# On the refined 3 -enumeration of alternating sign matrices 

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#### Abstract

An explicit expression for the numbers $A(n, r ; 3)$ describing the refined 3-enumeration of alternating sign matrices is given. The derivation is based on the recent results of Stroganov for the corresponding generating function. As a result, $A(n, r ; 3)$ 's are represented as 1 -fold sums which can also be written in terms of terminating ${ }_{4} F_{3}$ series of argument $1 / 4$.


## 1. Introduction

An alternating sign matrix (ASM) is a matrix of 1 's, 0 's and -1 's such that in each row and in each column the first and the last nonzero entry is 1 , and all nonzero entries alternate in sign. There are many nice results concerning ASMs (for a review, see [1]). The most celebrated result gives the total number $A(n)$ of $n \times n$ ASMs. It was conjectured by Robbins, Mills and Rumsey [2,3] and subsequently proved by Zeilberger [4] and Kuperberg [5] that

$$
\begin{equation*}
A(n)=\prod_{k=1}^{n} \frac{(3 k-2)!(k-1)!}{(2 k-1)!(2 k-2)!}=\prod_{k=1}^{n} \frac{(3 k-2)!}{(2 n-k)!} . \tag{1.1}
\end{equation*}
$$

Yet another formula was conjectured in [2,3], and proved in [6], concerning the refined enumeration of ASMs, namely, that the number of $n \times n$ ASMs with their sole 1 of the first (or last) column (or row) at the $r$-th entry, denoted as $A(n, r)$, is

$$
\begin{equation*}
A(n, r)=\frac{\binom{n+r-2}{n-1}\binom{2 n-1-r}{n-1}}{\binom{3 n-2}{n-1}} A(n) . \tag{1.2}
\end{equation*}
$$

One more conjecture of Robbins, Mills and Rumsey, proved by Kuperberg [5], gives the number of 3 -enumerated ASMs. In general, $x$-enumeration counts ASMs with a weight $x^{k}$, where $k$ is the total number of -1 's in each matrix (the number $x$ here should not be confused with
the variable $x$ below). The total number of 3 -enumerated ASMs is denoted as $A(n ; 3)$ and the result reads
$A(2 m+1 ; 3)=3^{m(m+1)} \prod_{k=1}^{m}\left[\frac{(3 k-1)!}{(m+k)!}\right]^{2} \quad A(2 m+2 ; 3)=3^{m} \frac{(3 m+2)!m!}{[(2 m+1)!]^{2}} A(2 m+1 ; 3)$.
In the present paper we study the problem of the refined 3 -enumeration of ASMs, i.e., we derive the numbers $A(n, r ; 3)$. The main result can be summarized as follows.

Theorem (Main result). The refined 3-enumeration of $\operatorname{ASMs}, A(n, r ; 3)$, is given by

$$
\begin{align*}
& A(2 m+2, r ; 3)=\frac{b(m, r-1)+b(m, r-2)}{2} A(2 m+2 ; 3) \\
& A(2 m+3, r ; 3)=\frac{2 b(m, r-1)+5 b(m, r-2)+2 b(m, r-3)}{9} A(2 m+3 ; 3) \tag{1.4}
\end{align*}
$$

where the quantities $b(m, \alpha)$, obeying $b(m, \alpha)=b(m, 2 m-\alpha)$, are given by

$$
\begin{align*}
b(m, \alpha)=\frac{(2 m+1)!m!}{3^{m}(3 m+2)!} \sum_{\ell=\max (0, \alpha-m)}^{[\alpha / 2]}(2 m & +2-\alpha+2 \ell)\binom{3 m+3}{\alpha-2 \ell} \\
& \times\binom{ 2 m+\ell-\alpha+1}{m+1}\binom{m+\ell+1}{m+1} 2^{\alpha-2 \ell} \tag{1.5}
\end{align*}
$$

for $\alpha=0,1, \ldots, 2 m$, while they are assumed to be zero for all other values of $\alpha$.
As a comment to the main result it is to be mentioned that the numbers $A(n, r ; 3)$ appear not to be writable as a single hypergeometric term, i.e., not to be 'round' (or 'smooth'), contrarily to the expressions for $A(n), A(n, r)$ and $A(n ; 3)$. For instance, direct inspection of the results of computer experiments for $A(n, r ; 3)$ shows the appearance of large prime factors in their prime factorizations, strongly suggesting that no answer for $A(n, r ; 3)$ can be given in the form of a single hypergeometric term. This explains why no conjecture concerning $A(n, r ; 3)$ was given previously, and it might moreover imply that any closed expression for $A(n, r ; 3)$ could be at best a sum of several 'round' terms. In view of this, the expression in formulae (1.4) and (1.5), even if not as elegant and neat as other results in the context of ASM enumerations, is probably the best that can be achieved: a 1-fold sum of 'round' terms.

Our proof of the Theorem is based on a certain representation for the generating function for the numbers $A(n, r ; 3)$, recently obtained by Stroganov [7]. In the next Section we recall the relevant formulae of paper [7] and shortly review their backgrounds. In Section 3 we apply standard relations for Gauss hypergeometric function to simplify significantly Stroganov's expression for the generating function of 3-enumerated ASMs. In Section 4, from this simplified expression we obtain explicit formulae for $A(n, r ; 3)$, thus proving the Theorem.

## 2. Preliminaries

In studying the refined 3 -enumeration of $n \times n$ ASMs it is convenient to define the generating function

$$
\begin{equation*}
H_{n}^{(3)}(t)=\frac{1}{A(n ; 3)} \sum_{r=1}^{n} A(n, r ; 3) t^{r-1}, \quad H_{n}^{(3)}(1)=1 . \tag{2.1}
\end{equation*}
$$

A simple property of the generating function is $H_{n}^{(3)}(t)=t^{n-1} H_{n}^{(3)}\left(t^{-1}\right)$, which follows from the relation $A(n, r ; 3)=A(n, n-r+1 ; 3)$ expressing left-right (top-bottom) symmetry within the set of $n \times n$ ASMs.

In paper [7] the following defining properties of the generating function have been established. The generating functions $H_{n}^{(3)}(t)$ for $n$ even, $n=2 m+2$, and for $n$ odd, $n=2 m+3$, are related by the formulae

$$
\begin{equation*}
H_{2 m+2}^{(3)}(t)=\frac{(t+1)}{2} B_{2 m}(t), \quad H_{2 m+3}^{(3)}(t)=\frac{(2 t+1)(t+2)}{9} B_{2 m}(t) \tag{2.2}
\end{equation*}
$$

where $B_{2 m}(t)$ is some polynomial of degree $2 m$. It is convenient to define $E_{m}(t):=t^{-m} B_{2 m}(t)$ so that $E_{m}(t)=E_{m}\left(t^{-1}\right)$. The function $E_{m}(t)$ can be related to another function $V_{m}(x)$, obeying, in turn, the relation $V_{m}(x)=V_{m}\left(x^{-1}\right)$, by means of the transformation

$$
\begin{equation*}
E_{m}(t)=\frac{V_{m}(x)}{\left(x-1+x^{-1}\right)^{m}}, \quad t=-\frac{x-q}{q x-1} \tag{2.3}
\end{equation*}
$$

or, inversely,

$$
\begin{equation*}
V_{m}(x)=\frac{3^{m} E_{m}(t)}{\left(t+1+t^{-1}\right)^{m}}, \quad x=\frac{t+q}{t q+1} \tag{2.4}
\end{equation*}
$$

where (and everywhere below) the following shorthand notation is used

$$
\begin{equation*}
q:=\exp (\mathrm{i} \pi / 3) \tag{2.5}
\end{equation*}
$$

The function

$$
\begin{equation*}
h_{m}(x):=\left(x-x^{-1}\right)^{2 m+1}\left(x+2+x^{-1}\right) V_{m}(x) \tag{2.6}
\end{equation*}
$$

is found to be

$$
\begin{equation*}
h_{m}(x)=c_{m}\left(g_{m}(x)+\frac{3 m+2}{3 m+1} f_{m}(x)\right) \tag{2.7}
\end{equation*}
$$

where the functions $f_{m}(x)$ and $g_{m}(x)$ are given by

$$
\begin{align*}
& g_{m}(x):=\sum_{k=0}^{m}\binom{m+2 / 3}{k}\binom{m-2 / 3}{m-k}\left(x^{3 m+2-6 k}-x^{-3 m-2+6 k}\right)  \tag{2.8}\\
& f_{m}(x):=\sum_{k=0}^{m}\binom{m+1 / 3}{k}\binom{m-1 / 3}{m-k}\left(x^{3 m+1-6 k}-x^{-3 m-1+6 k}\right) \tag{2.9}
\end{align*}
$$

and $c_{m}$ is some constant such that $V_{m}(1)=E_{m}(1)=1$. For the reader's convenience we note that the variable $x$ here and the variable $u$ of Ref. [7] are related by $\mathrm{e}^{2 \mathrm{i} u}=-x^{-1}$.

The result just described for the generating function of 3 -enumerated ASMs was obtained by analyzing Izergin's determinant formula [8] for the partition function of the six-vertex model with domain wall boundary conditions (also known as 'square ice'). The one-to-one correspondence of square ice states with ASMs was pointed out in [9] (see also [10]) and fruitfully applied by Kuperberg in [5]. The approach proposed in [11] for 1-enumerated ASMs and extended to 3-enumerated ASMs in [7], uses a certain functional equation for the square ice partition function. This functional equation is nothing but the Baxter's T-Q equation for the ground state of XXZ Heisenberg spin- $1 / 2$ chain at $\Delta=-1 / 2$ and with odd number of sites $N=2 m+1$, studied previously in [12].

In particular, expression (2.7) for $h_{m}(x)$ arises as the solution of the functional equation

$$
\begin{equation*}
y(x)+y\left(q^{2} x\right)+y\left(q^{-2} x\right)=0 \tag{2.10}
\end{equation*}
$$

which is supplemented by the condition that its solution $y(x)=h_{m}(x)$ must have the structure (2.6) where the function $V_{m}(x)$ is supposed to be such that $V_{m}\left(\mathrm{e}^{\mathrm{i} \varphi}\right)$ is an even trigonometric polynomial of degree $m$ in the variable $\varphi$. This condition allows one to reduce the functional equation to a finite set of linear algebraic equations. These equations can be solved explicitly thus leading to the formula for $h_{m}(x)$.

The functions $f_{m}(x)$ and $g_{m}(x)$ are also solutions of equation (2.10), and by construction they have the structure

$$
\begin{equation*}
f_{m}(x)=\left(x-x^{-1}\right)^{2 m+1} Q_{m}(x), \quad g_{m}(x)=\left(x-x^{-1}\right)^{2 m+1} P_{m+1}(x) \tag{2.11}
\end{equation*}
$$

where $Q_{m}(x)$ and $P_{m+1}(x)$ are such that $Q_{m}\left(\mathrm{e}^{\mathrm{i} \varphi}\right)$ and $P_{m+1}\left(\mathrm{e}^{\mathrm{i} \varphi}\right)$ are even trigonometric polynomials of degrees $m$ and $m+1$ respectively. These two functions are two independent solutions of T-Q equation. Note that as the second independent solution (i.e., that solution which is of higher degree) one can also regard the function $P_{m+1}(x)+\frac{3 m+2}{3 m+1} Q_{m}(x)$ as well as any function of the form $P_{m+1}(x)+\alpha Q_{m}(x)$, see [13]. Thus, (2.7) just implies that 3-enumerated ASMs are described by the second independent solution of T-Q equation. The first solution, $Q_{m}(x)$, was shown in [11] to be related to 1-enumerated ASMs.

Namely, in the case of 1-enumerated ASMs it was shown that

$$
\begin{equation*}
H_{n}^{(1)}(t)=\text { const } \times \frac{Q_{n-1}(x)}{\left(q x-q^{-1} x^{-1}\right)^{n-1}}, \quad t=\frac{q x^{-1}-q^{-1} x}{q x-q^{-1} x^{-1}} . \tag{2.12}
\end{equation*}
$$

where the generating function $H_{n}^{(1)}(t)$ is defined similarly to (2.1) by

$$
\begin{equation*}
H_{n}^{(1)}(t):=\frac{1}{A(n)} \sum_{r=1}^{n} A(n, r) t^{r-1}, \quad H_{n}^{(1)}(1)=1 \tag{2.13}
\end{equation*}
$$

To reproduce formulae (1.1) and (1.2) from (2.12) it was proposed to use the differential equation satisfied by $f_{m}(x)$, which reads

$$
\begin{equation*}
\left[\partial_{\varphi}^{2}-6 m \cot 3 \varphi \partial_{\varphi}-(3 m+1)(3 m-1)\right] f_{m}\left(\mathrm{e}^{\mathrm{i} \varphi}\right)=0 \tag{2.14}
\end{equation*}
$$

Fortunately, the corresponding equation on the generating function appears to be a hypergeometric equation

$$
\begin{equation*}
\left[(t-1) t \partial_{t}^{2}+2(t+n-1) \partial_{t}-n(n-1)\right] H_{n}^{(1)}(t)=0 \tag{2.15}
\end{equation*}
$$

and its polynomial solution gives

$$
H_{n}^{(1)}(t)=\frac{(2 n-1)!(2 n-2)!}{(3 n-2)!(n-1)!}{ }_{2} F_{1}\left(\left.\begin{array}{c}
-n+1, n  \tag{2.16}\\
-2 n+2
\end{array} \right\rvert\, t\right)
$$

Here the proper normalization, see (2.13), follows from the Chu-Vandermonde identity

$$
{ }_{2} F_{1}\left(\left.\begin{array}{c}
-m, b  \tag{2.17}\\
c
\end{array} \right\rvert\, 1\right)=\frac{(c-b)_{m}}{(c)_{m}} ; \quad(a)_{m}:=a(a+1) \cdots(a+m-1)
$$

Since $A(n, 1)=A(n-1)$, Eqn. (2.16) results in an elementary recursion

$$
\begin{equation*}
\frac{A(n-1)}{A(n)}=H_{n}^{(1)}(0)=\frac{(2 n-1)!(2 n-2)!}{(3 n-2)!(n-1)!} \tag{2.18}
\end{equation*}
$$

which, supplemented by the initial condition $A(1)=1$, gives formula (1.1). Expanding RHS of (2.16) in power series in $t$ and multiplying these coefficients by $A(n)$, one reproduces, in turn, formula (1.2).

Unfortunately, a similar procedure cannot be applied in the case of 3-enumerated ASMs. It can be verified directly that the function $h_{m}(x)$ solves the equation

$$
\begin{equation*}
\left\{\partial_{\varphi}^{2}-3\left[(2 m+1) \cot 3 \varphi-\frac{1}{\sin 3 \varphi}\right] \partial_{\varphi}-(3 m+1)(3 m+2)\right\} h_{m}\left(\mathrm{e}^{\mathrm{i} \varphi}\right)=0 \tag{2.19}
\end{equation*}
$$

This equation leads to a rather bulky non-hypergeometric equation for the generating function $H_{n}^{(3)}(t)$, which can hardly be solved by standard means. Moreover, even though the function $Q_{m}(x)$ can easily be deduced from the equation for $f_{m}(x)$ above, the differential equation for $g_{m}(x)$ (for this equation, see [12]) does not allow one to find the function $P_{m+1}(x)$ in a similar way. As a matter of fact, only an implicit answer for $H_{n}^{(3)}(t)$ in terms of two recurrences, generated by three-terms recurrences for $f_{m}(x)$ and $g_{m}(x)$, was given in paper [7]. Here we would like just to mention that from differential equation (2.19) it can easily be seen that the operator $(\sin 3 \varphi)^{-1} \partial_{\varphi}$ plays the role of a 'lowering' operator in the set of trigonometric polynomials $\left\{h_{m}\left(\mathrm{e}^{\mathrm{i} \varphi}\right)\right\}$. It is thus straightforward to rewrite Eqn. (2.19) as a three term recurrence for $h_{m}(x)$
and, hence, to obtain a single recurrence for $H_{n}^{(3)}(t)$. However this point of view will not be developed further, since it amounts merely to a reformulation of the problem. Instead, in the next Section we show how to overcome the difficulty and find the function $H_{n}^{(3)}(t)$ explicitly.

As a last comment here we would like to mention that, although the previous results on the refined 3-enumeration of ASMs run over papers [7,11,12], the present research have originated from our study of the square ice boundary correlators obtained in [14]. Within our study the refined enumerations of ASMs arise as solutions of some differential equations rather than functional ones, in particular, the expression for $h_{m}(x)$ arises as a solution of (2.19). The illustration of this approach is out of the scope of the present paper and will be given elsewhere [15]; here we restrict ourselves to obtaining the numbers $A(n, r ; 3)$ from the known $h_{m}(x)$.

## 3. The generating function

We begin with noticing that the functions $f_{m}(x)$ and $g_{m}(x)$ can be also written as follows

$$
\begin{align*}
& g_{m}(x)=\frac{\Gamma(m+1 / 3)}{m!\Gamma(1 / 3)}\left[x^{3 m+2}{ }_{2} F_{1}\binom{-m,-m-2 / 3 \mid x^{-6}}{1 / 3}\right. \\
&\left.-x^{-3 m-2}{ }_{2} F_{1}\left(\left.\begin{array}{c}
-m,-m-2 / 3 \\
1 / 3
\end{array} \right\rvert\, x^{6}\right)\right],  \tag{3.1}\\
& f_{m}(x)=\frac{\Gamma(m+4 / 3)}{m!\Gamma(4 / 3)}\left[x^{-3 m+1}{ }_{2} F_{1}\left(\left.\begin{array}{c}
-m,-m+1 / 3 \mid \\
4 / 3
\end{array} \right\rvert\, x^{6}\right)\right. \\
&\left.-x^{3 m-1}{ }_{2} F_{1}\left(\left.\begin{array}{c}
-m,-m+1 / 3 \\
4 / 3
\end{array} \right\rvert\, x^{-6}\right)\right] . \tag{3.2}
\end{align*}
$$

Since the parameters of the hypergeometric functions entering these expressions differ by integers one can expect that $g_{m}(x)$ and $f_{m}(x)$ are connected by some three-term relations via Gauss relations (see, e.g., Sect. 2.8 of [16]). Indeed, using Gauss relations it can be shown that

$$
\begin{align*}
{ }_{2} F_{1}\left(\left.\begin{array}{c}
-m,-m-2 / 3 \mid z \\
1 / 3
\end{array} \right\rvert\, z\right)=\frac{3 m+4}{2} & { }_{2} F_{1}\left(\left.\begin{array}{c}
-m-1,-m-2 / 3 \mid z \\
4 / 3
\end{array} \right\rvert\,\right. \\
& -\frac{3 m+2}{2}(1+z){ }_{2} F_{1}\binom{-m,-m+1 / 3 \mid z}{4 / 3} \tag{3.3}
\end{align*}
$$

and therefore we have the following relation

$$
\begin{equation*}
g_{m}(x)=\frac{3 m+2}{2(3 m+1)}\left(x^{3}+x^{-3}\right) f_{m}(x)-\frac{3(m+1)}{2(3 m+1)} f_{m+1}(x) . \tag{3.4}
\end{equation*}
$$

Substituting (3.4) into (2.7) we obtain

$$
\begin{equation*}
h_{m}(x)=c_{m} \frac{3 m+2}{2(3 m+1)}\left[\left(x^{3}+2+x^{-3}\right) f_{m}(x)-\frac{3 m+3}{3 m+2} f_{m+1}(x)\right] . \tag{3.5}
\end{equation*}
$$

Comparing Eqns. (3.4) and (3.5) with Eqns. (2.6) and (2.11) we find that

$$
\begin{equation*}
P_{m+1}(x)=\frac{3 m+2}{2(3 m+1)}\left(x^{3}+x^{-3}\right) Q_{m}(x)-\frac{3(m+1)}{2(3 m+1)}\left(x-x^{-1}\right)^{2} Q_{m+1}(x) \tag{3.6}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{m}(x)=c_{m} \frac{3 m+2}{2(3 m+1)}\left[\left(x-1+x^{-1}\right)^{2} Q_{m}(x)-\frac{3 m+3}{3 m+2}\left(x-2+x^{-1}\right) Q_{m+1}(x)\right] . \tag{3.7}
\end{equation*}
$$

Hence all functions in question are expressed in terms of $Q_{m}(x)$ 's.
The advantage of this approach is based on the fact that for the function $Q_{m}(x)$ the following explicit formula can be found

$$
Q_{m}(x)=\frac{(2 m)!}{3^{m}(m!)^{2}}\left(\frac{q x^{-1}-q^{-1} x}{q-q^{-1}}\right)^{m}{ }_{2} F_{1}\left(\begin{array}{c}
-m, m+1  \tag{3.8}\\
-2 m
\end{array} \left\lvert\, \frac{q x-q^{-1} x^{-1}}{q x^{-1}-q^{-1} x}\right.\right) .
$$

As already mentioned in our preliminary comments, the function $Q_{m}(x)$ can be deduced from the differential equation for $f_{m}(x)$ above (the proper normalization, for instance, can be found using the three-term relation for $f_{m}(x)$ 's). Here we give a proof of (3.8) by a straightforward transformation of the function $f_{m}(x)$ to the form $f_{m}(x)=\left(x-x^{-1}\right)^{2 m+1} Q_{m}(x)$.

The key identity which is to be used here is the so-called cubic transformation of Gauss hypergeometric function [16] which in its most symmetric form reads:

$$
\begin{gather*}
\frac{\Gamma(a)}{\Gamma(2 / 3)}{ }_{2} F_{1}\left(\left.\begin{array}{c}
a+1 / 3, a \\
2 / 3
\end{array} \right\rvert\, z^{3}\right)-\omega^{-1} z \frac{\Gamma(a+2 / 3)}{\Gamma(4 / 3)}{ }_{2} F_{1}\left(\left.\begin{array}{c}
a+1 / 3, a+2 / 3 \\
4 / 3
\end{array} \right\rvert\, z^{3}\right) \\
=3^{-3 a+1}\left(\frac{1-z}{1-\omega}\right)^{-3 a} \frac{\Gamma(3 a)}{\Gamma(2 a+2 / 3)}{ }_{2} F_{1}\left(\left.\begin{array}{c}
a+1 / 3,3 a \\
2 a+2 / 3
\end{array} \right\rvert\, \omega \frac{z-\omega}{1-z}\right) . \tag{3.9}
\end{gather*}
$$

Here $\omega$ is a primitive cubic root of unity, $\omega=\exp ( \pm 2 \mathrm{i} \pi / 3)$, and $a$ is arbitrary parameter. To show that indeed the cubic transformation is relevant to our case, let us rewrite (3.2) in the form consistent with LHS of (3.9). Taking into account that

$$
{ }_{2} F_{1}\left(\left.\begin{array}{c}
-m,-m+1 / 3  \tag{3.10}\\
4 / 3
\end{array} \right\rvert\, z\right)=\frac{\Gamma(1 / 3) \Gamma(4 / 3)}{\Gamma(-m+1 / 3) \Gamma(m+4 / 3)}(-z)^{m}{ }_{2} F_{1}\left(\left.\begin{array}{c}
-m,-m-1 / 3 \\
2 / 3
\end{array} \right\rvert\, z^{-1}\right)
$$

and

$$
\begin{equation*}
\frac{\Gamma(1 / 3)}{\Gamma(m+4 / 3)}=(-1)^{m+1} \frac{\Gamma(-m-1 / 3)}{\Gamma(2 / 3)}, \quad \frac{\Gamma(m+4 / 3)}{\Gamma(-m+1 / 3)}=(-1)^{m} \frac{(3 m+1)!}{3^{3 m+1} m!} \tag{3.11}
\end{equation*}
$$

it is easy to see that (3.2) can be rewritten in the form

$$
\begin{array}{r}
f_{m}(x)=\frac{(-1)^{m+1}(3 m+1)!}{3^{3 m+1}(m!)^{2}} x^{3 m+1}\left[\frac{\Gamma(-m-1 / 3)}{\Gamma(2 / 3)}{ }_{2} F_{1}\left(\left.\begin{array}{c}
-m,-m-1 / 3 \\
2 / 3
\end{array} \right\rvert\, x^{-6}\right)\right. \\
\left.+x^{-2} \frac{\Gamma(-m+1 / 3)}{\Gamma(4 / 3)}{ }_{2} F_{1}\left(\left.\begin{array}{c}
-m,-m+1 / 3 \\
4 / 3
\end{array} \right\rvert\, x^{-6}\right)\right] . \tag{3.12}
\end{array}
$$

Clearly, both terms in the brackets are the same as in LHS of (3.9) provided the parameter $a$ is specialized to the value $a=-m-1 / 3$.

To apply the cubic transformation to (3.12) we first define

$$
W(a ; z):=\frac{\Gamma(a)}{\Gamma(2 / 3)}{ }_{2} F_{1}\left(\left.\begin{array}{c}
a+1 / 3, a  \tag{3.13}\\
2 / 3
\end{array} \right\rvert\, z^{3}\right)+z \frac{\Gamma(a+2 / 3)}{\Gamma(4 / 3)}{ }_{2} F_{1}\left(\left.\begin{array}{c}
a+1 / 3, a+2 / 3 \\
4 / 3
\end{array} \right\rvert\, z^{3}\right)
$$

so that

$$
\begin{equation*}
f_{m}(x)=\frac{(-1)^{m+1}(3 m+1)!}{3^{3 m+1}(m!)^{2}} x^{3 m+1} W\left(-m-1 / 3 ; x^{-2}\right) \tag{3.14}
\end{equation*}
$$

Next, we note that for a sum of two terms one can always write

$$
\begin{equation*}
X+Y=\frac{q}{q-q^{-1}}\left(X-q^{-2} Y\right)-\frac{q^{-1}}{q-q^{-1}}\left(X-q^{2} Y\right) \tag{3.15}
\end{equation*}
$$

and if $q=\exp (\mathrm{i} \pi / 3)$, which is exactly the case, one can set $\omega=q^{2}$ for the first pair of terms and $\omega=q^{-2}$ for the second one. This recipe allows one to apply the cubic transformation, that gives

$$
\begin{align*}
W(a ; z)=\frac{3^{-3 a+1} \Gamma(3 a)}{\Gamma(2 a+2 / 3)} \frac{(1-z)^{-3 a}}{q\left(1-q^{2}\right)^{-3 a+1}} & {\left[{ }_{2} F_{1}\left(\left.\begin{array}{c}
a+1 / 3,3 a \mid \\
2 a+2 / 3
\end{array} \right\rvert\, q^{2} \frac{z-q^{2}}{1-z}\right)\right.} \\
& \left.+q^{3 a+1}{ }_{2} F_{1}\left(\left.\begin{array}{c}
a+1 / 3,3 a \\
2 a+2 / 3
\end{array} \right\rvert\, q^{-2} \frac{z-q^{-2}}{1-z}\right)\right] . \tag{3.16}
\end{align*}
$$

To obtain a new formula for $f_{m}(x)$ via (3.14) we have to evaluate now the limit $a \rightarrow-m-1 / 3$ of (3.16). The limit of the pre-factor can be easily found due to

$$
\begin{equation*}
\lim _{a \rightarrow-m-1 / 3} \frac{\Gamma(3 a)}{\Gamma(2 a+2 / 3)}=\frac{2}{3} \frac{(-1)^{m+1}(2 m)!}{(3 m+1)!} . \tag{3.17}
\end{equation*}
$$

To find the limit of the expression in the brackets in (3.16) we note that

$$
\begin{equation*}
q^{2} \frac{z-q^{2}}{1-z}=: s \quad q^{-2} \frac{z-q^{-2}}{1-z}=1-s \tag{3.18}
\end{equation*}
$$

and hence the following formula can be used

$$
\lim _{a \rightarrow-m-\frac{1}{3}}\left[{ }_{2} F_{1}\left(\left.\begin{array}{c}
a+\frac{1}{3}, 3 a  \tag{3.19}\\
2 a+\frac{2}{3}
\end{array} \right\rvert\, s\right)+q^{3 a+1}{ }_{2} F_{1}\left(\left.\begin{array}{c}
a+\frac{1}{3}, 3 a \\
2 a+\frac{2}{3}
\end{array} \right\rvert\, 1-s\right)\right]=\frac{3}{2}{ }_{2} F_{1}\left(\left.\begin{array}{c}
-m,-3 m-1 \\
-2 m
\end{array} \right\rvert\, s\right) .
$$

Formula (3.19) can be proved, for instance, by virtue of standard analytic continuation formulae for the hypergeometric function (see, e.g., Eqns. (1) and (2) in $\S 2.10$ of [16]). Collecting formulae we arrive to the expression

$$
W(-m-1 / 3 ; z)=-\frac{3^{2 m+1}(2 m)!}{(3 m+1)!} \frac{(1-z)^{3 m+1}}{\left(q-q^{-1}\right)^{m}}{ }_{2} F_{1}\left(\left.\begin{array}{c}
-m,-3 m-1  \tag{3.20}\\
-2 m
\end{array} \right\rvert\, q^{2} \frac{z-q^{2}}{1-z}\right) .
$$

Finally, substituting this expression into (3.14) and using the identity

$$
{ }_{2} F_{1}\left(\left.\begin{array}{c}
a, b  \tag{3.21}\\
c
\end{array} \right\rvert\, z\right)=(1-z)^{-a}{ }_{2} F_{1}\left(\left.\begin{array}{c}
a, c-b \\
c
\end{array} \right\rvert\, \frac{z}{z-1}\right)
$$

we obtain

$$
f_{m}(x)=\frac{(2 m)!}{3^{m}(m!)^{2}}\left(x-x^{-1}\right)^{2 m+1}\left(\frac{q x^{-1}-q^{-1} x}{q-q^{-1}}\right)^{m}{ }_{2} F_{1}\left(\begin{array}{c}
-m, m+1  \tag{3.22}\\
-2 m
\end{array} \left\lvert\, \frac{q x-q^{-1} x^{-1}}{q x^{-1}-q^{-1} x}\right.\right) .
$$

Obviously, this expression leads directly to (3.8) which is thus proved.
Hence, we have just shown that a particular solution of equation (2.10), the function $f_{m}(x)$, Eqn. (2.9), is connected to the 'first' solution of the Baxter T-Q equation, $Q_{m}(x)$, Eqn. (3.8), via the cubic transformation. Formulae (3.4) and (3.5) mean that the same transformation is responsible for relationship of $g_{m}(x)$ and $h_{m}(x)$ with the second independent solution of TQ equation, which can be chosen either as $P_{m+1}(x)$, in the case of $g_{m}(x)$, or as $P_{m+1}(x)+$ $\frac{3 m+2}{3 m+1} Q_{m}(x)=\left(x+2+x^{-1}\right) V_{m}(x)$, in the case of $h_{m}(x)$.

To write down the resulting expressions for functions $P_{m+1}(x)$ and $V_{m}(x)$, and for the purpose of subsequent transformation of these expressions, we define

$$
\Phi_{m}^{(k)}(x):=\left(\frac{q x^{-1}-q^{-1} x}{q-q^{-1}}\right)^{m}{ }_{2} F_{1}\left(\begin{array}{c}
-m, k+1  \tag{3.23}\\
-m-k
\end{array} \left\lvert\, \frac{q x-q^{-1} x^{-1}}{q x^{-1}-q^{-1} x}\right.\right) .
$$

Note that for positive integer $m$, function $\Phi_{m}^{(k)}(x)$ possesses the property $\Phi_{m}^{(k)}(x)=\Phi_{m}^{(k)}\left(x^{-1}\right)$ and thus, obviously, $\Phi_{m}^{(k)}(x)$ is a polynomial of degree $m$ in the variable $u:=x+x^{-1}$.

In particular, Eqn. (3.8) now reads

$$
\begin{equation*}
Q_{m}(x)=\frac{(2 m)!}{3^{m}(m!)^{2}} \Phi_{m}^{(m)}(x) . \tag{3.24}
\end{equation*}
$$

Formulae (3.6) and (3.7) give

$$
\begin{align*}
& P_{m+1}(x)=\frac{3 m+2}{2(3 m+1)} \frac{(2 m)!}{3^{m}(m!)^{2}}\left[\left(x^{3}+x^{-3}\right) \Phi_{m}^{(m)}(x)\right. \\
&\left.\quad-\frac{2(2 m+1)}{3 m+2}\left(x-x^{-1}\right)^{2} \Phi_{m+1}^{(m+1)}(x)\right]  \tag{3.25}\\
& V_{m}(x)=\frac{(2 m)!(2 m+1)!}{m!(3 m+1)!}[(x-1\left.+x^{-1}\right)^{2} \Phi_{m}^{(m)}(x) \\
&\left.-\frac{2(2 m+1)}{3 m+2}\left(x-2+x^{-1}\right) \Phi_{m+1}^{(m+1)}(x)\right] . \tag{3.26}
\end{align*}
$$

Here the expression for $V_{m}(x)$ is written according to the proper normalization of this function, $V_{m}(1)=1$. The normalization can be verified by virtue of Chu-Vandermonde identity (2.17)
and it corresponds to the choice

$$
\begin{equation*}
c_{m}=(3 m+1) \frac{3^{m+1} m!(2 m+2)!}{(3 m+3)!} \tag{3.27}
\end{equation*}
$$

Formulae (3.25) and (3.26) for functions $P_{m+1}(x)$ and $V_{m}(x)$ are not the final expressions for them yet since they can be notably simplified. Indeed, again using Gauss relations one can prove, for instance, the following relations for the functions (3.23)

$$
\begin{align*}
\Phi_{m+1}^{(k)}(x) & =\left(x+x^{-1}\right) \Phi_{m}^{(k)}(x)-\frac{m(m+2 k+1)}{3(m+k+1)(m+k)}\left(x^{2}+1+x^{-2}\right) \Phi_{m-1}^{(k)}(x)  \tag{3.28}\\
\Phi_{m}^{(k+1)}(x) & =\frac{m+2 k+2}{2(m+k+1)} \Phi_{m}^{(k)}(x)+\frac{m}{2(m+k+1)}\left(x+x^{-1}\right) \Phi_{m-1}^{(k+1)}(x) . \tag{3.29}
\end{align*}
$$

These two relations can be regarded as basic relations; many other tree-term relations connecting $\Phi_{m}^{(k)}(x)$ 's can be derived from (3.28) and (3.29). Using these three-term relations we find that for the function $P_{m+1}(x)$, in particular, the following formula is valid

$$
\begin{equation*}
P_{m+1}(x)=\frac{(2 m)!}{3^{m} m!(m+1)!}\left[(3 m+2) \Phi_{m+1}^{(m-1)}(x)-(2 m+1) \Phi_{m+1}^{(m)}(x)\right] \tag{3.30}
\end{equation*}
$$

Similarly, for the function $V_{m}(x)$ we obtain

$$
\begin{equation*}
V_{m}(x)=\frac{(2 m)!(2 m+2)!}{(m+1)!(3 m+2)!}\left[(2 m+1) \Phi_{m}^{(m+1)}(x)-m\left(x-1+x^{-1}\right) \Phi_{m-1}^{(m+1)}(x)\right] \tag{3.31}
\end{equation*}
$$

This formula for $V_{m}(x)$ can be further simplified, due to (3.29), to be written solely in terms of $\Phi_{m}(k)$ 's, but (3.31) is more convenient for the purpose of obtaining the function $E_{m}(t)$, which is of primary interest for 3 -enumerated ASMs. Applying (2.4) we obtain

$$
\begin{align*}
E_{m}(t)=\frac{(2 m)!(2 m+2)!}{3^{m}(m+1)!(3 m+2)!} & {\left[(2 m+1)(t+2)^{m}{ }_{2} F_{1}\left(\left.\begin{array}{c}
-m, m+2 \\
-2 m-1
\end{array} \right\rvert\, \frac{1+2 t}{t(t+2)}\right)\right.} \\
& \left.-3 m(t+2)^{m-1}{ }_{2} F_{1}\left(\left.\begin{array}{c}
-m+1, m+2 \\
-2 m
\end{array} \right\rvert\, \frac{1+2 t}{t(t+2)}\right)\right] . \tag{3.32}
\end{align*}
$$

Formulae (3.30), (3.31) and, especially, (3.32) are the main results of this Section.

## 4. The numbers $A(n, r ; 3)$

Let us begin with showing how Eqn. (1.3) for $A(n ; 3)$ is recovered from the just obtained expression for the generating function. Here the property $A(n, 1 ; 3)=A(n-1 ; 3)$ is to be used. It implies

$$
\begin{equation*}
\frac{A(2 m+1 ; 3)}{A(2 m+2 ; 3)}=H_{2 m+2}^{(3)}(0)=\frac{1}{2} B_{2 m}(0), \quad \frac{A(2 m+2 ; 3)}{A(2 m+3 ; 3)}=H_{2 m+3}^{(3)}(0)=\frac{2}{9} B_{2 m}(0) \tag{4.1}
\end{equation*}
$$

Hence, one has the recurrences

$$
\begin{array}{ll}
A(2 m+3 ; 3)=\frac{9}{\left[B_{2 m}(0)\right]^{2}} A(2 m+1 ; 3), & A(1 ; 3)=1 \\
A(2 m+2 ; 3)=\frac{9}{B_{2 m}(0) B_{2 m-2}(0)} A(2 m ; 3), & A(2 ; 3)=2 \tag{4.3}
\end{array}
$$

Recall that $B_{2 m}(t)=t^{m} E_{m}(t)$ in (2.2). From (3.32) one finds

$$
\begin{equation*}
B_{2 m}(0)=\frac{(2 m+1)!(2 m+2)!}{3^{m}(m+1)!(3 m+2)!} \tag{4.4}
\end{equation*}
$$

Substituting (4.4) into (4.2) and (4.3), and solving the recurrences one obtains (1.3).
Let us now turn to our main target, $A(n, r ; 3)$, which describes the refined 3-enumeration of ASMs. Formulae (2.2) imply that

$$
\begin{align*}
& A(2 m+2, r ; 3)=\frac{b(m, r-1)+b(m, r-2)}{2} A(2 m+2 ; 3) \\
& A(2 m+3, r ; 3)=\frac{2 b(m, r-1)+5 b(m, r-2)+2 b(m, r-3)}{9} A(2 m+3 ; 3) \tag{4.5}
\end{align*}
$$

where $b(m, \alpha)$ are defined as

$$
\begin{equation*}
B_{2 m}(t)=t^{m} E_{m}(t)=: \sum_{\alpha=0}^{2 m} b(m, \alpha) t^{\alpha} \tag{4.6}
\end{equation*}
$$

and are assumed to be zero if $\alpha$ is out of the range of values $0,1, \ldots, 2 m$. To find these coefficients we expand (3.32) in power series in $t$, thus expressing $B_{2 m}(t)$ as a triple sum. Chu-Vandermonde summation formula can now be applied to the sum with respect to the index defining the hypergeometric series in (3.32), thus expressing $B_{2 m}(t)$ as a double sum. These two summations can be rearranged in such a way that one of them becomes with respect to $\alpha$ while the other one defines the coefficients of power expansion in $t$. We obtain

$$
\begin{align*}
b(m, \alpha)=\frac{(2 m+1)!m!}{3^{m}(3 m+2)!} \sum_{\ell=\max (0, \alpha-m)}^{[\alpha / 2]}(2 m & +2-\alpha+2 \ell)\binom{3 m+3}{\alpha-2 \ell} \\
& \times\binom{ 2 m+\ell-\alpha+1}{m+1}\binom{m+\ell+1}{m+1} 2^{\alpha-2 \ell} . \tag{4.7}
\end{align*}
$$

Here $[\alpha / 2]$ denotes integer part of $\alpha / 2$. Formulae (4.5) and (4.7) complete the proof of the Theorem, which constitutes the main result of the present paper. Here we would like to mention that in terms of terminating hypergeometric series the last formula may be rewritten, for
instance, as follows

$$
\begin{gather*}
b(m, \alpha)=\frac{2^{\alpha}\binom{3 m+3}{\alpha}\binom{2 m+1-\alpha}{m+1}}{3^{m}\binom{3 m+2}{m+1}}\left[2{ }_{4} F_{3}\left(\left.\begin{array}{c}
-(\alpha-1) / 2,-\alpha / 2, m+2,2 m+2-\alpha \\
(3 m+4-\alpha) / 2,(3 m+5-\alpha) / 2, m-\alpha+1
\end{array} \right\rvert\, \frac{1}{4}\right)\right. \\
\left.-\frac{\alpha}{m+1}{ }_{4} F_{3}\left(\left.\begin{array}{c}
-(\alpha-1) / 2,-\alpha / 2+1, m+2,2 m+2-\alpha \\
(3 m+4-\alpha) / 2,(3 m+5-\alpha) / 2, m-\alpha+1
\end{array} \right\rvert\, \frac{1}{4}\right)\right] . \tag{4.8}
\end{gather*}
$$

This formula is valid for $\alpha=0,1, \ldots, m$ (a similar expression for $\alpha=m+1, m+2, \ldots, 2 m$ can be simply obtained through the replacement $\alpha \rightarrow 2 m-\alpha$ in RHS of (4.8)). The two ${ }_{4} F_{3}$ in (4.8) can be further combined into a single ${ }_{5} F_{4}$. Analogous formulae for $b(m, \alpha)$ in terms of terminating hypergeometric series of argument 4 may be written down as well. Analyzing these expressions we were anyway unable to perform the sum in (4.7) in a closed form, even if very suggestive similarities can be found with known summation formulae, see $\S \S 7.5$ and 7.6 , especially §7.6.4, of Ref. [17]. We therefore regard Eqns. (4.5) and (4.7) as the most compact formulae for the numbers $\mathrm{A}(\mathrm{n}, \mathrm{r} ; 3)$ available at the present moment.

To conclude, let us comment the just obtained result in application to the square ice. The quantity of interest here is $n A(n, r ; 3) / A(n ; 3)$ which plays the role of spatial derivative of the boundary polarization of the square ice with a particular choice of the vertex weights. It is interesting to study the behavior of this quantity in the large $n$ and the large $r$ limits, with the ratio $\xi=r / n$ kept fixed, $0<\xi<1$. In the context of the square ice this scaling limit corresponds to the continuous limit, with $1 / n$ playing the role of a lattice spacing. For the square ice in the disordered regime (which corresponds to $x$-enumerations with $0<x<4$ ) a Heaviside step-function behavior $\theta(\xi-1 / 2)$ of the boundary polarization is expected [14]. In ASM enumeration language this corresponds to

$$
\begin{equation*}
\lim _{\substack{n, r \rightarrow \infty \\ r / n=\xi}} n \frac{A(n, r ; x)}{A(n ; x)}=\delta(\xi-1 / 2) \quad \text { for } \quad 0<x<4 \tag{4.9}
\end{equation*}
$$

For instance, in the case of 1-enumeration of ASMs (i.e., for $x=1$ ) the validity of this formula can be verified directly from (1.2) by virtue of Stirling formula. This result is also valid for the simple case of 2-enumerated ASMs whose refined enumeration is just $A(n, r ; 2) / A(n ; 2)=$ $\binom{n-1}{r-1} / 2^{n-1}$; for the result for the whole free-fermion line see [14]. Formulae (4.5) and 4.7) allows one to study the scaling limit of the expression $n A(n, r ; 3) / A(n ; 3)$ as well. In this limit the sum in 4.7) turns into an integral and standard saddle point method can be applied. In this way we find

$$
\begin{equation*}
\lim _{\substack{n, r \rightarrow \infty \\ r / n=\xi}} n \frac{A(n, r ; 3)}{A(n ; 3)}=\delta(\xi-1 / 2) \tag{4.10}
\end{equation*}
$$

This confirms (4.9) also in the $x=3$ case.

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