# Generalizations of Khovanskiu's theorems on growth of sumsets in abelian semigroups 

Vít Jelínek and Martin Klazar*

November 20, 2018


#### Abstract

We show that if $P$ is a lattice polytope in the nonnegative orthant of $\mathbb{R}^{k}$ and $\chi$ is a coloring of the lattice points in the orthant such that the color $\chi(a+b)$ depends only on the colors $\chi(a)$ and $\chi(b)$, then the number of colors of the lattice points in the dilation $n P$ of $P$ is for large $n$ given by a polynomial (or, for rational $P$, by a quasipolynomial). This unifies a classical result of Ehrhart and Macdonald on lattice points in polytopes and a result of Khovanskiĭ on sumsets in semigroups. We also prove a strengthening of multivariate generalizations of Khovanskiì's theorem. Another result of Khovanskiĭ states that the size of the image of a finite set after $n$ applications of mappings from a finite family of mutually commuting mappings is for large $n$ a polynomial. We give a combinatorial proof of a multivariate generalization of this theorem.


## 1 Introduction

In many classes of enumerative combinatorial problems, every counting function is equal - usually for sufficiently large arguments - to a polynomial or to a quasipolynomial. In this article, we consider several classes of problems with this property, (re)derive their polynomiality in a more uniform manner, and generalize and strengthen existing results. We begin with three important examples.

[^0]
### 1.1 Lattice polytopes, sumsets in semigroups, ideals in a poset

For $n \in \mathbb{N}$ and a lattice polytope $P \subset \mathbb{R}^{k}$, which is a convex hull of a finite set of points from $\mathbb{Z}^{k}$, denote by $i(P, n)$ the number of the lattice points lying in the dilation $n P=\{n x: x \in P\}$ of $P$,

$$
i(P, n)=\left|n P \cap \mathbb{Z}^{k}\right|
$$

Ehrhart and Macdonald obtained the following result.
Theorem 1.1 (Ehrhart [5], Macdonald [13, 14]). The number $i(P, n)$ of the lattice points in $n P$ is for all $n \in \mathbb{N}$ given by a polynomial.

More generally, if $P$ is a rational polytope (its vertices have rational coordinates), then $i(P, n)$ is for all $n \in \mathbb{N}$ given by a quasipolynomial (the definition of a quasipolynomial is recalled in Section 1.3). See Stanley [20, Section 4.6] for more information.

For a commutative semigroup $(G,+)$ and subsets $A, B \subset G$, consider the sumsets

$$
n * A=\left\{a_{1}+\cdots+a_{n}: a_{i} \in A\right\} \text { and } A+B=\{a+b: a \in A, b \in B\} .
$$

For a (typically infinite) set $X$, its subset $B \subset X$, and a family $\mathcal{F}$ of mutually commuting mappings $f: X \rightarrow X$, the $n$th iterated image of $B$ by $\mathcal{F}$ is

$$
\mathcal{F}^{(n)}(B)=\bigcup_{f_{i} \in \mathcal{F}}\left(f_{1} \circ \cdots \circ f_{n}\right)(B),
$$

where $f(B)$ denotes the set $\{f(x): x \in B\}$. The following three theorems are due to Khovanskiĭ.

Theorem 1.2 (Khovanskiĭ [9]). Let $A$ and $B$ be finite sets in a commutative semigroup.

1. For large $n$, the cardinality of the sumset $|n * A|$ is given by a polynomial.
2. For large $n$, the cardinality of the sumset $|n * A+B|$ is given by a polynomial.

Theorem 1.3 (Khovanskiĭ [10]). Let $G=(G,+)$ be a commutative semigroup, $A, B \subset G$ be two finite subsets, and $\psi: G \rightarrow \mathbb{C}$ be an additive character of $G$ (i.e., $\psi(a+b)=\psi(a) \psi(b))$. Then there exist polynomials $p_{a}(x)$, $a \in A$, such that for large $n$ one has

$$
\sum_{a \in n * A+B} \psi(a)=\sum_{a \in A} p_{a}(n) \psi(a)^{n} .
$$

Theorem 1.4 (Khovanskiĭ [9]). If $B$ is a finite subset of $X$ and $\mathcal{F}$ is finite family of mutually commuting mappings from $X$ to itself, then the cardinality of the iterated image $\mathcal{F}^{(n)}(B)$ is for large $n$ given by a polynomial in $n$.

Khovanskiĭ stated and proved just part 2 of Theorem 1.2 (as a corollary of Theorem 1.4); however, part 2 immediately implies part 1 which we state explicitly for the purpose of later reference. Both Theorem 1.3 and Theorem 1.4 include part 2 of Theorem 1.2 as a particular case: set $\psi \equiv 1$, respectively set $X=G$ and consider the mappings $\mathcal{F}=\left\{s_{a}: a \in A\right\}$ where $s_{a}(x)=x+a$.

Let us now consider the poset $\left(\mathbb{N}_{0}^{k}, \leq\right), \mathbb{N}_{0}=\{0,1,2, \ldots\}$, with componentwise ordering:

$$
a=\left(a_{1}, \ldots, a_{k}\right) \leq b=\left(b_{1}, \ldots, b_{k}\right) \Longleftrightarrow a_{i} \leq b_{i}, i=1, \ldots, k
$$

A lower ideal $S \subset \mathbb{N}_{0}^{k}$, is a set satisfying the condition $a \leq b, b \in S \Rightarrow a \in S$. The following result was first posed as a problem in the American Mathematical Monthly, see also [20, Exercise 6 in Chapter 4].

Theorem 1.5 (Stanley [19]). For a lower ideal $S$ in the poset $\left(\mathbb{N}_{0}^{k}, \leq\right)$, the number of the elements $a=\left(a_{1}, \ldots, a_{k}\right) \in S$ with $\|a\|_{1}=a_{1}+\cdots+a_{k}=n$ is for large $n$ given by a polynomial.

We prove all five theorems (Theorem 1.1 in a weaker form for large $n$ only) in the framework of more general results in Section 2.

### 1.2 Our results

At first, we wanted to understand the connection between Theorems 1.1 and 1.2, and to find reasons for polynomiality of these two and other classes. This turned into a goal to explain the above results on polynomiality in a uniform manner, and to give combinatorial proofs of these combinatorial results; some of the above theorems were originally proved by somewhat opaque algebraic arguments. We succeeded in this to large extent for the five theorems. In Section 2, we demonstrate that Theorems 1.11 .4 (Theorem 1.1 for large $n$ only) follow as corollaries of Stanley's Theorem 1.5 or of its natural extensions stated in Theorems 2.2 and 2.15. We will give multivariate generalizations of Theorems 1.2 1.4. Theorem 2.15 can be used to prove polynomiality of further classes of enumerative problems, which we briefly mention in Section 3 and will discuss in details in [7].

We build on the results of Khovanskiĭ [9, 10, Nathanson and Ruzsa [17] and Stanley [19]. Khovanskiì's original proof of part 2 of Theorem 1.2 as a corollary of Theorem 1.4 in [9] was algebraic, by means of the Hilbert polynomial of graded
modules. In [10], he gave a combinatorial proof of part 2 as a corollary of Theorem 1.3. Extending Khovanskiì's algebraic argument, Nathanson [16] proved a multivariate generalization of part 2 (see Theorem 2.5). Then Nathanson and Ruzsa [17] gave a simple combinatorial proof for a multivariate generalization of part 1 (see Theorem 2.4).

Our contribution is a common strengthening of these generalizations in Theorem 2.10, If $A_{1}, \ldots, A_{l}$ are finite sets in a commutative semigroup $(G,+)$ and

$$
p\left(n_{1}, \ldots, n_{l}\right):=\left|n_{1} * A_{1}+\cdots+n_{l} * A_{l}\right|
$$

then there is a constant $c>0$ such that for any $l$-tuple of arguments $n_{1}, \ldots, n_{l}$, if the arguments $n_{i}$ not exceeding $c$ are fixed, then $p\left(n_{1}, \ldots, n_{l}\right)$ is a polynomial function in the remaining arguments $n_{i}$ bigger than $c$. We characterize such eventually strongly polynomial functions in Proposition 2.9.

In Theorems 2.1 and 2.8, we prove our next result, a common generalization of a weaker form of Theorem 1.1 and part 1 of Theorem 1.2. We prove that if $P$ is a lattice polytope in the nonnegative orthant of $\mathbb{R}^{k}$, and $\chi$ is a coloring of the lattice points in the orthant such that $\chi(a+b)$ depends only on the colors $\chi(a)$ and $\chi(b)$, then the number of colors

$$
\left|\chi\left(n P \cap \mathbb{Z}^{k}\right)\right|
$$

used on the points $n P \cap \mathbb{Z}^{k}$ is a polynomial in $n$ for large $n$. More generally, if $P$ is a rational polytope, then the number of colors is for large $n$ a quasipolynomial (Theorem 2.8). This includes Theorem 1.1 (in a weaker form for large $n$ ) and part 1 of Theorem 1.2 as particular cases. We want to remark that our Theorem 2.1 is to some extent hinted to already by Khovanskiĭ [9, paragraph 5] who derives, as an application of part 2 of Theorem 1.2, the weaker form of Theorem 1.1, We also obtain Theorem 2.1 as a corollary of part 2 of Theorem 1.2 and a geometric lemma.

Our third result are multivariate generalizations of Theorems 1.3 and 1.4, presented in Theorems 2.11 and 2.12, respectively. We give combinatorial proofs. The proof of Theorem 2.11 on additive characters is a simple extension of the combinatorial proof of Theorem [2.10 and we only give a sketch of the proof. The proof of Theorem 2.12 on iterated images is more interesting. We derive it from Theorem 2.15 which extends Stanley's Theorem 1.5 on lower ideals. Theorem 2.15 characterizes the sets $S \subset \mathbb{N}_{0}^{k}$ for which Theorem 1.5 holds.

Our combinatorial approach is based on expressing counting problems in terms of colorings $\chi$ of $\mathbb{N}_{0}^{k}$ and on counting the color classes of $\chi$ via appropriate representatives, so called substantial points. We have learned both techniques from Nathanson and Ruzsa [17]. A new ingredient is the representation of counting
functions in a compact and convenient way by their generating power series (which play almost no role in [9, 10, 16, 17]). We recall some results on them in the next subsection.

In Section 3, we give some concluding remarks and references to further examples of polynomial classes of enumerative problems.

### 1.3 Notation and results on power series

We fix notation and recall some useful results on power series. $\mathbb{N}$ is the set of natural numbers $\{1,2, \ldots\}$ and $\mathbb{N}_{0}$ is the set $\{0\} \cup \mathbb{N}$. The symbols for number sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ have their usual meanings. For $n \in \mathbb{N}$, the set $\{1,2, \ldots, n\}$ is denoted by $[n]$. We call the elements of $\mathbb{Z}^{k}$ lattice points. All semigroups in this article are commutative. We will use the lexicographic ordering of $\mathbb{N}_{0}^{k}$, which is a total ordering: $a<_{\text {lex }} b$ iff $a_{1}=b_{1}, \ldots, a_{i}=b_{i}, a_{i+1}<b_{i+1}$ for some $i, 0 \leq i<k$.

A quasipolynomial is a function $f: \mathbb{Z} \rightarrow \mathbb{C}$ for which there are $d$ polynomials $p_{1}(x), \ldots, p_{d}(x)$ such that $f(n)=p_{i}(n)$ if $n \equiv i \bmod d ; d$ is the period of $f$. Equivalently, $f(n)=a_{k}(n) n^{k}+\cdots+a_{1}(n) n+a_{0}(n)$ where $a_{i}: \mathbb{Z} \rightarrow \mathbb{C}$ are periodic functions. The term quasipolynomial is sometimes (e.g., in [10]) used also for linear combinations of exponentials with polynomial coefficients (as in Theorem 1.3); we use it in the present sense.

We shall use formal power series

$$
F\left(x_{1}, \ldots, x_{k}\right)=\sum_{a \in \mathbb{N}_{0}^{k}} \alpha(a) x_{1}^{a_{1}} \ldots x_{k}^{a_{k}}
$$

with real coefficients $\alpha(a)=\alpha\left(a_{1}, \ldots, a_{k}\right)$ and several variables $x_{1}, \ldots, x_{k}$; their set is denoted by $\mathbb{R}\left[\left[x_{1}, \ldots, x_{k}\right]\right]$. The symbol

$$
\left[x_{1}^{a_{1}} \ldots x_{k}^{a_{k}}\right] F
$$

denotes the coefficient $\alpha\left(a_{1}, \ldots, a_{k}\right)$ of $x_{1}^{a_{1}} \ldots x_{k}^{a_{k}}$ in $F$. For a subset $A \subset \mathbb{N}_{0}^{k}$, $F_{A}(x)=F_{A}\left(x_{1}, \ldots, x_{k}\right) \in \mathbb{R}\left[\left[x_{1}, \ldots, x_{k}\right]\right]$ is the power series

$$
F_{A}\left(x_{1}, \ldots, x_{k}\right)=\sum_{a \in A} x_{1}^{a_{1}} \ldots x_{k}^{a_{k}}
$$

i.e., $\alpha(a)$ is the characteristic function of $A$.

Lemma 1.6. Let $F \in \mathbb{R}\left[\left[x_{1}, \ldots, x_{k}\right]\right]$ be a rational power series of the form

$$
F\left(x_{1}, \ldots, x_{k}\right)=\frac{r\left(x_{1}, \ldots, x_{k}\right)}{\left(1-x_{1}\right)^{e_{1}} \ldots\left(1-x_{k}\right)^{e_{k}}}
$$

where $r \in \mathbb{R}\left[x_{1}, \ldots, x_{k}\right]$ is a polynomial and $e_{i} \in \mathbb{N}_{0}$. Then for every $l \in \mathbb{N}_{0}, l \leq k$, and every l-tuple $\left(a_{1}, \ldots, a_{l}\right) \in \mathbb{N}_{0}^{l}$, there exist a constant $c>0$ and a polynomial $p \in \mathbb{R}\left[x_{l+1}, \ldots, x_{k}\right]$ (for $l=k$ we understand $p$ as a real constant) such that if $n_{l+1}, \ldots, n_{k} \in \mathbb{N}$ are all bigger than $c$, then

$$
\left[x_{1}^{a_{1}} \ldots x_{l}^{a_{l}} x_{l+1}^{n_{l+1}} \ldots x_{k}^{n_{k}}\right] F=p\left(n_{l+1}, \ldots, n_{k}\right) .
$$

Proof. Let us check that the claim holds when $k=1,0 \leq l \leq 1$, and $r\left(x_{1}\right)=$ $r(x)=x^{b}$. By the binomial expansion,

$$
\frac{x^{b}}{(1-x)^{e}}=\sum_{n \geq 0}\binom{n+e-1}{e-1} x^{b+n}=\sum_{n \geq b}\binom{n+e-1-b}{e-1} x^{n}
$$

The general case reduces to this by expressing $F$ as a finite linear combination of terms of the type

$$
\frac{x_{1}^{b_{1}} \ldots x_{k}^{b_{k}}}{\left(1-x_{1}\right)^{e_{1}} \ldots\left(1-x_{k}\right)^{e_{k}}}=\prod_{i=1}^{k} \frac{x_{i}^{b_{i}}}{\left(1-x_{i}\right)^{e_{i}}}
$$

We add three comments to the lemma. If the polynomial $r\left(x_{1}, \ldots, x_{k}\right)$ has rational coefficients, then $p\left(x_{l+1}, \ldots, x_{k}\right)$ has rational coefficients as well. Also, Lemma 1.6 holds more generally for any subset of the set of variables $x_{1}, \ldots, x_{k}$ (we have chosen the subset $x_{l+1}, \ldots, x_{k}$ only for the convenience of notation). Finally, Lemma 1.6 can be strengthened by selecting the constant $c$ first and thus making it independent on the $l$-tuples $\left(a_{1}, \ldots, a_{l}\right)$. We return to this matter in Proposition 2.9.

Let $F \in \mathbb{R}\left[\left[x_{1}, \ldots, x_{k}\right]\right]$ be a power series and $P=\left\{P_{1}, \ldots, P_{l}\right\}$ be a partition of the index set $[k]$ into $l$ blocks. The substitution $x_{i}:=y_{j}$, where $1 \leq i \leq k$ and $j$ is the unique index satisfying $i \in P_{j}$, turns $F$ into the power series $G \in \mathbb{R}\left[\left[y_{1}, \ldots, y_{l}\right]\right]$ with the coefficients

$$
\left[y_{1}^{n_{1}} \ldots y_{l}^{n_{l}}\right] G=\sum\left[x_{1}^{a_{1}} \ldots x_{k}^{a_{k}}\right] F
$$

where we sum over all $a \in \mathbb{N}_{0}^{k}$ satisfying $\sum_{i \in P_{j}} a_{i}=n_{j}, 1 \leq j \leq l$. We call a substitution of this kind $P$-substitution. It is immediate that $P$-substitutions preserve the class of rational power series considered in Lemma 1.6.

Lemma 1.7. If $F \in \mathbb{R}\left[\left[x_{1}, \ldots, x_{k}\right]\right]$ has the form $F=r\left(1-x_{1}\right)^{-e_{1}} \ldots\left(1-x_{k}\right)^{-e_{k}}$, where $r \in \mathbb{R}\left[x_{1}, \ldots, x_{k}\right]$ and $e_{i} \in \mathbb{N}_{0}$, and $G \in \mathbb{R}\left[\left[y_{1}, \ldots, y_{l}\right]\right]$ is obtained from $F$ by a $P$-substitution, then $G=s\left(1-y_{1}\right)^{-f_{1}} \ldots\left(1-y_{l}\right)^{-f_{l}}$, where $s \in \mathbb{R}\left[y_{1}, \ldots, y_{l}\right]$ and $f_{i} \in \mathbb{N}_{0}$.

## 2 Generalizations of Khovanskiu's theorems

This section is devoted to the proofs of our main results, which are Theorems 2.1, 2.8, 2.10, 2.11, 2.12, and 2.15.

### 2.1 Additive colorings

We shall work with the semigroup $\left(\mathbb{N}_{0}^{k},+\right)$, where the addition of $k$-tuples is defined componentwise. For a (possibly infinite) set of colors $X$, we say that a coloring $\chi: \mathbb{N}_{0}^{k} \rightarrow X$ is additive if

$$
\chi(a+b)=\chi(c+d) \text { whenever } \chi(a)=\chi(c) \text { and } \chi(b)=\chi(d)
$$

that is, if the color of every sum depends only on the colors of summands. The coloring $\chi$ then can be viewed as a homomorphism between the semigroups (in fact monoids) $\left(\mathbb{N}_{0}^{k},+\right)$ and $(X,+)$. The additivity of $\chi$ is equivalent to the seemingly weaker property of shift-stability, which only requires that

$$
\chi(a+b)=\chi(c+b) \text { for every } b \text { whenever } \chi(a)=\chi(c) .
$$

Indeed, if $\chi$ is shift-stable and $a, b, c, d \in \mathbb{N}_{0}^{k}$ are arbitrary elements satisfying $\chi(a)=\chi(c)$ and $\chi(b)=\chi(d)$, then $\chi(a+b)=\chi(a+d)$ and $\chi(a+d)=\chi(c+d)$, so $\chi(a+b)=\chi(c+d)$.

Let $(G,+)$ be a (commutative) semigroup, we may assume that it has a neutral element and is a monoid. If $A=\left(a_{1}, \ldots, a_{k}\right)$ is a sequence of (possibly repeating) elements from $G$, then the associated coloring

$$
\chi: \mathbb{N}_{0}^{k} \rightarrow G, \chi(v)=\chi\left(\left(v_{1}, \ldots, v_{k}\right)\right)=v_{1} a_{1}+\cdots+v_{k} a_{k},
$$

is additive. In terms of this coloring, the cardinality of the sumset

$$
n * A=\left\{n_{1} a_{1}+\cdots+n_{k} a_{k}: n_{1}+\cdots+n_{k}=n\right\}
$$

equals to the number of colors $\left|\chi\left(n P \cap \mathbb{Z}^{d}\right)\right|$ appearing on the lattice points in the dilation of the unit simplex

$$
P=\left\{x \in \mathbb{R}^{k}: x_{i} \geq 0, x_{1}+\cdots+x_{k}=1\right\} .
$$

We prove the following common generalization of a weaker form of Theorem 1.1 (for large $n$ only) and part 1 of Theorem 1.2.

Theorem 2.1. Let $P$ be a polytope in $\mathbb{R}^{k}$ with vertices in $\mathbb{N}_{0}^{k}$ and let $\chi: \mathbb{N}_{0}^{k} \rightarrow X$ be an additive coloring. Then, for $n \in \mathbb{N}$ sufficiently large, the number of colors

$$
\left|\chi\left(n P \cap \mathbb{Z}^{k}\right)\right|=\left|\chi\left(n P \cap \mathbb{N}_{0}^{k}\right)\right|
$$

is given by a polynomial.
For large $n$, Theorem 1.1 corresponds to the case when $\chi$ is injective (hence additive) and $P$ is a general polytope, while part 1 of Theorem 1.2 corresponds to the case when $\chi$ is a general additive coloring and $P$ is the unit simplex.

We begin with proving a formally stronger version of Theorem 1.5; our proof is a straightforward adaptation of that in [19]. Recall that $S \subset \mathbb{N}_{0}^{k}$ is a lower ideal in the poset $\left(\mathbb{N}_{0}^{k}, \leq\right)$ if for every $a \in \mathbb{N}_{0}^{k}$ we have $a \in S$ whenever $a \leq b$ for some $b \in S$. Upper ideals are defined similarly. The proof rests on the well-known result, sometimes called Dickson's lemma, which states that all antichains (sets with elements mutually incomparable by $\leq$ ) in $\left(\mathbb{N}_{0}^{k}, \leq\right)$ are finite. This lemma is a corollary of the more general fact that if $\left(P, \leq_{P}\right)$ and $\left(Q, \leq_{Q}\right)$ are two posets which have no infinite antichains and no infinite strictly descending chains, then this property carries over to the product poset $\left(P \times Q, \leq_{P \times Q}\right)$ (see, e.g., Kruskal [11]).

Theorem 2.2. Let $S \subset \mathbb{N}_{0}^{k}$ be a lower or an upper ideal in the poset $\left(\mathbb{N}_{0}^{k}, \leq\right)$. Then

$$
F_{S}\left(x_{1}, \ldots, x_{k}\right)=\frac{r\left(x_{1}, \ldots, x_{k}\right)}{\left(1-x_{1}\right) \ldots\left(1-x_{k}\right)}
$$

where $r\left(x_{1}, \ldots, x_{k}\right)$ is an integral polynomial.
Proof. Since every upper ideal $S$ has as its complement $T=\mathbb{N}_{0}^{k} \backslash S$ a lower ideal and vice versa, and

$$
F_{S}(x)+F_{T}(x)=F_{\mathbb{N}_{0}^{k}}(x)=\frac{1}{\left(1-x_{1}\right) \ldots\left(1-x_{k}\right)},
$$

it suffices to prove the result only for ideals of one kind. Let $S$ be an upper ideal. If $M \subset S$ is the set of the minimal elements in $S$, then

$$
S=\bigcup_{a \in M} O_{a}
$$

where $O_{a}=\left\{b \in \mathbb{N}_{0}^{k}: b \geq a\right\}$. Being an antichain, $M$ is finite by Dickson's lemma and $S$ is a finite union of the orthants $O_{a}, a \in M$. For any finite set $T$ of points in $\mathbb{N}_{0}^{k}$ we have

$$
\bigcap_{t \in T} O_{t}=O_{s},
$$

where $s=\left(s_{1}, s_{2}, \ldots, s_{k}\right)$ is the componentwise maximum of the points $t \in T$. Thus, by the principle of inclusion and exclusion, the characteristic function of $S$ is a linear combination, with coefficients $\pm 1$, of characteristic functions of finitely many orthants $O_{s}$. Since each of them has generating function

$$
F_{O_{s}}(x)=\frac{x_{1}^{s_{1}} \cdots x_{k}^{s_{k}}}{\left(1-x_{1}\right) \cdots\left(1-x_{k}\right)},
$$

we have $F_{S}(x)=r /\left(\left(1-x_{1}\right) \ldots\left(1-x_{k}\right)\right)$ for some integral polynomial $r$.
Theorem 1.5 now follows as a corollary, with the help of Lemmas 1.6 and 1.7 and the $P$-substitution $P=\{\{1, \ldots, k\}\}$.

Next, we prove the multivariate generalizations of Theorem 1.2 from [16] and [17]; this is necessary, since we need part 2 of Theorem 1.2 for the proof of Theorem 2.1. In Corollary 2.3 we lift the result of Nathanson and Ruzsa to the level of generating functions.

Suppose that $P$ is a partition of $[k]$ into $l$ blocks and $\chi: \mathbb{N}_{0}^{k} \rightarrow X$ is a coloring. For $x \in \mathbb{N}_{0}^{k}$ we define $\|x\|_{P}$ to be the $l$-tuple $\left(c_{1}, \ldots, c_{l}\right) \in \mathbb{N}_{0}^{l}$, where $c_{i}=\sum_{j \in P_{i}} x_{j}$ is the sum of the coordinates with indices in the $i$ th block. Using the notion introduced in [17], we say that a point $a \in \mathbb{N}_{0}^{k}$ is $P$-substantial (with respect to $\chi$ ) if it is the lexicographically minimum element in the set

$$
\left\{b \in \mathbb{N}_{0}^{k}: \chi(b)=\chi(a),\|b\|_{P}=\|a\|_{P}\right\}
$$

Note that every nonempty intersection of a color class with the set $\left\{x \in \mathbb{N}_{0}^{k}\right.$ : $\left.\|x\|_{P}=\left(n_{1}, \ldots, n_{l}\right)\right\}$ (for $l=1$ this is the dilation $n_{1} P$ where $P$ is the unit simplex) contains exactly one $P$-substantial point. $P$-substantial points are representatives which enable us to count the color classes.

Corollary 2.3. Let $P$ be a partition of $[k]$ into $l$ blocks, $\chi: \mathbb{N}_{0}^{k} \rightarrow X$ be an additive coloring and $S \subset \mathbb{N}_{0}^{k}$ be the set of $P$-substantial points. Then

$$
F_{S}\left(x_{1}, \ldots, x_{k}\right)=\frac{r\left(x_{1}, \ldots, x_{k}\right)}{\left(1-x_{1}\right) \cdots\left(1-x_{k}\right)}
$$

where $r\left(x_{1}, \ldots, x_{k}\right)$ is an integral polynomial.
Proof. In view of the previous theorem, it suffices to show that $P$-substantial points form a lower ideal or, equivalently, that their complement is an upper ideal. The latter way is a more natural choice. Let $b \in \mathbb{N}_{0}^{k}$ be any point such that $b \geq a$ for a non- $P$-substantial point $a \in \mathbb{N}_{0}^{k}$. There is a point $a^{\prime} \in \mathbb{N}_{0}^{k}$ satisfying $\chi\left(a^{\prime}\right)=\chi(a),\left\|a^{\prime}\right\|_{P}=\|a\|_{P}$, and $a^{\prime}<_{l e x} a$. Consider the point $b^{\prime}=a^{\prime}+(b-a)$. We have $\chi\left(b^{\prime}\right)=\chi(b)$ by the additivity (indeed, shift-stability) of $\chi$, and $\left\|b^{\prime}\right\|_{P}=$ $\left\|a^{\prime}\right\|_{P}+\|b-a\|_{P}=\|a\|_{P}+\|b-a\|_{P}=\|b\|_{P}$ and $b^{\prime}<_{l e x} b$ by the properties of addition in $\left(\mathbb{N}_{0}^{k},+\right)$. Thus $b$ is not $P$-substantial either.

Theorem 2.4 (Nathanson and Ruzsa [17]). Let $A_{1}, \ldots, A_{l}$ be finite sets in a semigroup $(G,+)$. There exist a constant $c>0$ and an integral polynomial $p \in$ $\mathbb{Z}\left[x_{1}, \ldots, x_{l}\right]$ such that if $n_{1}, \ldots, n_{l} \in \mathbb{N}$ are all bigger than $c$, then

$$
\left|n_{1} * A_{1}+\cdots+n_{l} * A_{l}\right|=p\left(n_{1}, \ldots, n_{l}\right) .
$$

Proof. Let $A=\left(a_{1}, \ldots, a_{k}\right)$ be a fixed ordering of all elements appearing in the sets $A_{1}, \ldots, A_{l}$ (taken with their multiplicities, so $k=\left|A_{1}\right|+\cdots+\left|A_{l}\right|$ ) and $P$ be the corresponding partition of $[k]$ into $l$ blocks. Let $\chi: \mathbb{N}_{0}^{k} \rightarrow G$ be the coloring associated with $A$ and $S \subset \mathbb{N}_{0}^{k}$ be the corresponding set of $P$-substantial points. Let $G \in \mathbb{R}\left[\left[y_{1}, \ldots, y_{l}\right]\right]$ be the power series obtained from $F_{S}\left(x_{1}, \ldots, x_{k}\right)$ by the $P$-substitution. Then

$$
\left|n_{1} * A_{1}+\cdots+n_{l} * A_{l}\right|=\left[y_{1}^{n_{1}} \ldots y_{l}^{n_{l}}\right] G .
$$

The result now follows by Corollary 2.3 and by Lemmas 1.6 and 1.7 .
Extending Khovanskiǐ's original algebraic argument, Nathanson [16] proved a multivariate generalization of part 2 of Theorem 1.2,

Theorem 2.5 (Nathanson [16]). Let $A_{1}, \ldots, A_{l+1}$ be finite sets in a semigroup $(G,+)$. There exist a constant $c>0$ and a polynomial $p \in \mathbb{Z}\left[x_{1}, \ldots, x_{l}\right]$ such that if $n_{1}, \ldots, n_{l} \in \mathbb{N}$ are all bigger than $c$, then

$$
\left|n_{1} * A_{1}+\cdots+n_{l} * A_{l}+A_{l+1}\right|=p\left(n_{1}, \ldots, n_{l}\right) .
$$

Proof. The proof is almost identical to the proof of Theorem 2.4. We again see that

$$
\left|n_{1} * A_{1}+\cdots+n_{l} * A_{l}+A_{l+1}\right|=\left[y_{1}^{n_{1}} \ldots y_{l}^{n_{l}} y_{l+1}\right] G
$$

and use Corollary 2.3 and Lemmas 1.6 and 1.7 .
The last ingredient needed for the proof of Theorem 2.1 is a geometric lemma. Before we state the lemma, let us point out some observations about multiples of polytopes. Let $P \subset \mathbb{R}^{k}$ be a polytope, $n \in \mathbb{N}_{0}$, and $\alpha_{1}, \ldots, \alpha_{n} \in \mathbb{R}$ be nonnegative coefficients. Clearly, $n P \subset n * P$. On the other hand, representing points in $P$ as convex combinations of the vertices of $P$, we deduce the following set inclusion

$$
\begin{equation*}
\alpha_{1} P+\cdots+\alpha_{n} P \subset\left(\alpha_{1}+\cdots+\alpha_{n}\right) P \tag{1}
\end{equation*}
$$

In particular, $n * P \subset n P$ and thus $n * P=n P$. As a corollary, we obtain another set inclusion

$$
\begin{equation*}
\left(\alpha_{1} P \cap \mathbb{Z}^{k}\right)+\cdots+\left(\alpha_{n} P \cap \mathbb{Z}^{k}\right) \subset\left(\alpha_{1}+\cdots+\alpha_{n}\right) P \cap \mathbb{Z}^{k} \tag{2}
\end{equation*}
$$

In particular, $n *\left(P \cap \mathbb{Z}^{k}\right) \subset n P \cap \mathbb{Z}^{k}$. The opposite inclusion in general does not hold. To get equality in some form also for lattice points, we use Carathéodory's theorem. This theorem says that if a point $a$ in $\mathbb{R}^{k}$ is in the convex hull of a set of points $M$, then $a$ can be expressed as a convex combination of at most $k+1$ points of the set $M$ (see, e.g., Matoušek [15]).

Lemma 2.6. Let $k \in \mathbb{N}$ and $P \subset \mathbb{R}^{k}$ be a lattice polytope. Then for every $n \in \mathbb{N}$, $n \geq k$, we have in $\left(\mathbb{Z}^{k},+\right)$ the identity

$$
n P \cap \mathbb{Z}^{k}=(n-k) *\left(P \cap \mathbb{Z}^{k}\right)+\left(k P \cap \mathbb{Z}^{k}\right)
$$

Proof. Let $v_{1}, \ldots, v_{r}$ be the vertices of $P$ and let $p \in n P \cap \mathbb{Z}^{k}$ with $n \in \mathbb{N}$ and $n \geq k$. Clearly, $p$ is in the convex hull of the points $n v_{1}, \ldots, n v_{r}$. By Carathéodory's theorem, $p$ is a convex combination of at most $k+1$ of these points. Hence

$$
\begin{aligned}
p & =\beta_{1} n w_{1}+\cdots+\beta_{j} n w_{j}, \quad \text { where } \beta_{i} \geq 0 \text { and } \beta_{1}+\cdots+\beta_{j}=1 \\
& =n_{1} w_{1}+\cdots+n_{j} w_{j}+w
\end{aligned}
$$

where $n_{i}=\left\lfloor\beta_{i} n\right\rfloor \in \mathbb{N}_{0}, j \leq k+1, w_{1}, \ldots, w_{j}$ are some distinct vertices of $v_{1}, \ldots, v_{r}$, and

$$
w=\alpha_{1} w_{1}+\cdots+\alpha_{j} w_{j}, \quad \text { where } \alpha_{i}=\beta_{i} n-\left\lfloor\beta_{i} n\right\rfloor \in[0,1) \text {. }
$$

Since $w=p-\left(n_{1} w_{1}+\cdots+n_{j} w_{j}\right)$, we see that $w$ is a lattice point. By (1), $w \in\left(\alpha_{1}+\cdots+\alpha_{j}\right) P=c P$. We have $0 \leq c=\alpha_{1}+\cdots+\alpha_{j}<j \leq k+1$ and $c=\alpha_{1}+\cdots+\alpha_{j}=n-\left(n_{1}+\cdots+n_{j}\right) \in \mathbb{N}_{0}$. Thus $c \leq k$. We conclude that $w \in c P \cap \mathbb{Z}^{k}$ where $c \in \mathbb{N}_{0}, c=n-\left(n_{1}+\cdots+n_{j}\right)$, and $c \leq k$.

We split $n_{1} w_{1}+\cdots+n_{j} w_{j}$ in the individual $n_{1}+\cdots+n_{j}=n-c$ summands, each of them equal to some $w_{i}$, and merge $k-c$ of them with $w$ so that we obtain a point $z \in k P \cap \mathbb{Z}^{k}$ (using the inclusion (2) above). Thus we get the expression

$$
p=z_{1}+\cdots+z_{n-k}+z
$$

where $z_{i} \in P \cap \mathbb{Z}^{k}$ (in fact, $z_{i} \in\left\{v_{1}, \ldots, v_{r}\right\}$ ) and $z \in k P \cap \mathbb{Z}^{k}$. This shows that

$$
n P \cap \mathbb{Z}^{k} \subset(n-k) *\left(P \cap \mathbb{Z}^{k}\right)+\left(k P \cap \mathbb{Z}^{k}\right)
$$

The opposite inclusion follows from (2).
We are ready to prove Theorem 2.1.
Proof of Theorem 2.1. We consider the semigroup of color classes $\left(\chi\left(\mathbb{N}_{0}^{k}\right),+\right)$ and its subsets $A=\chi\left(P \cap \mathbb{N}_{0}^{k}\right)$ and $B=\chi\left(k P \cap \mathbb{N}_{0}^{k}\right)$. By Lemma 2.6,

$$
\left|\chi\left(n P \cap \mathbb{N}_{0}^{k}\right)\right|=|(n-k) * A+B|
$$

By part 2 of Theorem 1.2 (or by Theorem 2.5 or by Theorem 2.10 in the next subsection), this quantity is for big $n$ a polynomial in $n-k$ and hence a polynomial in $n$.

We generalize Theorem 2.1 to rational polytopes. Our argument is based on the following generalization of Lemma 2.6.

Lemma 2.7. Let $k \in \mathbb{N}$ and let $P \subset \mathbb{R}^{k}$ be a rational polytope. Let $m \in \mathbb{N}$ be such that $m P$ is a lattice polytope. If $n \in \mathbb{N}$ satisfies $n \geq m k$ and is congruent to $r \in\{0,1, \ldots, m-1\}$ modulo $m$, then we have the identity

$$
n P \cap \mathbb{Z}^{k}=\frac{n-m k-r}{m} *\left(m P \cap \mathbb{Z}^{k}\right)+\left((m k+r) P \cap \mathbb{Z}^{k}\right)
$$

Proof. The proof is an extension of that for Lemma [2.6 and we proceed more briefly. Again, it suffices to prove the set inclusion " $\subseteq$ ", the opposite one is trivial. Fix a point $p \in n P \cap \mathbb{Z}^{k}$ with $n \geq m k$ congruent to $r$ modulo $m$. As in the proof of Lemma [2.6, only replacing the integral part $n_{i}=\left\lfloor\beta_{i} n\right\rfloor$ with the largest multiple of $m$ not exceeding $\beta_{i} n$, we write $p$ as

$$
p=\sum_{i=1}^{j} n_{i} w_{i}+\sum_{i=1}^{j} \alpha_{i} w_{i}
$$

where $j \leq k+1, w_{i}$ are some vertices of $P, n_{i} \in \mathbb{N}_{0}$ are multiples of $m, \alpha_{i} \in[0, m)$, and $c=\alpha_{1}+\cdots+\alpha_{j}=n-\left(n_{1}+\cdots+n_{j}\right) \in \mathbb{N}_{0}$ is congruent to $r$ modulo $m$. So $c \leq m k+r$. Moving several multiples of $m$ from $n_{i}$ to the corresponding $\alpha_{i}$, we may assume that $c=m k+r$. It follows that the first sum of the right-hand side is equal to an element of $\frac{n-m k-r}{m} *\left(m P \cap \mathbb{Z}^{k}\right)$, while the second sum belongs to $(m k+r) P \cap \mathbb{Z}^{k}$.

Using this lemma and part 2 of Theorem [1.2, we get the following theorem in the same way as we got Theorem [2.1. We omit the proof.

Theorem 2.8. Let $P$ be a polytope in $\mathbb{R}^{k}$ with vertices in $\mathbb{Q}_{\geq 0}^{k}$, let $m \in \mathbb{N}$ be such that the vertices of $m P$ lie in $\mathbb{N}_{0}^{k}$, and let $\chi: \mathbb{N}_{0}^{k} \rightarrow X$ be an additive coloring. Then, for $n \in \mathbb{N}$ sufficiently large, the number of colors

$$
\left|\chi\left(n P \cap \mathbb{N}_{0}^{k}\right)\right|
$$

is given by a quasipolynomial with period $m$.

### 2.2 Strongly eventually polynomial functions

Theorems 2.4 and 2.5 say nothing about the values of the corresponding functions when some argument $n_{i}$ is not bigger than $c$. In Theorem 2.10 we give a stronger formulation using other notion of an eventually polynomial function in several variables, which is suggested by power series.

For $k, c \in \mathbb{N}$ we define $V(k, c)=([0, c] \cup\{\infty\})^{k}$; the elements of $V(k, c)$ are the $(c+2)^{k}$ words $w=w_{1} w_{2} \ldots w_{k}$ of length $k$ such that every entry $w_{i}$ is $0, \ldots, c$ or $\infty$. We say that a function

$$
f: \mathbb{N}_{0}^{k} \rightarrow \mathbb{R}
$$

is strongly eventually polynomial if there exist a $c \in \mathbb{N}$ and $(c+2)^{k}$ polynomials $p_{w} \in \mathbb{R}\left[x_{1}, \ldots, x_{k}\right]$ indexed by the words $w \in V(k, c)$ so that for every $k$-tuple $n=\left(n_{1}, \ldots, n_{k}\right) \in \mathbb{N}_{0}^{k}$ and the unique $w=w(n) \in V(k, c)$ determined by $w_{i}=n_{i}$ if $n_{i} \leq c$ and $w_{i}=\infty$ if $n_{i}>c$, we have

$$
f\left(n_{1}, \ldots, n_{k}\right)=p_{w(n)}\left(n_{1}, \ldots, n_{k}\right)
$$

Said more briefly, there is a constant $c \in \mathbb{N}$ such that for any selection of arguments $n_{i}$, when we fix arguments not exceeding $c, f\left(n_{1}, \ldots, n_{k}\right)$ is a polynomial function in the remaining arguments (which are all bigger than $c$ ). Note that for $k=1$ this notion is identical with the usual notion of an eventually polynomial function $f: \mathbb{N}_{0} \rightarrow \mathbb{R}$ (there is a constant $c>0$ and a polynomial $p \in \mathbb{R}[x]$ such that $f(n)=p(n)$ for $n>c)$. Note also that if $f: \mathbb{N}_{0}^{k} \rightarrow \mathbb{R}$ is strongly eventually polynomial for a constant $c$, then it is strongly eventually polynomial for any larger constant.

We give a stronger version of Lemma 1.6.
Proposition 2.9. A function $f: \mathbb{N}_{0}^{k} \rightarrow \mathbb{R}$ is strongly eventually polynomial if and only if

$$
F\left(x_{1}, \ldots, x_{k}\right)=\sum_{n \in \mathbb{N}_{0}^{k}} f(n) x_{1}^{n_{1}} \cdots x_{k}^{n_{k}}=\frac{r\left(x_{1}, \ldots, x_{k}\right)}{\left(1-x_{1}\right)^{e_{1}} \cdots\left(1-x_{k}\right)^{e_{k}}}
$$

for some $r \in \mathbb{R}\left[x_{1}, \ldots, x_{k}\right]$ and $e_{i} \in \mathbb{N}_{0}$.
Proof. If $f$ is strongly eventually polynomial and is represented by the polynomials $p_{v}, v \in V(k, c)$, we have

$$
F(x)=\sum_{n \in \mathbb{N}_{0}^{k}} f(n) x_{1}^{n_{1}} \cdots x_{k}^{n_{k}}=\sum_{v \in V(k, c)} \sum_{\substack{n \\ w(n)=v}} p_{v}(n) x_{1}^{n_{1}} \ldots x_{k}^{n_{k}}
$$

Each inner sum is a power series which can be transformed in the form $r(1-$ $\left.x_{1}\right)^{-e_{1}} \ldots\left(1-x_{k}\right)^{-e_{k}}$ for some $r \in \mathbb{R}\left[x_{1}, \ldots, x_{k}\right]$ and $e_{i} \in \mathbb{N}_{0}$. Thus $F(x)$ has the stated form.

Suppose that $F(x)$ has the stated form. As in the proof of Lemma 1.6, we write it as a linear combination of terms of the type

$$
\prod_{i=1}^{k} \frac{x_{i}^{b_{i}}}{\left(1-x_{i}\right)^{e_{i}}}
$$

where $b_{i}, e_{i} \in \mathbb{N}_{0}$. The coefficients of the power series $x^{b} /(1-x)^{e}$ form a univariate strongly eventually polynomial function. It is easy to see that the concatenative product $h: \mathbb{N}_{0}^{k+l} \rightarrow \mathbb{R}$ of two strongly eventually polynomial functions $f: \mathbb{N}_{0}^{k} \rightarrow \mathbb{R}$ and $g: \mathbb{N}_{0}^{l} \rightarrow \mathbb{R}$, defined by

$$
h\left(n_{1}, \ldots, n_{k+l}\right)=f\left(n_{1}, \ldots, n_{k}\right) g\left(n_{k+1}, \ldots, n_{k+l}\right)
$$

is strongly eventually polynomial as well (as we know, we may assume that the constant $c$ is the same for $f$ and $g$ ). The same holds for the linear combination $\alpha f+\beta g: \mathbb{N}_{0}^{k} \rightarrow \mathbb{R}$ of two strongly eventually polynomial functions $f, g: \mathbb{N}_{0}^{k} \rightarrow \mathbb{R}$. From the expression of $F(x)$ as a linear combination of the mentioned products, it follows that the function $\left(n_{1}, \ldots, n_{k}\right) \mapsto\left[x_{1}^{n_{1}} \ldots x_{k}^{n_{k}}\right] F\left(x_{1}, \ldots, x_{k}\right)$ is a finite linear combination of concatenative products of strongly eventually polynomial (univariate) functions. Thus it is strongly eventually polynomial as well.

The following theorem is a common strengthening of Theorems 2.4 and 2.5 , which cancels the distinction between the projective and affine formulations (parts 1 and 2 of Theorem (1.2).
Theorem 2.10. Let $A_{1}, \ldots, A_{l}$ be finite sets in a semigroup $(G,+)$. Then

$$
\left(n_{1}, \ldots, n_{l}\right) \mapsto\left|n_{1} * A_{1}+\cdots+n_{l} * A_{l}\right|
$$

is a strongly eventually polynomial function from $\mathbb{N}_{0}^{l}$ to $\mathbb{N}_{0}$.
Proof. The proof is almost identical to the proof of Theorem 2.4, only we use Proposition 2.9 in place of Lemma 1.6.

### 2.3 Multivariate generalizations of Theorems 1.3 and 1.4

Recall that for $l, c \in \mathbb{N}$, the set $V(l, c)$ consists of the $(c+2)^{l}$ words of length $l$ over the alphabet $\{0, \ldots, c, \infty\}$ and that for $n=\left(n_{1}, \ldots, n_{l}\right) \in \mathbb{N}_{0}^{l}$ the word $w(n)=w_{1} \ldots w_{l} \in V(l, c)$ is defined by $w_{i}=n_{i}$ if $n_{i} \leq c$ and $w_{i}=\infty$ if $n_{i}>c$. The next theorem generalizes Theorems 1.3 and 2.10 (and thus in turn Theorems 1.2, 2.4, and (2.5).

Theorem 2.11. For finite sets $A_{1}, \ldots, A_{l}$ in a semigroup $G=(G,+)$ and a character $\psi: G \rightarrow \mathbb{C}$, there exist a constant $c \in \mathbb{N}$ and $(c+2)^{l}\left|A_{1}\right| \cdots\left|A_{l}\right|$ polynomials $p_{w, a_{1}, \ldots, a_{l}} \in \mathbb{C}\left[x_{1}, \ldots, x_{l}\right]$, where $w \in V(l, c)$ and $a_{i} \in A_{i}$, such that for every $l$-tuple $n=\left(n_{1}, \ldots, n_{l}\right) \in \mathbb{N}_{0}^{l}$ and the corresponding word $w(n) \in V(l, c)$, we have

$$
\sum_{a \in n_{1} * A_{1}+\cdots+n_{l} * A_{l}} \psi(a)=\sum_{a_{1} \in A_{1}, \ldots, a_{l} \in A_{l}} p_{w(n), a_{1}, \ldots, a_{l}}\left(n_{1}, \ldots, n_{l}\right) \psi\left(a_{1}\right)^{n_{1}} \ldots \psi\left(a_{l}\right)^{n_{l}} .
$$

Proof (Sketch). We pull $\psi$ back to the semigroup $\left(\mathbb{N}_{0}^{k},+\right)$ with the associated coloring and for $X \subset \mathbb{N}_{0}^{k}$ work with the power series

$$
F_{X, \psi}(x)=\sum_{n \in X} \psi(n) x_{1}^{n_{1}} \ldots x_{k}^{n_{k}}
$$

For an orthant $O_{s} \subset \mathbb{N}_{0}^{k}$ we then have, denoting the $k$ basic unit vectors by $u_{i}$,

$$
F_{O_{s}, \psi}(x)=\frac{\psi(s) x_{1}^{s_{1}} \ldots x_{k}^{s_{k}}}{\left(1-\psi\left(u_{1}\right) x_{1}\right) \ldots\left(1-\psi\left(u_{k}\right) x_{k}\right)}
$$

Thus, arguing as in the proof of Theorem [2.2, if $X \subset \mathbb{N}_{0}^{k}$ is a lower or an upper ideal, then

$$
F_{X, \psi}(x)=\frac{r\left(x_{1}, \ldots, x_{k}\right)}{\left(1-\psi\left(u_{1}\right) x_{1}\right) \ldots\left(1-\psi\left(u_{k}\right) x_{k}\right)}
$$

where $r$ is a polynomial whose coefficients are finite sums of $\pm$ values of $\psi$. It follows that

$$
\sum_{a \in n_{1} * A_{1}+\cdots+n_{l} * A_{l}} \psi(a)=\left[y_{1}^{n_{1}} \ldots y_{l}^{n_{l}}\right] G
$$

where $G(y)$ is obtained from such $F_{X, \psi}(x)$ by a $P$-substitution. The theorem now follows by a version of Proposition 2.9 for rational power series of the form $r /\left(\left(1-\alpha_{1} x_{1}\right)^{e_{1}} \ldots\left(1-\alpha_{k} x_{k}\right)^{e_{k}}\right)$.

In the multivariate generalization of Theorem 1.4 we refine the iterated image $\mathcal{F}^{(n)}(B)$ by partitioning $\mathcal{F}$. For a (typically infinite) set $X$, its subset $B \subset X$, a family $\mathcal{F}$ of mutually commuting mappings $f: X \rightarrow X$, and a partition $P=$ $\left\{P_{1}, \ldots, P_{l}\right\}$ of $\mathcal{F}$ into nonempty blocks, we let $\mathcal{F}^{\left(n_{1}, \ldots, n_{l}\right)}$ denote the set of all the functions that can be obtained by composing $l$ functions $f_{1} \circ f_{2} \circ \cdots \circ f_{l}$, where each $f_{i}$ is itself a composition of $n_{i}$ functions belonging to the block $P_{i}$, and set

$$
\mathcal{F}^{\left(n_{1}, \ldots, n_{l}\right)}(B)=\bigcup_{f \in \mathcal{F}^{\left(n_{1}, \ldots, n_{l}\right)}} f(B)
$$

The next theorem generalizes Theorems 1.4 and 2.10 (and thus in turn Theorems 1.2, 2.4, and 2.5).

Theorem 2.12. If $B$ is a finite subset of $X, \mathcal{F}$ is finite family of mutually commuting mappings from $X$ to itself, and $P=\left\{P_{1}, \ldots, P_{l}\right\}$ is a partition of $\mathcal{F}$, then

$$
\left(n_{1}, \ldots, n_{l}\right) \mapsto\left|\mathcal{F}^{\left(n_{1}, \ldots, n_{l}\right)}(B)\right|
$$

is a strongly eventually polynomial function from $\mathbb{N}_{0}^{l}$ to $\mathbb{N}_{0}$.
For the combinatorial proof we need an extension of Theorem 2.2 to sets more general than lower or upper ideals. For $k \in \mathbb{N}, I \subset[k]$, and $s \in \mathbb{N}_{0}^{k}$, the generalized orthant $O_{s, I} \subset \mathbb{N}_{0}^{k}$ is defined by

$$
O_{s, I}=\left\{x \in \mathbb{N}_{0}^{k}: i \in I \Rightarrow x_{i}=s_{i}, i \notin I \Rightarrow x_{i} \geq s_{i}\right\} .
$$

An empty set is also a generalized orthant. A subset $S \subset \mathbb{N}_{0}^{k}$ is simple if it is a finite union of generalized orthants. In particular, every finite set is simple. So is every upper ideal and, as we shall see in a moment, every lower ideal.

Lemma 2.13. Intersection of any system of generalized orthants is a generalized orthant. Complement of a generalized orthant to $\mathbb{N}_{0}^{k}$ is a simple set.

Proof. A $k$-tuple $x$ of $\mathbb{N}_{0}^{k}$ lies in the intersection of the system $O_{s(j), I(j)}, j \in J$, of nonempty generalized orthants iff for every $i \in[k]$ the $i$ th coordinate $x_{i}$ satisfies for every $j \in J$ the condition imposed by the membership $x \in O_{s(j), I(j)}$. These conditions have form $x_{i} \in\left\{s_{i, j}\right\}$ or $x_{i} \in\left[s_{i, j},+\infty\right)$ for some $s_{i, j} \in \mathbb{N}_{0}$. Intersection (conjunction) of these conditions over all $j \in J$ is a condition of the type $x_{i} \in \emptyset$ or $x_{i} \in\left\{s_{i}\right\}$ or $x_{i} \in\left[s_{i},+\infty\right)$ for some $s_{i} \in \mathbb{N}_{0}$. This is true for every $i \in[k]$. Thus $\bigcap_{j \in J} O_{s(j), I(j)}$ is an empty set or a nonempty generalized orthant.

Let $O=O_{s, I} \subset \mathbb{N}_{0}^{k}$ be a generalized orthant. We have $x \in \mathbb{N}_{0}^{k} \backslash O$ iff there exists an $i \in[k]$ such that (i) $i \in I$ and $x_{i}$ satisfies $x_{i} \in\left[s_{i}+1,+\infty\right)$ or $x_{i} \in\left[0, s_{i}-1\right]$ or such that (ii) $i \notin I$ and $x_{i}$ satisfies $x_{i} \in\left[0, s_{i}-1\right]$. Let $u(i, j) \in \mathbb{N}_{0}^{k}$, for $i \in[k]$ and $j \in \mathbb{N}_{0}$, denote the $k$-tuple with all coordinates zero except the $i$ th one which is equal to $j$. It follows that $\mathbb{N}_{0}^{k} \backslash O$ is the union of the generalized orthants

$$
O_{u\left(i, s_{i}+1\right), \emptyset}, i \in I ; O_{u\left(i, j_{i}\right),\{i\}}, i \in[k] \text { and } j_{i} \in\left[0, s_{i}-1\right]
$$

(if $s_{i}=0$, no $O_{u\left(i, j_{i}\right),\{i\}}$ is needed). Thus $\mathbb{N}_{0}^{k} \backslash O$ is a simple set.
Corollary 2.14. The family of simple sets in $\mathbb{N}_{0}^{k}$ contains the sets $\emptyset$ and $\mathbb{N}_{0}^{k}$ and is closed under taking finite unions, finite intersections, and complements. Hence it forms a boolean algebra.

Proof. This follows by the previous lemma and by elementary set identities involving unions, intersections and complements.

The family of simple sets is in general not closed to infinite unions nor to infinite intersections.

The next theorem is an extension of Theorems 1.5 and 2.2. It characterizes the sets $S \subset \mathbb{N}_{0}^{K}$, for which these theorems hold.

Theorem 2.15. If $S \subset \mathbb{N}_{0}^{k}$ is a simple set, then

$$
F_{S}\left(x_{1}, \ldots, x_{k}\right)=\frac{r\left(x_{1}, \ldots, x_{k}\right)}{\left(1-x_{1}\right) \cdots\left(1-x_{k}\right)}
$$

where $r\left(x_{1}, \ldots, x_{k}\right)$ is an integral polynomial. If $S \subset \mathbb{N}_{0}^{k}$ is a set such that

$$
F_{S}\left(x_{1}, \ldots, x_{k}\right)=\frac{r\left(x_{1}, \ldots, x_{k}\right)}{\left(1-x_{1}\right) \cdots\left(1-x_{k}\right)}
$$

where $r\left(x_{1}, \ldots, x_{k}\right)$ is an integral polynomial, then $S$ is a simple set.
Proof. Suppose that $S \subset \mathbb{N}_{0}^{k}$ is simple and $S=O_{1} \cup \cdots \cup O_{r}$ for some generalized orthants $O_{i}$. By the principle of inclusion and exclusion, $F_{S}\left(x_{1}, \ldots, x_{k}\right)$ is a sum of the $2^{r}$ terms $(-1)^{|X|} F_{O(X)}\left(x_{1}, \ldots, x_{k}\right), X \subset[r]$, where

$$
O(X)=\bigcap_{i \in X} O_{i}
$$

By Lemma 2.13, each $O(X)$ is again a generalized orthant. For a generalized orthant $O=O_{s, I}$,

$$
F_{O}\left(x_{1}, \ldots, x_{k}\right)=\frac{x_{1}^{s_{1}} \cdots x_{k}^{s_{k}}}{\prod_{i \in[k] \backslash I}\left(1-x_{i}\right)}
$$

The first claim follows.
Suppose that $S \subset \mathbb{N}_{0}^{k}$ and $F_{S}\left(x_{1}, \ldots, x_{k}\right)=r /\left(\left(1-x_{1}\right) \cdots\left(1-x_{k}\right)\right)$ where $r \in \mathbb{Z}\left[x_{1}, \ldots, x_{k}\right]$. Hence $F_{S}\left(x_{1}, \ldots, x_{k}\right)$ is an $l$-term integral linear combination

$$
\sum_{s \in T} \frac{c_{s} x_{1}^{s_{1}} \cdots x_{k}^{s_{k}}}{\left(1-x_{1}\right) \cdots\left(1-x_{k}\right)}
$$

where $T \subset \mathbb{N}_{0}^{k},|T|=l$, and $c_{s} \in \mathbb{Z}$. Every summand is in fact equal to $c_{s} F_{O_{s}}\left(x_{1}, \ldots, x_{k}\right)$. The characteristic function of $S$ is an integral linear combination of the characteristic functions of the $l$ (full-dimensional) orthants $O_{s}=O_{s, \emptyset}$, $s \in T$. With $X$ running through the $2^{l}$ subsets of $T$, we partition $\mathbb{N}_{0}^{k}$ in the $2^{l}$ cells

$$
\bigcap_{s \in X} O_{s} \cap \bigcap_{s \in T \backslash X} \mathbb{N}_{0}^{k} \backslash O_{s}
$$

The characteristic function of $S$ is an integral linear combination of the characteristic functions of these cells. Since the cells are pairwise disjoint, it follows that $S$ is a union of some of these cells. Each cell is a simple set by Corollary 2.14 and therefore $S$ is a simple set as well.

Proof of Theorem 2.12. Let $X, B, \mathcal{F}$, and $P=\left\{P_{1}, \ldots, P_{l}\right\}$ be as stated. Enlarging $\mathcal{F}$ by repeating some mappings and enlarging $B$ by repeating some elements does not affect the set $\mathcal{F}^{\left(n_{1}, \ldots, n_{l}\right)}(B)$. Therefore, we may assume that $|\mathcal{F}|=|B|=k$, $\mathcal{F}=\left\{f_{1}, \ldots, f_{k}\right\}$ and $B=\left\{b_{1}, \ldots, b_{k}\right\}$. We set $K=k^{2}$ and define a partial coloring

$$
\chi: \mathbb{N}_{0}^{K}=\mathbb{N}_{0}^{k^{2}} \rightarrow X \cup\{u\}
$$

as follows: the elements $x$ with $\chi(x)=u$ are regarded as "uncolored"; for $i \in[k]$ and $x \in \mathbb{N}_{0}^{K}$ such that $z_{1}:=x_{(i-1) k+1}, \ldots, z_{k}:=x_{(i-1) k+k}$ are positive but all other coordinates of $x$ are zero, we set

$$
\chi(x)=\left(f_{1}^{z_{1}-1} \circ \cdots \circ f_{k}^{z_{k}-1}\right)\left(b_{i}\right) .
$$

Note that if $z_{1}=\cdots=z_{k}=1$, then $\chi(x)=b_{i}$. We denote the set of all these points $x$ by $C_{i}$. The set of colored points is $C=C_{1} \cup \cdots \cup C_{k}$. The points in $\mathbb{N}_{0}^{K} \backslash C$ are uncolored. Each $C_{i}$ is a generalized orthant. If $x \in C_{i}$ and $x^{\prime} \in C_{j}$ for $i<j$, then $x$ and $x^{\prime}$ are incomparable by $\leq$ but $x^{\prime}<_{l e x} x$. For $x \in \mathbb{N}_{0}^{K}$ with all coordinates different from $(i-1) k+1, \ldots,(i-1) k+k$ equal to zero (e.g., if $x \in C_{i}$ ) and $j \in[k]$, we define $x(j)$ by shifting the $k$-term block of possibly nonzero coordinate values to the coordinates $(j-1) k+1, \ldots,(j-1) k+k$. The key property of $\chi$ is the following:

$$
\text { if } x, y \in C_{i}, x \leq y, x^{\prime} \in C_{j} \text {, and } \chi(x)=\chi\left(x^{\prime}\right) \text {, then } \chi(y)=\chi\left(x^{\prime}+(y-x)(j)\right)
$$

Indeed, if $\chi(x)=\chi\left(x^{\prime}\right)=c \in X$ and the coordinates $k(i-1)+1, \ldots, k(i-1)+k$ of $y-x$ are $z_{1}, \ldots, z_{k}$, then $\chi(y)=\chi(x+(y-x))=\left(f_{1}^{z_{1}} \circ \cdots \circ f_{k}^{z_{k}}\right)(c)=$ $\chi\left(x^{\prime}+(x-y)(j)\right)$.
$P$ induces naturally a partition of $[K]$ into $l$ blocks which we again denote $P=\left\{P_{1}, \ldots, P_{l}\right\}:$ for $f_{j} \in P_{r}$ we put in the $P_{r} \subset[K]$ all $k$ elements $j, j+k, j+$ $2 k, \ldots, j+(k-1) k$. Note that for $n_{1}, \ldots, n_{l} \in \mathbb{N}$ we have (recall the definition of $\|x\|_{P}$ before the proof of Corollary (2.3)

$$
\left|\chi\left(\left\{x \in \mathbb{N}_{0}^{K}:\|x\|_{P}=\left(n_{1}, \ldots, n_{l}\right)\right\}\right) \backslash\{u\}\right|=\left|\mathcal{F}^{\left(n_{1}-1, \ldots, n_{l}-1\right)}(B)\right|
$$

We call a point $x \in \mathbb{N}_{0}^{K} P$-substantial if it is colored and is the lexicographically minimum element in the set

$$
\left\{y \in \mathbb{N}_{0}^{K}: \chi(y)=\chi(x),\|y\|_{P}=\|x\|_{P}\right\}
$$

As before, $P$-substantial points are representatives of the nonempty intersections of the color classes of $\chi$ with the simplex $\|x\|_{P}=\left(n_{1}, \ldots, n_{l}\right)$. Thus

$$
\left|\mathcal{F}^{\left(n_{1}-1, \ldots, n_{l}-1\right)}(B)\right|=\left[y_{1}^{n_{1}} \ldots y_{1}^{n_{l}}\right] G
$$

where $G\left(y_{1}, \ldots, y_{l}\right)$ is obtained by the $P$-substitution from $F_{S}\left(x_{1}, \ldots, x_{K}\right)$ and $S$ is the set of all $P$-substantial points in $\mathbb{N}_{0}^{K}$. Now the theorem follows as before by Proposition 2.9, Lemma 1.7 and Theorem 2.15. provided that we show that $S$ is a simple set.

To prove that $S$ is simple we consider the complement $\mathbb{N}_{0}^{K} \backslash S$. We have that

$$
\mathbb{N}_{0}^{K} \backslash S=\left(\mathbb{N}_{0}^{K} \backslash C\right) \cup C^{*}
$$

where $C^{*}$ consists of all colored points that are not $P$-substantial. The set $\mathbb{N}_{0}^{K} \backslash C$ is simple by Corollary 2.14 because $C$ is simple (as a union of the generalized orthants $C_{i}$ ). Now $C^{*}=C_{1}^{*} \cup \cdots \cup C_{k}^{*}$ where $C_{i}^{*}=C^{*} \cap C_{i}$. We show that each $C_{i}^{*}$ is an upper ideal in $\left(C_{i}, \leq\right)$. Then, by Dickson's lemma, $C_{i}^{*}$ is a finite union of generalized orthants, which implies that $C_{i}^{*}$ and $C^{*}$ are simple. So $\mathbb{N}_{0}^{K} \backslash S$ is simple and $S$ is simple.

Thus suppose that $x \in C_{i}^{*}$ and $y \in C_{i}$ with $x \leq y$. It follows that there is a colored point $x^{\prime} \in \mathbb{N}_{0}^{K}$ with $\chi\left(x^{\prime}\right)=\chi(x),\left\|x^{\prime}\right\|_{P}=\|x\|_{P}$, and $x^{\prime}<_{\text {lex }} x$. Let $x^{\prime} \in C_{j}$. Consider the point $y^{\prime}=x^{\prime}+(y-x)(j)$. By the property of $\chi$ we have $\chi\left(y^{\prime}\right)=\chi(y)$. Since $\|y-x\|_{P}=\|(y-x)(j)\|_{P}$ (by the definition of $P$ ), we have $\left\|y^{\prime}\right\|_{P}=\left\|x^{\prime}+(y-x)(j)\right\|_{P}=\|x\|_{P}+\|(y-x)(j)\|_{P}=\|x\|_{P}+\|y-x\|_{P}=\|y\|_{P}$. If $i=j$, then $y-x=(y-x)(j)$ and $y^{\prime}=x^{\prime}+(y-x)<_{l e x} x+(y-x)=y$. If $i \neq j$, we must have $i<j$ because $x^{\prime} \in C_{j}, x \in C_{i}$, and $x^{\prime}<_{l e x} x$. But $y^{\prime} \in C_{j}$ and $y \in C_{i}$, so again $y^{\prime}<_{\text {lex }} y$. Thus $\chi\left(y^{\prime}\right)=\chi(y),\left\|y^{\prime}\right\|_{P}=\|y\|_{P}$, and $y^{\prime}<_{\text {lex }} y$, which shows that $y \in C_{i}^{*}$. We have shown that $C_{i}^{*}$ is an upper ideal in $\left(C_{i}, \leq\right)$, which concludes the proof.

## 3 Concluding remarks

In [7], we plan to look from general perspective at further polynomial and quasipolynomial classes of enumerative problems. A natural question, for example, is about the multivariate generalization of Theorem [2.1; generalization of Theorem 1.1 to several variables was considered by Beck [3, 4]. Theorem 2.1] is related in spirit to results of Lisoněk [12] who counts orbits of group actions on lattice points in polytopes. It would be interesting to have an explicit description of the structure of an additive coloring $\chi: \mathbb{N}_{0}^{k} \rightarrow X$ because one may consider further statistics of $\chi$ on the points $n P \cap \mathbb{Z}^{k}$, such as the number of occurrences of a specified color. We
plan to investigate polynomial classes arising from counting permutations (e.g., Albert, Atkinson and Brignall [1], Huczynska and Vatter [6], Kaiser and Klazar [8]), graphs (e.g., Balogh, Bollobás and Morris [2]), relational structures (e.g., Pouzet and Thiéry [18]), and perhaps other.

## References

[1] M. H. Albert, M. D. Atkinson and R. Brignall: Permutation classes of polynomial growth, arXiv:math.CO/0603315.
[2] J. Balogh, B. Bollobás and R. Morris: Hereditary properties of ordered graphs, in: M. Klazar, J. Kratochvíl, M. Loebl, J. Matoušek, R. Thomas and P. Valtr (editors): Topics in Discrete Mathematics. Dedicated to Jarik Nešetřil on the Occasion of his 60th Birthday, Springer, Berlin, 2006, 179-213.
[3] M. Beck: A closer look at lattice points in rational simplices, Electronic J. Combin. 6 (1999), article R37.
[4] M. Beck: Multidimensional Ehrhart reciprocity, J. Combin. Theory, Ser. A 97 (2002), 187-194.
[5] E. Ehrhart: Sur les polyèdres rationnels homothétiques à $n$ dimensions, $C$. R. Acad. Sci. Paris 254 (1962), 616-618.
[6] S. Huczynska and V. Vatter: Grid classes and the Fibonacci dichotomy for restricted permutations, Electron. J. Combin. 13 (2006), article R54, 14 pages.
[7] V. Jelínek and M. Klazar: Polynomial and quasipolynomial counting, in preparation.
[8] T. Kaiser and M. Klazar: On growth rates of closed permutation classes, Electron. J. Combin. 9(2) (2003), article R10, 20 pages.
[9] A. G. Khovanskii: Newton polyhedron, Hilbert polynomial, and sums of finite sets, Functional. Anal. Appl. 26 (1992), 276-281.
[10] A. G. Khovanskiǐ: Sums of finite sets, orbits of commutative semigroups, and Hilbert functions, Functional. Anal. Appl. 29 (1995), 102-112.
[11] J.B. Kruskal: The theory of well-quasi-ordering: A frequently discovered concept, J. Combinatorial Theory Ser. A 13 (1972), 297-305.
[12] P. Lisoněk: Combinatorial families enumerated by quasi-polynomials, J. Combin. Theory, Ser. A, 114 (2007), 619-630.
[13] I. G. Macdonald: The volume of a lattice polyhedron, Proc. Camb. Phil. Soc. 59 (1963), 719-726.
[14] I. G. Macdonald: Polynomials associated with finite cell complexes, J. London Math. Soc. (2) 4 (1971), 181-192.
[15] J. Matoušek: Lectures on Discrete Geometry, Springer, 2002.
[16] M. B. Nathanson: Growth of sumsets in abelian semigroups, Semigroup Forum 61 (2000), 149-153.
[17] M. B. Nathanson and I. Ruzsa: Polynomial growth of sumsets in abelian semigroups, J. Théor. Nombres Bordeaux 14 (2002), 553-560.
[18] M. Pouzet and N. M. Thiéry: Some relational structures with polynomial growth and their associated algebras, arXiv:math.CO/0601256.
[19] R. P. Stanley: Problem E2546, Amer. Math. Monthly 82 (1975), 756; solution ibidem 83 (1976), 813-814.
[20] R. P. Stanley: Enumerative Combinatorics. Volume 1, Cambridge University Press, 2002.


[^0]:    *Institute for Theoretical Computer Science and Department of Applied Mathematics, Faculty of Mathematics and Physics of Charles University, Malostranské náměstí 25, 11800 Praha, Czech Republic. ITI is supported by the project 1 M 0021620808 of the Czech Ministry of Education. Email: \{jelinek, klazar\}@kam.mff.cuni.cz

