter and hence a strong filter.

∎

We can restate (a) as follows: the mapping $\psi \mapsto \hat{\psi}\psi$ on $\mathcal{O}_+(\mathcal{L})$ is a closing with domain of invariance the (inf-closed) set of all underfilters. A similar reformulation holds for (b), (e) and (f).

An important consequence of this result is that the set of all filters on $\mathcal{L}$ (under the ordering of $\mathcal{O}_+(\mathcal{L})$) defines a complete lattice. Namely, let ψ_i $(i \in I)$ be an arbitrary collection of filters, and define

$$\lambda := \bigwedge_{i \in I} \psi_i \quad \text{and} \quad \mu := \bigvee_{i \in I} \psi_i. \tag{7.3}$$

Then λ is an underfilter and μ is an overfilter. From the proposition we deduce that $\hat{\mu}\mu$ is a filter, the smallest filter which majorates all ψ_i, and therefore the supremum of ψ_i $(i \in I)$ in the lattice of all filters. Similarly, $\check{\lambda}\lambda$ is the infimum of ψ_i $(i \in I)$ within this lattice. Analogously it follows that the set of all strong filters forms a complete lattice if the underlying lattice $\mathcal{L}$ is modular. Using the same notation as in (7.3), $\mu\hat{\mu}$ is the supremum of the ψ_i and $\lambda\check{\lambda}$ the infimum.

The considerations above indicate that it is important to develop tools for the explicit computation of $\check{\psi}$ and $\hat{\psi}$. Below we show that under appropriate continuity assumptions such an explicit computation can be made by iteration of $\mathrm{id} \wedge \psi$ respectively $\mathrm{id} \vee \psi$.

Assume now that ψ is increasing and $\psi \leq \mathrm{id}$ (if the latter condition does not hold we replace ψ with $\mathrm{id} \wedge \psi$). Then

$$\ldots \leq \psi^{n+1} \leq \psi^n \leq \psi^{n-1} \leq \ldots \leq \psi \leq \mathrm{id}.$$

Does $\psi^\infty := \bigwedge_{n\geq 1} \psi^n$ define an opening? Or in other words, is $\check{\psi} = \psi^\infty$? It is easily seen that this is indeed the case if and only if

$$\psi\psi^\infty = \psi^\infty.$$

Below we shall see that the answer is affirmative under the assumption that ψ is $\downarrow$-continuous. In [5] we have constructed a counterexample which shows that the $\downarrow$-continuity is essential: see also [19]. This example will be reproduced here for convenience.

Example 7.3. Let the increasing, translation-invariant operator $\psi : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ be given by $\psi(X) = (X \oplus A) \cap X$, where $A = \{\ldots, -5, -3, -1, 2\}$. Evidently, ψ is anti-extensive. If $X = \{0, 1, 3, 5, \ldots\}$ then, by a straightforward calculation, $\psi^n(X) = \{0, 2n+1, 2n+3, 2n+5, \ldots\}$, and so $\psi^\infty(X) = \{0\}$. But then $\psi\psi^\infty(X) = \psi(\{0\}) = \emptyset \neq \psi^\infty(X)$.

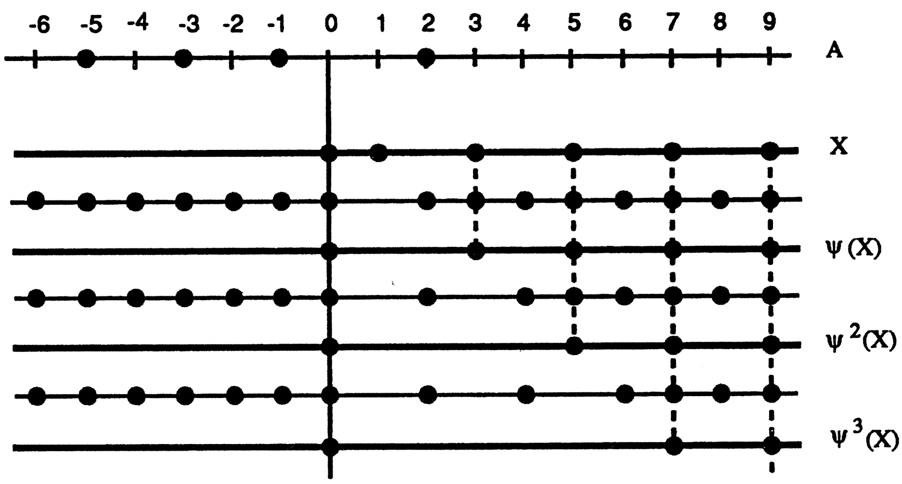

FIGURE 3. *Iteration of an anti-extensive operator does not always yield an idempotent operator. In this case $\psi^\infty(X) \neq \psi^{\infty+1}(X)$.*

For an arbitrary operator ψ on on $\mathcal{L}$ we define

$$\psi^\infty(X) = \limsup \psi^n(X)$$
$$\psi_\infty(X) = \liminf \psi^n(X).$$

If $\psi_\infty = \psi^\infty$ then we write $\psi^n \to \psi^\infty$. In particular, if $\psi^2 \leq \psi$ (resp. $\psi^2 \geq \psi$) then $\psi_\infty = \psi^\infty = \bigwedge_{n\geq 1} \psi^n$ (resp. $\psi_\infty = \psi^\infty = \bigvee_{n\geq 1} \psi^n$).

Proposition 7.4. *Let ψ be an arbitrary operator on the complete lattice $\mathcal{L}$.*

(a) If ψ is $\downarrow$-continuous and $\psi^n \to \psi^\infty$ then $\psi^\infty \leq \psi\psi^\infty$.

(b) If ψ is $\uparrow$-continuous and $\psi^n \to \psi^\infty$ then $\psi^\infty \geq \psi\psi^\infty$.

(c) If ψ is continuous and $\psi^n \to \psi^\infty$ then $\psi^\infty = \psi\psi^\infty$ and ψ^∞ is idempotent.

PROOF. We only prove (a); then (b) follows by duality and (c) is an immediate consequence of (a) and (b).
If $\psi^n \to \psi^\infty$ and ψ is $\downarrow$-continuous then, for $X \in \mathcal{L}$,

$$\psi^\infty(X) = \limsup \psi(\psi^n(X)) \leq \psi(\psi^\infty(X)).$$

∎

Corollary 7.5. *Let ψ be an arbitrary operator on $\mathcal{L}$.*
(a) If $\psi \leq \mathsf{id}$ and $\downarrow$-continuous then $\psi^\infty = \bigwedge_{n\geq 1} \psi^n$ is idempotent.
(b) If $\psi \geq \mathsf{id}$ and $\uparrow$-continuous then $\psi^\infty = \bigvee_{n\geq 1} \psi^n$ is idempotent.

PROOF. We only prove (a). If $\psi \leq \mathsf{id}$, then $\psi^n \to \psi^\infty = \bigwedge_{n\geq 1} \psi^n$, and from Proposition 7.4(a) we find that $\psi^\infty \leq \psi\psi^\infty$. On the other hand, $\psi^\infty \geq \psi\psi^\infty$, and so $\psi\psi^\infty = \psi^\infty$ whence the result follows.

∎

In [23, Theorem 4] has proven a similar result for operators on the lattice of closed subsets of some topological space, provided with the hit-or-miss convergence.

From this point on we shall deal exclusively with increasing operators.

Proposition 7.6. *If ψ is an underfilter (resp. overfilter, sup-underfilter, inf-overfilter) then ψ^∞ is an underfilter (resp. overfilter, sup-underfilter, inf-overfilter).*

PROOF. We only prove the statement for underfilters and sup-underfilters. If ψ is an (sup-) underfilter then ψ^n is an (sup-) underfilter. Now the statement follows from the fact that $\psi^\infty = \bigwedge_{n\geq 1} \psi^n$ and the inf-closedness of the set of (sup-) underfilters.

∎

Proposition 7.7. *Let ψ be a $\downarrow$- continuous underfilter (resp. $\uparrow$- continuous overfilter). Then $\psi\psi^\infty = \psi^\infty$ and ψ^∞ is a $\downarrow$- continuous (resp. $\uparrow$- continuous) filter.*

PROOF. If ψ is an underfilter then $\psi^n \downarrow \psi^\infty$ and from Proposition 3.3 we conclude that $\psi\psi^n \downarrow \psi\psi^\infty$. But on the other hand $\psi\psi^n = \psi^{n+1} \downarrow \psi^\infty$, hence $\psi\psi^\infty = \psi^\infty$. In particular this yields that ψ^∞ is a filter. The $\downarrow$-continuity of ψ^∞ follows from Proposition 3.6 together with the observation that any iterate ψ^n is $\downarrow$- continuous.

Corollary 7.8. *Let ψ be an increasing operator.*
(a) If ψ is $\uparrow$-continuous then $\hat{\psi} = (\mathsf{id} \vee \psi)^\infty$.
(b) If ψ is $\downarrow$-continuous then $\check{\psi} = (\mathsf{id} \wedge \psi)^\infty$.

PROOF. We only prove (a). From the previous result we know that $(\mathsf{id} \vee \psi)^\infty$ is a filter which is $\geq \mathsf{id}$. So $(\mathsf{id} \vee \psi)^\infty$ is a closing. Since $\mathsf{id} \vee \psi \leq \hat{\psi}$ we get that $(\mathsf{id} \vee \psi)^n \leq \hat{\psi}^n = \hat{\psi}$, and therefore $(\mathsf{id} \vee \psi)^\infty \leq \hat{\psi}$. Since $\hat{\psi}$ is defined as the smallest closing majorating ψ we may conclude that $\hat{\psi} = (\mathsf{id} \vee \psi)^\infty$.

∎

Filters

We have seen that the class of all filters forms a complete lattice. Let ψ_i $(i \in I)$ be filters and let $\lambda := \bigwedge_{i\in I} \psi_i$ and $\mu := \bigvee_{i\in I} \psi_i$. Then $\check{\lambda}\lambda$ and $\hat{\mu}\mu$ are the infimum respectively supremum of ψ_i within this lattice. Now suppose that every ψ_i is $\downarrow$-continuous. Then λ is an underfilter which, by Proposition 3.6, is $\downarrow$-continuous. We conclude from Proposition 7.7 that $\lambda^\infty = \bigwedge_{n\geq 1} \lambda^n$ is a filter. Moreover, if λ^* is a filter which minorates λ, then $\lambda^* \leq \lambda^n$ for all n, and $\lambda^* \leq \lambda^\infty$. Hence

$$\lambda^\infty = \check{\lambda}\lambda.$$

Note that it follows from Corollary 7.8 that

$$\check{\lambda} = (\mathrm{id} \wedge \lambda)^\infty.$$

Analogously, if every ψ_i is $\uparrow$-continuous, then μ is an $\uparrow$-continuous overfilter and we get that

$$\hat{\mu}\mu = \mu^\infty = \bigvee_{n\geq 1} \mu^n,$$

where $\hat{\mu} = (\mathrm{id} \vee \mu)^\infty$.

Openings

The set of all openings on $\mathcal{L}$ is a complete lattice with the supremum of ψ_i $(i \in I)$ given by μ and infimum $\check{\lambda}$. If every opening ψ_i is $\downarrow$-continuous then λ is $\downarrow$-continuous and we conclude from Corollary 7.8 that $\check{\lambda} = \lambda^\infty$. Moreover, $\mathrm{Inv}(\lambda^\infty) = \bigcap_{i\in I} \mathrm{Inv}(\psi_i)$: see also [19, Example 5.1].

Strong filters

We have seen that the strong filters form a complete lattice in case that the underlying lattice $\mathcal{L}$ is modular. The supremum and infimum of the family of strong filters ψ_i $(i \in I)$ are respectively given by $\mu\hat{\mu}$ and $\lambda\check{\lambda}$. We show that the modularity assumption can be replaced with a continuity assumption on the operators ψ_i.

Proposition 7.9. *Let ψ_i be $\downarrow$-continuous strong filters, and let $\lambda := \bigwedge_{i\in I} \psi_i$. Then the infimum of ψ_i in the lattice of strong filters is given by*

$$\lambda\check{\lambda} = \lambda^\infty = \bigwedge_{n\geq 1} \lambda^n,$$

which is a $\downarrow$-continuous operator.
Analogously, if any ψ_i is $\uparrow$-continuous and $\mu := \bigvee_{i\in I} \psi_i$ then the supremum is given by

$$\mu\hat{\mu} = \mu^\infty = \bigvee_{n\geq 1} \mu^n,$$

which is $\uparrow$-continuous.

Proof. We only prove the first assertion. Since every ψ_i is a sup-underfilter and since the class of sup-underfilters is inf-closed we get that λ is sup-underfilter. Furthermore, λ is $\downarrow$-continuous, and we deduce from Proposition 7.7 that $\lambda^\infty = \bigwedge_{n\geq 1}\lambda^n$ is both a filter and a sup-underfilter. In addition λ^∞ is $\downarrow$-continuous. It remains to show that λ^∞ is an inf-overfilter, or more precisely, that

$$\lambda^\infty(\mathrm{id}\wedge\lambda^\infty) \geq \lambda^\infty.$$

First we note that

$$\lambda^n(\mathrm{id}\wedge\lambda^n) \geq \lambda^n(\mathrm{id}\wedge\lambda) = \lambda^{n-1}\lambda(\mathrm{id}\wedge\lambda) = \lambda^n.$$

Therefore, λ^n is an inf-overfilter. Now

$$\begin{aligned}\lambda^\infty(\mathrm{id}\wedge\lambda^\infty) &= \lambda^\infty(\mathrm{id}\wedge\bigwedge_{n\geq 1}\lambda^n) = \lambda^\infty\big(\bigwedge_{n\geq 1}(\mathrm{id}\wedge\lambda^n)\big)\\ &= \bigwedge_{n\geq 1}\lambda^\infty(\mathrm{id}\wedge\lambda^n) = \bigwedge_{n\geq 1}\lambda^\infty\lambda^n(\mathrm{id}\wedge\lambda^n)\\ &= \bigwedge_{n\geq 1}\lambda^\infty\lambda^n = \lambda^\infty.\end{aligned}$$

Here we have used that λ^∞ as an infimum of $\downarrow$- continuous operators is $\downarrow$-continuous and that $\lambda^\infty = \lambda^\infty\lambda^n$.

∎

Centre and middle element

According to a known result in morphological filtering due to Matheron [22, Theorem 6.16], given a sup-underfilter ξ and an inf-overfilter η, both acting on a modular lattice $\mathcal{L}$, with $\xi \geq \eta$, there exists a strong filter μ, called *middle element* between ξ and η, such that

$$\mu = \check{\xi}\hat{\eta} = \hat{\eta}\check{\xi}, \quad \check{\mu} = \check{\xi}, \quad \hat{\mu} = \hat{\eta}$$

and

$$\eta \leq \eta\hat{\eta} \leq \mu \leq \xi\check{\xi} \leq \xi.$$

On the other hand, the notion of the *morphological center* β of a family $\{\psi_i\}$ of operators on $\mathcal{L}$ defined as

$$\beta = \big[\mathrm{id}\wedge(\vee\psi_i)\big]\vee(\wedge\psi_i),$$

gives, when it is applied to the two element family $\{\xi,\eta\}$:

$$\beta = (\mathrm{id}\wedge\xi)\vee\eta = (\mathrm{id}\vee\eta)\wedge\xi;$$

see [22, Chapter 8]. By introducing order continuity conditions, we get the following result.

Proposition 7.10. *Let ξ be a $\downarrow$-continuous sup-underfilter and η a $\uparrow$-continuous inf-overfilter on a modular lattice $\mathcal{L}$ with $\xi \geq \eta$. Then the morphological center $\beta = (\mathrm{id}\vee\eta)\wedge\xi$ between ξ and η converges under iteration towards their middle element, i.e.,*

$$\beta^n \to \beta^\infty = \mu = \check{\xi}\hat{\eta} = \hat{\eta}\check{\xi}.$$

PROOF. From $\xi \leq \xi(\mathrm{id} \vee \eta) \leq \xi(\mathrm{id} \vee \xi) = \xi$ we deduce that

$$\beta = (\mathrm{id} \vee \eta) \wedge \xi = (\mathrm{id} \vee \eta) \wedge \xi(\mathrm{id} \vee \eta) = (\mathrm{id} \wedge \xi)(\mathrm{id} \vee \eta),$$

and similarly that $\beta = (\mathrm{id} \vee \eta)(\mathrm{id} \wedge \xi)$. Consequently, we have for all $n < \infty$,

$$\beta^n = (\mathrm{id} \wedge \xi)^n(\mathrm{id} \vee \eta)^n = (\mathrm{id} \vee \eta)^n(\mathrm{id} \wedge \xi)^n.$$

Therefore,

$$\beta_\infty = \liminf \beta^n \geq \bigvee_{N\geq 1} \bigwedge_{n\geq N} (\mathrm{id} \vee \eta)^n \check{\xi} = \bigvee_{n\geq 1} (\mathrm{id} \vee \eta)^n \check{\xi} = \hat{\eta}\check{\xi} = \mu,$$

In the same way we derive that $\limsup \beta^n \leq \mu$, and so $\mu \leq \liminf \beta^n \leq \limsup \beta^n \leq \mu$, whence the result follows.

∎

Owing to the order convergence, Proposition 7.10 extends a result established in [22, p.174] for finite lattices to the infinite case. One should notice that, unlike ξ or η, β itself may be neither $\uparrow$-continuous nor $\downarrow$-continuous. However, it can easily be shown that

$$\mathrm{id} \wedge \beta^n = (\mathrm{id} \wedge \xi)^n \quad \text{and} \quad \mathrm{id} \vee \beta^n = (\mathrm{id} \vee \eta)^n;$$

see [22, Proposition 8.11]. These two relations enlighten the behaviour of β^n in the lattice of numerical functions $f : E \to \overline{\mathbb{R}}$. They indicate the following *pointwise monotonicity*: at any given point $x \in E$, the numerical values $(\beta^n f)(x)$ can only decrease or increase, as n increases, but not oscillate under iteration. This property indicates an essential difference between the morphological centers and the median filters used in image processing: see also [7]

Finally, by applying Corollary 7.8, we see that $\beta^n(f)$ tends towards $\hat{\eta}(f)$ at the points where this sequence increases, and towards $\check{\xi}(f)$ at the points where it decreases. Such a double sense convergence may also be interpreted as $\uparrow$-continuity by introducing the so-called *activity order* on the space of operators [22, Section 8.2].

8. Application to numerical functions

In this last section which is intended to illustrate the previous ones, we develop a Euclidean model for the increasing operators on the space of grey-level images. In other words, we want to exhibit a class of functions from $\mathbb{R}^d$ into $\overline{\mathbb{R}}$, and a class of operators acting on it, which (i) are realistic enough to model the actual situations met in practice, and which (ii) satisfy good theoretical properties. From a physical viewpoint, the continuous functions appear to be a good choice (Shannon's theorem), but unfortunately, the continuous functions do not generate a lattice, and so our theory does not apply to them, at least not directly. On the other hand, another challenge stems from the theory itself. It transpires from what we have seen up to now, namely that the order semi-continuity is an essential requirement. Among others, it governs the behaviour of the operators under iteration. However, when we want it to be brought into play, we are faced with the following alternative. Either we take the largest possible function lattice, i.e., that of all numerical functions $f : \mathbb{R}^d \to \overline{\mathbb{R}}$, and then the class of semi-continuous operators on it is very poor (mainly using finite structuring elements). Alternatively, we may require operators to be $\uparrow$- or $\downarrow$-continuous, thus restricting the possible functions to those which are upper or lower semi-continuous (see below). The *regular model* which we present below combines the two alternatives by considering at the same time three different function lattices.

(a) The class $\mathsf{Fun}(\mathbb{R}^d)$ of the functions mapping $\mathbb{R}^d$ into $\overline{\mathbb{R}}$ is a complete lattice with the usual pointwise supremum and infimum [6,8,22,24].

(b) The class $\mathsf{Fun}_u(\mathbb{R}^d)$ of the upper semi-continuous (u.s.c.) functions from $\mathbb{R}^d$ into $\overline{\mathbb{R}}$. A function f is said to be u.s.c. if for every $x \in \mathbb{R}^d$ and every $t > f(x)$ there exist a neighbourhood V of x such that $t > f(y)$ for all $y \in V$. Geometrically, this means that the points in $\mathbb{R}^d \times \overline{\mathbb{R}}$ on or below the graph of f form a closed set: see Figure 4.

(c) The class $\mathsf{Fun}_l(\mathbb{R}^d)$ of lower semi-continuous (l.s.c.) functions. A function f is called l.s.c. if for every $x \in \mathbb{R}^d$ and every $t < f(x)$ there exist a neighbourhood V of x such that $t < f(y)$ for all $y \in V$. A function f is l.s.c. if the points in $\mathbb{R}^d \times \overline{\mathbb{R}}$ on or below the graph of f form an open set: see Figure 4.

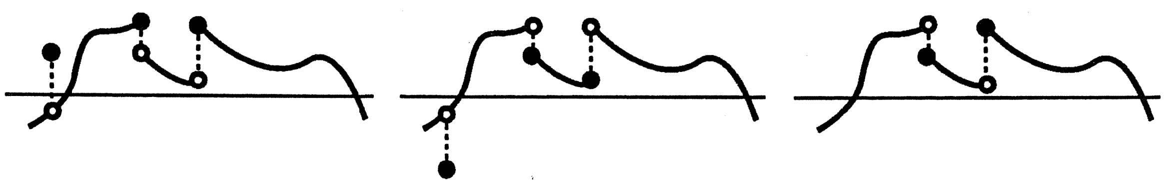

FIGURE 4. *From left to right: u.s.c. function, l.s.c. function, and a function which is neither u.s.c. nor l.s.c.*

It is obvious that upper semi-continuity and lower semi-continuity are dual notions: if f is u.s.c. then $-f$ is l.s.c. and vice versa. For functions, upper and lower semi-continuity play the same role as closedness respectively openness for sets. For example, analogous to the fact that intersections of closed sets yield closed sets we have that pointwise infima of u.s.c. functions yield u.s.c. functions. Analogous to the closure and interior of a subset of $\mathbb{R}^d$ one can define the upper respectively lower envelope of a function in $\mathsf{Fun}(\mathbb{R}^d)$. The supremum with respect to the lattice $\mathsf{Fun}_u(\mathbb{R}^d)$ of a collection of u.s.c. functions f_i is the upper envelope of their pointwise supremum. In this way one can easily show that both $\mathsf{Fun}_u(\mathbb{R}^d)$ and $\mathsf{Fun}_l(\mathbb{R}^d)$ form complete lattices. For more details concerning u.s.c and l.s.c. functions we refer to [3] and [21, p.426].

Now we are ready to introduce the class of regular operators.

Definition 8.1. An increasing operator ψ on $\mathsf{Fun}(\mathbb{R}^d)$ is said to be regular if the following conditions are astisfied:

(i) $\mathsf{Fun}_u(\mathbb{R}^d)$ and $\mathsf{Fun}_l(\mathbb{R}^d)$ are invariant under ψ;

(ii) ψ is $\downarrow$-continuous on $\mathsf{Fun}_u(\mathbb{R}^d)$ and $\uparrow$-continuous on $\mathsf{Fun}_l(\mathbb{R}^d)$.

So a regular operator is defined on the overall function lattice $\mathsf{Fun}(\mathbb{R}^d)$, but meantime, we restrict our demands concerning its $\uparrow$- or $\downarrow$-continuity to some particular subclasses of functions. In fact, the class of regular operators extends the class of *compact operators* introduced and studied by Matheron in the set framework in [14, p.158].

We will now show that the class of regular operators is quite large. It includes the min-max operators which appeared in the field of mathematical morphology in the seventies [20,16,25]. The first consistent theory due to Serra [21, Chapter XII] focusses on the semi-continuous functions. Recently, Heijmans [6] developped an approach which does not use the semi-continuity requirement.

In the sequel $\bigvee$ and $\bigwedge$ denote supremum respectively infimum in the function lattice $\mathsf{Fun}(\mathbb{R}^d)$ unless otherwise stated. For the case of sets Matheron [14] has shown that dilation by a compact set A maps any closed (open) set onto a closed (open) set. Furthermore, in Example 3.13 we have seen that such a dilation is order continuous on the closed subsets of $\mathbb{R}^d$. One can also show that it is $\uparrow$-continuous on the complete lattice of open subsets of $\mathbb{R}^d$. Defining the dilation of a function f by a set A as $f \oplus A = \bigvee_{h \in A} f_h$ (and erosion by $f \ominus A = \bigwedge_{h \in A} f_{-h}$), these results easily carry over to the lattices of u.s.c. and l.s.c. functions described above, thus showing the following proposition.

Proposition 8.2. *Let A be a compact structuring element in $\mathbb{R}^d$. Then the dilation $f \to f \oplus A$ and the erosion $f \to f \ominus A$ are regular operators. Moreover this dilation and erosion are order continuous on $\mathsf{Fun}_u(\mathbb{R}^d)$ respectively $\mathsf{Fun}_l(\mathbb{R}^d)$.*

In fact, it suffices to prove this result for dilations, since then the result for erosions follows by duality; see also the next result. If ψ is an operator on $\mathsf{Fun}(\mathbb{R}^d)$, then we define the dual operator ψ^* by

$$\psi^*(f) = -\psi(-f).$$

One easily derives the following result.

Proposition 8.3. *Let ψ be an increasing operator on $\mathsf{Fun}(\mathbb{R}^d)$. Then ψ is regular if and only if ψ^* is regular.*

Proposition 8.4.

(a) A finite composition of regular operators is again regular.

(b) A finite supremum and infimum of regular operators on $\mathsf{Fun}(\mathbb{R}^d)$ is regular.

The first statement of this last proposition follows from Proposition 3.9, and the second from Propositions 3.6 and 3.8 and their opposites. The Propositions 8.2–8.4 cover most examples of increasing translation invariant operators based on flat structuring elements that are met in practice. In particular, it includes the morphological openings, their finite supremum, say γ, the corresponding closing, say ϕ, and also the filters $\gamma\phi$, $\phi\gamma$, $\gamma\phi\gamma$, $\phi\gamma\phi$, etc., see [17,22,24].

It remains to analyze the behaviour of compact increasing operators under iteration. The most interesting result is obtained on the space of continuous functions. Note that this space is precisely the intersection of the spaces $\mathsf{Fun}_u(\mathbb{R}^d)$ and $\mathsf{Fun}_l(\mathbb{R}^d)$.

Proposition 8.5. *Let f be a continuous function mapping $\mathbb{R}^d$ into $\overline{\mathbb{R}}$, and let ψ be a regular increasing operator. Then*

$$\hat{\psi}(f) = \bigvee_{n\geq 1} (\mathrm{id} \vee \psi)^n(f) \quad \text{and} \quad \check{\psi}(f) = \bigwedge_{n\geq 1} (\mathrm{id} \wedge \psi)^n(f).$$

PROOF. The statement is an immediate consequence of Corollary 7.8 and the observation that f is both u.s.c. and l.s.c., and that the usual sup (resp. inf) coincides with the sup (resp. inf) of the l.s.c. (resp. u.s.c.) functions.

∎

We conclude this section with a specialization of Proposition 7.10 to continuous functions.

Corollary 8.6. *Let ξ be a sup-underfilter and η an inf-overfilter on the lattice $\mathrm{Fun}(\mathbb{R}^d)$ with $\xi \geq \eta$, and assume that both operators are regular. Then, if f is a continuous function, the middle element $\mu(f) = \check{\xi}\hat{\eta}(f)$ is obtained by the iterative formula*

$$\mu(f) = [(\mathrm{id} \vee \eta) \wedge \xi]^n(f).$$

PROOF. First we note that the lattice $\mathrm{Fun}(\mathbb{R}^d)$ is modular, and, just as in the proof of Proposition 7.10, we obtain that $\beta^n = (\mathrm{id} \wedge \xi)^n(\mathrm{id} \vee \eta)^n$, where β is the centre $\beta = (\mathrm{id} \wedge \xi) \vee \eta$. Then $\liminf \beta^n \geq \bigvee_{n\geq 1} \check{\xi}(\mathrm{id} \vee \eta)^n$. Now, if f is continuous, then the functions $(\mathrm{id} \vee \eta)^n(f)$ are l.s.c. for all n. Since $\check{\xi}$ is $\uparrow$-continuous on $\mathrm{Fun}_l(\mathbb{R}^d)$ we get that $\liminf \beta^n(f) \geq \check{\xi}\big(\bigvee_{n\geq 1}(\mathrm{id} \vee \eta)^n(f)\big)$, and from Corollary 7.8 we obtain that $\liminf \beta^n(f) \geq \check{\xi}\hat{\eta}(f) = \mu(f)$.

Similarly, we obtain that $\limsup \beta^n(f) \leq \mu(f)$ from which the result follows.

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