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H.J.A.M. Heijmans, C. Ronse

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Part I: Dilations and erosions

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The Algebraic Basis of Mathematical Morphology
Part I: Dilations and Erosions

H.J.A.M. Heijmans

Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

C. Ronse

Philips Research Laboratory Brussels
Avenue Van Becelaere 2, Box 8, B-1170 Brussels, Belgium

Mathematical morphology is a theory of image transformations and functionals deriving its tools from set theory and integral geometry. This paper deals with a general algebraic approach which both reveals the mathematical structure of morphological operations and unifies several examples into one framework. The main assumption is that the object space is a complete lattice and that the transformations of interest are invariant under a given abelian group of automorphisms on that lattice. It turns out that the basic operations called dilation and erosion are adjoints of each other in a very specific lattice sense and can be completely characterized if the automorphism group is assumed to be transitive on a sup-generating subset of the complete lattice. The abstract theory is illustrated by a large variety of examples and applications.

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1. Introduction and Mathematical Background

1.1. *What is mathematical morphology?*

Mathematical morphology is a particular discipline in the field of image processing, which has been applied to analyse the structure of materials in various disciplines such as mineralogy, petrography, angiography, cytology, etc.. It was born in 1964 when Matheron started to investigate the relationships between the geometry of porous media and their permeabilities, and Serra was asked to quantify the petrography of iron ores in order to predict their milling properties. This initial research led to the formation of a team at the Paris School of Mines

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at Fontainebleau, the 'Centre de Morphologie Mathématique', which combined theoretical work with the design of practical applications, such as the 'texture analyser'.

Following the publication of [14] and [23], and also the work of Sternberg in the USA [25,26], this discipline gained an increasing popularity inside the image processing community, as is witnessed by a special issue of the journal *Computer Vision, Graphics, and Image Processing* devoted to mathematical morphology [27], and the increasing number of articles in technical journals referring to it (for example the tutorial [6]).

While some recent contributions to the subject do not always give any new insight, it is interesting to have a deeper reflexion on the nature of mathematical morphology and its basis. What is it? What are its methods?

Broadly speaking, one can consider that mathematical morphology is an approach for the analysis of structure based on set-theoretic concepts. It has three aspects: an algebraic one, dealing with image transformations derived from set-theoretical operations, a probabilistic one, dealing with models of random sets applicable to the selection of small samples of materials, and an integral geometrical one, dealing with image functionals.

This paper addresses the first aspect: the algebraic study of a body of image transformations based on operations similar to those of set theory, which are in particular non-linear. This raises several questions: What is the importance of transformations in image analysis? What have algebraic properties to do with this? Why non-linear operations? Why the similarity with set theory?

1.1.1. The importance of transformations and their algebraic properties. One of the basic intuitions of mathematical morphology is that the the analysis of an image does not reduce to a simple measurement. Instead, it relies on a succession of operators which transform it in order to make certain features apparent. As says Serra [23], "to perceive an image, is to transform it". Indeed, a picture usually contains an unstructured wealth of information, most of which is of no use to us. From it we have to extract what interests us, obtaining thus a structure which is in fact a simplified sketch (a caricature) of the original image. In particular, its recognition involves a controlled loss of information, since we eliminate irrelevant features from it. For example in optical character recognition, one can simplify the task by first performing on a binary digital image representing a typed text a *skeletonization*, which reduces each connected component to a one-pixel thick skeleton retaining its shape; this discards all (useless) information about the thickness of characters, and the reduced amount of information contained in such an image makes further recognition steps quicker and easier.

Many transformations have been applied to the analysis of images: linear filtering (in Fourier analysis), median filtering, skeletonization, histogram equalization, etc.. An often overlooked fact is that simple algebraic properties can be important criteria for assessing the validity of transformations involved in the analysis of images. Let us return to the example given above of the skeletonization of binary images. A good skeletonization algorithm must transform a shape in the same way whatever its position in the plane, in other words the skeletonization operator S must *commute* with any translation T of the plane: $T \cdot S = S \cdot T$. It must not change a connnected component which is itself a skeleton, in other words S must be *idempotent*: $S \cdot S = S$. If that algorithm is based on *thinning*, that is a succession of stages where border pixels are removed, we must have a garantee that on any given digital image, the algorithm will stop after a finite number of thinning steps, in other words we have to know the convergence properties of these steps. Other important algebraic properties will be considered later, for example increasingness, extensivity, etc., and we will give in the second part of this study [21] practical examples of their usefulness in digital subtraction angiography.

A more detailed discussion on the importance of transformations in image analysis can be found in [17] and in the first pages of [23].

1.1.2. Why non-linear transformations? This problem certainly deserves more discussion than we can afford here, and it will be considered in detail in [20]. Let us only give here a brief explanation.

The whole field dealing with the processing of sound signals (in particular speech) is based on linearity. Indeed, sound signals combine linearly by superposition. Therefore commercial 'Hi-Fi' devices for the rendition of sound signals must preserve the ratio between different sound sources, in other words they are based on the constraint of linearity. It is thus no wonder that the two basic operations in the theory of sound processing, the convolution and the Fourier transform, are linear.

Workers in image processing have attempted to apply linear techniques to the analysis of images. One thought for example that the global structure of an image would be derived from a low-pass filtering, and the finer details from a high-pass filtering.

Such a simple view did not pass the challenge of experimental practice, and was soon abandoned. As said Marr [13], "these ideas based on Fourier theory are *like* what is wanted, but they are not *what* is wanted". Indeed arguments have been advanced from the point of view of both human psychophysics and texture analysis to show that Fourier analysis is inappropriate for modeling visual processes.

So why should the analysis of images not be based on the same linear operations as the processing of sound signals? Arguments have been advanced by Serra [24] about the difference of behavior between light and sound waves when interacting with objects. But there is a more fundamental reason: the two human senses of vision and audition have not the same purpose. The goal of audition is the analysis of sound waves, while the goal of vision is not the analysis of light waves, but rather the recognition of objects and scenes in three-dimensional space from the light waves that they reflect. In particular vision is a more difficult task than audition, since the information we wish to obtain (the shape of objects) is only indirectly provided by an image. The difference in purpose between these two senses leads therefore to differences in the processes they sustend.

1.1.3. Mathematical morphology based on set theory. We can generally identify a scene from a (two-tone) pencil drawing of it, where we have only simple elements such as contours, bars, blobs, shadows, etc.. It is also widely accepted that one of the earliest goals of low-level visual perception is the detection of significant features in images, such as edges (that is, lines formed by points of significant intensity changes), and the construction from them of elementary structural primitives.

It should be noted that the 'pencil drawing', that is the image representing elementary features (contours, edges, bars, blobs, etc.), and also the more elaborate images built from it in order to represent structural primitives, are Boolean images. Therefore any further processing of them cannot be linear, but must be related to Boolean algebra. For example if object X is behind object Y, this means that in the 'pencil drawing' we see the contour of X minus Y (in the set-theoretical sense) [24]. Or the shadow of a union of objects will be the union of their shadows.

This line of analysis shows the relationship between certain properties of images and set-theoretic concepts, and explains why image transformations can be based on the Boolean algebra of set operations. This is more evident in the case of binary images, as we have just explained above. In fact mathematical morphology was first developed for the analysis of binary images, and its extension to grey-level images was a later development. Let us thus for the time being concentrate on this particular case.

In a simplified way, let us say that one can analyse the structure of a binary image by looking at patterns of a certain form at various places. For example in the study of texture discrimination in preattentive vision, Julesz and Bergen [10] proposed to characterize textures by elementary primitives called *textons* having a simple shape: a bar, a cross, a T-junction, etc..

This idea of describing structure by linking similar patterns at various locations is quantified in mathematical morphology by the concept of a *structuring element*. We consider Boolean images as subsets of a Euclidean or digital space $\mathcal{E}$. A structuring element is basically a subset B of $\mathcal{E}$. Now we suppose that we have fixed the origin o in $\mathcal{E}$; then to each point p of $\mathcal{E}$ corresponds the translation mapping o to p, and this translation maps B onto B_p, the translate of B by p. For a structuring element B, we consider in fact all its translates B_p. This is because we assume that the space $\mathcal{E}$ in which Boolean images are represented is homogeneous under the group of translations. Note that other groups of transformations than the one of translations can be considered, according to the type of structure one is studying. (This will be discussed in Sections 3 and 4 of this paper).

Given a subset X of $\mathcal{E}$, we can see how the translates B_p of a structuring element B interact with X. For this purpose two basic operations are introduced by mathematical morphology. The first one derives from X and B the set $X \oplus B$ defined by

$$X \oplus B = \{x + b \mid x \in X, b \in B\} = \bigcup_{x \in X} B_x = \bigcup_{b \in B} X_b. \tag{1.1}$$

This operation dates in fact from Minkowski [18], and it is thus called the *Minkowski addition*. The second one, its *dual*, was introduced by Hadwiger [4,5] under the name of

Minkowski subtraction. It associates to X and B the set $X \ominus B$ defined by

$$X \ominus B = \{z \in \mathcal{E} \mid B_z \subseteq X\} = \bigcap_{b \in B} X_{-b}. \tag{1.2}$$

The transformation of X into $X \oplus B$ is called a *dilation*, and the transformation of X into $X \ominus B$ is called an *erosion*. (N.B. In the works of Matheron and Serra, $X \ominus B$ is defined in a slightly different way, while in those by Sternberg, it is defined as here. These two types of notation differ in that for some operations one has to replace B by $\check{B} = \{-b \mid b \in B\}$, see Section 4 for more details.) We illustrate these two operations in Figure 1.

In [23] it is shown how a large class of transformations of subsets of a digital space $\mathcal{E}$ can be built from these two basic operations using simple structuring elements: skeletonization, thinning, thickening, connected component extraction, etc.. We give here only two well-known examples. The erosion by B followed by the dilation by B gives the *opening by B*, an idempotent operation which transforms every set X into X_B, the union of all translates of B contained in X. It can be used to delete narrow portions of a set. On the other hand the dilation by B followed by the erosion by B gives a *closing*, another idempotent operation, which can be used to fill narrow holes in a set.

At this point, mathematical morphology can be seen as a set of tools for analysing Boolean images by the use of set-theoretic transformations based on dilations and erosions. The object space, the space of all Boolean images, can be represented by $\mathcal{P}(\mathcal{E})$, the set of all subsets of $\mathcal{E}$. Often one is only interested in a smaller object space, e.g. the space $Conv(\mathcal{E})$ of all convex subsets of $\mathcal{E}$ (where $\mathcal{E}$ is the d-dimensional Euclidean space $\mathbf{R}^d$ or the discrete space $\mathbf{Z}^d$ with a notion of convexity defined on it). The intersection of convex sets is again convex but the union is not, in general. To a great extent, however, the basic notions of mathematical morphology carry over to the smaller object space $Conv(\mathcal{E})$ by replacing $\bigcup_j X_j$ by $CH(\bigcup_j X_j)$, where for a subset A of $\mathcal{E}$, $CH(A)$ denotes the convex hull of A.

1.1.4. Grey-level images. Meyer [16] and Sternberg [25] were among the first to extend mathematical morphology to grey-level images. In this subsection we shall present the underlying ideas. A more thorough discussion including some of the problems which arise if the set of grey-levels is finite can be found in Section 4.

Let $\mathcal{E}$ be the Euclidean or discrete space and let $\mathcal{G}$ be a set of grey-levels. In this subsection we assume that $\mathcal{G} = \overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$ or $\mathcal{G} = \overline{\mathbf{Z}} = \mathbf{Z} \cup \{-\infty, +\infty\}$. A grey-level image is represented by a function $F : \mathcal{E} \to \mathcal{G}$, i.e. an element of the object space $\mathcal{G}^{\mathcal{E}}$. If both F and G are members of $\mathcal{G}^{\mathcal{E}}$, then we can define the addition $F \oplus G$ and subtraction $F \ominus G$ as follows: (a) the graph of $F \oplus G$ (which is a subset of $\mathcal{E} \times \mathcal{G}$) is obtained by associating to each point $(x, F(x))$ a translate of the graph of G, and taking the upper envelope of this set of translates; (b) the graph of $F \ominus G$ is obtained by associating to each point $(y, F(y))$ a translate of the graph of $\check{G}$ (defined by $\check{G}(x) = -G(-x)$), and taking the lower envelope of this set of translates. In other words,

$$(F \oplus G)(x) = \sup_{h \in \mathcal{E}} \big(F(x-h) + G(h)\big); \tag{1.3}$$

$$(F \ominus G)(x) = \inf_{h \in \mathcal{E}} \big(F(x+h) - G(h)\big). \tag{1.4}$$

These two operations are illustrated in Figure 2. (N.B. Again, in the works of Matheron and Serra, $F \ominus G$ is defined in a slightly different way, while in those by Sternberg, it is defined as here. This will be discussed in Section 4 in more detail.)

In practice, the structuring function G will have finite values on a compact support S and will be equal to $-\infty$ outside S. It is obvious that the operations $F \mapsto F \oplus G$, the dilation by G, and $F \mapsto F \ominus G$, the erosion by G, are invariant under translations on the space $\mathcal{E}$ and on the set $\mathcal{G}$ of grey-levels (see Section 4).

In the literature one often gives a geometric interpretation of these operations by considering so-called *umbras*. An umbra is a subset U of $\mathcal{E} \times \mathcal{G}$ satisfying: $(x,t) \in U$ if and only if $(x,s) \in U$ for every $s < t$. The umbra $U(F)$ of the function F is defined as:

$$U(F) = \{(x,t) \in \mathcal{E} \times \mathcal{G} \mid t \leq F(x)\},$$

and (1.3) restated in terms of umbras has the form:

$$U(F \oplus G) = U(F) \oplus U(G),$$

where the second $\oplus$ is the original Minkowski set addition. Exploiting the fact that the space of umbras is homeomorphic to the space $\mathcal{G}^{\mathcal{E}}$ one can alternatively use either characterization. In this paper we have chosen to work with functions only.

Morphological operators can be applied to the analysis of several types of grey-level images, those for which local configurations of grey-levels directly represent material properties of the objects pictured. One such type of images occurs in X-ray angiography. A contrast-enhancing product is injected in blood vessels (for example coronary arteries) before taking a X-ray photograph of them. Then narrow lines brighter than their surrounding are likely to correspond to these blood vessels, and they can be extracted with the use of structuring functions modeling such lines. Another type is given by tomographic images, which display a flat section of a given material photographed under a constant illumination. Then clearly the grey-level of each point indicates the intrinsic brightness of the corresponding point in the object photographed, and represents thus directly a material property of it. For example in cytology, nuclei and walls of cells form particular configurations of grey-levels having a certain shape. Or in petrography, pores are represented by small dots having a different grey-level than the rest of the image. Or in mineralogy, blobs of different grey-levels can represent particles of various metals. All these can be extracted by the use of apropriate structuring functions.

It will be argued in [20] that mathematical morphology is not suited to the analysis in depth of three-dimensional scenes from their two-dimensional pictures.

1.2. From sets to complete lattices

From the above discussion, mathematical morphology appears as a set of tools for analysing images by the use of transformations based on set-theoretical operations which are invariant under translations. This is, however, a rather vague description. A more precise description would involve (*i*) a characterization of the object space, and (*ii*) a choice of the basic operations (e.g. translations) which should commute with the transformations one wishes to consider. There are several choices of the object space conceivable. We have already seen some of them: $\mathcal{P}(\mathcal{E})$, the space of all subsets of $\mathcal{E}$, $Conv(\mathcal{E})$ the space of all convex subsets, and $\mathcal{G}^{\mathcal{E}}$, the space of all grey-level functions with values in $\mathcal{G}$. Another important example is $\mathcal{F}(\mathcal{E})$, the space of all *closed* subsets of $\mathcal{E}$ (here $\mathcal{E}$ must be a topological space). This latter space naturally arises if one wishes to supply the space $\mathcal{P}(\mathcal{E})$ with a topology: see [14] or [23]. In all these cases it is possible to define transformations like in (1.1) and (1.2) although the algebraic structure of the underlying spaces is quite different. There is, however, one major resemblance: all given spaces form so-called *complete lattices*. A complete lattice is a space $\mathcal{L}$ on which a partial order relation is defined such that every subset of $\mathcal{L}$ has a supremum (least upper bound) and infimum (greatest lower bound) in $\mathcal{L}$ (see Subsection 1.3). In the case where $\mathcal{L} = \mathcal{P}(\mathcal{E})$, the supremum coincides with the union whereas the infimum coincides with the intersection. Here dilations and erosions are precisely the transformations which commute respectively with the union and the intersection. Indeed, for every family of subsets X_j of $\mathcal{E}$ (even a void one) and any structuring element B we have

$$(\bigcup_j X_j) \oplus B = \bigcup_j (X_j \oplus B) \qquad \text{and} \qquad (\bigcap_j X_j) \ominus B = \bigcap_j (X_j \ominus B). \tag{1.5}$$

The 'special' relation which links the Minkowski addition and subtraction is that for every subsets X, Y of $\mathcal{E}$ and any structuring element B we have

$$X \oplus B \subseteq Y \quad \Longleftrightarrow \quad X \subseteq Y \ominus B. \tag{1.6}$$

Moreover (1.5) follows from (1.6), as we will see in Section 2. It will be a consequence of Section 3 that the Minkowski addition and subtraction are the only translation-invariant dilations and erosions of a Euclidean or digital space $\mathcal{E}$. Formulas (1.5) and (1.6) form the basis for an abstract definition of dilation and erosion on an arbitrary complete lattice, and general properties of these operations do not depend on their particular form, but on general properties pertaining to the order relation and the two operations of supremum and infimum.

Such abstract definitions also allow us to define new types of dilations and erosions which share the properties of the standard ones. Heijmans [8] for example considers dilations and erosions on the Euclidean plane invariant under rotations and scalar multiplications, and Herman [9] has introduced a new type of dilations and erosions for grey-level images based on multiplicative structuring functions: see also Section 4. This is our main motivation for the generalization of mathematical morphology to complete lattices: it unifies a number of particular examples into one abstract mathematical framework. A second motivation intimately connected to the previous one is that an abstract approach provides a deeper insight into the essence of the theory (which assumptions are minimally required to have certain properties?) and links it to other, sometimes rather old, mathematical disciplines. For example, a theorem by Matheron on the decomposition of an increasing operator as a supremum of erosions will contain as particular case a well-known theorem in switching theory on the decomposition of an increasing Boolean function as a maximum of minima.

In [19] it is shown with concrete examples that even a practical engineering approach to the computer implementation of morphological operations must take into account the complete lattice structure of the object space.

The expression of mathematical morphology in the general framework of complete lattices was initiated by Matheron [15] and Serra [24]. Our aim is to pursue this work in a systematic way, and to link it with classical results of lattice theory [1,3]. By this we hope to give a sound basis to current research on mathematical morphology, to allow the design and validation of new types of morphological operators, and also to help preventing the periodic 'reinvention of the wheel' happening too often in applied mathematics and engineering, where 'new ideas' are sometimes particular cases of 'old ideas' in pure mathematics.

This paper represents the first part of our work. It is devoted to basic concepts concerning complete lattices and operators, to the analysis of dilations and erosions, and to the generalization of the property of translation-invariance. In a second paper [21] we will deal with openings and closings, including their relations with translation-invariance.

The rest of this section is devoted to the recall of basic definitions and results concerning complete lattices, and to the introduction of our notation. In Section 2 we give a general analysis of dilations and erosions, which contains important results from [24], but also well-known ones from [3]. Section 3 is to our knowledge entirely new. In it we show how to generalize the Euclidean notion of translation-invariance to complete lattices having an abelian group of automorphisms with certain properties. This will allow us to introduce a generalization of the Minkowski addition and subtraction, and to characterize in terms of them dilations and erosions which commute with that group. In Section 4 we will illustrate our approach with several examples and applications.

1.3. Complete lattices

In this subsection we will introduce the mathematical background needed for the rest of the paper. As the notion of a complete lattice is not familiar to the image processing community, it is worth recalling its definition and main properties in some detail. For a thorough exposition, the reader is referred to [1], especially Chapters 1 and 5.

1.3.1. Basic notions. Consider a set $\mathcal{L}$; a binary relation $\leq$ on $\mathcal{L}$ is called a *partial order relation* if it is

(*i*) *reflexive*: for any $X \in \mathcal{L}$, $X \leq X$;

(*ii*) *antisymmetric*: for any $X, Y \in \mathcal{L}$, if $X \leq Y$ and $Y \leq X$, then $X = Y$;

(*iii*) *transitive*: for any $X, Y, Z \in \mathcal{L}$, if $X \leq Y$ and $Y \leq Z$, then $X \leq Z$.

We say then that $(\mathcal{L}, \leq)$ is a *partially ordered set*, or in brief a *poset*. The reverse relation $\geq$ (defined by $X \geq Y$ if and only if $Y \leq X$) is also a partial order relation. When $X \leq Y$ and $X \neq Y$, we write $X < Y$ or $Y > X$. The negation of $X \leq Y$ is written $X \not\leq Y$ or $Y \not\geq X$.

Given $L, U \in \mathcal{L}$ and $\mathcal{K} \subseteq \mathcal{L}$, we say that U is an *upper bound* of $\mathcal{K}$ if for any $K \in \mathcal{K}$ we have $U \geq K$, and that L is a *lower bound* of $\mathcal{K}$ if for any $K \in \mathcal{K}$ we have $L \leq K$. A *supremum* of $\mathcal{K}$ in $(\mathcal{L}, \leq)$ is a *least upper bound* of $\mathcal{K}$, in other words an upper bound X of $\mathcal{K}$ such that $X \leq U$ for any other upper bound U of $\mathcal{K}$. Conversely, an *infimum* of $\mathcal{K}$ in $(\mathcal{L}, \leq)$ is a *greatest lower bound* of $\mathcal{K}$, in other words a lower bound Y of $\mathcal{K}$ such that $Y \geq L$ for any other lower bound L of $\mathcal{L}$. By the antisymmetry of $\leq$, the supremum and infimum of $\mathcal{K}$ in $(\mathcal{L}, \leq)$ are unique whenever they exist.

The supremum of $\mathcal{K}$ in $(\mathcal{L}, \leq)$ will be written $\sup \mathcal{K}$ or $\bigvee \mathcal{K}$, while the infimum will be written $\inf \mathcal{K}$ or $\bigwedge \mathcal{K}$. When there can be an ambiguity on the poset in which the supremum and infimum are taken, we can be more precise and write $\sup_{\mathcal{L}}$ or even $\sup_{(\mathcal{L}, \leq)}$ for the supremum, and similarly for the infimum.

Now we will say that the poset $\mathcal{L}$ is a *complete lattice* if every nonvoid subset $\mathcal{K}$ of $\mathcal{L}$ has a supremum and an infimum. Two elements of the complete lattice $\mathcal{L}$ are important: the *universal bounds*. They are the *greatest* element I and the *least* element O, defined by $O \leq X \leq I$ for every $X \in \mathcal{L}$. Their existence and uniqueness follow from the equalities $I = \sup \mathcal{L}$ and $O = \inf \mathcal{L}$. Let us give some examples of complete lattices:

(1°) $\overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty, -\infty\}$, with the usual order $\leq$, infimum, supremum, and with universal bounds $+\infty$ and $-\infty$.

(2°) The set of natural integers, ordered by the relation 'divides'; the supremum is the lowest common multiple, the infimum the highest common divisor, and the greatest and least elements are 0 and 1 respectively.

(3°) The set of parts of a set E, ordered by set inclusion, where the supremum and infimum are the union and the intersection respectively, and the universal bounds are $\emptyset$ and E.

(4°) If E is a topological space and $\mathcal{L}$ is the set of closed sets of E, then the infimum and supremum of a subset $\mathcal{K}$ of $\mathcal{L}$ are $\bigcap \mathcal{K}$ and $\overline{\bigcup \mathcal{K}}$ respectively, where for every $X \subseteq E$, $\overline{X}$ is the topological closure of X; the universal bounds are E and $\emptyset$.

In a complete lattice $\mathcal{L}$, any element of $\mathcal{L}$ is both an upper and a lower bound of the empty subset $\emptyset$ of $\mathcal{L}$. Thus the least upper bound of $\emptyset$ is the least element, and the greatest lower bound of $\emptyset$ is the greatest element. In other words,

$$O = \bigvee \emptyset \qquad \text{and} \qquad I = \bigwedge \emptyset. \tag{1.7}$$

Thus *every* subset of $\mathcal{L}$ has a supremum and an infimum, not only nonvoid ones.

Given the complete lattice $\mathcal{L}$, there are some other usual conventions for the notation. If a subset $\mathcal{K}$ of $\mathcal{L}$ is written under the form of a set of expressions satisfying some condition, then $\bigvee \mathcal{K}$ can be written with the condition under the $\bigvee$ sign, for example

$$\bigvee_{j \in J} X_j \qquad \text{for} \qquad \bigvee \{X_j \mid j \in J\},$$

and similarly for $\bigwedge \mathcal{K}$. When $\mathcal{K}$ is finite and we have $\mathcal{K} = \{X_1, \ldots, X_n\}$, we will write $X_1 \vee \cdots \vee X_n$ and $X_1 \wedge \cdots \wedge X_n$ respectively. These two expressions use the binary operations $\vee$ and $\wedge$ (the supremum and infimum of two elements of $\mathcal{L}$).

We have the following well-known characterization of complete lattices (see Theorem 3 in Chapter 5 of [1]):

PROPOSITION 1.1. *Let $(\mathcal{L}, \leq)$ be a poset. Then the following three statements are equivalent:*

(i) $\mathcal{L}$ is a complete lattice.

(ii) $\mathcal{L}$ has a least element O and every subset of $\mathcal{L}$ has a supremum.

(iii) $\mathcal{L}$ has a greatest element I and every subset of $\mathcal{L}$ has an infimum.

Note that in (*ii*) $O = \sup \emptyset$ and in (*iii*) $I = \inf \emptyset$. Moreover in (*ii*) the infimum of a subset $\mathcal{K}$ of $\mathcal{L}$ is defined as the supremum of the set $\mathrm{LB}(\mathcal{K})$ of its lower bounds, while in (*iii*) the supremum of $\mathcal{K}$ is defined as the infimum of the set $\mathrm{UB}(\mathcal{K})$ of its upper bounds.

Given a set E, the set $\mathcal{L}^E$ of functions $E \to \mathcal{L}$ inherits the complete lattice structure of $\mathcal{L}$. For any $X, Y \in \mathcal{L}^E$, we set

$$X \leq Y \qquad \Longleftrightarrow \qquad \forall e \in E, \quad X(e) \leq Y(e).$$

Then $\mathcal{L}^E$ is a complete lattice, with the supremum and infimum given by

$$(\sup_{\mathcal{L}^E} \mathcal{K})(e) = \sup_{\mathcal{L}} \{X(e) \mid X \in \mathcal{K}\}$$
$$\text{and} \qquad (\inf_{\mathcal{L}^E} \mathcal{K})(e) = \inf_{\mathcal{L}} \{X(e) \mid X \in \mathcal{K}\}$$

for any $\mathcal{K} \subseteq \mathcal{L}^E$ and $e \in E$. This power structure intervenes for example in the set $\mathcal{G}^{\mathcal{E}}$ of grey-level functions from the Euclidean or digital space $\mathcal{E}$ to a complete lattice $\mathcal{G}$ of grey-levels, and in the rest of the paper we will consider the complete lattice $\mathcal{O} = \mathcal{L}^{\mathcal{L}}$ of transformations of $\mathcal{L}$.

Finally, given two complete lattices $(\mathcal{L}, \leq)$ and $(\mathcal{L}', \leq)$, an *isomorphism* from $\mathcal{L}$ to $\mathcal{L}'$ is a bijection $\psi : \mathcal{L} \to \mathcal{L}'$ which induces a bijection between their order relations, in other words such that for any $X, Y \in \mathcal{L}$, $X \leq Y$ if and only if $\psi(X) \leq \psi(Y)$. Clearly ψ commutes then with $\bigvee$ and $\bigwedge$:

$$\psi(\bigvee_{j \in J} X_j) = \bigvee_{j \in J} \psi(X_j) \qquad \text{and} \qquad \psi(\bigwedge_{j \in J} X_j) = \bigwedge_{j \in J} \psi(X_j).$$

An isomorphism $\mathcal{L} \to \mathcal{L}$ is called an *automorphism* of $\mathcal{L}$.

1.3.2. The principle of duality. We said above that the reverse $\geq$ of an order relation $\leq$ is itself an order relation. This reversion extends then to the supremum and infimum, since we have for any $\mathcal{K} \subseteq \mathcal{L}$:

$$\sup_{(\mathcal{L}, \geq)} \mathcal{K} = \inf_{(\mathcal{L}, \leq)} \mathcal{K};$$
$$\inf_{(\mathcal{L}, \geq)} \mathcal{K} = \sup_{(\mathcal{L}, \leq)} \mathcal{K}.$$

The universal bounds of $(\mathcal{L}, \geq)$ are those of $(\mathcal{L}, \leq)$, but interchanged.

Thus if $(\mathcal{L}, \leq)$ is a complete lattice, with supremum $\bigvee$, infimum $\bigwedge$, least element O, and greatest element I, then $(\mathcal{L}, \geq)$ is also a complete lattice, but this time with supremum $\bigwedge$, infimum $\bigvee$, least element I, and greatest element O. We call it the *dual lattice* of $(\mathcal{L}, \leq)$. Note that conversely $(\mathcal{L}, \leq)$ is the dual lattice of $(\mathcal{L}, \geq)$.

Thus to every definition, property, or statement on $(\mathcal{L}, \leq)$ corresponds a *dual* one on $(\mathcal{L}, \geq)$, where we interchange $\leq$ and $\geq$, $\bigvee$ and $\bigwedge$, O and I. This trivial but important fact is called the *duality principle*. It will be illustrated throughout the sequel. For example we will see in Section 2 that dilations and erosions are dual concepts.

Given a second lattice $(\mathcal{L}', \leq)$, a *dual isomorphism* from $\mathcal{L}$ to $\mathcal{L}'$ is a bijection $\psi : \mathcal{L} \to \mathcal{L}'$ such that for any $X, Y \in \mathcal{L}$, $X \leq Y$ if and only if $\psi(X) \geq \psi(Y)$. This makes $(\mathcal{L}, \leq)$ isomorphic to the dual $(\mathcal{L}', \geq)$ of $(\mathcal{L}', \leq)$. For example the set complementation on a topological space $\mathcal{E}$ induces a dual isomorphism between the complete lattice $\mathcal{F}(\mathcal{E})$ of closed subsets of $\mathcal{E}$ and the complete lattice $\mathcal{G}(\mathcal{E})$ of open subsets of $\mathcal{E}$, both ordered by inclusion.

1.3.3. Closed subsets and complete sublattices. Given a complete lattice $\mathcal{L}$, a subset $\mathcal{M}$ of $\mathcal{L}$ is called *inf-closed* if for any subset $\mathcal{U}$ of $\mathcal{M}$ (including $\emptyset$), $\bigwedge \mathcal{U} \in \mathcal{M}$. In particular it contains $\bigwedge \emptyset = I$. The dual concept is that of a *sup-closed* subset, it is defined similarly with $\bigvee$ instead of $\bigwedge$.

From Proposition 1.1, an inf-closed subset $\mathcal{M}$ is itself a complete lattice, with the same infimum as in $\mathcal{L}$, but with a different supremum than $\bigvee$. For $\mathcal{K} \subseteq \mathcal{M}$ we have

$$\inf_{\mathcal{M}} \mathcal{K} = \bigwedge \mathcal{K} \qquad \text{and} \qquad \sup_{\mathcal{M}} \mathcal{K} = \bigwedge \mathrm{UB}_{\mathcal{M}}(\mathcal{K}),$$

where $\mathrm{UB}_{\mathcal{M}}(\mathcal{K})$ is the set of upper bounds of $\mathcal{K}$ in $\mathcal{M}$. Moreover $\mathcal{M}$ has $I = \bigwedge \emptyset$ as greatest element, but its least element is not necessarily O.

Consider for example a vector space V; let $\mathcal{L}$ be the set of parts of V, ordered by inclusion, and let $\mathcal{M}$ be the set of vector subspaces of V. Then $\mathcal{L}$ is a complete lattice with the union as supremum and the intersection as infimum, and $\mathcal{M}$ is a $\bigcap$-closed subset in $\mathcal{L}$. In other words V is a vector subspace of V, and the intersection of vector subspaces of V is itself a vector subspace of V. Then $\mathcal{M}$ is a complete lattice, with again the intersection as infimum, but with a supremum which is not the union, but the sum of vector subspaces. Moreover, $\mathcal{M}$ has the same greatest element V as $\mathcal{L}$, but not the same least element: the null vector space $\{0\}$ instead of $\emptyset$.

What we have said here can be dually transposed to sup-closed subsets.

A subset $\mathcal{M}$ of $\mathcal{L}$ is called a *complete sublattice* of $\mathcal{L}$ if it is a complete lattice with $\bigvee$ and $\bigwedge$ as supremum and infimum, in other words if it is both sup- and inf-closed.

1.3.4. Generating families and atomic lattices. We end this subsection with some concepts that will be used extensively in Sections 3 and 4.

A subset ℓ of $\mathcal{L}$ is called *sup-generating* if every element of $\mathcal{L}$ can be written as a supremum of elements of ℓ. We use lower-case letters for elements of ℓ. For every $X \in \mathcal{L}$ we set $\ell(X) = \{x \in \ell \mid x \leq X\}$, and then $X = \bigvee \ell(X)$. We have the following general properties:

$$\begin{aligned} \ell(\bigwedge_{j\in J} X_j) &= \bigcap_{j\in J} \ell(X_j), \\ \ell(\bigvee_{j\in J} X_j) &\supseteq \bigcup_{j\in J} \ell(X_j), \\ \bigvee(\bigcup_{j\in J} \ell(X_j)) &= \bigvee_{j\in J} X_j. \end{aligned} \tag{1.8}$$

The dual concept is that of a *inf-generating* subset, which is defined in an analogous way.

For example if $(\mathcal{L}, \subseteq)$ is the set of parts of a set E, the set ℓ of singletons is $\bigcup$-generating, and $\ell(X)$ is the set of singletons corresponding to elements of X. This is a particular case of an important class of lattices for which a sup-generating family exists, namely the so-called *atomic* complete lattices. An element $A \neq O$ of $\mathcal{L}$ is called a *point* or an *atom* if for any $Y \in \mathcal{L}$, $O \leq Y \leq A$ implies that $Y = O$ or $Y = A$, in other words if there is no $Y \in \mathcal{L}$ such that $O < Y < A$. Note that atoms are always members of any sup-generating subset of $\mathcal{L}$. Now the complete lattice $\mathcal{L}$ is called *atomic* if the set of its atoms is sup-generating, in other words if every element of $\mathcal{L}$ is the supremum of the atoms less than or equal to it.

The complete lattice $\mathcal{L}$ is *Boolean* if it satisfies the distributivity laws $X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z)$ and $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$ for all $X, Y, Z \in \mathcal{L}$, and if every element X has a complement X', defined by $X' \vee X = I$ and $X' \wedge X = O$. By the distributivity laws, such a complement is *unique* (see Theorem 10 in Chapter 1 of [1]).

The set $\mathcal{P}(E)$ of parts of a set E is an atomic Boolean complete lattice. It can be shown that an atomic complete lattice $\mathcal{L}$ in which every element X has a *unique* complement X', is isomorphic to the set of parts $\mathcal{P}(A)$ of the set A of atoms of $\mathcal{L}$ (see Theorem 18 in Chapter 5 of [1]). In other words, an atomic complete lattice is Boolean if and only if it is isomorphic to the set of parts of its set of atoms.

2. Operators, Dilations, and Erosions

We take a complete lattice $\mathcal{L}$ with the order relation $\leq$, supremum $\bigvee$, infimum $\bigwedge$, least element O and greatest element I. Elements of $\mathcal{L}$ will be written as capital letters X, Y, Z, etc.. In practice $\mathcal{L}$ will correspond to a particular set of pictures we work with. For example, if we consider binary images on a Euclidean or digital space $\mathcal{E}$, we take $\mathcal{L} = \mathcal{P}(\mathcal{E})$, and if we consider grey-level images on $\mathcal{E}$, we take $\mathcal{L} = \mathcal{G}^{\mathcal{E}}$, the set of functions $\mathcal{E} \rightarrow \mathcal{G}$, where the set $\mathcal{G}$ of grey-levels is itself a complete lattice.

We consider the set $\mathcal{O}$ of all transformations on $\mathcal{L}$, in other words the set $\mathcal{L}^{\mathcal{L}}$ of functions $\mathcal{L} \rightarrow \mathcal{L}$. Given $X \in \mathcal{L}$ and $\theta \in \mathcal{O}$, θ maps X to $\theta(X)$, which will be called the *transform of X by θ*. Elements of $\mathcal{O}$ will be called *operators*. Three particular operators are worth mentioning right now:

— the identity **id** defined by $\mathbf{id}(X) = X$ for every $X \in \mathcal{L}$;

— the constant operators $\mathbf{O}$ and $\mathbf{I}$, defined by $\mathbf{O}(X) = O$ and $\mathbf{I}(X) = I$ for every $X \in \mathcal{L}$. Other operators will be written by lowercase greek letters β, γ, etc., with the letters α, δ, ε, φ, τ being reserved to openings, dilations, erosions, closings, and 'translations'.

Since $\mathcal{O}$ is a power of $\mathcal{L}$, the complete lattice structure of $\mathcal{L}$ extends to $\mathcal{O}$, as we have explained at the end of Section 1. The order relation $\leq$ on $\mathcal{L}$ can be extended to an order relation on $\mathcal{O}$ by setting for $\eta, \theta \in \mathcal{O}$ (*cfr.* Subsection 1.3):

$$\eta \leq \theta \qquad \Longleftrightarrow \qquad \forall X \in \mathcal{L}, \quad \eta(X) \leq \theta(X). \tag{2.1}$$

Then $\mathcal{O}$ is a complete lattice, and we will write $\bigvee$ and $\bigwedge$ for the supremum and infimum in $\mathcal{O}$, as we do in $\mathcal{L}$. For any $X \in \mathcal{L}$ and $\mathcal{Q} \subseteq \mathcal{O}$ we have (*cfr.* Subsection 1.3):

$$\begin{aligned} &(\bigvee \mathcal{Q})(X) = \bigvee_{\eta \in \mathcal{Q}} \eta(X) \\ \text{and} \qquad &(\bigwedge \mathcal{Q})(X) = \bigwedge_{\eta \in \mathcal{Q}} \eta(X). \end{aligned} \tag{2.2}$$

Moreover $(\mathcal{O}, \leq)$ has least element $\mathbf{O}$ and greatest element $\mathbf{I}$.

The *composition* $\eta\theta$ of the operator θ by the operator η is defined by setting

$$\eta\theta(X) = \eta\big(\theta(X)\big) \tag{2.3}$$

for every $X \in \mathcal{L}$. This operation is associative, in other words $\beta(\eta\theta) = (\beta\eta)\theta$ for any $\beta, \eta, \theta \in \mathcal{O}$. Whenever a composition of operators appears in an expression, it must be considered as forming a group, as if it was surrounded by parentheses.

From (2.1) it follows that for any $\beta, \eta, \theta \in \mathcal{O}$,

$$\eta \leq \theta \implies \eta\beta \leq \theta\beta. \tag{2.4}$$

It also is an easily shown consequence of (2.2) that for any $\beta \in \mathcal{O}$ and $\mathcal{Q} \subseteq \mathcal{O}$ we have

$$\begin{aligned} &(\bigvee \mathcal{Q})\beta = \bigvee_{\eta \in \mathcal{Q}} \eta\beta \\ \text{and} \qquad &(\bigwedge \mathcal{Q})\beta = \bigwedge_{\eta \in \mathcal{Q}} \eta\beta. \end{aligned} \tag{2.5}$$

2.1. Increasing operators, dilations, and erosions

We will now introduce three classes of operators having certain properties related to the order $\leq$ and the operations $\bigvee$ and $\bigwedge$:

DEFINITION 2.1. Let $\beta \in \mathcal{O}$. Then we say that:

(a) β is *increasing* if for every $X, Y \in \mathcal{L}$, $X \leq Y$ implies that $\beta(X) \leq \beta(Y)$.

(b) β is a *dilation* if for every $\mathcal{T} \subseteq \mathcal{L}$, $\beta(\bigvee \mathcal{T}) = \bigvee_{X \in \mathcal{T}} \beta(X)$.

(c) β is an *erosion* if for every $\mathcal{T} \subseteq \mathcal{L}$, $\beta(\bigwedge \mathcal{T}) = \bigwedge_{X \in \mathcal{T}} \beta(X)$.

Note that in (b) and (c) we must also take into account the case where $\mathcal{T}$ is empty. Thus (by (1.7)) a dilation preserves O and an erosion preserves I. Dilations will be written $\delta, \delta', \delta_1$, etc., while erosions will be written $\varepsilon, \varepsilon', \varepsilon_1$, etc.. This general definition of dilations and erosions is due to Serra [24]. It includes as particular cases the Minkowski operations (see (1.5)) and the grey-level dilations and erosions by structuring functions.

The concept of increasingness is its own dual, while the concept of a dilation is the dual of that of an erosion. An operator β is an automorphism if and only if β is a bijection and both β and β^{-1} are increasing. Note that an automorphism β is both a dilation and an erosion.

It is easily seen (by (2.1) and (2.2)) that for any $\beta \in \mathcal{O}$ the following hold:

— If β is increasing, then for any $\eta, \theta \in \mathcal{O}$, $\eta \leq \theta \implies \beta\eta \leq \beta\theta$.

— If β is a dilation, then for every $\mathcal{Q} \subseteq \mathcal{O}$, $\beta(\bigvee \mathcal{Q}) = \bigvee_{\gamma \in \mathcal{Q}} \beta\gamma$.

— If β is an erosion, then for every $\mathcal{Q} \subseteq \mathcal{O}$, $\beta(\bigwedge \mathcal{Q}) = \bigwedge_{\gamma \in \mathcal{Q}} \beta\gamma$.

These equalities represent in some-way a mirror-image property of (2.4) and (2.5).

Let us first give a few elementary properties of increasing operators. Afterwards we will deal with dilations and erosions.

LEMMA 2.1. *Let $\beta \in \mathcal{O}$. Then the following three statements are equivalent:*

(*i*) *β is increasing.*

(*ii*) *For every $\mathcal{T} \subseteq \mathcal{L}$, $\beta(\bigvee \mathcal{T}) \geq \bigvee_{X \in \mathcal{T}} \beta(X)$.*

(*iii*) *For every $\mathcal{T} \subseteq \mathcal{L}$, $\beta(\bigwedge \mathcal{T}) \leq \bigwedge_{X \in \mathcal{T}} \beta(X)$.*

In particular dilations and erosions are increasing.

PROOF. We only show that (*i*) is equivalent to (*ii*), the equivalence between (*i*) and (*iii*) follows then by duality.

(*i*) *implies* (*ii*): Given $\mathcal{T} \subseteq \mathcal{L}$, for every $X \in \mathcal{T}$ we have $\bigvee \mathcal{T} \geq X$, and as β is increasing, $\beta(\bigvee \mathcal{T}) \geq \beta(X)$. By definition of the supremum, $\beta(\bigvee \mathcal{T}) \geq \bigvee_{X \in \mathcal{T}} \beta(X)$.

(*ii*) *implies* (*i*): Let $Y, Z \in \mathcal{L}$ with $Y \geq Z$. Take $\mathcal{T} = \{Y, Z\}$. Then $Y = Z \vee Y = \bigvee \mathcal{T}$, and by (*ii*) we have

$$\beta(Y) = \beta(\bigvee \mathcal{T}) \geq \bigvee_{X \in \mathcal{T}} \beta(X) = \beta(Y) \vee \beta(Z),$$

which implies that $\beta(Y) \geq \beta(Z)$.

As dilations satisfy (*ii*) and erosions satisfy (*iii*), they are increasing. ∎

PROPOSITION 2.2. *The set of increasing operators is*

(*i*) *closed under composition, and it contains* id;

(*ii*) *a complete sublattice of $\mathcal{O}$.*

PROOF. Take $X, Y \in \mathcal{L}$ such that $X \leq Y$.

(*i*) It is obvious that id is increasing. Let η, θ be increasing operators. As θ is increasing, we get $\theta(X) \leq \theta(Y)$, and as η is increasing, we obtain $\eta(\theta(X)) \leq \eta(\theta(Y))$. Thus $\eta\theta$ is increasing.

(ii) It is obvious that $\mathbf{O}$ and $\mathbf{I}$ are increasing. Consider now a non-empty set $\mathcal{Q}$ of increasing operators. Let us show that $\beta = \bigvee \mathcal{Q}$ is increasing. For every $\eta \in \mathcal{Q}$, as η is increasing, we have $\eta(X) \leq \eta(Y)$. Now $\eta \leq \beta$ (since η intervenes in the $\bigvee$-decomposition of β), and so $\eta(Y) \leq \beta(Y)$. Thus $\eta(X) \leq \beta(Y)$. Hence

$$\beta(X) = \bigvee_{\eta \in \mathcal{Q}} \eta(X) \leq \beta(Y),$$

and so β is increasing.

By duality, $\bigwedge \mathcal{Q}$ is also increasing. Hence the set of increasing operators is a complete sublattice of $\mathcal{O}$. ∎

Let us now turn to dilations and erosions. By duality, it is sufficient to prove results about dilations only, and dual results about erosions will follow immediately. The following analogue of Proposition 2.2 comes from [24]:

PROPOSITION 2.3. *The set of dilations is*

(i) closed under composition, and it contains id;

(ii) a sup-closed subset of $\mathcal{O}$.

PROOF. (i) Consider two dilations δ, δ'. For any $\mathcal{T} \subseteq \mathcal{L}$,

$$(\delta\delta')(\bigvee \mathcal{T}) = \delta(\delta'(\bigvee \mathcal{T})) = \delta(\bigvee_{X \in \mathcal{T}} \delta'(X)) = \bigvee_{X \in \mathcal{T}} \delta(\delta'(X)) = \bigvee_{X \in \mathcal{T}} (\delta\delta')(X),$$

and so $\delta\delta'$ is a dilation. Now clearly

$$\mathrm{id}(\bigvee \mathcal{T}) = \bigvee \mathcal{T} = \bigvee_{X \in \mathcal{T}} X = \bigvee_{X \in \mathcal{T}} \mathrm{id}(X),$$

and so id is a dilation.

(ii) We must show that for any set $\mathcal{Q}$ of dilations (including a void one) $\bigvee \mathcal{Q}$ is a dilation, in other words that for any $\mathcal{T} \subseteq \mathcal{L}$,

$$(\bigvee \mathcal{Q})(\bigvee \mathcal{T}) = \bigvee_{X \in \mathcal{T}} \big((\bigvee \mathcal{Q})(X)\big).$$

Indeed we have

$$\begin{aligned}
(\bigvee \mathcal{Q})(\bigvee \mathcal{T}) &= \bigvee_{\delta \in \mathcal{Q}} \delta(\bigvee \mathcal{T}) && \text{(by definition, see(2.2));}\\
&= \bigvee_{\delta \in \mathcal{Q}} (\bigvee_{X \in \mathcal{T}} \delta(X)) && \text{(since each } \delta \in \mathcal{Q} \text{ is a dilation);}\\
&= \bigvee_{X \in \mathcal{T}} (\bigvee_{\delta \in \mathcal{Q}} \delta(X)) && \text{(by the commutativity of } \textstyle\bigvee);\\
&= \bigvee_{X \in \mathcal{T}} \big((\bigvee \mathcal{Q})(X)\big) && \text{(by (2.2)).}
\end{aligned}$$

(Note that this argument is also valid if $\mathcal{Q}$ or $\mathcal{T}$ is empty.) ∎

The statement of the dual result concerning erosions is left to the reader.

A particular consequence of Proposition 2.3 (and its dual) is that the set of dilations and the set of erosions are complete lattices (thanks to Proposition 1.1, see Subsection 1.3). The least dilation is $\mathbf{O}$, while the greatest one fixes O and transforms every other $X \in \mathcal{L}$ into I. The supremum for dilations is $\bigvee$, but the infimum is not $\bigwedge$. Similarly, the greatest erosion is $\mathbf{I}$, while the least one fixes I and transforms every other $X \in \mathcal{L}$ into O. The infimum for erosions is $\bigwedge$, but the supremum is not $\bigvee$.

Let us illustrate Proposition 2.3 in the case of Minkowski operations on a Euclidean space $\mathcal{E}$. For any structuring element $B \subseteq \mathcal{E}$, write δ_B for the dilation $X \to X \oplus B$, and ε_B for the erosion $X \to X \ominus B$. Write o for the origin. Take any $X, B, B' \subseteq \mathcal{E}$. We have the following:

(a) $X \oplus \{o\} = X \ominus \{o\} = X$, in other words $\delta_{\{o\}} = \varepsilon_{\{o\}} = \mathrm{id}$.

(b) $(X \oplus B) \oplus B' = X \oplus (B \oplus B')$ and $(X \ominus B) \ominus B' = X \ominus (B \oplus B')$, that is $\delta_B \delta_{B'} = \delta_{B \oplus B'}$ and $\varepsilon_B \varepsilon_{B'} = \varepsilon_{B \oplus B'}$.

Hence (i) holds for Minkowski addition and subtraction. Take now a family of structuring elements B_j, where $j \in J$. We have the following:

(c) $X \oplus (\bigcup_{j\in J} B_j) = \bigcup_{j \in J}(X \oplus B_j)$ and $X \ominus (\bigcup_{j\in J} B_j) = \bigcap_{j\in J}(X \ominus B_j)$, in other words $\delta_{\bigcup_{j\in J} B_j} = \bigvee_{j\in J} \delta_{B_j}$ and $\varepsilon_{\bigcup_{j\in J} B_j} = \bigwedge_{j\in J} \varepsilon_{B_j}$.

(d) $X \oplus \emptyset = \emptyset$ and $X \ominus \emptyset = \mathcal{E}$, that is $\delta_\emptyset = \mathbf{O}$ and $\varepsilon_\emptyset = \mathbf{I}$.

Therefore (ii) holds for Minkowski addition and subtraction.

2.2. *Matheron's Theorem*

Matheron proved [14] that if ψ is an increasing translation-invariant transformation of the set $\mathcal{P}(\mathcal{E})$ of parts of the Euclidean space $\mathcal{E}$ (with origin o), then for every $X \subseteq \mathcal{E}$ we have

$$\psi(X) = \bigcup_{B \in \mathcal{V}[\psi]} (X \ominus B), \qquad \text{where} \qquad \mathcal{V}[\psi] = \{B \subseteq \mathcal{E} \mid o \in \psi(B)\}. \tag{2.6}$$

In other words an increasing translation-invariant transformation of $\mathcal{E}$ is a supremum of translation-invariant erosions. By duality, it is also an infimum of translation-invariant dilations.

In Section 3 we will generalize this result to complete lattices having a certain type of abelian group of automorphisms generalizing translations. However, in the case where we omit translation-invariance, the corresponding result for complete lattices was obtained by Serra [24]. We will give here a slightly modified proof of Serra's result.

THEOREM 2.4. *Let $\psi \in \mathcal{O}$. Then the following two statements are equivalent:*

(i) ψ is increasing and $\psi(I) = I$.

(ii) ψ is the supremum of a non-empty set of erosions.

PROOF. *(ii) implies (i):* Let $\mathcal{Q}$ be a non-empty set of erosions. First, we have

$$(\bigvee \mathcal{Q})(I) = \bigvee_{\varepsilon \in \mathcal{Q}} \varepsilon(I) = \bigvee_{\varepsilon \in \mathcal{Q}} I = I.$$

(The last equality follows from the fact that the set in which one takes the supremum is non-empty). Second, Proposition 2.1 implies that every $\varepsilon \in \mathcal{Q}$ is increasing, and so by Proposition 2.2 *(ii)* $\bigvee \mathcal{Q}$ is increasing.

(i) implies (ii): Suppose that ψ is increasing and preserves I. For any $B \in \mathcal{L}$, consider the two operators ε_B^0 and ε_B^1 defined as follows:

$$\varepsilon_B^0(Z) = \begin{cases} I & \text{if } Z = I, \\ \psi(B) & \text{if } Z < I, \end{cases} \qquad \varepsilon_B^1(Z) = \begin{cases} I & \text{if } Z \geq B, \\ O & \text{if } Z \not\geq B. \end{cases}$$

It is easily checked that they are erosions. Thus their infimum $\varepsilon_B = \varepsilon_B^0 \wedge \varepsilon_B^1$ is an erosion by the dual of Proposition 2.3 *(ii)*. In fact, the two previous equalities imply that for $Z \in \mathcal{L}$ we have

$$\varepsilon_B(Z) = \begin{cases} I & \text{if } Z = I, \\ \psi(B) & \text{if } I > Z \geq B, \\ O & \text{if } Z \not\geq B. \end{cases} \tag{2.7}$$

Let $\gamma = \bigvee_{B \in \mathcal{L}} \varepsilon_B$. We must show that $\gamma = \psi$.

First it is clear by (2.7) that $\varepsilon_B(I) = I$ for any $B \in \mathcal{L}$, and so that $\gamma(I) = I = \psi(I)$. Take now $B, Z \in \mathcal{L}$ such that $Z < I$. If $Z \geq B$, then by (2.7) $\varepsilon_B(Z) = \psi(B)$; as ψ is increasing, this implies that $\varepsilon_B(Z) = \psi(B) \leq \psi(Z)$. If $Z \not\geq B$, then by (2.7) $\varepsilon_B(Z) = O \leq \psi(Z)$. Thus $\varepsilon_B(Z) \leq \psi(Z)$ for any $B \in \mathcal{L}$, and as $\varepsilon_Z(Z) = \psi(Z)$, we have $\gamma(Z) = \bigvee_{B \in \mathcal{L}} \varepsilon_B(Z) = \psi(Z)$. Hence $\gamma = \psi$ and so ψ is a supremum of erosions. ∎

There is of course a dual result with an infimum of dilations. In Subsection 2.4 we will show how Matheron's Theorem contains as a particular case a fundamental result in the theory of Boolean functions. In Section 4 we will illustrate it in the case of Euclidean translation-invariant operators. Let us give here an example of Theorem 2.4 in a complete lattice unrelated to the Euclidean plane. Let $\mathrm{N} = \{0, 1, 2, \ldots\}$, the set of natural integers, ordered by the relation 'divides'. We write $a \backslash b$ for 'a divides b'. Then N is a complete lattice with the lowest common multiple as supremum and the highest common divisor as infimum, 1 as least element, and 0 as greatest element. Let Π be the set of primes, and for every $n \in \mathrm{N}$, let $\pi(n) = \{p \in \Pi \mid p \backslash n\}$ (the set of prime divisors of n). Let ψ be the map defined by $\psi(0) = 0$, and $\psi(n) = \prod \pi(n)$ (the product of prime divisors of n) for an integer $n > 0$. Clearly $\psi(m) \backslash \psi(n)$ when $m \backslash n$, in other words ψ is increasing. For every prime p, the map ε_p corresponding to (2.7) is defined by $\varepsilon(0) = 0$ and $\varepsilon_p(n) = p \wedge n$ for $n > 0$. It is an erosion, and for every $n > 0$ we have $\psi(n) = \bigvee_{p \in \Pi} (p \wedge n)$.

2.3. Adjunctions and Galois connections

We said above that dilation and erosion are dual concepts from the lattice point of view. We will show that for any complete lattice $\mathcal{L}$, we always have a dual isomorphism between the complete lattice of dilations on $\mathcal{L}$ and the complete lattice of erosions on $\mathcal{L}$. This dual isomorphism is called by Serra [24] the *morphological duality*. In fact it is linked to what one calls *Galois connections* in lattice theory [1,3], as we will see at the end of this subsection.

2.3.1. Adjunctions. We take the following definition from [3]:

Definition 2.2. Let $\delta, \varepsilon \in \mathcal{O}$. Then we will say that (ε, δ) is an *adjunction* if for every $X, Y \in \mathcal{L}$, we have

$$\delta(X) \leq Y \quad \Longleftrightarrow \quad X \leq \varepsilon(Y). \tag{2.8}$$

In an adjunction (ε, δ), ε will be called the *upper adjoint* and δ the *lower adjoint*.

For example in the case of Minkowski operations on Boolean images, the erosion and dilation by a structuring element B form an adjunction (compare (1.6) with (2.8)). The grey-level erosion and dilation defined in (1.3) and (1.4) also form an adjunction.

Note that (2.8) can be expressed in a dual form with $\geq$ instead of $\leq$:

$$\varepsilon(Y) \geq X \quad \Longleftrightarrow \quad Y \geq \delta(X). \tag{2.9}$$

Thus (ε, δ) is an adjunction in $(\mathcal{L}, \leq)$ if and only if (δ, ε) is an adjunction in the dual lattice $(\mathcal{L}, \geq)$. Hence δ and ε will play dual roles. As is hinted by our notation and our examples above, in an adjunction (ε, δ), δ will be a dilation, and ε an erosion. This is shown by our next result, which comes from [3] (see their Theorem 3.3):

Proposition 2.5. *Let $\delta, \varepsilon \in \mathcal{O}$. If (ε, δ) is an adjunction, then δ is a dilation and ε is an erosion.*

Proof. We have only to show that δ is a dilation. The fact that ε is an erosion follows then by duality. As we have $O \leq \varepsilon(O)$ anyway, (2.8) implies that $\delta(O) \leq O$, and so $\delta(O) = O$. Take now a non-empty $\mathcal{T} \subseteq \mathcal{L}$, and let Y be any element of $\mathcal{L}$. We obtain the following succession of equivalent statements:

$$\begin{array}{ll}
\bigvee_{X \in \mathcal{T}} \delta(X) \leq Y; & \\
\forall X \in \mathcal{T}, \quad \delta(X) \leq Y & \text{(by definition of } \bigvee); \\
\forall X \in \mathcal{T}, \quad X \leq \varepsilon(Y) & \text{(by (2.8))}; \\
\bigvee \mathcal{T} \leq \varepsilon(Y) & \text{(by definition of } \bigvee); \\
\delta(\bigvee \mathcal{T}) \leq Y & \text{(by (2.8))}.
\end{array}$$

Thus for any $Y \in \mathcal{L}$, $\bigvee_{X \in \mathcal{T}} \delta(X) \leq Y$ if and only if $\delta(\bigvee \mathcal{T}) \leq Y$. Taking successively $Y = \bigvee_{X \in \mathcal{T}} \delta(X)$ and $Y = \delta(\bigvee \mathcal{T})$, we obtain $\bigvee_{X \in \mathcal{T}} \delta(X) = \delta(\bigvee \mathcal{T})$, in other words δ is a dilation. ∎

This result explains why for Minkowski operations on Boolean images, (1.6) implies (1.5), as we said in Subsection 1.2. In [24] an adjunction is called a *morphological duality*, because the set of adjunctions will constitute a dual isomorphism between the two complete lattices of dilations and of erosions. We will give equivalent definitions of adjunctions. But this requires a further definition:

Definition 2.3. Given $\eta \in \mathcal{O}$, we define $\dot{\eta} \in \mathcal{O}$ by setting

$$\dot{\eta}(Y) = \bigvee \{Z \in \mathcal{L} \mid \eta(Z) \leq Y\} \tag{2.10}$$

for every $Y \in \mathcal{L}$, and we define $\underset{\cdot}{\eta} \in \mathcal{O}$ by setting

$$\underset{\cdot}{\eta}(X) = \bigwedge\{Z \in \mathcal{L} \mid X \leq \eta(Z)\} \tag{2.11}$$

for every $X \in \mathcal{L}$.

Note that $\theta = \dot{\eta}$ in $(\mathcal{L}, \leq)$ if and only if $\theta = \underset{\cdot}{\eta}$ in $(\mathcal{L}, \geq)$. Thus the two definitions (2.10) and (2.11) are dual. We will see that in an adjunction (ε, δ), $\varepsilon = \dot{\delta}$ and $\delta = \underset{\cdot}{\varepsilon}$. Let us indeed give equivalent definitions of an adjunction:

Proposition 2.6. *Let $\delta, \varepsilon \in \mathcal{O}$. If (ε, δ) is an adjunction, then the following four statements hold:*

(i) ε is increasing and $\mathrm{id} \leq \varepsilon\delta$.
(i') $\varepsilon = \dot{\delta}$ (see (2.10)*).*
(ii) δ is increasing and $\delta\varepsilon \leq \mathrm{id}$.
(ii') $\delta = \underset{\cdot}{\varepsilon}$ (see (2.11)*).*
Conversely, if (i) or (i') holds, and (ii) or (ii') holds, then (ε, δ) is an adjunction.

Proof. It is sufficient to show that: (1°) if (ε, δ) is an adjunction, then (i') and (ii') hold; (2°) (i') implies (i), and (ii') implies (ii); (3°) if (i) and (ii) hold, then (ε, δ) is an adjunction.
(1°) *If (ε, δ) is an adjunction, then (i') and (ii') hold:* It is obvious that for every $Y \in \mathcal{L}$,

$$\varepsilon(Y) = \bigvee\{Z \in \mathcal{L} \mid Z \leq \varepsilon(Y)\}.$$

Now by the adjunction (ε, δ) the inequality $Z \leq \varepsilon(Y)$ is equivalent to $\delta(Z) \leq Y$ (see (2.8)). Thus the previous equation becomes:

$$\varepsilon(Y) = \bigvee\{Z \in \mathcal{L} \mid \delta(Z) \leq Y\}$$

for every $Y \in \mathcal{L}$, in other words $\varepsilon = \dot{\delta}$, that is ($i'$). Finally ($ii'$) is obtained a similar way (or follows by duality).
(2°) *(i') implies (i), and (ii') implies (ii):* We only show that (i') implies (i); the fact that (ii') implies (ii) follows by duality. For every $Y \in \mathcal{L}$, we define

$$E(Y) = \{Z \in \mathcal{L} \mid \delta(Z) \leq Y\}.$$

Then by (i') $\varepsilon(Y) = \dot{\delta}(Y) = \bigvee E(Y)$. Now if $Y \leq Y'$, then clearly $E(Y) \subseteq E(Y')$, and so $\bigvee E(Y) \leq \bigvee E(Y')$. Thus ε is increasing. Moreover, for every $Y \in \mathcal{L}$, $\delta(Y) \leq \delta(Y)$, and so $Y \in E(\delta(Y))$. Thus $Y \leq \bigvee E(\delta(Y)) = \varepsilon(\delta(Y))$, and so $\mathrm{id} \leq \varepsilon\delta$.
(3°) *If (i) and (ii) hold, then (ε, δ) is an adjunction:* Suppose that (i) holds and $\delta(X) \leq Y$. Then the fact that ε is increasing implies that $\varepsilon\delta(X) \leq \varepsilon(Y)$, and the fact that $\mathrm{id} \leq \varepsilon\delta$ implies that $X \leq \varepsilon\delta(X)$. Thus $\delta(X) \leq Y$ implies $X \leq \varepsilon(Y)$ when (i) holds. One shows similarly that $X \leq \varepsilon(Y)$ implies $\delta(X) \leq Y$ when (ii) holds. Thus when (i) and (ii) hold together, (ε, δ) is an adjunction. ∎

In [3], it is shown that the fact that (ε, δ) is an adjunction is equivalent to (i) and (ii'), to (i') and (ii), and to (i) and (ii) (see their Theorems 3.2 and 3.6).

Let us illustrate (i') in the case of Minkowski operations on Boolean images. Here (i') can be expressed as:

$$X \ominus B = \bigcup \{Z \subseteq \mathcal{E} \mid Z \oplus B \subseteq X\}.$$

It is easy to see that this is equivalent to the definition given in (1.2), namely:

$$X \ominus B = \{z \in \mathcal{E} \mid B_z \subseteq X\}.$$

We can now state our main result on the relation between dilations and erosions induced by adjunctions:

THEOREM 2.7. *The set of adjunctions constitutes a dual isomorphism between the two complete lattices of erosions and dilations. In other words:*

(i) *For any dilation δ, there is exactly one erosion ε such that (ε, δ) is an adjunction. We have $\varepsilon = \dot{\delta}$.*

(ii) *For any erosion ε, there is exactly one dilation δ such that (ε, δ) is an adjunction. We have $\delta = \underset{\cdot}{\varepsilon}$.*

(iii) *Given two dilations δ, δ' and two erosions $\varepsilon, \varepsilon'$ such that (ε, δ) and (ε', δ') are adjunctions, we have $\delta \leq \delta'$ if and only if $\varepsilon \geq \varepsilon'$.*

In particular this implies that:

(iv) *If $(\varepsilon_j, \delta_j)$ is an adjunction for every $j \in J$, then $(\bigwedge_{j \in J} \varepsilon_j, \bigvee_{j \in J} \delta_j)$ is an adjunction.*

Moreover, this dual isomorphism reverses the law of composition. In other words:

(v) *Given two dilations δ, δ' and two erosions $\varepsilon, \varepsilon'$ such that (ε, δ) and (ε', δ') are adjunctions, $(\varepsilon'\varepsilon, \delta\delta')$ is an adjunction.*

PROOF. (i) Let δ be a dilation. Take $\varepsilon = \dot{\delta}$ (see (2.10)). Clearly δ is increasing, and for every $Y \in \mathcal{L}$ the fact that δ is a dilation implies that

$$\delta\varepsilon(Y) = \delta(\dot{\delta}(Y)) = \delta(\bigvee\{Z \in \mathcal{L} \mid \delta(Z) \leq Y\}) = \bigvee\{\delta(Z) \mid Z \in \mathcal{L}, \delta(Z) \leq Y\} \leq Y,$$

in other words $\delta\varepsilon \leq \mathrm{id}$. Thus δ and ε satisfy conditions (i') and (ii) of Proposition 2.6, and so (ε, δ) is an adjunction. By Proposition 2.5 ε is an erosion. Given another erosion ε' such that (ε', δ) is an adjunction, then $\varepsilon' = \dot{\delta} = \varepsilon$ by condition (i') of Proposition 2.6.

(ii) is proved in the same way as (i), or can be deduced from it by duality.

(iii) The following statements are equivalent:

$\delta \leq \delta'$;

$\forall X \in \mathcal{L}, \quad \delta(X) \leq \delta'(X)$;

$\forall X, Y \in \mathcal{L}, \quad \delta'(X) \leq Y \implies \delta(X) \leq Y$;

$\forall X, Y \in \mathcal{L}, \quad X \leq \varepsilon'(Y) \implies X \leq \varepsilon(Y)$ (by definition of adjunctions);

$\forall Y \in \mathcal{L}, \quad \varepsilon'(Y) \leq \varepsilon(Y)$;

$\varepsilon' \leq \varepsilon$.

(iv) As the dual isomorphism reverses the order, it associates to a greatest lower bound of erosions a least upper bound of dilations. By Proposition 2.3, the greatest upper bound of erosions ε_j is $\bigwedge_{j \in J} \varepsilon_j$ and the least upper bound of dilations δ_j is $\bigvee_{j \in J} \delta_j$.

(v) For any $X, Y \in \mathcal{L}$ we have by (2.8):

$$\delta\delta'(X) \leq Y \iff \delta'(X) \leq \varepsilon(Y) \iff X \leq \varepsilon'\varepsilon(Y),$$

in other words $(\varepsilon'\varepsilon, \delta\delta')$ is an adjunction. ∎

Note in particular that $(\mathbf{I}, \mathbf{O})$ is an adjunction. Serra [24] proved (i), (ii) and (iii); there the dual isomorphism linking dilations and erosions is called *morphological duality*, and the upper adjoint of a dilation is called its *morphological dual.* The existence of a lower adjoint for every erosion, and of an upper adjoint for every dilation is also shown in Theorem 3.4 and Corollary 3.5 of [3].

The following result is well-known (see [24] or Theorem 3.6 of [3]):

PROPOSITION 2.8. *Given an adjunction* (ε, δ), $\delta\varepsilon\delta = \delta$ *and* $\varepsilon\delta\varepsilon = \varepsilon$.

PROOF. By Proposition 2.6 (i) and (ii) we have $\mathbf{id} \leq \varepsilon\delta$, $\delta\varepsilon \leq \mathbf{id}$, and δ is increasing. Hence

$$\delta = \delta\,\mathbf{id} \leq \delta(\varepsilon\delta) = (\delta\varepsilon)\delta \leq \mathbf{id}\,\delta = \delta,$$

that is $\delta\varepsilon\delta = \delta$. The other equality $\varepsilon\delta\varepsilon = \varepsilon$ follows by duality. ∎

We said in Subsection 2.1 that an automorphism is both a dilation and an erosion. We can thus describe its upper and lower adjoints. The following result will be used in Section 3:

PROPOSITION 2.9. *Given an automorphism* ψ *of* $\mathcal{L}$, ψ *is both a dilation and an erosion, and* $\dot{\psi} = \underset{\bullet}{\psi} = \psi^{-1}$

The proof is left to the reader. It can be achieved with (2.8) or with Proposition 2.6.

2.3.2. The relation with Galois connections. As we have seen above, several results proved here about adjunctions can also be found in sources dealing with the mathematical theory of complete lattices [3], although they are expressed there in very different terms. This is no mere coincidence. Indeed, the concept of adjunctions between dilations and erosions in mathematical morphology is closely linked to the concept of *Galois connections* between posets, which has been investigated for more than 40 years (bibliographic references on Galois connections can be found in [3], particularly in p. 29).

We will explicit this link here. Readers interested only in mathematical morphology can skip this portion and go directly to Subsection 2.4.

In Galois theory one considers a field K, an extension K' of K, and a group G of automorphisms of K' fixing all elements of K. Usually K' is the extension of K generated by the roots of an irreducible polynomial on K, and G is the group of permutations of these roots. We have the complete lattice $\mathcal{K}$ of subfields of K' containing K, and the complete lattice $\mathcal{G}$ of subgroups of G. The two maps $\gamma : \mathcal{K} \to \mathcal{G}$ and $\kappa : \mathcal{G} \to \mathcal{K}$ are defined by

$$\begin{aligned} \forall L \in \mathcal{K}, \quad & \gamma(L) = \{g \in G \mid \forall k \in L, g(k) = k\}, \\ \forall H \in \mathcal{G}, \quad & \kappa(H) = \{k \in K' \mid \forall g \in H, g(k) = k\}. \end{aligned}$$

Then for every $L \in \mathcal{K}$ and $H \in \mathcal{G}$ we have

$$L \subseteq \kappa(H) \iff \forall k \in L, \forall g \in H, g(k) = k \iff H \subseteq \gamma(L).$$

The pair of maps γ and κ is the *Galois connection* between $\mathcal{K}$ and $\mathcal{G}$.

This concept can be generalized to arbitrary posets [1]. We consider two posets $(\mathcal{P}, \leq)$ and $(\mathcal{Q}, \leq)$, and two maps $\eta : \mathcal{P} \to \mathcal{Q}$ and $\zeta : \mathcal{Q} \to \mathcal{P}$. Then this pair of maps forms a *Galois connection* between $(\mathcal{P}, \leq)$ and $(\mathcal{Q}, \leq)$ if for every $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ we have

$$P \leq \zeta(Q) \iff Q \leq \eta(P). \tag{2.12}$$

This is a symmetric relation between $\mathcal{P}$ and $\mathcal{Q}$. It can be shown that η and ζ form a Galois connection if and only if for every $P, P' \in \mathcal{P}$ and $Q, Q' \in \mathcal{Q}$ we have:

$$\begin{aligned} P \leq P' & \implies \eta(P) \geq \eta(P'), \\ Q \leq Q' & \implies \zeta(Q) \geq \zeta(Q'), \\ & P \leq \zeta\eta(P), \\ \text{and} \qquad & Q \leq \eta\zeta(Q). \end{aligned} \tag{2.13}$$

Let us now return to our complete lattice $\mathcal{L}$. Then clearly an *adjunction* in $\mathcal{L}$ is a Galois connection between $(\mathcal{L}, \leq)$ and its dual $(\mathcal{L}, \geq)$ (indeed, compare (2.12) and (2.8)). In this context, statements (i) and (ii) in Proposition 2.6 correspond together to (2.13).

Note that in [3] the term 'adjunction' is considered as a synonym of 'Galois connection', but this does not correspond to the definition of Galois connections given above in accordance with [Birkhoff].

2.4. *The decomposition of dilations and erosions*

So far we have limited ourselves to operators $\psi : \mathcal{L} \to \mathcal{L}$. But most of what we have said can be generalized to operators mapping one lattice $\mathcal{L}_1$ to a second one $\mathcal{L}_2$. Of course, one has to take care that expressions still make sense. For instance, in an adjunction (ε, δ), if $\varepsilon : \mathcal{L}_1 \to \mathcal{L}_2$ then $\delta : \mathcal{L}_2 \to \mathcal{L}_1$. If $\mathcal{L}_1$ or $\mathcal{L}_2$ or both can be decomposed as a product of lattices (in particular, if $\mathcal{L}_1$ or $\mathcal{L}_2$ is a power lattice) then a dilation, and hence an erosion, can be decomposed. We illustrate this by means of the following situation.

Let $\mathcal{E}_1, \mathcal{E}_2$ be arbitrary sets, and let $\mathcal{G}$ be a complete lattice. Let $\delta : \mathcal{G}^{\mathcal{E}_1} \to \mathcal{G}^{\mathcal{E}_2}$ be a dilation. For $x \in \mathcal{E}_1$ and $t \in \mathcal{G}$ (notice the different notation for elements of $\mathcal{G}$), we define $f_{x,t} \in \mathcal{G}^{\mathcal{E}_1}$ as:

$$f_{x,t}(y) = \begin{cases} t & \text{if } y = x, \\ O & \text{if } y \neq x. \end{cases} \tag{2.14}$$

Here O is the least element of $\mathcal{G}$. Every $F \in \mathcal{G}^{\mathcal{E}_1}$ can be written as:

$$F = \bigvee_{x \in \mathcal{E}} f_{x,F(x)}, \tag{2.15}$$

in other words, $\{f_{x,t} \mid x \in \mathcal{E}_1,\ t \in \mathcal{G}\}$ is a sup-generating family in $\mathcal{G}^{\mathcal{E}_1}$ (see the end of Subsection 1.3). For $x \in \mathcal{E}_1$ and $y \in \mathcal{E}_2$ we define $\delta_{y,x} : \mathcal{G} \to \mathcal{G}$ as:

$$\delta_{y,x}(t) = \delta(f_{x,t})(y), \quad t \in \mathcal{G}. \tag{2.16}$$

We show that every $\delta_{y,x}$ is a dilation on $\mathcal{G}$. Obviously, $\delta_{y,x}(O) = O$. Now let $t_j \in \mathcal{G},\ j \in J$. Then

$$\delta_{y,x}(\sup_{j \in J} t_j) = \delta(f_{x,\sup_{j\in J} t_j})(y) = \delta(\sup_{j \in J} f_{x,t_j})(y) = \sup_{j \in J} \delta(f_{x,t_j})(y) = \sup_{j \in J} \delta_{y,x}(t_j).$$

This proves the assertion. Now let $F \in \mathcal{G}^{\mathcal{E}_1}$. Then for $y \in \mathcal{E}_2$,

$$\delta(F)(y) = \delta(\sup_{x \in \mathcal{E}_1} f_{x,F(x)})(y) = \sup_{x \in \mathcal{E}_1} \delta(f_{x,F(x)})(y) = \sup_{x \in \mathcal{E}_1} \delta_{y,x}(F(x)).$$

Conversely, every operator $\delta : \mathcal{G}^{\mathcal{E}_1} \to \mathcal{G}^{\mathcal{E}_2}$ which has this form is a dilation by Proposition 2.3 *(ii)*.

PROPOSITION 2.10. *Let $\mathcal{E}_1, \mathcal{E}_2$ be arbitrary sets and let $\mathcal{G}$ be a complete lattice. Then the operator $\delta : \mathcal{G}^{\mathcal{E}_1} \to \mathcal{G}^{\mathcal{E}_2}$ is a dilation if and only if for every $x \in \mathcal{E}_1$ and $y \in \mathcal{E}_2$ there exists a dilation $\delta_{x,y} : \mathcal{G} \to \mathcal{G}$ such that for $F_1 \in \mathcal{G}^{\mathcal{E}_1}$ and $y \in \mathcal{E}_2$:*

$$\delta(F_1)(y) = \bigvee_{x \in \mathcal{E}_1} \delta_{y,x}(F_1(x)). \tag{2.17}$$

The upper adjoint erosion $\varepsilon : \mathcal{G}^{\mathcal{E}_2} \to \mathcal{G}^{\mathcal{E}_1}$ is given by

$$\varepsilon(F_2)(x) = \bigwedge_{y \in \mathcal{E}_2} \varepsilon_{x,y}(F_2(y)), \tag{2.18}$$

for $F_2 \in \mathcal{G}^{\mathcal{E}_2}$ and $x \in \mathcal{E}_1$, where $\varepsilon_{x,y}$ is the upper adjoint of $\delta_{y,x}$.

PROOF. Apart from the statement about the adjoint erosion we have proved this proposition above. Let $F_1 \in \mathcal{G}^{\mathcal{E}_1}$ and $F_2 \in \mathcal{G}^{\mathcal{E}_2}$. We show that

$$\delta(F_1) \leq F_2 \iff F_1 \leq \varepsilon(F_2),$$

where δ, ε are given by (2.17) and (2.18) respectively.

$$\begin{aligned}
\delta(F_1) \leq F_2 &\iff \forall y \in \mathcal{E}_2, \quad \sup_{x \in \mathcal{E}_1} \delta_{y,x}(F_1(x)) \leq F_2(y) \\
&\iff \forall y \in \mathcal{E}_2, \forall x \in \mathcal{E}_1, \quad \delta_{y,x}(F_1(x)) \leq F_2(y) \\
&\iff \forall x \in \mathcal{E}_1, \forall y \in \mathcal{E}_2, \quad F_1(x) \leq \varepsilon_{x,y}(F_2(y)) \\
&\iff \forall x \in \mathcal{E}_1, \quad F_1(x) \leq \inf_{y \in \mathcal{E}_2} \varepsilon_{x,y}(F_2(y)) \\
&\iff F_1 \leq \varepsilon(F_2). \ \blacksquare
\end{aligned}$$

This result can in particular be applied to grey-level functions. Here $\mathcal{E} = \mathcal{E}_1 = \mathcal{E}_2$ is an arbitrary set (e.g. $\mathcal{E} = \mathbf{R}^d$) and $\mathcal{G}$ is the set of grey-levels (e.g. $\mathcal{G} = \overline{\mathbf{R}}$). For example in the grey-level dilation and erosion by a structuring function G given in (1.3) and (1.4), we have $\delta_{x,y}(t) = t + G(x-y)$ and $\varepsilon_{y,x}(t) = t - G(x-y)$.

As a second example we consider *Boolean functions* $\psi : \mathcal{B}^{\mathcal{E}} \to \mathcal{B}$ where $\mathcal{B} = \{0,1\}$. Note that every dilation $\mathcal{B} \to \mathcal{B}$ is either constant 0 or the identity mapping $\mathbf{id}$. From Proposition 2.10 it follows that every dilation $\delta : \mathcal{B}^{\mathcal{E}} \to \mathcal{B}$ can be written as

$$\delta(F) = \bigvee_{x \in \mathcal{E}} \delta_x(F(x)), \quad F \in \mathcal{B}^{\mathcal{E}},$$

where $\delta_x : \mathcal{B} \to \mathcal{B}$ is a dilation for every $x \in \mathcal{E}$. Let $\mathcal{E}'$ be the subset of $\mathcal{E}$ such that $\delta_x = \mathbf{id}$ for $x \in \mathcal{E}'$. Then

$$\delta(F) = \bigvee_{x \in \mathcal{E}'} F(x).$$

In other words, a dilation is either constant 0 or a partial supremum function. Now Matheron's Theorem for dilations (Theorem 2.4 in its dual form) says that any increasing operator preserving the least element is an infimum of dilations. But it is easy to see that an increasing operator which does not preserve the least element is constant 1. Moreover, in an infimum of dilations, if at least one of them is constant 0, then the infimum is also constant 0. In other words, we have shown that every non-constant increasing function $\mathcal{B}^{\mathcal{E}} \to \mathcal{B}$ is an infimum of partial suprema, or if $\mathcal{E}$ is finite, a minimum of partial maxima: we obtain in this way a well-known result in the theory of Boolean functions (see e.g. [7], page 189).

3. Translation-Invariance

In [14] and [23] mathematical morphology in the Euclidean space was studied in the framework of *translation-invariance*, in other words every morphological operator was required to commute with any translation. We want to generalize this property to our general framework, that of an arbitrary complete lattice $\mathcal{L}$. Readily, the role of translations of the Euclidean space will be played by certain automorphisms of $\mathcal{L}$. If we do not impose any particular conditions on these automorphisms, we can prove only generalities (Subsection 3.1). On the other hand, if we assume a sup-generating family ℓ of $\mathcal{L}$ and an abelian group $\mathbf{T}$ of automorphisms of $\mathcal{L}$ which acts transitively on ℓ (for example in the Euclidean case, $\mathbf{T}$ is the group of all translations, and ℓ is the set of all singletons), then $\mathbf{T}$-invariant dilations and erosions will take a form analogous to Minkowski operations, and we can generalize certain results, such as Matheron's theorem for erosions: see Subsection 3.2.

3.1. *Generalities*

Write $Aut(\mathcal{L})$ for the set of automorphisms of $\mathcal{L}$. Given an automorphism $\tau \in Aut(\mathcal{L})$ and an operator $\eta \in \mathcal{O}$, we will say that η *commutes with* τ, or that η *is* τ*-invariant*, if $\eta\tau = \tau\eta$. Given a subset $\mathbf{T}$ of $Aut(\mathcal{L})$, we will say that η *is* $\mathbf{T}$*-invariant* if η commutes with every $\tau \in \mathbf{T}$.

It is an elementary fact from group theory that $Aut(\mathcal{L})$ is a group, and that for any $\mathcal{Q} \subseteq \mathcal{O}$, the automorphisms of $\mathcal{L}$ with which every element of $\mathcal{Q}$ commutes, form a subgroup of $Aut(\mathcal{L})$. Thus when we consider $\mathbf{T}$-invariant operators for some subset $\mathbf{T}$ of $Aut(\mathcal{L})$, we can assume that $\mathbf{T}$ is in fact a group, since '$\mathbf{T}$-invariant' is equivalent to '$\langle\mathbf{T}\rangle$-invariant', where $\langle\mathbf{T}\rangle$ is the subgroup of $Aut(\mathcal{L})$ generated by $\mathbf{T}$.

Given a subset (or rather subgroup) $\mathbf{T}$ of $Aut(\mathcal{L})$, we will use the prefix '$\mathbf{T}$-' for '$\mathbf{T}$-invariant'. We will speak thus of $\mathbf{T}$-operators, $\mathbf{T}$-dilations, $\mathbf{T}$-erosions, etc.. The structure of the set of $\mathbf{T}$-operators is summarized in the following result:

PROPOSITION 3.1. *Given a group $\mathbf{T}$ of automorphisms of $\mathcal{L}$, the set of $\mathbf{T}$-operators is*

(i) closed under composition, and it contains **id**;

(ii) a complete sublattice of $\mathcal{O}$.

PROOF. *(i)* is straightforward. Let us prove *(ii)*. Let $\mathcal{O}_{\mathbf{T}}$ be the set of $\mathbf{T}$-operators. Take any $\tau \in \mathbf{T}$ and any subset $\mathcal{Q}$ of $\mathcal{O}_{\mathbf{T}}$. By (2.5) we have $\left(\bigvee \mathcal{Q}\right)\tau = \sup_{\eta\in\mathcal{Q}} \eta\tau$. As τ is a dilation, we have (see the remark after Definition 2.1): $\tau\left(\bigvee \mathcal{Q}\right) = \sup_{\eta\in\mathcal{Q}} \tau\eta$. As $\eta\tau = \tau\eta$ for $\eta \in \mathcal{Q}$, the two preceding equalities imply that $(\bigvee \mathcal{Q})\tau = \tau(\bigvee \mathcal{Q})$, in other words $\mathcal{Q}$ is τ-invariant for any $\tau \in \mathbf{T}$. Hence $\bigvee \mathcal{Q} \in \mathcal{O}_{\mathbf{T}}$. We prove similarly that $\bigwedge \mathcal{Q} \in \mathcal{O}_{\mathbf{T}}$. Thus $\mathcal{O}_{\mathbf{T}}$ is a complete sublattice of $\mathcal{O}$. ∎

Comparing this result with Propositions 2.2 and 2.3, it is easy to see that they remain true if we replace 'increasing operators' and 'dilations' by 'increasing $\mathbf{T}$-operators' and '$\mathbf{T}$-dilations'. An interesting thing is that $\mathbf{T}$-invariance is preserved by adjunctions:

PROPOSITION 3.2. *Given an adjunction (ε, δ) and a group $\mathbf{T}$ of automorphisms of $\mathcal{L}$, δ is $\mathbf{T}$-invariant if and only if ε is $\mathbf{T}$-invariant.*

PROOF. Suppose that ε commutes with every $\tau \in \mathbf{T}$. Then it commutes with τ^{-1}. So for any $X, Y \in \mathcal{L}$ we have:

$$\begin{aligned} &\delta\tau(X) \leq Y \iff \tau(X) \leq \varepsilon(Y) \iff X \leq \tau^{-1}\varepsilon(Y) \\ \iff &X \leq \varepsilon\tau^{-1}(Y) \iff \delta(X) \leq \tau^{-1}(Y) \iff \tau\delta(X) \leq Y. \end{aligned}$$

Thus $\delta\tau(X) = \tau\delta(X)$, and so δ commutes with τ (for every $\tau \in \mathbf{T}$). We have shown that if ε is $\mathbf{T}$-invariant, then δ is $\mathbf{T}$-invariant. The reverse implication is proved in a similar way. ∎

We will thus call a $\mathbf{T}$-*adjunction* an adjunction (ε, δ) such that δ is a $\mathbf{T}$-dilation and ε a $\mathbf{T}$-erosion. By Proposition 3.2, Theorem 2.7 remains valid if we replace in it 'dilations', 'erosions', and 'adjunctions', by '$\mathbf{T}$-dilations', '$\mathbf{T}$-erosions', and '$\mathbf{T}$-adjunctions'. In particular, an operator δ is a $\mathbf{T}$-dilation if and only if there is some $\varepsilon \in \mathcal{O}$ such that (ε, δ) is a $\mathbf{T}$-adjunction, and an operator ε is a $\mathbf{T}$-erosion if and only if there is some $\delta \in \mathcal{O}$ such that (ε, δ) is a $\mathbf{T}$-adjunction.

In the case where $\mathcal{L} = \mathcal{P}(\mathcal{E})$, the set of parts of a Euclidean space $\mathcal{E}$, translation-invariant dilations and erosions can be built with translations. If we fix the origin o, every point $a \in \mathcal{E}$ determines the unique translation τ_a defined by $\tau_a(o) = a$. For $X \subseteq \mathcal{E}$, $\tau_a(X) = X_a$. Then for every $X, A \subseteq \mathcal{E}$, (1.1) and (1.2) mean that

$$X \oplus A = \bigcup_{a \in A} \tau_a(X) \qquad \text{and} \qquad X \ominus A = \bigcap_{a \in A} \tau_a^{-1}(X). \tag{3.1}$$

Thus the dilation $\delta_A : X \mapsto X \oplus A$ and the erosion $\varepsilon_A : X \mapsto X \ominus A$ have the following decomposition in terms of translations:

$$\delta_A = \bigvee_{a \in A} \tau_a \qquad \text{and} \qquad \varepsilon_A = \bigwedge_{a \in A} \tau_a^{-1}. \tag{3.2}$$

It is not hard to see that all translation-invariant dilations and erosions in the Euclidean space $\mathcal{E}$ take this form.

It would be interesting to see how far this can be generalized to an arbitrary complete lattice $\mathcal{L}$. Given a subset $\mathbf{A}$ of $Aut(\mathcal{L})$, we define

$$\mathbf{A}^{-1} = \{\tau^{-1} \mid \tau \in \mathbf{A}\}.$$

By Proposition 2.9, (τ^{-1}, τ) is an adjunction for every automorphism τ. Hence by Theorem 2.7 *(iv)*, $(\bigwedge \mathbf{A}^{-1}, \bigvee \mathbf{A})$ is an adjunction for every $\mathbf{A} \subseteq Aut(\mathcal{L})$. Now if $\mathbf{T}$ is an *abelian* subgroup of $Aut(\mathcal{L})$ (that is, every two elements of $\mathbf{T}$ commute), then Proposition 3.1 implies that an adjunction

$$(\bigwedge \mathbf{A}^{-1}, \bigvee \mathbf{A}) \qquad \text{with} \quad \mathbf{A} \subseteq \mathbf{T}, \tag{3.3}$$

is in fact a $\mathbf{T}$-adjunction. Here we have the analogy with (3.2).

However, the converse is not always true. Given an abelian group $\mathbf{T}$ of automorphisms of $\mathcal{L}$, not every $\mathbf{T}$-adjunction takes the form (3.3). An extreme case is when $\mathbf{T}$ is reduced to the identity, and every adjunction is a $\mathbf{T}$-adjunction, but only the trivial adjunctions $(\mathbf{I}, \mathbf{O})$ and $(\mathbf{id}, \mathbf{id})$ take the form (3.3). As we will see in the next subsection, we need an assumption on $\mathbf{T}$ which corresponds to the fact that in a Euclidean space the group of translations is transitive on the set of points.

3.2. *Transitivity on a sup-generating family*

In a Euclidean space, we have two particular features: first, all subsets can be built from singletons with the union operation; second, the group of translations is abelian and transitive on the set of singletons. This leads to the following generalization. We consider the complete lattice $\mathcal{L}$ and an *abelian* group $\mathbf{T}$ of automorphisms of $\mathcal{L}$, and make the:

BASIC ASSUMPTION. *$\mathcal{L}$ has a sup-generating subset ℓ such that:*

(i) $\mathbf{T}$ *leaves ℓ invariant, in other words for every $\tau \in \mathbf{T}$ and $x \in \ell$, $\tau(x) \in \ell$;*

(ii) $\mathbf{T}$ *is transitive on ℓ, in other words for every $x, y \in \ell$, there exists $\tau \in \mathbf{T}$ such that $\tau(x) = y$.*

Elements of ℓ will be written as lower-case letters x, y, z, etc.. We recall that for any $X \in \mathcal{L}$ we define

$$\ell(X) = \{x \in \ell \mid x \leq X\},$$

and so the fact that ℓ is sup-generating means that $X = \bigvee \ell(X)$. In particular (1.8) holds.

The above assumption is satisfied in the Euclidean case by taking ℓ to be the set of singletons. Other examples will be given in Section 4, in particular the set of grey-level functions $\mathbf{R}^n \to \overline{\mathbf{R}}$.

Now, as in the Euclidean case, the basic assumption implies that $\mathbf{T}$ acts regularly on ℓ, in other words that for every $x, y \in \ell$, there is a *unique* $\tau \in \mathbf{T}$ such that $\tau(x) = y$. Indeed, suppose that $\tau_1(x) = \tau_2(x) = y$; then $\tau_1^{-1}\tau_2(x) = x$. Now for any $z \in \ell$, there is some $\tau_3 \in \ell$ such that $\tau_3(x) = z$, and so $\tau_1^{-1}\tau_2(z) = (\tau_1^{-1}\tau_2)\tau_3(x) = \tau_3(\tau_1^{-1}\tau_2)(x) = \tau_3(x) = z$. Thus $\tau_1^{-1}\tau_2$ fixes every element of ℓ, and as $\tau_1^{-1}\tau_2$ commutes with the supremum and ℓ is sup-generating, this means that $\tau_1^{-1}\tau_2$ fixes every element of $\mathcal{L}$, that is $\tau_1 = \tau_2$.

Thanks to this fact, we can build a bijection between ℓ and $\mathbf{T}$. Indeed, we fix some $o \in \ell$, and for every $x \in \ell$ we define τ_x as the unique element of $\mathbf{T}$ such that $\tau_x(o) = x$. Now with this bijection we can endow ℓ with the structure of a group isomorphic to $\mathbf{T}$. We define the binary addition $+$ on ℓ by

$$x + y = \tau_x\tau_y(o) = \tau_x(y) = \tau_y(x).$$

This gives indeed $\tau_{x+y} = \tau_x\tau_y$, i.e., the isomorphism with $\mathbf{T}$. In particular $(\ell, +)$ is an abelian group with neutral element o. We write $-y$ for the inverse of y in this group, that is $\tau_{-y} = \tau_y^{-1}$. We define then the subtraction $-$ by

$$x - y = x + (-y) = \tau_x\tau_y^{-1}(o) = \tau_x(-y) = \tau_y^{-1}(x).$$

Note that $o - y = -y$.

REMARK 3.1. In fact in a Euclidean space $\mathcal{E}$, the group $\mathbf{T}$ of translations acting on $\mathcal{P}(\mathcal{E})$ is first of all a group of translations on the set $\mathcal{E}$ of points, which forms naturally an additive abelian group. Here the action of $\mathbf{T}$ on $\mathcal{P}(\mathcal{E})$ is derived from this basic group structure. More generally if $\mathcal{L} = \mathcal{P}(S)$, the set of parts of an arbitrary set S, and if S is an abelian group, then we can view it as a transitive abelian group of permutations of itself, and so it becomes naturally a group of automorphisms of $\mathcal{L}$ satisfying the basic assumption. However if $\mathcal{L}$ is not isomorphic to some $\mathcal{P}(S)$, then it may be inadequate to start with a group structure on ℓ, since a permutation of ℓ does not necessarily extend into an automorphism of $\mathcal{L}$. Take for example the Euclidean plane $\mathcal{E}$, and let the set $\mathcal{L}$ consist of $\mathcal{E}$, all lines passing through the origin o, all singletons $\{p\}$ in $\mathcal{E}$, and $\emptyset$. It is easy to see that $\mathcal{L}$, ordered by inclusion, is a complete lattice. Here the set ℓ of singletons is sup-generating, and has the translation group structure of $\mathcal{E}$; now this group of permutations of ℓ does not extend to a group of automorphisms of $\mathcal{L}$, because a translation does not preserve the set of lines passing through the origin.

Our next goal is to use this group structure to define on $\mathcal{L}$ a generalization of the Minkowski operations. But we need the following two lemmas in order to lay a sound basis for this generalization:

LEMMA 3.3. *Let $\mathcal{X}, \mathcal{Y} \subseteq \ell$ and let $X = \sup \mathcal{X}$ and $Y = \sup \mathcal{Y}$. Then*

$$\bigvee_{x \in \mathcal{X}} \tau_x(Y) = \bigvee_{x \in \mathcal{X}, y \in \mathcal{Y}} (x + y) = \bigvee_{y \in \mathcal{Y}} \tau_y(X).$$

PROOF. We only prove the second identity, the first one is shown in the same way. For every $y \in \mathcal{Y}$ we have

$$\tau_y(X) = \tau_y(\bigvee_{x \in \mathcal{X}} x) = \bigvee_{x \in \mathcal{X}} \tau_y(x) = \bigvee_{x \in \mathcal{X}} (x + y),$$

which leads to the result. ∎

LEMMA 3.4. *Let $z_j \in \ell$ $(j \in J)$ and $z \in \ell$. Then the following three statements are equivalent:*

(*i*) $\quad z = \bigvee_{j \in J} z_j.$

(*ii*) $\quad \tau_z = \bigvee_{j \in J} \tau_{z_j}.$

(*iii*) $\quad \tau_z^{-1} = \bigwedge_{j \in J} \tau_{z_j}^{-1}.$

PROOF. (*i*) *implies* (*ii*): Let $Y \in \mathcal{L}$ and $\mathcal{Y} = \ell(Y)$. Then $Y = \sup \mathcal{Y}$, and so applying Lemma 3.3 with $\mathcal{X} = \{z\}$ and $X = z$, we get $\tau_z(Y) = \sup_{y \in \ell(Y)} \tau_y(z)$. As $z = \sup_{j \in J} z_j$, applying Lemma 3.3 with $\mathcal{X} = \{z_j \mid j \in J\}$ and $X = z$, we get $\sup_{j \in J} \tau_{z_j}(Y) = \sup_{y \in \ell(Y)} \tau_y(z)$. Combining both equalities, we obtain $\tau_z(Y) = \sup_{j \in J} \tau_{z_j}(Y)$. As Y was arbitrary, (*ii*) follows.

(*ii*) *implies* (*i*): If $\tau_z = \sup_{j \in J} \tau_{z_j}$, then $z = \tau_z(o) = \sup_{j \in J} \tau_{z_j}(o) = \sup_{j \in J} z_j$.

(*ii*) *is equivalent to* (*iii*): Clearly (τ_z^{-1}, τ_z) and $(\inf_{j \in J} \tau_{z_j}^{-1}, \sup_{j \in J} \tau_{z_j})$ are adjunctions (see (3.3)). By Theorem 2.7 the two upper adjoints are equal if and only if the two lower adjoints are equal. ∎

For $X \in \mathcal{L}$ and $h \in \ell$ we set

$$X_h = \tau_h(X)$$

(as in the Euclidean case). We now define the binary operations $\oplus$ and $\ominus$ on $\mathcal{L}$ by:

$$X \oplus Y = \bigvee_{y \in \ell(Y)} X_y; \tag{3.4}$$

$$X \ominus Y = \bigwedge_{y \in \ell(Y)} X_{-y}. \tag{3.5}$$

Compare with (1.1) and (1.2). The following basic properties hold:

Proposition 3.5. *For $X, Y \in \mathcal{L}$ and $a, b \in \ell$ we have:*

(*i*) $a \oplus b = a + b, \quad a \ominus b = a - b.$

(*ii*) $X \oplus b = X_b, \quad X \ominus b = X_{-b}.$

(*iii*) $X \oplus Y = Y \oplus X = \bigvee\{x + y \mid x \in \ell(X),\ y \in \ell(Y)\}.$

(*iv*) $X \ominus Y = \bigvee\{z \in \ell \mid Y_z \leq X\}.$

Proof. (*i*) and (*ii*) follow from Lemma 3.4, and (*iii*) from Lemma 3.3. We prove (*iv*): We set $W = \sup\{z \in \ell \mid Y_z \leq X\}$. Let $z \in \ell$ such that $Y_z \leq X$. Then for every $y \in \ell(Y)$, $z_y = y + z = y_z \leq Y_z \leq X$, and so $z = (z_y)_{-y} \leq X_{-y}$; by (3.5) this means then that $z \leq X \ominus Y$. We have thus shown that $W \leq X \ominus Y$. Now let $z \in \ell(X \ominus Y)$. Then $z \leq X_{-y}$ for every $y \in \ell(Y)$ (see (3.5)), and so $y_z = y + z = z_y \leq (X_{-y})_y = X$; as $Y_z = \sup_{y \in \ell(Y)} y_z$, we get $Y_z \leq X$, and so $z \leq W$. We have thus shown that $X \ominus Y \leq W$. Hence the equality follows. ∎

Statements (*iii*) and (*iv*) generalize similar expressions in (1.1) and (1.2). For $A \in \mathcal{L}$ we define the operators $\delta_A, \varepsilon_A \in \mathcal{O}$ by

$$\delta_A = \bigvee_{a \in \ell(A)} \tau_a \qquad \text{and} \qquad \varepsilon_A = \bigwedge_{a \in \ell(A)} \tau_a^{-1}. \tag{3.6}$$

(Cfr. (3.2) in the case where $\mathcal{L} = \mathcal{P}(\mathcal{E})$.) By (3.4) and (3.5) we have obviously

$$\delta_A(X) = X \oplus A \qquad \text{and} \qquad \varepsilon_A(X) = X \ominus A$$

for any $X \in \mathcal{L}$. We are now ready to prove the main theorem of this section:

Theorem 3.6. *For any $A \in \mathcal{L}$, $(\varepsilon_A, \delta_A)$ is a $\mathbb{T}$-adjunction. Moreover, any $\mathbb{T}$-adjunction has this form.*

Proof. Given $A \in \mathcal{L}$, if $\mathbf{A} = \{\tau_a \mid a \in \ell(A)\}$, then by (3.6) the pair $(\varepsilon_A, \delta_A)$ is as in (3.3), and so it is a $\mathbb{T}$-adjunction.

Conversely assume that (ε, δ) is a $\mathbb{T}$-adjunction on $\mathcal{L}$. Let $A = \delta(o)$. We show that $\delta = \delta_A$. For $x \in \ell$,

$$\delta(x) = \delta\big(\tau_x(o)\big) = \tau_x\big(\delta(o)\big) = \tau_x(A).$$

Hence, for $X \in \mathcal{L}$,

$$\delta(X) = \delta\big(\sup \ell(X)\big) = \sup_{x \in \ell(X)} \delta(x) = \sup_{x \in \ell(X)} \tau_x(A) = A \oplus X = X \oplus A = \delta_A(X).$$

By the uniqueness of the upper adjoint of a dilation, this implies also that $\varepsilon = \varepsilon_A$. ∎

Proposition 3.7. *For $X, Y, Z \in \mathcal{L}$ we have:*

(*i*) $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z) = \bigvee\{x + y + z \mid x \in \ell(X),\ y \in \ell(Y),\ z \in \ell(Z)\}.$

(*ii*) $(X \ominus Y) \ominus Z = X \ominus (Y \oplus Z).$

PROOF. (i): We have the following decomposition:

$$\begin{aligned}(X \oplus Y) \oplus Z &= \sup_{z \in \ell(Z)} \tau_z(X \oplus Y);\\ &= \sup_{z \in \ell(Z)} \tau_z\big(\sup\{x + y \mid x \in \ell(X),\ y \in \ell(Y)\}\big);\\ &= \sup_{z \in \ell(Z)} \sup\{\tau_z(x + y) \mid x \in \ell(X),\ y \in \ell(Y)\};\\ &= \sup\{x + y + z \mid x \in \ell(X),\ y \in \ell(Y),\ z \in \ell(Z)\}.\end{aligned}$$

By the commutativity of $+$, $X \oplus (Y \oplus Z) = (Y \oplus Z) \oplus X$ has the same decomposition.
(ii): Clearly (i) means that $\delta_Z \delta_Y = \delta_{Y \oplus Z}$, and so $\delta_Y \delta_Z = \delta_{Z \oplus Y} = \delta_{Y \oplus Z}$. Now ε_Y, ε_Z, and $\varepsilon_{Y \oplus Z}$ are the upper adjoints of δ_Y, δ_Z, and $\delta_{Y \oplus Z}$ respectively; by Theorem 2.7 (v), $\varepsilon_Z \varepsilon_Y$ is the upper adjoint of $\delta_Y \delta_Z$. By the unicity of the upper adjoint, $\delta_Y \delta_Z = \delta_{Y \oplus Z}$ implies that $\varepsilon_Z \varepsilon_Y = \varepsilon_{Y \oplus Z}$. Applying both terms to X, we get the result. ∎

We know that the set of $\mathbf{T}$-dilations is sup-closed in $\mathcal{O}$. It is therefore a complete lattice with $\bigvee$ as supremum operation, but with an infimum operation distinct from $\bigwedge$ (see Subsection 1.3). As each $\mathbf{T}$-dilation is of the form δ_A for some $A \in \mathcal{L}$, the following result is not surprising:

THEOREM 3.8. *The map $A \mapsto \delta_A$ defines an isomorphism between $\mathcal{L}$ and the complete lattice of $\mathbf{T}$-dilations. In particular, given $A_j \in \mathcal{L}$, $(j \in J)$, we have*

$$\bigvee_{j \in J} \delta_{A_j} = \delta_{\sup_{j \in J} A_j}. \tag{3.7}$$

PROOF. Write $\mathcal{D}_{\mathbf{T}}(\mathcal{L})$ for the set of $\mathbf{T}$-dilations. The map $\Delta : \mathcal{D}_{\mathbf{T}}(\mathcal{L}) \to \mathcal{L}$ defined by $\Delta(A) = \delta_A$ is surjective by Theorem 3.6. As $\delta_A(o) = A$ for any $A \in \mathcal{L}$, this implies that $\delta_A \neq \delta_B$ for $A \neq B$, in other words Δ is injective. It is thus a bijection between $\mathcal{L}$ and $\mathcal{D}_{\mathbf{T}}(\mathcal{L})$. Now for $A, B \in \mathcal{L}$, if $A \leq B$, then $\ell(A) \subseteq \ell(B)$, and so by (3.6) we have $\delta_A \leq \delta_B$. On the other hand, if $\delta_A \leq \delta_B$, then $A = \delta_A(o) \leq \delta_B(o) = B$. Thus $A \leq B \iff \delta_A \leq \delta_B$, in other words Δ is an isomorphism. In particular, it must preserve the supremum operation $\bigvee$. Thus (3.7) holds. ∎

Equation (3.7) means that

$$\bigvee_{j \in J} (X \oplus A_j) = X \oplus \big(\bigvee_{j \in J} A_j\big)$$

for all $X \in \mathcal{L}$. By the dual isomorphism $\delta_A \mapsto \varepsilon_A$ between the two complete lattices of $\mathbf{T}$-dilations and $\mathbf{T}$-erosions, we derive from Theorem 3.8 the following:

COROLLARY 3.9. *The map $A \mapsto \varepsilon_A$ defines a dual isomorphism between $\mathcal{L}$ and the complete lattice of $\mathbf{T}$-erosions. In particular, given $A_j \in \mathcal{L}$, $(j \in J)$, we have*

$$\bigwedge_{j \in J} \varepsilon_{A_j} = \varepsilon_{\sup_{j \in J} A_j}. \tag{3.8}$$

Equation (3.8) means that

$$\bigwedge_{j\in J}(X \ominus A_j) = X \ominus (\bigvee_{j\in J} A_j)$$

for all $X \in \mathcal{L}$. As we explained in the proof of Proposition 3.7, the following additional properties hold:

$$\begin{aligned} \delta_A\delta_B &= \delta_B\delta_A = \delta_{A\oplus B} \\ \varepsilon_A\varepsilon_B &= \varepsilon_B\varepsilon_A = \varepsilon_{A\oplus B}. \end{aligned} \tag{3.9}$$

We shall now be concerned with increasing $\mathbf{T}$-operators and Matheron's theorem. We will indeed prove a generalization of (2.6) for increasing $\mathbf{T}$-operators. But we need first the following:

LEMMA 3.10. *Let $\psi \in \mathcal{O}$ be an increasing $\mathbf{T}$-operator. Then*

$$\psi(X) \oplus B \leq \psi(X \oplus B) \qquad \text{and} \qquad \psi(X) \ominus B \geq \psi(X \ominus B)$$

for $X, B \in \mathcal{L}$.

PROOF. Let $X, B \in \mathcal{L}$. Then by Lemma 2.1 and $\mathbf{T}$-invariance we have:

$$\psi(X \oplus B) = \psi(\sup_{b\in\ell(B)} \tau_b(X)) \geq \sup_{b\in\ell(B)} \psi(\tau_b(X)) = \sup_{b\in\ell(B)} \tau_b\psi(X) = \psi(X) \oplus B.$$

The other inequality is proved in the same way. ∎

Given an increasing $\mathbf{T}$-operator ψ, the *kernel* $\mathcal{V}[\psi]$ of ψ is defined by

$$\mathcal{V}[\psi] = \{X \in \mathcal{L} \mid o \leq \psi(X)\}. \tag{3.10}$$

It is not difficult to see that $\mathcal{V}[\psi]$ is *sup-hereditary*, that is, $X \in \mathcal{V}[\psi]$ and $X \leq Y$ implies that $Y \in \mathcal{V}[\psi]$ as well (see [14], p.218).

THEOREM 3.11. *Let $\psi \in \mathcal{O}$ be an increasing $\mathbf{T}$-operator with kernel $\mathcal{V}[\psi]$. Then*

$$\psi = \bigvee_{A\in\mathcal{V}[\psi]} \varepsilon_A,$$

in other words

$$\psi(X) = \bigvee_{A\in\mathcal{V}[\psi]} (X \ominus A)$$

for every $X \in \mathcal{L}$.

PROOF. Given $X \in \mathcal{L}$, let $Z = \sup_{A\in\mathcal{V}[\psi]}(X \ominus A)$.

(*a*) $\psi(X) \geq Z$: Let $A \in \mathcal{V}[\psi]$, that is, $o \leq \psi(A)$. By Proposition 2.6 (*ii*) we have $\delta_A\varepsilon_A \leq \mathrm{id}$, and so $(X \ominus A) \oplus A \leq X$. Then by Lemma 3.10 we get:

$$X \ominus A = (X \ominus A) \oplus o \leq (X \ominus A) \oplus \psi(A) \leq \psi((X \ominus A) \oplus A) \leq \psi(X).$$

Therefore $Z \leq \psi(X)$.

(*b*) $\psi(X) \leq Z$: Take $x \in \ell(\psi(X))$. As $x \leq \psi(X)$, we have $o \leq \psi(X)_{-x} = \psi(X_{-x})$, and so $X_{-x} \in \mathcal{V}[\psi]$. As $x \oplus X_{-x} = X_{-x+x} = X$, we obtain by adjunction $x \leq X \ominus X_{-x}$. As $X_{-x} \in \mathcal{V}[\psi]$, we have $X \ominus X_{-x} \leq Z$. Hence $x \leq Z$ for $x \in \ell(\psi(X))$, and so $\psi(X) \leq Z$. ∎

REMARK 3.2. (*i*) It is not always necessary to take the whole kernel of ψ. Indeed, given a subset $\mathcal{V}'$ of $\mathcal{V}[\psi]$ such that for every $A \in \mathcal{V}[\psi]$, there is some $A' \in \mathcal{V}'$ with $A' \leq A$, then we have also $\psi(X) = \sup_{A' \in \mathcal{V}'}(X \ominus A')$. Indeed, for $A \in \mathcal{V}[\psi] \setminus \mathcal{V}'$, given $A' \in \mathcal{V}'$ with $A' \leq A$, we have $X \ominus A' \geq X \ominus A$, and so A is redundant in the decomposition of $\psi(X)$. This question is discussed in [12] in the Euclidean case.
(*ii*) In Theorem 2.4 we required that $\psi(I) = I$ in order to decompose ψ into the supremum of a *non-empty* set of erosions. Here we do not make any such requirement, because if $\psi(I) < I$, then $\psi = \mathbf{O}$ and $\mathcal{V}[\psi] = \emptyset$. Indeed, as $I = \sup \ell$, $\psi(I) < I$ implies that there is some $x \in \ell$ such that $x \not\leq \psi(I)$; by $\mathbf{T}$-invariance this means that for every $y \in \ell$, $y = x + (y - x) \not\leq \psi(I)_{y-x} = \psi(I_{y-x}) = \psi(I)$, that is $\psi(I) = O$, and so $\psi = \mathbf{O}$. But then $o \not\leq \psi(X)$ for every $X \in \mathcal{L}$, and so $\mathcal{V}[\psi] = \emptyset$.
(*iii*) If $O \in \mathcal{V}[\psi]$, then $\mathcal{V}[\psi] = \mathcal{L}$ and $\psi = \mathbf{I}$. Indeed $o \leq \psi(O)$ implies, by the increasingness of ψ, that $\mathcal{V}[\psi] = \mathcal{L}$. By $\mathbf{T}$-invariance, we have $x \leq \psi(O)_x = \psi(O_x) = \psi(O)$ for every $x \in \ell$, that is $\psi(O) = I$, and so $\psi = \mathbf{I}$.
(*iv*) A corresponding version of Matheron's theorem for $\mathbf{T}$-dilations does not hold in general. In Section 4 we will give an example of an increasing $\mathbf{T}$-operator which is not decomposable as an infimum of $\mathbf{T}$-dilations. From the duality principle it is clear that such a corresponding result for $\mathbf{T}$-dilations is obtained if one assumes the dual of the basic assumption, namely the existence of an *inf-generating* family on which $\mathbf{T}$ is transitive. If $\mathcal{L}$ is a Boolean lattice then both the basic assumption and its dual are equivalent, and Matheron's theorem holds in both versions (see Section 4).

We end this section by considering a minor problem: are there other automorphisms of $\mathcal{L}$ with which all $\mathbf{T}$-dilations commute? Given $\sigma \in Aut(\mathcal{L})$, by Proposition 3.1 and Theorem 3.6, the following three statements are equivalent:

— Every $\mathbf{T}$-dilation is σ-invariant.
— Every element of $\mathbf{T}$ commutes with σ.
— Every $\mathbf{T}$-erosion is σ-invariant.

We will see in Section 4 that in certain cases this is possible for some $\sigma \notin \mathbf{T}$. However it is impossible in the case where ℓ is the set of atoms of $\mathcal{L}$, for example if $\mathcal{L} = \mathcal{P}(\mathcal{E})$, the set of parts of a Euclidean space, with the usual group of translations. We can generalize this negative fact as follows:

PROPOSITION 3.12. *Suppose that $\ell \cup \{O, I\}$ is inf-closed. If $\sigma \in Aut(\mathcal{L})$ and σ commutes with every element of $\mathbf{T}$, then $\sigma \in \mathbf{T}$.*

PROOF. Clearly σ is a $\mathbf{T}$-erosion, and we have thus $\sigma = \varepsilon_A$ for some $A \in \mathcal{L}$. Then $\sigma(o) = o \ominus A = \inf\{-a \mid a \in \ell(A)\}$. As $\ell \cup \{O, I\}$ is inf-closed, we have $\sigma(o) \in \ell \cup \{O, I\}$. As $O < o < I$ and σ is an automorphism, $O < \sigma(o) < I$, and so $\sigma(o) = x \in \ell$. Now σ is also a $\mathbf{T}$-dilation, and so $\sigma = \delta_B$ for some $B \in \mathcal{L}$. In fact $B = \sigma(o) = x$, and so $\sigma = \delta_x = \tau_x \in \mathbf{T}$. ∎

4. EXAMPLES AND APPLICATIONS

In this section we apply the abstract results of the previous two sections to some practical examples. In Subsection 4.1 we study the Boolean case. Here we have an extra duality relation between dilations and erosions deriving from the fact that in a Boolean lattice every element has a unique complement. In Subsection 4.2 we consider the complete atomic lattice consisting of the closed subsets of $\mathbf{R}^d$. There we give also an example of an increasing translation-invariant transformation which cannot be written as an infimum of translation-invariant dilations. Finally, in Subsection 4.3 we give a characterization of dilations and erosions on grey-level functions which are invariant under spatial or under both spatial and grey-level translations. There we deal also with the so-called *multiplicative structuring functions* studied by Herman [9].

4.1. *The Boolean case*

Suppose that $\mathcal{L}$ is a complete lattice and that the basic assumption of Subsection 3.2 is satisfied. Before restricting to the Boolean case we give alternative characterizations of dilations and erosions by the so-called *reflected structuring element*, which is defined as follows:

$$\check{A} = \bigvee\{-a \mid a \in \ell(A)\} \tag{4.1}$$

for $A \in \mathcal{L}$.

LEMMA 4.1. *For $A, X \in \mathcal{L}$ we have*

$$\begin{aligned} X \oplus \check{A} &= \bigvee_{a\in\ell(A)} X_{-a} \\ \text{and} \quad X \ominus \check{A} &= \bigwedge_{a\in\ell(A)} X_a. \end{aligned} \tag{4.2}$$

PROOF. Let us show the first equality. By (4.1), $\check{A} = \sup_{a\in\ell(A)}(-a)$, and so by (3.7) we have $\delta_{\check{A}} = \sup_{a\in\ell(A)} \delta_{-a}$. Aplying this to X we get:

$$X \oplus \check{A} = \delta_{\check{A}}(X) = \bigvee_{a\in\ell(A)} \delta_{-a}(X) = \bigvee_{a\in\ell(A)} X_{-a}.$$

The other equality in (4.2) is proved in the same way (using (3.8)), or is derived by adjunction (see (3.3) and Theorem 3.6). ∎

Now let for the rest of this subsection $\mathcal{L}$ be a Boolean complete lattice. We denote by X' the complement of the element X. Clearly $\sigma(X') = \sigma(X)'$ for every automorphism σ of $\mathcal{L}$, and in particular $(X')_a = (X_a)'$ for $a \in A$. We have the following:

PROPOSITION 4.2. *Let $\mathcal{L}$ be Boolean and satisfying the basic assumption of Subsection 3.2. Then for every $A \in \mathcal{L}$, $\check{\check{A}} = A$,*

$$\begin{aligned} (X' \oplus A)' &= X \ominus \check{A}, \\ \text{and} \quad (X' \ominus A)' &= X \oplus \check{A}. \end{aligned} \tag{4.3}$$

PROOF. By (4.2) we have:

$$X' \oplus A = \sup_{a \in \ell(A)} (X')_a = \sup_{a \in \ell(A)} (X_a)' = (\inf_{a \in \ell(A)} X_a)' = (X \ominus \check{A})'.$$

This proves the first equality of (4.3), and the second one is shown in the same way. But then

$$X \oplus \check{\check{A}} = (X' \ominus \check{A})' = (X'' \oplus A)'' = X \oplus A,$$

which implies that $\check{\check{A}} = A$. ∎

By (4.1) the map $A \mapsto \check{A}$ is increasing, and as it is its own inverse, it is an automorphism of $\mathcal{L}$. We will now see that Theorem 3.11 can be expressed in the dual form. If $\psi : \mathcal{L} \to \mathcal{L}$ is an increasing $\mathbf{T}$-operator, then ψ' defined by

$$\psi'(X) = \big(\psi(X')\big)', \quad X \in \mathcal{L}, \tag{4.4}$$

is an increasing $\mathbf{T}$-operator as well. It follows then from Theorem 3.11 that

$$\psi'(X) = \bigvee_{A \in \mathcal{V}'} (X \ominus A),$$

where $\mathcal{V}'$ is the kernel of ψ'. This yields that

$$\begin{aligned} \psi(X) &= \big(\psi'(X')\big)' = \big(\bigvee_{A \in \mathcal{V}'} (X' \ominus A)\big)' \\ &= \bigwedge_{A \in \mathcal{V}'} (X' \ominus A)' = \bigwedge_{A \in \mathcal{V}'} (X \oplus \check{A}). \end{aligned} \tag{4.5}$$

We have thus the following:

COROLLARY 4.3. *Suppose that $\mathcal{L}$ is Boolean and satisfies the basic assumption of Subsection 3.2. Then every increasing $\mathbf{T}$-operator can be written as a supremum of $\mathbf{T}$-erosions and, alternatively, as an infimum of $\mathbf{T}$-dilations.*

In Subsection 4.2 we construct an example which shows that in this corollary the assumption that $\mathcal{L}$ is Boolean cannot be dropped.

4.1.1. Translation-invariance. Let $\mathcal{L} = P(\mathbf{R}^d)$, the space of all subsets of $\mathbf{R}^d$. Then $\mathcal{L}$ is a complete Boolean lattice, the complement of an element $X \in \mathcal{L}$ being given by the ordinary set complement X^c. The lattice $\mathcal{L}$ is atomic, the family ℓ of atoms being given by the set of singletons:

$$\ell = \big\{\{\xi\} \mid \xi \in \mathbf{R}^d\big\}.$$

In particular, ℓ is a sup-generating family.

For $h \in \mathbf{R}^d$ we define the operator τ_h on $\mathcal{L}$ by

$$\tau_h(X) = X_h = \{\xi + h \mid \xi \in X\}.$$

Thus τ_h translates every set X along the vector h. It is clear that every τ_h is an automorphism on $\mathcal{L}$, and that $\mathbf{T} = \{\tau_h \mid h \in \mathbf{R}^d\}$ is an abelian group of automorphisms on $\mathcal{L}$ which is transitive on ℓ. The corresponding group operation on $\mathbf{R}^d$ is the vector addition. The extension of $+$ to $\mathcal{L}$ as defined in (3.4) is

$$X \oplus A = \{x + a \mid x \in X, a \in A\} = \bigcup_{a \in A} X_a,$$

which is the Minkowski addition defined in Subsection 1.1.3. Furthermore

$$X \ominus A = \bigcap_{a \in A} X_{-a},$$

which is the Minkowski subtraction. Thus in this example $\mathbf{T}$-erosions and $\mathbf{T}$-dilations correspond to the notions of erosion and dilation defined by Serra [23] in the translation-invariant case. Here $\check{A} = \{-a \mid a \in A\}$.

However, as we mentioned already in Subsection 1.1.3, our notation is slightly different from the notation used by Serra [23] and Matheron [14] for the Boolean complete lattice $\mathcal{P}(\mathbf{R}^d)$. Their definition of the Minkowski addition $\oplus$ is the same as ours, but they call the operator $X \to X \oplus \check{A}$ the dilation by A. Moreover, their definition of the Minkowski subtraction differs from ours in the sense that they put $X \ominus A = \bigcap_{a \in A} X_a$, which is in fact $X \ominus \check{A}$ in our notation. Furthermore, they call $X \to X \ominus \check{A}$ the erosion by A and this coincides with our nomenclature (although we write $X \ominus A$). Our definition coincides with Hadwiger's original definition (see [4,5]; to our knowledge, Minkowski never defined the subtraction) and with the terminology used by Sternberg [25]. Our motivation for our conventions lies in the fact that the duality relation between dilations and erosions induced by adjunctions is a general one, whereas the duality obtained by taking complements in the Boolean case (see (4.3); in Serra's notation this would amount to the formula $X \ominus A = (X' \oplus A)'$) cannot be extended to the non-Boolean case.

In the digital case we replace the continuous space $\mathbf{R}^d$ by the digital grid $\mathbf{Z}^d$, and everything works in the same way.

4.1.2. Invariance under rotations and scalar multiplications. Certain types of images do not correspond to a translation-invariant material structure, but rather to a circular one, which is invariant under rotations. An example is given in Figure 1.8 of [23], which represents the amount of sunshine in a forest, obtained by making a photograph of the sky from the ground. Another example is given by radar displays.

In order to express such a structure in the framework of Section 3, we take $\mathcal{L}$ to be the complete atomic Boolean lattice $\mathcal{P}(\Pi)$, where $\Pi = \mathbf{R}^2 \setminus \{0\}$, whose points will be indexed by polar coordinates (r, θ), with r being the radius ($r > 0$) and θ the angle (taken modulo 2π). Let ℓ be the set of atoms (singletons). We must build an abelian group $\mathbf{T}$ of automorphisms of $\mathcal{L}$ which is transitive on ℓ. This can be achieved by endowing Π with the structure of an abelian group (see Remark 3.1). Let $\mathbf{R}^+ = \{r \in \mathbf{R} \mid r > 0\}$, and suppose that $\mathbf{R}^+$ is an abelian group for a binary operation $*$. Then Π is a group for the product $\cdot$ defined by

$$(r,\theta)\cdot(s,\varphi) = (r * s, \theta + \varphi), \tag{4.6}$$

where $\theta + \varphi$ is taken modulo 2π. In group-theoretical terms, Π is the direct product of $\mathbf{R}^+$ and of the group of rotations. Here $\mathbf{T}$ is the set of automorphisms of $\mathcal{L}$ of the form $\tau_{r,\theta}$, where we have

$$\tau_{r,\theta}(X) = \{(r,\theta)\cdot(s,\varphi) \mid (s,\varphi) \in X\}$$

for $X \in \mathcal{L}$.

As a particular case, we can take $*$ to be the ordinary multiplication in $\mathbf{R}^+$. Then each $\tau_{r,\theta} \in \mathbf{T}$ is the composition of a scalar multiplication relative to the origin with a factor r, and a rotation around the origin by an angle θ. If to each $(r,\theta) \in \Pi$ we associate the complex number $re^{i\theta}$, then the product operation $\cdot$ on Π (see (4.6)) corresponds to the multiplication in the complex plane. We can define the Minkowski operations as in the previous example of Subsection 4.1.1, with $\cdot$ instead of $+$. Although the resulting formulas for $\mathbf{T}$-erosions and $\mathbf{T}$-dilations are the same, their action is completely different: see Figure 3.

If we want to implement such a circular morphology on a machine, we must replace the continuous space $\Pi = \mathbf{R}^2 \setminus \{0\}$ by a discrete *polar grid* (in the same way as we replace the continuous space $\mathbf{R}^d$ by the digital grid $\mathbf{Z}^d$ in the translation-invariant case). It is built as follows. We choose an integer $s > 1$ and an element $r > 1$ of $\mathbf{R}^+$; then the polar grid consists of all points $p[m,n]$ having polar coordinates $(r^m, n\frac{2\pi}{s})$ for $m, n \in \mathbf{Z}$ (with n taken modulo s). We illustrate such a grid in Figure 4.

Polar grids are used in theoretical neurology to model the topography of the layer of retinal ganglion cells (the neurons in the retina which feed the optic nerve). Corresponding visual areas in the cerebral cortex are modeled as a rectangular grid where pixel (m,n) corresponds approximately (for m large enough) to the point $p[m,n]$ of the polar grid with polar coordinates $(r^m, n\frac{2\pi}{s})$. See [22,28] for more details.

4.2. *The complete lattice of closed subsets of* $\mathbf{R}^d$

Define $\mathcal{L} = \mathcal{F}(\mathbf{R}^d)$, the family of all closed subsets of $\mathbf{R}^d$. Then $\mathcal{L}$ is a complete lattice and the supremum and infimum of the collection $X_j \in \mathcal{L}$, where j runs through the (finite or infinite) index set J, are respectively given by

$$\bigvee_{j\in J} X_j = \overline{\bigcup_{j\in J} X_j}$$

$$\text{and} \quad \bigwedge_{j\in J} X_j = \bigcap_{j\in J} X_j.$$

(N.B. We write $\overline{U}$ and U° respectively for the closure and the interior of a Euclidean set U.) Again $\mathcal{L}$ is atomic, the atoms being the singletons of $\mathbf{R}^d$ (essentially, one uses that $\mathbf{R}^d$ is a topological T_1-space: see Birkhoff [1] or Kelley [11]). However, $\mathcal{L}$ is not Boolean.

Let $\mathbf{T}$ be the automorphism group of translations. Then $\mathbf{T}$ is transitive on ℓ. Again, the group operation induced on ℓ is the ordinary vector addition. The operations in $\mathcal{L}$ corresponding to $\oplus$ and $\ominus$ in $\mathcal{P}(\mathbf{R}^d)$ are written $\overline{\oplus}$ and $\overline{\ominus}$ (to avoid confusion); they are defined by setting for $X, A \in \mathcal{L}$:

$$X \mathbin{\overline{\oplus}} A = \bigvee_{a \in A} X_a = \overline{\bigcup_{a \in A} X_a}$$

$$\text{and} \quad X \mathbin{\overline{\ominus}} A = \bigwedge_{a \in A} X_{-a} = \bigcap_{a \in A} X_{-a}.$$

Note that $X \mathbin{\overline{\oplus}} A = \overline{X \oplus A}$, while $X \mathbin{\overline{\ominus}} A = X \ominus A$.

Theorem 3.11 states that every increasing $\mathbf{T}$-operator on $\mathcal{L}$ can be obtained as a supremum of $\mathbf{T}$-erosions. We now show by means of a counterexample that Matheron's theorem for $\mathbf{T}$-dilations (see Corollary 4.3) is not valid in this case: we define an increasing $\mathbf{T}$-operator ψ on $\mathcal{L} = \mathcal{F}(\mathbf{R}^d)$ which cannot be obtained as an infimum of $\mathbf{T}$-dilations.

For $r > 0$ we define $F_r \in \mathcal{L}$ by:

$$F_r = \{x \in \mathbf{R}^d \mid \|x\| \geq r\}.$$

For every closed subset X of $\mathbf{R}^d$ we define $\rho(X)$ as the radius of the smallest circumscribing circle (which is $+\infty$ if X is unbounded). Let $A \subseteq \mathbf{R}^d$ be closed. We have the following equivalence:

$$F_r \mathbin{\overline{\oplus}} A = \mathbf{R}^d \qquad \text{if and only if} \qquad \rho(A) \geq r. \tag{4.7}$$

Indeed, by the symmetry of F_r and $\mathbf{R}^d$, $F_r \mathbin{\overline{\oplus}} A = \mathbf{R}^d$ if and only if $F_r \mathbin{\overline{\oplus}} \check{A} = \mathbf{R}^d$. Now let

$$B_r = \{x \in \mathbf{R}^d \mid \|x\| < r\}$$

be the complement of F_r; it is an open disk of radius r. Using Proposition 4.2 in $\mathcal{P}(\mathbf{R}^d)$ we get:

$$(F_r \mathbin{\overline{\oplus}} \check{A})^c = \left(\overline{F_r \oplus \check{A}}\right)^c = ((F_r \oplus \check{A})^c)^\circ = ((F_r)^c \ominus A)^\circ = (B_r \ominus A)^\circ.$$

Hence $F_r \mathbin{\overline{\oplus}} A = \mathbf{R}^d$ if and only if $(B_r \ominus A)^\circ = \emptyset$. If $\rho(A) < r$, then $A \leq \tau_a(B_s)$ for some $s < r$ and some point a, and so

$$B_r \ominus A \geq B_r \ominus \tau_a(B_s) = \tau_{-a}(B_r \ominus B_s).$$

Now $\tau_{-a}(B_r \ominus B_s)$ is a closed disk of radius $r - s$, and its interior is not empty. Thus $(B_r \ominus A)^\circ \neq \emptyset$ when $\rho(A) < r$, and in this case $F_r \mathbin{\overline{\oplus}} A \neq \mathbf{R}^d$. Suppose now that $\rho(A) \geq r$. Then no translate of A is a subset of B_r: this is obvious if $\rho(A) > r$, while if $\rho(A) = r$, this follows from the fact that A is *closed* and the disk B_r is *open*. By Proposition 3.5 (*iv*), $B_r \ominus A$ is the set of points $z \in \mathbf{R}^d$ such that $A_z \subseteq B$, and so $B_r \ominus A = \emptyset$ when $\rho(A) \geq r$, which implies that $F_r \mathbin{\overline{\oplus}} A = \mathbf{R}^d$. This shows (4.7).

We define now our counterexample. Let the operator ψ be given by

$$\psi(X) = \bigcap\{(F_1)_h \mid h \in \mathbf{R}^d \text{ and } X \subseteq (F_1)_h\} \tag{4.8}$$

for $X \in \mathcal{F}(\mathbf{R}^d)$. In other words $\psi(X)$ is the intersection of all translates of F_1 containing it. Clearly ψ is increasing and $\mathbf{T}$-invariant. (In Part II we will pay more attention to this operator which turns out to be an example of what we will call a *structural closing*).

Now suppose that ψ is an infimum of a family of $\mathbf{T}$-dilations, and let δ_A be a member of this family; then $\delta_A \geq \psi$. Obviously, $\psi(F_r) = \mathbf{R}^d$ if $r < 1$. Hence $F_r \oplus A = \delta_A(F_r) = \mathbf{R}^d$ if $r < 1$. From (4.7) we conclude that $\rho(A) \geq r$ if $r < 1$, and therefore $\rho(A) \geq 1$. Applying (4.7) once more we get that $\delta_A(F_1) = \mathbf{R}^d$. Since this holds for every dilation greater than ψ we may conclude that $\psi(F_1) = \mathbf{R}^d$, a contradiction since $\psi(F_1) = F_1$. Thus ψ cannot be written as an infimum of $\mathbf{T}$-dilations.

4.3. *Grey-level functions*

Although the formalism of Section 3 was built by analogy with the Boolean case, it can also be applied to grey-level functions. We obtain in this way both the usual translation-invariant dilation and erosion given in (1.3) and (1.4), but also Herman's operators [9]. We finally consider a more general type of dilations and erosions, which do not satisfy the basic assumption of Subsection 3.2, but are invariant under spatial translations.

4.3.1. Additive structuring functions. Let $\mathcal{L}$ be the complete lattice of all functions mapping $\mathbf{R}^d$ into $\overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty, -\infty\}$. Clearly $\mathcal{L}$ is a power lattice (see Subsection 1.3), and so the supremum and infimum of a family F_j, $j \in J$, are as follows:

$$(\bigvee_{j\in J} F_j)(x) = \sup_{j\in J} F_j(x),\ x \in \mathbf{R}^d$$

$$(\bigwedge_{j\in J} F_j)(x) = \inf_{j\in J} F_j(x),\ x \in \mathbf{R}^d.$$

Let ℓ be the family of functions $f_{x,t}$ ($x \in \mathbf{R}^d$, $t \in \mathbf{R}$) given by

$$f_{x,t}(y) = \begin{cases} t, & \text{if } y = x; \\ -\infty, & \text{if } y \neq x. \end{cases}$$

(Cfr. (2.14).) It is obvious that ℓ is a sup-generating family. For $h \in \mathbf{R}^d$ and $v \in \mathbf{R}$ ('h' stands for horizontal and 'v' for vertical) we define the automorphism $\tau_{h,v}$ on $\mathcal{L}$ as follows:

$$\big(\tau_{h,v}(F)\big)(x) = F(x-h) + v \qquad \text{for} \quad F \in \mathcal{L},$$

i.e., $\tau_{h,v}$ translates the graph of a function along the vector (h, v). Then

$$\mathbf{T} = \{\tau_{h,v} \mid h \in \mathbf{R}^d, v \in \mathbf{R}\}$$

is an abelian automorphism group on $\mathcal{L}$, and moreover $\mathbf{T}$ is transitive on ℓ. To be precise, $\tau_{h,v}(f_{x,t}) = f_{x+h,t+v}$. The corresponding group operation on ℓ is given by:

$$f_{x_1,t_1} + f_{x_2,t_2} = f_{x_1+x_2,t_1+t_2}.$$

This leads to the operations $\oplus$ and $\ominus$ on $\mathcal{L}$ defined in (1.3) and (1.4), namely:

$$\begin{aligned} (F \oplus G)(x) &= \sup_{h\in\mathbf{R}^d} \big(F(x-h) + G(h)\big) \\ \text{and} \quad (F \ominus G)(x) &= \inf_{h\in\mathbf{R}^d} \big(F(x+h) - G(h)\big), \end{aligned} \tag{4.9}$$

with the further conventions, in cases of ambiguous expressions of the form $+\infty - \infty$, that $F(x-h)+G(h) = -\infty$ when $F(x-h) = -\infty$ or $G(h) = -\infty$, and that $F(x+h)-G(h) = +\infty$ when $F(x+h) = +\infty$ or $G(h) = -\infty$. The operations $F \mapsto F \oplus G$ and $F \mapsto F \ominus G$ are the so-called *grey-level dilation* and *erosion* respectively. Both are invariant under translations in the spatial (=horizontal) and grey-level (=vertical) direction.

Although the complete lattice of grey-level functions $\mathbf{R}^d \to \overline{\mathbf{R}}$ is not Boolean, some of the ideas of Subsection 4.1 can still be applied here. We do not have the complementation, but the grey level inversion $F \mapsto -F$ is a dual automorphism of $\mathcal{L}$. For any $G \in \mathcal{L}$ we define

$$(\overline{G})(x) = G(-x),$$

and $\overline{G}$ takes the role of the reflected structuring element $\check{A}$. Indeed we have $\overline{\overline{G}} = G$, and the following relations hold:

$$\begin{aligned} -\big((-F) \oplus G\big) &= F \ominus \overline{G} \\ \text{and} \quad -\big((-F) \ominus G\big) &= F \oplus \overline{G}. \end{aligned} \tag{4.10}$$

Compare with Proposition 4.2. We can apply the same argument as in the proof of Corollary 4.3 (see (4.4) and (4.5)), to show that in this case too Matheron's theorem holds both for $\mathbf{T}$-erosions and $\mathbf{T}$-dilations. Note also that grey-level inversion transforms ℓ into an inf-generating family ℓ' on which $\mathbf{T}$ acts transitively, so that the results of Section 3 can be applied in their dual form, leading in particular to Matheron's theorem for $\mathbf{T}$-dilations.

We mentioned above in Subsection 4.1.1 that for the Boolean complete lattice $\mathcal{P}(\mathbf{R}^d)$ Serra and Matheron derived erosions from dilations by complementation, leading to a different notation as ours, while Sternberg defined erosions as us. Similarly in [23], Serra defines the addition $F \oplus G$ of grey-level functions as we do here, but the subtraction $F \ominus G$ is defined by addition and grey-level inversion, in other words as $-\big((-F) \oplus G\big)$. Hence $F \ominus G$ in Serra's nomenclature corresponds to $F \ominus \overline{G}$ in ours. On the other hand Sternberg's notations [25,26] are in concordance with ours.

In Proposition 3.12, we have seen that under the assumption that $\ell \cup \{O, I\}$ is inf-closed, the only automorphisms which commute with every $\mathbf{T}$-dilation and every $\mathbf{T}$-erosion are the elements of $\mathbf{T}$. The following example shows that the condition on ℓ may not be dropped. Suppose that we take

$$\ell = \{f_{x,t} \mid x \in \mathbf{R}^d, t \in \mathbf{Q}\} \qquad \text{and} \qquad \mathbf{T} = \{\tau_{h,v} \mid h \in \mathbf{R}^d, v \in \mathbf{Q}\}.$$

Then the basic assumption of Subsection 3.2 is again satisfied, however every $\tau_{h,v}$ with $v \notin \mathbf{Q}$ is an automorphism on $\mathcal{L}$ which commutes with every $\mathbf{T}$-dilation and every $\mathbf{T}$-erosion, but which is not in $\mathbf{T}$.

REMARK 4.1. One can define a topology on the space of grey-level functions if one restricts oneself to the complete lattice of *upper semi-continuous* (in brief, *u.s.c.*) functions mapping $\mathbf{R}^d$ into $\overline{\mathbf{R}}$. These functions generalize the closed sets in the sense that F is u.s.c. if and only if its umbra $U(F)$ is closed: see [2]. Note that the functions $f_{x,t}$ are u.s.c. and that the family ℓ defined above is sup-generating in the complete lattice of u.s.c. functions. So our theory also works in this case.

4.3.2. Multiplicative structuring functions. In this subsection we assume that $\mathcal{L}$ is the complete lattice of functions mapping $\mathbf{R}^d$ into $[0,+\infty]$. Let $\mathbf{R}^+ = \{r \in \mathbf{R} \mid r > 0\}$. The family

$$\ell = \{f_{x,t} \mid x \in \mathbf{R}^d, t \in \mathbf{R}^+\}$$

is sup-generating, and we choose

$$\mathbf{T} = \{\dot{\tau}_{h,v} \mid h \in \mathbf{R}^d, v \in \mathbf{R}^+\},$$

where

$$\big(\dot{\tau}_{h,v}(F)\big)(x) = v \cdot F(x-h) \qquad \text{for} \quad F \in \mathcal{L},$$

i.e., $\tau_{h,v}$ translates the graph of a function by h in the spatial domain, and magnifies it by a factor v in the grey-level domain. Then $\mathbf{T}$ is an abelian automorphism group on $\mathcal{L}$ which is transitive on ℓ. In fact, $\dot{\tau}_{h,v}(f_{x,t}) = f_{x+h,t\cdot v}$. The group operation $\dot{+}$ induced on ℓ is given by

$$f_{x_1,t_1} \dot{+} f_{x_2,t_2} = f_{x_1+x_2,t_1\cdot t_2}.$$

This leads to the operations $\dot{\oplus}$ and $\dot{\ominus}$ on $\mathcal{L}$ given by

$$\begin{aligned} (F \dot{\oplus} G)(x) &= \sup_{h\in\mathbf{R}^d} \big((F(x-h)\cdot G(h)\big) \\ \text{and} \quad (F \dot{\ominus} G)(x) &= \inf_{h\in\mathbf{R}^d} \big((F(x+h)/G(h)\big), \end{aligned} \tag{4.11}$$

with the further conventions, in cases of ambiguous expressions of the form $0\cdot\infty$, ∞/∞, or $0/0$, that $F(x-h)\cdot G(h) = 0$ when $F(x-h) = 0$ or $G(h) = 0$, and that $F(x+h)/G(h) = +\infty$ when $F(x+h) = +\infty$ or $G(h) = 0$. These two operations lead to a morphology for grey-tone functions based on *multiplicative structuring functions*. See [9]. The relation with $\oplus$ and $\ominus$ defined in Subsection 4.3.1 is given by:

$$F \dot{\oplus} G = e^{\log F \oplus \log G} \qquad \text{for} \quad F, G \in \mathcal{L},$$

and a similar relation holds for $\dot{\ominus}$ and $\ominus$. Given $\overline{G}(x) = G(-x)$, we have

$$\begin{aligned} F \dot{\ominus} \overline{G} &= \frac{1}{1/F \oplus G} \\ \text{and} \quad F \dot{\oplus} \overline{G} &= \frac{1}{1/F \dot{\ominus} G}. \end{aligned} \tag{4.12}$$

Compare with (4.10). We can apply the same reasoning as there, and so Matheron's theorem holds both for $\mathbf{T}$-erosions and $\mathbf{T}$-dilations.

Multiplicative structuring functions can be used as an alternative to additive ones, and Herman [9] applied them for edge enhancement in X-ray images.

4.3.3. A general class of grey-level dilations and erosions. When implementing grey-level morphological operations on a computer, one has to choose the grey-level set $\mathcal{G}$ finite, say $\mathcal{G} = \{0, 1, 2, \ldots, m-1\}$, or at best $\mathcal{G}$ can contain approximations of real numbers in a *bounded* interval. For many practicioners this does not present a real problem, they simply apply formulas like (4.9) with grey-levels belonging to a bounded interval of $\mathbf{R}$ or $\mathbf{Z}$. However it is easy to see that when the grey-levels of the function F are bounded by a maximum M and/or by a minimum m, the functions $F \oplus G$ and $F \ominus G$ given by (4.9) will be similarly bounded if and only if the structuring function G satisfies the following condition:

$$\sup_{h \in \mathbf{R}^d} G(h) = 0. \tag{4.13}$$

In particular everything works well with a *flat structuring function*, that is a function G defined on a restricted support S, where $G(h) = 0$ for each $h \in S$. Similarly, if one uses multiplicative structuring functions, formulas given in (4.11) require that:

$$\sup_{h \in \mathbf{R}^d} G(h) = 1. \tag{4.14}$$

On the other hand, applying formulas (4.9) or (4.11) with arbitrary structuring functions, and truncating the grey-levels of resulting functions whenever they exceed the bounds is not serious.

We might also attempt to apply Section 3 to the object space $\mathcal{G}^{\mathcal{E}}$, where $\mathcal{E} = \mathbf{R}^d$ and $\mathcal{G}$ is a bounded interval of $\mathbf{Z}$ or of $\mathbf{R}$ (for example $\{0, 1, 2, \ldots, m-1\}$). A short moment of reflection learns that this leads to an essential difficulty: how to define an abelian group operation $\star$ on $\mathcal{G}$ which preserves the ordering of $\mathcal{G}$, *i.e.*, if $g, g_1, g_2 \in \mathcal{G}$ and $g_1 \leq g_2$, then $g \star g_1 \leq g \star g_2$. The addition modulo m (or one of its permutations) does not have this property. Confronted with this problem the only way out is to drop the invariance in the "grey-level direction" and to require invariance in the spatial direction only. This is not as bad as it may seem. In contrast with the spatial translation-invariance there is no physical reason for requiring additive (if $\mathcal{G} = \overline{\mathbf{R}}$) or multiplicative (if $\mathcal{G} = [0, \infty]$) invariance. Even more, such a requirement only restricts the class of operations which are permitted. In this subsection we investigate the class of dilations and erosions which is obtained if only spatial invariance is required. We assume for simplicity that $\mathcal{G} = \overline{\mathbf{R}}$, but the approach works equally well for any complete lattice $\mathcal{G}$.

Let $\mathcal{L}$ be the complete lattice of grey-level functions $F : \mathbf{R}^d \to \overline{\mathbf{R}}$, let ℓ be the sup-generating family of Subsection 4.3.1, i.e., $\ell = \{f_{x,t} \mid x \in \mathbf{R}^d, t \in \mathbf{R}\}$, and let $\mathbf{T}$ be the automorphism group of spatial translations:

$$\mathbf{T} = \{\tau_{h,0} \mid h \in \mathbf{R}^d\}.$$

Note that $\mathbf{T}$ is *not* transitive on ℓ. Nevertheless we will be able to give a characterization of all $\mathbf{T}$-dilations and $\mathbf{T}$-erosions on $\mathcal{L}$. First we remark that in an adjunction (e, d) on the complete lattice $\overline{\mathbf{R}}$, d is an increasing function on $\overline{\mathbf{R}}$ with $d(-\infty) = -\infty$ which is continuous from the left, and e is an increasing function on $\overline{\mathbf{R}}$ with $e(+\infty) = +\infty$ which is continuous from the right. From (2.10) and (2.11) it follows that

$$\begin{aligned} & e(t) = \sup\{s \in \overline{\mathbf{R}} \mid d(s) \le t\} \\ \text{and} \quad & d(t) = \inf\{s \in \overline{\mathbf{R}} \mid t \le e(s)\}. \end{aligned}$$

THEOREM 4.4. *The pair (ε, δ) is a $\mathbf{T}$-adjunction on $\mathcal{L}$ if and only if there exists for every $h \in \mathbf{R}^d$ an adjunction (e_h, d_h) on $\overline{\mathbf{R}}$ such that for $F \in \mathcal{L}$*

$$\begin{aligned} & \delta(F)(x) = \sup_{h \in \mathbf{R}^d} d_h\big(F(x-h)\big) \\ \text{and} \quad & \varepsilon(F)(x) = \inf_{h \in \mathbf{R}^d} e_h\big(F(x+h)\big). \end{aligned} \tag{4.15}$$

PROOF. (*a*) *if*: Let (e_h, d_h) be an adjunction on $\overline{\mathbf{R}}$ for every $h \in \mathbf{R}^d$, and let δ, ε be defined by (4.15). It is clear that ε, δ are $\mathbf{T}$-invariant. We prove that

$$\delta(F) \le G \iff F \le \varepsilon(G)$$

for all $F, G \in \mathcal{L}$:

$$\begin{aligned} \delta(F) \le G & \iff \sup_{h \in \mathbf{R}^d} d_h\big(F(x-h)\big) \le G(x) \\ & \iff \forall h \in \mathbf{R}^d, \quad d_h\big(F(x-h)\big) \le G(x) \\ & \iff \forall h \in \mathbf{R}^d, \quad F(x-h) \le e_h\big(G(x)\big) \\ & \iff \forall h \in \mathbf{R}^d, \quad F(x) \le e_h\big(G(x+h)\big) \\ & \iff F \le \varepsilon(G). \end{aligned}$$

(*b*) *only if*: We only have to prove that any $\mathbf{T}$-dilation is of the form (4.15). Let δ be a $\mathbf{T}$-dilation and define $d_h : \overline{\mathbf{R}} \to \overline{\mathbf{R}}$ by

$$d_h(t) = \delta(f_{0,t})(h).$$

Then

$$d_h(\sup_{j \in J} t_i) = \delta(f_{0,\sup_{j\in J} t_j})(h) = \delta(\sup_{j\in J} f_{0,t_j})(h) = \sup_{j\in J}\Big(\delta(f_{0,t_j})(h)\Big) = \sup_{j\in J} d_h(t_j),$$

hence d_h is a dilation on $\overline{\mathbf{R}}$. From the observation that any $F \in \mathcal{L}$ can be written as

$$F = \sup_{y \in \mathbf{R}^d} f_{y,F(y)},$$

we get

$$\begin{aligned}
\delta(F)(x) &= \delta\big(\sup_{y\in\mathbf{R}^d} f_{y,F(y)}\big)(x) = \delta\big(\sup_{y\in\mathbf{R}^d} f_{y-x,F(y)}\big)(0) = \delta\big(\sup_{h\in\mathbf{R}^d} f_{-h,F(x-h)}\big)(0) \\
&= \sup_{h\in\mathbf{R}^d}\Big(\delta(f_{-h,F(x-h)})(0)\Big) = \sup_{h\in\mathbf{R}^d}\Big(\delta(f_{0,F(x-h)})(h)\Big) = \sup_{h\in\mathbf{R}^d} d_h(F(x-h)). \blacksquare
\end{aligned}$$

REMARK 4.2. We can shorten the proof given above considerably by using the results of Subsection 2.4, in particular Proposition 2.10. This result says that δ can be written as

$$\delta(F)(x) = \bigvee_{y\in\mathbf{R}^d} \delta_{x,y}(F(y)),$$

where every $\delta_{x,y}$ is a dilation on $\overline{\mathbf{R}}$. Now the spatial translation-invariance implies that $\delta_{x,y} = \delta_{x-y,0}$. Substitution of $d_h = \delta_{h,0}$ yields the result.

If we choose $d_h(t) = t + G(h)$, where $G \in \mathcal{L}$, then $\delta(F) = F \oplus G$ and we are back in the translation-invariant case considered in Subsection 4.3.1. Note that δ given by (4.15) is invariant under $\tau_{0,v}$ ($v \in \mathbf{R}$) if and only if

$$d_h(t+v) = d_h(t) + v, \quad t, v \in \mathbf{R},$$

which yields that

$$d_h(t) = t + G(h), \quad t \in \mathbf{R},$$

where $G(h) = d_h(0)$.

If we choose $d_h(t) = t \cdot G(h)$, where G maps $\mathbf{R}^d$ into $\mathbf{R}^+$, then we end up with multiplicative structuring functions: note however that here negative grey-levels are also allowed, whereas they had to be positive in Subsection 4.3.2.

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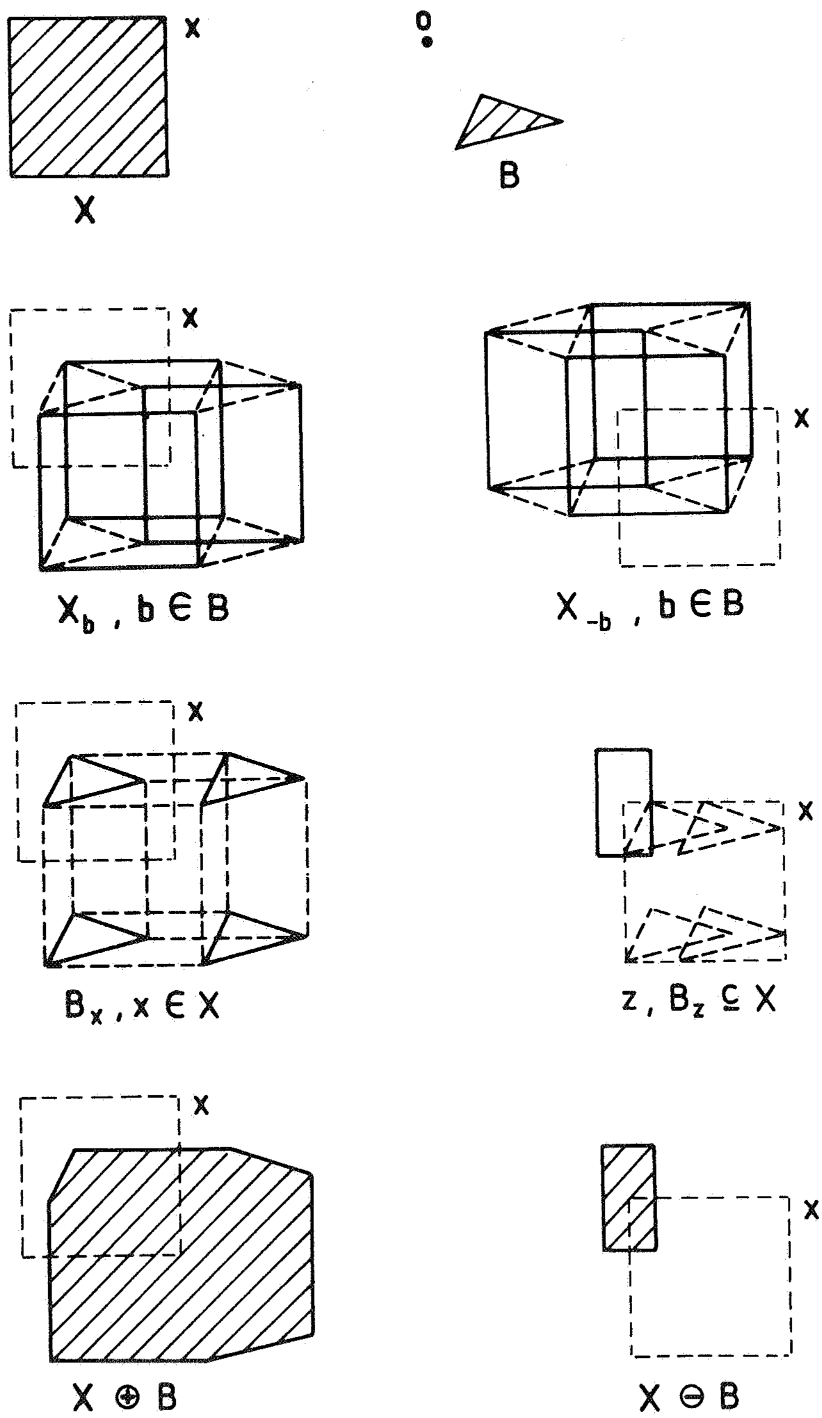


FIGURE 1. *Dilation and erosion of a set X by a structuring element B.*

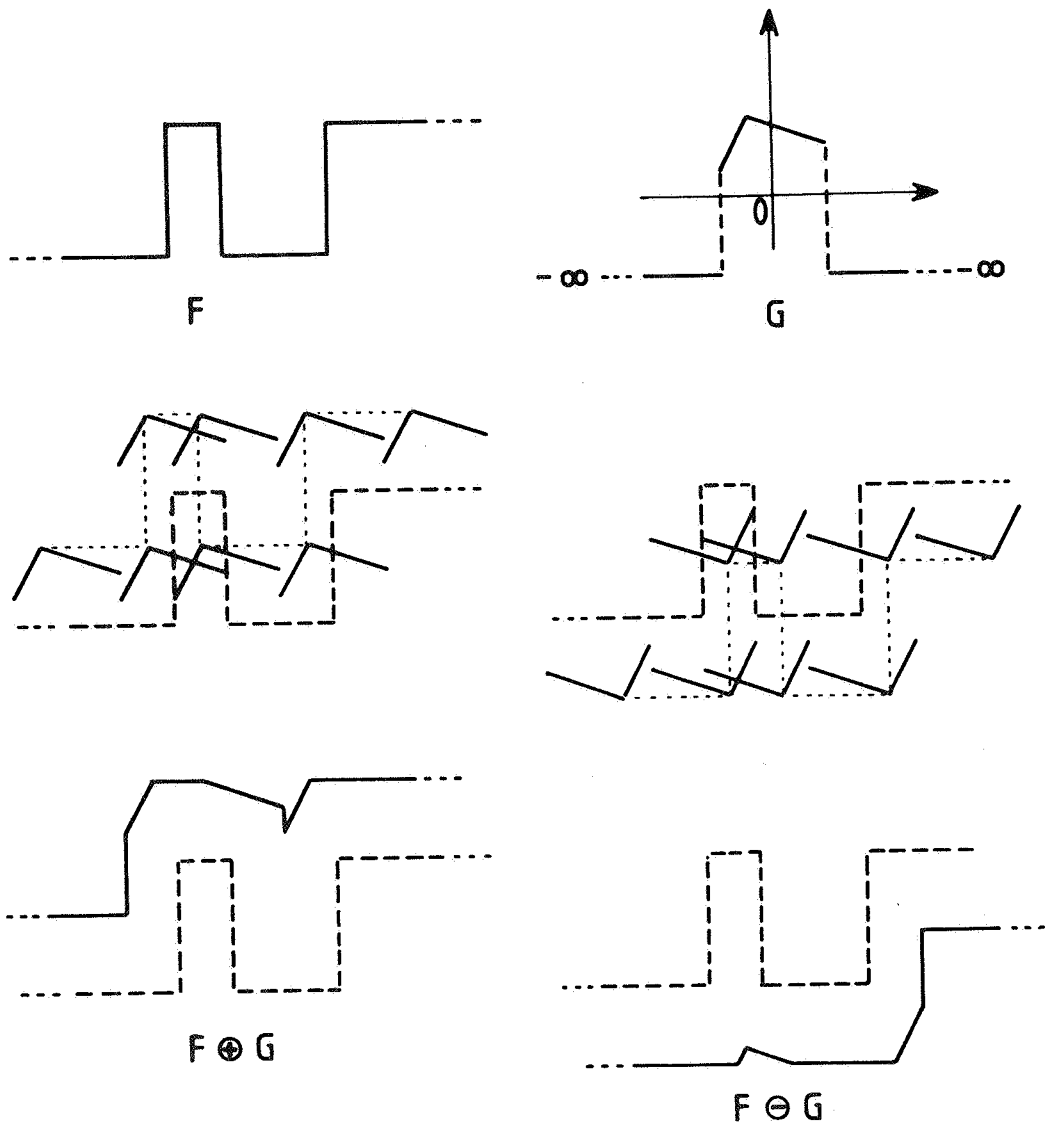
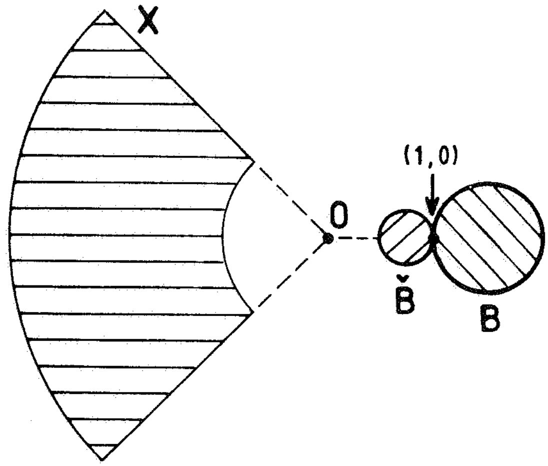
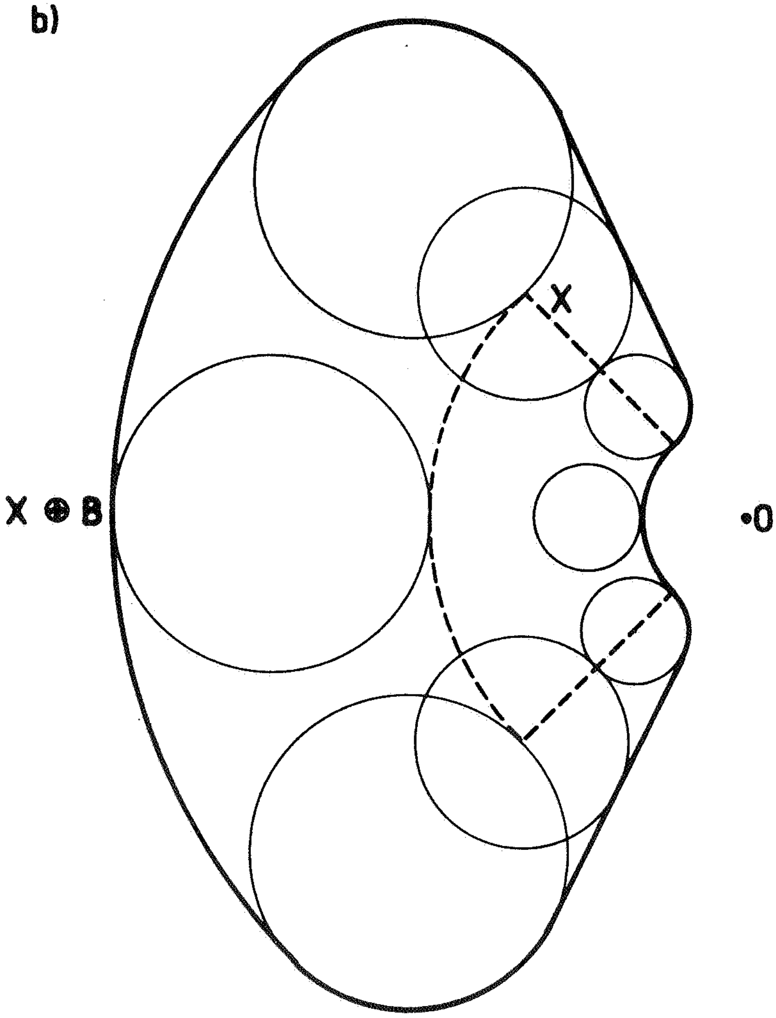


FIGURE 2. *Dilation and erosion of a function F by a structuring function G.*

a)



b)



c)

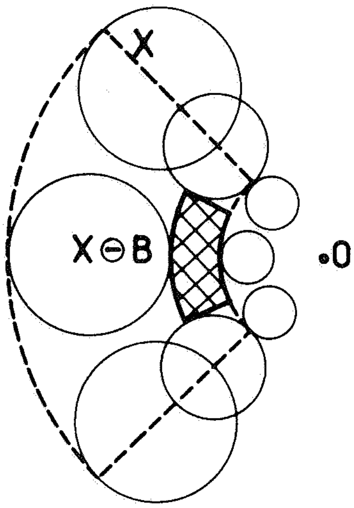


FIGURE 3. *Dilation and erosion invariant under rotations and scalar multiplications: (a) a set X, a structuring element B, and reflected one $\check{B}$; (b) $X \oplus B = \bigcup\{B_x \mid x \in X\}$; (c) $X \ominus B = (X^c \oplus \check{B})^c$.*

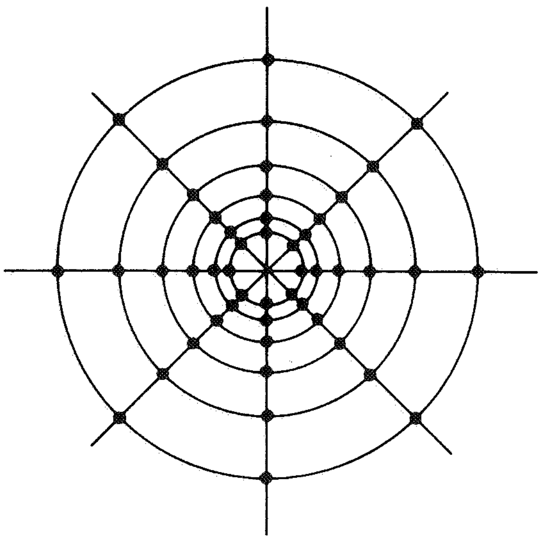


FIGURE 4. *Polar grid made of points $p[m,n]$ having polar coordinates $(r^m, n\frac{2\pi}{s})$, where $r = \sqrt{2}$ and $s = 8$.*