The Cauchy Operator for
Basic Hypergeometric Series

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Abstract

We introduce the Cauchy augmentation operator for basic hypergeometric series. Heine's $_2\phi_1$ transformation formula and Sears' $_3\phi_2$ transformation formula can be easily obtained by the symmetric property of some parameters in operator identities. The Cauchy operator involves two parameters, and it can be considered as a generalization of the operator $T(bD_q)$. Using this operator, we obtain extensions of the Askey-Wilson integral, the Askey-Roy integral, Sears' two-term summation formula, as well as the q-analogues of Barnes' lemmas. Finally, we find that the Cauchy operator is also suitable for the study of the bivariate Rogers-Szegö polynomials, or the continuous big q-Hermite polynomials.

Keywords: q-difference operator, the Cauchy operator, the Askey-Wilson integral, the Askey-Roy integral, basic hypergeometric series, parameter augmentation.

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1   Introduction

In an attempt to find efficient q-shift operators to deal with basic hypergeometric series identities in the framework of the q-umbral calculus [2, 18], Chen and Liu [14, 15] introduced two q-exponential operators for deriving identities from their special cases. This method is called parameter augmentation. In this paper, we continue the study of parameter augmentation by defining a new operator called the Cauchy augmentation operator which is suitable for certain transformation and integral formulas.

Recall that Chen and Liu [14] introduced the augmentation operator

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q;q)_n} \tag{1.1}$$

as the basis of parameter augmentation which serves as a method for proving q-summation and integral formulas from special cases for which some parameters are set to zero.

The main idea of this paper is to introduce the Cauchy augmentation operator, or simply the Cauchy operator,

$$T(a, b; D_q) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} (bD_q)^n, \tag{1.2}$$

which is reminiscent of the Cauchy q-binomial theorem [17, Appendix II.3]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1. \tag{1.3}$$

For the same reason, the operator $T(aD_q)$ should be named the Euler operator in view of Euler's identity [17, Appendix II.1]

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \qquad |z| < 1. \tag{1.4}$$

Compared with $T(bD_q)$, the Cauchy operator (1.2) involves two parameters. Clearly, the operator $T(bD_q)$ can be considered as a special case of the Cauchy operator (1.2) for $a = 0$. In order to utilize the Cauchy operator to basic hypergeometric series, several operator identities are deduced in Section 2. As to the applications of the Cauchy operator, we show that many classical results on basic hypergeometric series easily fall into this framework. Heine's $_2\phi_1$ transformation formula [17, Appendix III.2] and Sears' $_3\phi_2$ transformation formula [17, Appendix III.9] can be easily obtained by the symmetric property of some parameters in two operator identities for the Cauchy operator.

In Section 3 and Section 4, we use the Cauchy operator to generalize the Askey-Wilson integral and the Askey-Roy integral. In [20], Ismail, Stanton, and Viennot derived an integral named the Ismail-Stanton-Viennot integral which took the Askey-Wilson integral as a special case. It is easy to see that our extension of the Askey-Wilson integral is also an extension of the Ismail-Stanton-Viennot integral. In [16], Gasper discovered an integral which was a generalization of the Askey-Roy integral. We observe that Gasper's formula is a special case of the formula obtained by applying the Cauchy operator directly to the Askey-Roy integral. Furthermore, we find that the Cauchy operator can be applied to Gasper's formula to derive a further extension of the Askey-Roy integral.

In Section 5, we present that the Cauchy operator is suitable for the study of bivariate Rogers-Szegö polynomials. It can be used to derive the corresponding Mehler's and the Rogers formulas for the bivariate Rogers-Szegö polynomials, which can be stated in the equivalent forms in terms of the continuous big q-Hermite polynomials. Mehler's formula in this case turns out to be a special case of the nonsymmetric Poisson kernel formula for the continuous big q-Hermite polynomials due to Askey, Rahman, and Suslov [6]. Finally, in Section 6 and Section 7, we employ the Cauchy operator to deduce extensions of Sears' two-term summation formula [17, Eq. (2.10.18)] and the q-analogues of Barnes' lemmas [17, Eqs. (4.4.3), (4.4.6)].

As usual, we follow the notation and terminology in [17]. For $|q| < 1$, the q-shifted factorial is defined by

$$(a;q)_\infty = \prod_{k=0}^\infty (1 - aq^k) \quad \text{and} \quad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \text{ for } n \in \mathbb{Z}.$$

For convenience, we shall adopt the following notation for multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n,$$

where n is an integer or infinity.

The q-binomial coefficients, or the Gauss coefficients, are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}. \tag{1.5}$$

The (unilateral) basic hypergeometric series $_r\phi_s$ is defined by

$$_r\phi_s \left[\begin{array}{cccc} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{array} ; q, z \right] = \sum_{k=0}^\infty \frac{(a_1, a_2, \ldots, a_r; q)_k}{(q, b_1, b_2, \ldots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} z^k. \tag{1.6}$$

2 Basic Properties

In this section, we give some basic identities involving the Cauchy operator $T(a, b; D_q)$ and demonstrate that Heine's $_2\phi_1$ transformation formula and Sears' $_3\phi_2$ transformation formula are implied in the symmetric property of some parameters in two operator identities.

We recall that the q-difference operator, or Euler derivative, is defined by

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}, \tag{2.1}$$

and the Leibniz rule for D_q is referred to the following identity

$$D_q^n\{f(a)g(a)\} = \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_q^k\{f(a)\} D_q^{n-k}\{g(aq^k)\}. \tag{2.2}$$

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The following relations are easily verified.

Proposition 2.1 *Let k be a nonnegative integer. Then we have*

$$D_q^k \left\{ \frac{1}{(at;q)_\infty} \right\} = \frac{t^k}{(at;q)_\infty},$$

$$D_q^k \left\{ (at;q)_\infty \right\} = (-t)^k q^{\binom{k}{2}} (atq^k;q)_\infty,$$

$$D_q^k \left\{ \frac{(av;q)_\infty}{(at;q)_\infty} \right\} = t^k (v/t;q)_k \frac{(avq^k;q)_\infty}{(at;q)_\infty}.$$

Now, we are ready to give some basic identities for the Cauchy operator $T(a,b;D_q)$. We assume that $T(a,b;D_q)$ acts on the parameter c. The following identity is an easy consequence of the Cauchy q-binomial theorem (1.3).

Theorem 2.2 *We have*

$$T(a,b;D_q) \left\{ \frac{1}{(ct;q)_\infty} \right\} = \frac{(abt;q)_\infty}{(bt,ct;q)_\infty}, \tag{2.3}$$

provided $|bt| < 1$.

Proof. By Proposition 2.1, the left hand side of (2.3) equals

$$\sum_{n=0}^\infty \frac{(a;q)_n b^n}{(q;q)_n} D_q^n \left\{ \frac{1}{(ct;q)_\infty} \right\} = \frac{1}{(ct;q)_\infty} \sum_{n=0}^\infty \frac{(a;q)_n (bt)^n}{(q;q)_n},$$

which simplifies to the right hand side of (2.3) by the Cauchy q-binomial theorem (1.3). ∎

Theorem 2.3 *We have*

$$T(a,b;D_q) \left\{ \frac{1}{(cs,ct;q)_\infty} \right\} = \frac{(abt;q)_\infty}{(bt,cs,ct;q)_\infty} \, {}_2\phi_1 \left[\begin{matrix} a, & ct \\ & abt \end{matrix} ; q, bs \right], \tag{2.4}$$

provided $\max\{|bs|, |bt|\} < 1$.

Proof. In view of the Leibniz formula for D_q^n, the left hand side of (2.4) can be

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expanded as follows

$$\sum_{n=0}^{\infty} \frac{(a;q)_n b^n}{(q;q)_n} \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_q^k \left\{ \frac{1}{(cs;q)_\infty} \right\} D_q^{n-k} \left\{ \frac{1}{(ctq^k;q)_\infty} \right\}$$

$$= \sum_{n=0}^{\infty} \frac{(a;q)_n b^n}{(q;q)_n} \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} \frac{s^k}{(cs;q)_\infty} \frac{(tq^k)^{n-k}}{(ctq^k;q)_\infty}$$

$$= \frac{1}{(cs,ct;q)_\infty} \sum_{k=0}^{\infty} \frac{(ct;q)_k (bs)^k}{(q;q)_k} \sum_{n=k}^{\infty} \frac{(a;q)_n (bt)^{n-k}}{(q;q)_{n-k}}$$

$$= \frac{1}{(cs,ct;q)_\infty} \sum_{k=0}^{\infty} \frac{(a,ct;q)_k (bs)^k}{(q;q)_k} \sum_{n=0}^{\infty} \frac{(aq^k;q)_n (bt)^n}{(q;q)_n}$$

$$= \frac{(abt;q)_\infty}{(bt,cs,ct;q)_\infty} {}_2\phi_1 \left[\begin{matrix} a, & ct \\ & abt \end{matrix} ; q, bs \right],$$

as desired. ∎

Notice that when $a = 0$, the ${}_2\phi_1$ series on the right hand side of (2.4) can be summed by employing the Cauchy q-binomial theorem (1.3). In this case (2.4) reduces to

$$T(bD_q) \left\{ \frac{1}{(cs,ct;q)_\infty} \right\} = \frac{(bcst;q)_\infty}{(bs,bt,cs,ct;q)_\infty}, \qquad |bs|, |bt| < 1, \qquad (2.5)$$

which was derived by Chen and Liu in [14].

As an immediate consequence of the above theorem, we see that Heine's ${}_2\phi_1$ transformation formula [17, Appendix III.2] is really about the symmetry in s and t while applying the operator $T(a,b;q)$.

Corollary 2.4 (Heine's transformation) *We have*

$$ {}_2\phi_1 \left[\begin{matrix} a, & b \\ & c \end{matrix} ; q, z \right] = \frac{(c/b, bz;q)_\infty}{(c,z;q)_\infty} {}_2\phi_1 \left[\begin{matrix} abz/c, & b \\ & bz \end{matrix} ; q, \frac{c}{b} \right], \qquad (2.6)$$

where $\max\{|z|, |c/b|\} < 1$.

Proof. The symmetry in s and t on the left hand side of (2.4) implies that

$$\frac{(abt;q)_\infty}{(bt,cs,ct;q)_\infty} {}_2\phi_1 \left[\begin{matrix} a, & ct \\ & abt \end{matrix} ; q, bs \right] = \frac{(abs;q)_\infty}{(bs,ct,cs;q)_\infty} {}_2\phi_1 \left[\begin{matrix} a, & cs \\ & abs \end{matrix} ; q, bt \right], \qquad (2.7)$$

where $\max\{|bs|, |bt|\} < 1$.

Replacing a, b, c, s, t by $b, a, a^2b/c, z/a, c/ab$ in (2.7), respectively, we may easily express the above identity in the form of (2.6). ∎

Remark 2.5 *A closer look at the proof of Theorem 2.3 reveals that the essence of Heine's transformation lies in the symmetry of f and g in Leibniz's formula (2.2).*

We should note that we must be cautious about the convergence conditions while utilizing the Cauchy operator. In general, it would be safe to apply the Cauchy operator if the resulting series is convergent. However, it is possible that from a convergent series one may obtain a divergent series after employing the Cauchy operator. For example, let us consider Corollary 2.4. The resulting series (2.7) can be obtained by applying the Cauchy operator $T(a, b; D_q)$ to $1/(cs, ct; q)_\infty$ which is convergent for all t. However, the resulting series on the left hand side of (2.7) is not convergent for $|t| > 1/|b|$.

Combining Theorem 2.2 and the Leibniz rule (2.2), we obtain the following identity which implies Theorem 2.3 by setting $v = 0$. Sears' $_3\phi_2$ transformation formula [17, Appendix III.9] is also a consequence of Theorem 2.6.

Theorem 2.6 *We have*

$$T(a, b; D_q) \left\{ \frac{(cv; q)_\infty}{(cs, ct; q)_\infty} \right\} = \frac{(abs, cv; q)_\infty}{(bs, cs, ct; q)_\infty} {}_3\phi_2 \left[\begin{matrix} a, & cs, & v/t \\ & abs, & cv \end{matrix} ; q, b\,t \right], \qquad (2.8)$$

provided $\max\{|bs|, |b\,t|\} < 1$.

Proof. In light of Leibniz's formula, the left hand side of (2.8) equals

$$\sum_{n=0}^\infty \frac{(a; q)_n b^n}{(q; q)_n} D_q^n \left\{ \frac{(cv; q)_\infty}{(cs, ct; q)_\infty} \right\}$$

$$= \sum_{n=0}^\infty \frac{(a; q)_n b^n}{(q; q)_n} \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_q^k \left\{ \frac{(cv; q)_\infty}{(ct; q)_\infty} \right\} D_q^{n-k} \left\{ \frac{1}{(csq^k; q)_\infty} \right\}$$

$$= \sum_{n=0}^\infty \frac{(a; q)_n b^n}{(q; q)_n} \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} \frac{t^k (v/t; q)_k (cvq^k; q)_\infty}{(ct; q)_\infty} D_q^{n-k} \left\{ \frac{1}{(csq^k; q)_\infty} \right\}$$

$$= \sum_{k=0}^\infty \frac{(v/t; q)_k (cvq^k; q)_\infty t^k}{(q; q)_k (ct; q)_\infty} \sum_{n=k}^\infty \frac{b^n q^{k(k-n)} (a; q)_n}{(q; q)_{n-k}} D_q^{n-k} \left\{ \frac{1}{(csq^k; q)_\infty} \right\}$$

$$= \sum_{k=0}^\infty \frac{(a, v/t; q)_k (cvq^k; q)_\infty (b\,t)^k}{(q; q)_k (ct; q)_\infty} \sum_{n=0}^\infty \frac{(bq^{-k})^n (aq^k; q)_n}{(q; q)_n} D_q^n \left\{ \frac{1}{(csq^k; q)_\infty} \right\}$$

$$= \sum_{k=0}^\infty \frac{(a, v/t; q)_k (cvq^k; q)_\infty (b\,t)^k}{(q; q)_k (ct; q)_\infty} T(aq^k, bq^{-k}; D_q) \left\{ \frac{1}{(csq^k; q)_\infty} \right\}.$$

By Theorem 2.2, the above sum equals

$$\sum_{k=0}^{\infty} \frac{(a, v/t; q)_k (cvq^k; q)_{\infty} (b\,t)^k}{(q; q)_k (ct; q)_{\infty}} \frac{(absq^k; q)_{\infty}}{(bs, csq^k; q)_{\infty}}$$

$$= \frac{(cv; q)_{\infty}}{(cs, ct; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(a, cs, v/t; q)_k (b\,t)^k}{(q, cv; q)_k} \frac{(absq^k; q)_{\infty}}{(bs; q)_{\infty}}$$

$$= \frac{(abs, cv; q)_{\infty}}{(bs, cs, ct; q)_{\infty}} {}_3\phi_2 \left[\begin{array}{ccc} a, & cs, & v/t \\ & abs, & cv \end{array} ; q, b\,t \right], \qquad (2.9)$$

as desired. ∎

Corollary 2.7 (Sears' transformation) *We have*

$$_3\phi_2 \left[\begin{array}{ccc} a, & b, & c \\ & d, & e \end{array} ; q, \frac{de}{abc} \right] = \frac{(e/a, de/bc; q)_{\infty}}{(e, de/abc; q)_{\infty}} {}_3\phi_2 \left[\begin{array}{ccc} a, & d/b, & d/c \\ & d, & de/bc \end{array} ; q, \frac{e}{a} \right], \qquad (2.10)$$

where $\max\{|de/abc|, |e/a|\} < 1$.

Proof. Based on the symmetric property of the parameters s and t on the left hand side of (2.8), we find that

$$\frac{(abs, cv; q)_{\infty}}{(bs, cs, ct; q)_{\infty}} {}_3\phi_2 \left[\begin{array}{ccc} a, & cs, & v/t \\ & abs, & cv \end{array} ; q, b\,t \right] = \frac{(abt, cv; q)_{\infty}}{(bt, ct, cs; q)_{\infty}} {}_3\phi_2 \left[\begin{array}{ccc} a, & ct, & v/s \\ & abt, & cv \end{array} ; q, bs \right],$$

where $\max\{|bs|, |b\,t|\} < 1$.

Making the substitutions $c \to ab^2/e$, $v \to de/ab^2$, $s \to e/ab$, and $t \to de/ab^2c$, we get the desired formula. ∎

We see that the essence of Sears' transformation also lies in the symmetry of s and t in the application of Leibniz rule.

3 An Extension of the Askey-Wilson Integral

The Askey-Wilson integral [8] is a significant extension of the beta integral. Chen and Liu [14] presented a treatment of the Askey-Wilson integral via parameter augmentation. They first got the usual Askey-Wilson integral with one parameter by the orthogonality relation obtained from the Cauchy q-binomial theorem (1.3) and the Jacobi triple product identity [17, Appendix II.28], and then they applied the operator $T(bD_q)$ three times to deduce the Askey-Wilson integral involving four parameters [5, 19–21, 23, 28]

$$\int_0^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}; q)_{\infty} d\theta}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_{\infty}}$$

$$= \frac{2\pi (abcd; q)_{\infty}}{(q, ab, ac, ad, bc, bd, cd; q)_{\infty}}, \qquad (3.1)$$

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where $\max\{|a|, |b|, |c|, |d|\} < 1$.

In this section, we derive an extension of the Askey-Wilson integral (3.1) which contains the following Ismail-Stanton-Viennot's integral [20] as a special case:

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}, ge^{i\theta}, ge^{-i\theta}; q)_\infty}$$

$$= \frac{2\pi(abcg, abcd; q)_\infty}{(q, ab, ac, ad, ag, bc, bd, bg, cd, cg; q)_\infty} {}_3\phi_2 \left[\begin{matrix} ab, & ac, & bc \\ & abcg, & abcd \end{matrix} ; q, dg \right], \quad (3.2)$$

where $\max\{|a|, |b|, |c|, |d|, |g|\} < 1$.

Theorem 3.1 (Extension of the Askey-Wilson integral) *We have*

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}, fge^{i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}, ge^{i\theta}; q)_\infty}$$

$$\times {}_3\phi_2 \left[\begin{matrix} f, & ae^{i\theta}, & be^{i\theta} \\ & fge^{i\theta}, & ab \end{matrix} ; q, ge^{-i\theta} \right] d\theta$$

$$= \frac{2\pi(cfg, abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd, cg; q)_\infty} {}_3\phi_2 \left[\begin{matrix} f, & ac, & bc \\ & cfg, & abcd \end{matrix} ; q, dg \right], \quad (3.3)$$

where $\max\{|a|, |b|, |c|, |d|, |g|\} < 1$.

Proof. The Askey-Wilson integral (3.1) can be written as

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_\infty} \frac{(ab; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}; q)_\infty} d\theta$$

$$= \frac{2\pi}{(q, bc, bd, cd; q)_\infty} \frac{(abcd; q)_\infty}{(ac, ad; q)_\infty}. \quad (3.4)$$

Before applying the Cauchy operator to an integral, it is necessary to show that the Cauchy operator commutes with the integral. This fact is implicit in the literature. Since this commutation relation depends on some technical conditions in connection with the integrands, here we present a complete proof.

First, it can be easily verified that the q-difference operator D_q commutes with the integral. By the definition of D_q (2.1), it is clear that

$$D_q \left\{ \int_C f(\theta, a) d\theta \right\} = \int_C D_q \{f(\theta, a)\} d\theta. \quad (3.5)$$

Consequently, the operator D_q^n commutes with the integral. Given a Cauchy operator $T(f, g; D_q)$, we proceed to prove that it commutes with the integral. From the well-known fact that, for a sequence of continuous functions $u_n(\theta)$ on a curve C, the sum commutes with the integral in

$$\sum_{n=0}^\infty \int_C u_n(\theta) d\theta$$

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provided that $\sum_{n=0}^{\infty} u_n(\theta)$ is uniformly convergent. It is sufficient to check the convergence condition for the continuity is obvious. This can be done with the aid of the Weierstrass M-Test [4]. Using the Cauchy operator $T(f, g; D_q)$ to the left hand side of (3.4), we find that

$$T(f, g; D_q) \left\{ \int_0^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}; q)_{\infty}}{(be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_{\infty}} \frac{(ab; q)_{\infty}}{(ae^{i\theta}, ae^{-i\theta}; q)_{\infty}} d\theta \right\} \qquad (3.6)$$

$$= \sum_{n=0}^{\infty} \frac{(f; q)_n}{(q; q)_n} (gD_q)^n \int_0^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}; q)_{\infty}}{(be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_{\infty}} \frac{(ab; q)_{\infty}}{(ae^{i\theta}, ae^{-i\theta}; q)_{\infty}} d\theta$$

$$= \sum_{n=0}^{\infty} \int_0^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}; q)_{\infty}}{(be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_{\infty}} \frac{(f; q)_n g^n}{(q; q)_n} D_q^n \left\{ \frac{(ab; q)_{\infty}}{(ae^{i\theta}, ae^{-i\theta}; q)_{\infty}} \right\} d\theta.$$

Let $U_n(\theta)$ denote the integrand in the last line of the above equation. We make the assumption $0 < q < 1$ so that, for $0 \le \theta \le \pi$,

$$|(|x|; q)_{\infty}| \le |(xe^{\pm i\theta}; q)_{\infty}| \le (-|x|; q)_{\infty} \qquad (3.7)$$

and

$$|(e^{\pm 2i\theta}; q)_{\infty}| \le (-1; q)_{\infty}. \qquad (3.8)$$

Now we rewrite the series $\sum_{n=0}^{\infty} U_n(\theta)$ into another form $\sum_{n=0}^{\infty} V_n(\theta)$ in order to prove its uniform convergence. In the proof of Theorem 2.6, one sees that the absolute convergence of the $_3\phi_2$ series under the condition $|bs|, |bt| < 1$ implies the absolute convergence of the sum

$$\sum_{n=0}^{\infty} \frac{(a; q)_n b^n}{(q; q)_n} D_q^n \left\{ \frac{(cv; q)_{\infty}}{(cs, ct; q)_{\infty}} \right\}.$$

Therefore, under the condition $|g| < 1$, it follows from Theorem 2.6 that

$$\sum_{n=0}^{\infty} \frac{(f; q)_n g^n}{(q; q)_n} D_q^n \left\{ \frac{(ab; q)_{\infty}}{(ae^{i\theta}, ae^{-i\theta}; q)_{\infty}} \right\}$$

$$= \frac{(fge^{i\theta}, ab; q)_{\infty}}{(ge^{i\theta}, ae^{i\theta}, ae^{-i\theta}; q)_{\infty}} {}_3\phi_2 \left[\begin{array}{ccc} f, & ae^{i\theta}, & be^{i\theta} \\ & fge^{i\theta}, & ab \end{array} ; q, ge^{-i\theta} \right]. \qquad (3.9)$$

Hence

$$\sum_{n=0}^{\infty} U_n(\theta) = \frac{(e^{2i\theta}, e^{-2i\theta}, fge^{i\theta}, ab; q)_{\infty}}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}, ge^{i\theta}; q)_{\infty}}$$

$$\times {}_3\phi_2 \left[\begin{array}{ccc} f, & ae^{i\theta}, & be^{i\theta} \\ & fge^{i\theta}, & ab \end{array} ; q, ge^{-i\theta} \right]$$

$$= \frac{(e^{2i\theta}, e^{-2i\theta}, fge^{i\theta}, ab; q)_{\infty}}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}, ge^{i\theta}; q)_{\infty}}$$

$$\times \sum_{n=0}^{\infty} \frac{(f, ae^{i\theta}, be^{i\theta}; q)_n}{(q, fge^{i\theta}, ab; q)_n} \left(ge^{-i\theta} \right)^n. \qquad (3.10)$$

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Now, let

$$V_n(\theta) \;=\; \frac{(e^{2i\theta}, e^{-2i\theta}, fge^{i\theta}, ab; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}, ge^{i\theta}; q)_\infty}$$

$$\times \frac{(f, ae^{i\theta}, be^{i\theta}; q)_n}{(q, fge^{i\theta}, ab; q)_n} \left(ge^{-i\theta}\right)^n. \tag{3.11}$$

By the Weierstrass M-Test, it remains to find a convergent series $\sum_{n=0}^{\infty} M_n$, where M_n is independent of θ, such that $|V_n(\theta)| \le M_n$. For $\max\{|a|, |b|, |c|, |d|, |g|\} < 1$, we may choose

$$M_n = \left(\frac{(-1; q)_\infty}{(|a|, |b|, |c|, |d|; q)_\infty}\right)^2 \frac{(-|fg|, ab; q)_\infty}{(|g|; q)_\infty} \frac{(-|f|, -|a|, -|b|; q)_n |g|^n}{|(q, |fg|, ab; q)_n|}. \tag{3.12}$$

It is easy to see that $\sum_{n=0}^{\infty} M_n$ is convergent when $|g| < 1$. It follows that the Cauchy operator commutes with the integral in (3.6), so (3.6) can be written as

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_\infty} \sum_{n=0}^{\infty} \frac{(f; q)_n g^n}{(q; q)_n} D_q^n \left\{ \frac{(ab; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}; q)_\infty} \right\} d\theta$$

$$= \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_\infty} T(f, g; D_q) \left\{ \frac{(ab; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}; q)_\infty} \right\} d\theta.$$

Finally, we may come to the general condition $|q| < 1$ by the argument of analytic continuation. Hence, under the condition $\max\{|a|, |b|, |c|, |d|, |g|\} < 1$, we have shown that it is valid to exchange the Cauchy operator and the integral when we apply the Cauchy operator to (3.4).

Now, applying $T(f, g; D_q)$ to (3.4) with respect to the parameter a gives

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_\infty} \frac{(fge^{i\theta}, ab; q)_\infty}{(ge^{i\theta}, ae^{i\theta}, ae^{-i\theta}; q)_\infty}$$

$$\times {}_3\phi_2 \left[\begin{array}{ccc} f, & ae^{i\theta}, & be^{i\theta} \\ & fge^{i\theta}, & ab \end{array} ; q, ge^{-i\theta} \right] d\theta$$

$$= \frac{2\pi}{(q, bc, bd, cd; q)_\infty} \frac{(cfg, abcd; q)_\infty}{(cg, ac, ad; q)_\infty} {}_3\phi_2 \left[\begin{array}{ccc} f, & ac, & bc \\ & cfg, & abcd \end{array} ; q, dg \right], \tag{3.13}$$

where $\max\{|a|, |b|, |c|, |d|, |g|\} < 1$. This implies the desired formula. The proof is completed. ∎

In fact, the above proof also implies the convergence of the integral in Theorem 3.1. Once it has been shown that the sum commutes with the integral, one sees that the integral obtained from exchanging the sum and the integral is convergent.

Setting $f = ab$ in (3.3), by the q-Gauss sum [17, Appendix II.8]:

$$_2\phi_1 \left[\begin{array}{cc} a, & b \\ & c \end{array} ; q, \frac{c}{ab} \right] = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \quad |c/ab| < 1, \tag{3.14}$$

we arrive at the Ismail-Stanton-Viennot integral (3.2).

Setting $f = abcd$ in (3.3), by means of the q-Gauss sum (3.14) we find the following formula which we have not seen in the literature.

Corollary 3.2 *We have*

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}, abcdge^{i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}, ge^{i\theta}; q)_\infty}$$

$$\times {}_3\phi_2 \left[\begin{array}{ccc} abcd, & ae^{i\theta}, & be^{i\theta} \\ & abcdge^{i\theta}, & ab \end{array} ; q, ge^{-i\theta} \right] d\theta$$

$$= \frac{2\pi(abcd, acdg, bcdg; q)_\infty}{(q, ab, ac, ad, bc, bd, cd, cg, dg; q)_\infty}, \tag{3.15}$$

where $\max\{|a|, |b|, |c|, |d|, |g|\} < 1$.

4   A Further Extension of the Askey-Roy Integral

Askey and Roy [7] used Ramanujan's $_1\psi_1$ summation formula [17, Appendix II.29] to derive the following integral formula:

$$\frac{1}{2\pi} \int_{-\pi}^\pi \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho; q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}; q)_\infty} d\theta$$

$$= \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho; q)_\infty}{(q, ac, ad, bc, bd; q)_\infty}, \tag{4.1}$$

where $\max\{|a|, |b|, |c|, |d|\} < 1$ and $cd\rho \neq 0$, which is called the Askey-Roy integral.

In [16], Gasper discovered an integral formula

$$\frac{1}{2\pi} \int_{-\pi}^\pi \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdfe^{i\theta}; q)_\infty}{(ae^{i\theta}, be^{i\theta}, fe^{i\theta}, ce^{-i\theta}, de^{-i\theta}; q)_\infty} d\theta$$

$$= \frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, bcdf, acdf; q)_\infty}{(q, ac, ad, bc, bd, cf, df; q)_\infty}, \tag{4.2}$$

provided $\max\{|a|, |b|, |c|, |d|, |f|\} < 1$ and $cd\rho \neq 0$, which is an extension of the Askey-Roy integral. Note that Rahman and Suslov [24] found a proof of Gasper's formula

(4.2) based on the technique of iteration with respect to the parameters of $\rho(s)$ in the integral

$$\int_C \rho(s)q^{-s}ds,$$

where $\rho(s)$ is the solution of a Pearson-type first-order difference equation.

In this section, we first derive an extension of the Askey-Roy integral by applying the Cauchy operator. We see that Gasper's formula (4.2) is a special case of this extension (4.4). Moreover, a further extension of the Askey-Roy integral can be obtained by taking the action of the Cauchy operator on Gasper's formula.

Theorem 4.1 *We have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdfe^{i\theta}, ghe^{i\theta};q)_\infty}{(ae^{i\theta}, be^{i\theta}, fe^{i\theta}, he^{i\theta}, ce^{-i\theta}, de^{-i\theta};q)_\infty}$$

$$\times {}_3\phi_2\left[\begin{array}{ccc} g, & ae^{i\theta}, & fe^{i\theta} \\ & ghe^{i\theta}, & abcdfe^{i\theta} \end{array};q, bcdh\right]d\theta$$

$$=\frac{(abcd, \rho c/d, dq/\rho c, \rho, q/\rho, bcdf, acdf, cgh;q)_\infty}{(q, ac, ad, bc, bd, cf, ch, df;q)_\infty}$$

$$\times {}_3\phi_2\left[\begin{array}{ccc} g, & ac, & cf \\ & cgh, & acdf \end{array};q, dh\right], \tag{4.3}$$

where $\max\{|a|, |b|, |c|, |d|, |f|, |h|\} < 1$ *and* $cd\rho \neq 0$.

Proof. As in the proof of the extension of the Askey-Wilson integral, we can show that the Cauchy operator also commutes with the Aksey-Roy integral. So we may apply the Cauchy operator $T(f, g; D_q)$ to both sides of the Askey-Roy integral (4.1) with respect to the parameter a. It follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, fge^{i\theta};q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}, ge^{i\theta};q)_\infty}d\theta$$

$$=\frac{(abcd, cfg, \rho c/d, dq/\rho c, \rho, q/\rho;q)_\infty}{(q, ac, ad, bc, bd, cg;q)_\infty}{}_3\phi_2\left[\begin{array}{ccc} f, & ac, & bc \\ & cfg, & abcd \end{array};q, dg\right], \tag{4.4}$$

where $\max\{|a|, |b|, |c|, |d|, |g|\} < 1$ and $cd\rho \neq 0$.

Putting $f = abcd$ and $g = f$ in (4.4), by the q-Gauss sum (3.14), we get the formula (4.2) due to Gasper.

In order to apply the Cauchy operator to Gasper's formula (4.2), we rewrite it as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho;q)_\infty}{(be^{i\theta}, fe^{i\theta}, ce^{-i\theta}, de^{-i\theta};q)_\infty}\frac{(abcdfe^{i\theta};q)_\infty}{(ae^{i\theta}, abcd;q)_\infty}d\theta$$

$$=\frac{(\rho c/d, dq/\rho c, \rho, q/\rho, bcdf;q)_\infty}{(q, bc, bd, cf, df;q)_\infty}\frac{(acdf;q)_\infty}{(ac, ad;q)_\infty}. \tag{4.5}$$

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The proof is thus completed by employing the operator $T(g, h; D_q)$ with respect to the parameter a to the above identity. ∎

Replacing a, g by g, $cdfg$, respectively, and then taking $h = a$ in (4.3), we are led to the following identity due to Zhang and Wang [29].

Corollary 4.2 *We have*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\rho e^{i\theta}/d, qde^{-i\theta}/\rho, \rho ce^{-i\theta}, qe^{i\theta}/c\rho, abcdfge^{i\theta}, bcdfge^{i\theta}; q)_\infty}{(ae^{i\theta}, be^{i\theta}, fe^{i\theta}, ge^{i\theta}, ce^{-i\theta}, de^{-i\theta}; q)_\infty}$$

$$\times {}_3\phi_2 \left[\begin{matrix} fe^{i\theta}, & ge^{i\theta}, & gcdf \\ & acdfge^{i\theta}, & bcdfge^{i\theta} \end{matrix} ; q, abcd \right] d\theta$$

$$= \frac{(\rho c/d, dq/\rho c, \rho, q/\rho, acdf, acdg, bcdf, bcdg, cdfg; q)_\infty}{(q, ac, ad, bc, bd, cf, df, cg, dg; q)_\infty}, \tag{4.6}$$

where $\max\{|a|, |b|, |c|, |d|, |f|, |g|\} < 1$ *and* $cd\rho \neq 0$.

5   The Bivariate Rogers-Szegö Polynomials

In this section, we show that Mehler's formula and the Rogers formula for the bivariate Rogers-Szegö polynomials can be easily derived from the application of the Cauchy operator. The bivariate Rogers-Szegö polynomials are closely related to the continuous big q-Hermite polynomials. However, it seems that the following form of the bivariate Rogers-Szegö polynomials are introduced by Chen, Fu and Zhang [12], as defined by

$$h_n(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x, y), \tag{5.1}$$

where the Cauchy polynomials are given by

$$P_k(x, y) = x^k(y/x; q)_k = (x - y)(x - qy) \cdots (x - q^{n-1}y),$$

which naturally arise in the q-umbral calculus. Setting $y = 0$, the polynomials $h_n(x, y|q)$ reduce to the classical Rogers-Szegö polynomials $h_n(x|q)$ defined by

$$h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k. \tag{5.2}$$

It should be noted that Mehler's formula for the bivariate Rogers-Szegö polynomials is due to Askey, Rahman, and Suslov [6, Eq. (14.14)]. They obtained the nonsymmetric Poisson kernel formula for the continuous big q-Hermite polynomials, often denoted by $H_n(x; a|q)$. The formula of Askey, Rahman, and Suslov can be easily formulated in

terms of $h_n(x, y|q)$. Recently, Chen, Saad, and Sun presented an approach to Mehler's formula and the Rogers formula for $h_n(x, y|q)$ by using the homogeneous difference operator D_{xy} introduced by Chen, Fu, and Zhang. As will be seen, the Cauchy operator turns out to be more efficient compared with the techniques used in [13].

We recall that the generating function of the bivariate Rogers-Szegö polynomials

$$\sum_{n=0}^{\infty} h_n(x, y|q) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(t, xt; q)_\infty},\tag{5.3}$$

where $\max\{|x|, |xt| < 1\}$, can be derived from the Euler identity (1.4) using the Cauchy operator.

A direct calculation shows that

$$D_q^k \{a^n\} = \begin{cases} a^{n-k}(q^{n-k+1}; q)_k, & 0 \le k \le n, \\ 0, & k > n. \end{cases}\tag{5.4}$$

From the identity (5.4), we can easily establish the following lemma.

Lemma 5.1 *We have*

$$T(a, b; D_q) \{c^n\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a; q)_k b^k c^{n-k}.\tag{5.5}$$

Applying $T(a, b; D_q)$ to the Euler identity (1.4) with respect to the parameter z, we get

$$\sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a; q)_k \left(\frac{b}{z}\right)^k = \frac{(ab; q)_\infty}{(b, z; q)_\infty},\tag{5.6}$$

which leads to (5.3) by suitable substitutions.

The reason that we employ the Cauchy operator to deal with the bivariate Rogers-Szegö polynomials is based on the following fact

$$h_n(x, y|q) = \lim_{c \to 1} T(y/x, x; D_q) \{c^n\}.\tag{5.7}$$

We are ready to describe how one can employ the Cauchy operator to derive Mehler's formula and the Rogers formula for $h_n(x, y|q)$.

Theorem 5.2 (Mehler's formula for $h_n(x, y|q)$) *We have*

$$\sum_{n=0}^{\infty} h_n(x, y|q) h_n(u, v|q) \frac{t^n}{(q; q)_n} = \frac{(ty, tv; q)_\infty}{(t, tu, tx; q)_\infty} {}_3\phi_2 \begin{bmatrix} t, & y/x, & v/u \\ & ty, & tv \end{bmatrix} ; q, tux \end{bmatrix},\tag{5.8}$$

where $\max\{|t|, |tu|, |tx|, |tux|\} < 1$.

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Proof. By Lemma 5.1, the left hand side of (5.8) can be written as

$$\sum_{n=0}^{\infty} h_n(x, y|q) \lim_{c \to 1} T(v/u, u; D_q) \{c^n\} \frac{t^n}{(q; q)_n}$$

$$= \lim_{c \to 1} T(v/u, u; D_q) \left\{ \sum_{n=0}^{\infty} h_n(x, y|q) \frac{(ct)^n}{(q; q)_n} \right\}.$$

In view of the generating function (5.3), the above sum equals

$$\lim_{c \to 1} T(v/u, u; D_q) \left\{ \frac{(cty; q)_\infty}{(ct, ctx; q)_\infty} \right\}$$

$$= \lim_{c \to 1} \left(\frac{(tv, cty; q)_\infty}{(tu, ct, ctx; q)_\infty} {}_3\phi_2 \left[\begin{array}{ccc} v/u, & ct, & y/x \\ & tv, & cty \end{array} ; q, tux \right] \right)$$

$$= \frac{(ty, tv; q)_\infty}{(t, tu, tx; q)_\infty} {}_3\phi_2 \left[\begin{array}{ccc} t, & y/x, & v/u \\ & ty, & tv \end{array} ; q, tux \right], \tag{5.9}$$

where $\max\{|t|, |tu|, |xt|, |tux|\} < 1$. This completes the proof. ∎

We see that (5.8) is equivalent to [13, Eq. (2.1)] in terms of Sears' transformation formula (2.10). Setting $y = 0$ and $v = 0$ in (5.8) and employing the Cauchy q-binomial theorem (1.3), we obtain Mehler's formula [14, 19, 25, 27] for the Rogers-Szegö polynomials.

Corollary 5.3 *We have*

$$\sum_{n=0}^{\infty} h_n(x|q) h_n(u|q) \frac{t^n}{(q; q)_n} = \frac{(t^2 ux; q)_\infty}{(t, tu, tx, tux; q)_\infty}, \tag{5.10}$$

where $\max\{|t|, |tu|, |tx|, |tux|\} < 1$.

Theorem 5.4 (The Rogers formula for $h_n(x, y|q)$) *We have*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{m+n}(x, y|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \frac{(ty; q)_\infty}{(s, t, tx; q)_\infty} {}_2\phi_1 \left[\begin{array}{cc} t, & y/x \\ & ty \end{array} ; q, sx \right], \tag{5.11}$$

where $\max\{|s|, |t|, |sx|, |tx|\} < 1$.

Proof. Using Lemma 5.1, the left hand side of (5.11) equals

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \lim_{c \to 1} T(y/x, x; D_q) \{c^{m+n}\} \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m}$$

$$= \lim_{c \to 1} T(y/x, x; D_q) \left\{ \sum_{n=0}^{\infty} \frac{(ct)^n}{(q; q)_n} \sum_{m=0}^{\infty} \frac{(cs)^m}{(q; q)_m} \right\} \tag{5.12}$$

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$$= \lim_{c \to 1} T(y/x, x; D_q) \left\{ \frac{1}{(cs, ct; q)_\infty} \right\}$$

$$= \frac{(ty; q)_\infty}{(s, t, tx; q)_\infty} {}_2\phi_1 \left[\begin{matrix} t, & y/x \\ & ty \end{matrix} ; q, sx \right], \tag{5.13}$$

where $\max\{|s|, |t|, |sx|, |tx|\} < 1$. ∎

Note that (5.11) is equivalent to [13, Eq. (3.1)] in terms of Heine's transformation formula [17, Appendix III.1]. Setting $y = 0$ in (5.11), by the Cauchy q-binomial theorem (1.3) we get the Rogers formula [14, 25, 26] for the Rogers-Szegö polynomials.

Corollary 5.5 *We have*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{m+n}(x|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \frac{(stx; q)_\infty}{(s, sx, t, tx; q)_\infty}, \tag{5.14}$$

where $\max\{|s|, |t|, |sx|, |tx|\} < 1$.

6   An Extension of Sears' Formula

In this section, we give an extension of the Sears two-term summation formula [17, Eq. (2.10.18)]:

$$\int_c^d \frac{(qt/c, qt/d, abcdet; q)_\infty}{(at, bt, et; q)_\infty} d_q t$$

$$= \frac{d(1 - q)(q, dq/c, c/d, abcd, bcde, acde; q)_\infty}{(ac, ad, bc, bd, ce, de; q)_\infty}, \tag{6.1}$$

where $\max\{|ce|, |de|\} < 1$.

From the Cauchy operator, we deduce the following extension of (6.1).

Theorem 6.1 *We have*

$$\int_c^d \frac{(qt/c, qt/d, abcdet, fgt; q)_\infty}{(at, bt, et, gt; q)_\infty} {}_3\phi_2 \left[\begin{matrix} f, & at, & et \\ & fgt, & abcdet \end{matrix} ; q, bcdg \right] d_q t$$

$$= \frac{d(1 - q)(q, dq/c, c/d, abcd, bcde, acde, cfg; q)_\infty}{(ac, ad, bc, bd, ce, cg, de; q)_\infty}$$

$$\times {}_3\phi_2 \left[\begin{matrix} f, & ac, & ce \\ & cfg, & acde \end{matrix} ; q, dg \right], \tag{6.2}$$

where $\max\{|bcdg|, |ce|, |cg|, |de|, |dg|\} < 1$.

Proof. We may rewrite (6.1) as

$$\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(bt, et; q)_\infty} \frac{(abcdet; q)_\infty}{(at, abcd; q)_\infty} \mathrm{d}_q t$$

$$= \frac{d(1-q)(q, dq/c, c/d, bcde; q)_\infty}{(bc, bd, ce, de; q)_\infty} \frac{(acde; q)_\infty}{(ac, ad; q)_\infty}. \tag{6.3}$$

Applying the operator $T(f, g; D_q)$ with respect to the parameter a, we obtain (6.2). ∎

As far as the convergence is concerned, the above integral is of the following form

$$\sum_{n=0}^\infty A(n) \sum_{k=0}^\infty B(n, k). \tag{6.4}$$

To ensure that the series (6.4) converges absolutely, we assume that the following two conditions are satisfied:

1. $\sum_{k=0}^\infty B(n, k)$ converges to $C(n)$, and $C(n)$ has a nonzero limit as $n \to \infty$.

2. $\lim\limits_{n\to\infty} |\frac{A(n)}{A(n-1)}| < 1$.

It is easy to see that under the above assumptions, (6.4) converges absolutely, since

$$\lim_{n\to\infty} \left| \frac{A(n)C(n)}{A(n-1)C(n-1)} \right| = \lim_{n\to\infty} \left| \frac{A(n)}{A(n-1)} \right| < 1.$$

It is easy to verify the double summations in (6.2) satisfy the two assumptions of (6.4), so the convergence is guaranteed.

7 Extensions of q-Barnes' Lemmas

In this section, we obtain extensions of the q-analogues of Barnes' lemmas. Barnes' first lemma [9] is an integral analogue of Gauss' $_2F_1$ summation formula. Askey and Roy [7] pointed out that Barnes' first lemma is also an extension of the beta integral. Meanwhile, Barnes' second lemma [10] is an integral analogue of Saalschütz's formula.

The following q-analogue of Barnes' first lemma is due to Watson, see [17, Eq. (4.4.3)]:

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(q^{1-c+s}, q^{1-d+s}; q)_\infty}{(q^{a+s}, q^{b+s}; q)_\infty} \frac{\pi q^s ds}{\sin \pi(c-s) \sin \pi(d-s)}$$

$$= \frac{q^c}{\sin \pi(c-d)} \frac{(q, q^{1+c-d}, q^{d-c}, q^{a+b+c+d}; q)_\infty}{(q^{a+c}, q^{a+d}, q^{b+c}, q^{b+d}; q)_\infty}. \tag{7.1}$$

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The q-analogue of Barnes' second lemma is due to Agarwal, see [1] and [17, Eq. (4.4.6)]:

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{(q^{1+s},q^{d+s},q^{1+a+b+c+s-d};q)_\infty}{(q^{a+s},q^{b+s},q^{c+s};q)_\infty}\,\frac{\pi q^s ds}{\sin\pi s\sin\pi(d+s)}$$

$$=\csc\pi d\,\frac{(q,q^d,q^{1-d},q^{1+b+c-d},q^{1+a+c-d},q^{1+a+b-d};q)_\infty}{(q^a,q^b,q^c,q^{1+a-d},q^{1+b-d},q^{1+c-d};q)_\infty},\qquad(7.2)$$

where $\mathrm{Re}\{s\log q-\log(\sin\pi s\sin\pi(d+s))\}<0$ for large $|s|$. Throughout this section, the contour of integration always ranges from $-i\infty$ to $i\infty$ so that the increasing sequences of poles of integrand lie to the right and the decreasing sequences of poles lie to the left of the contour, see [17, p. 119]. In order to ensure that the Cauchy operator commutes with the integral, we assume that $q=e^{-\omega}$, $\omega>0$.

We obtain the following extension of Watson's q-analogue of Barnes' first lemma.

Theorem 7.1 *We have*

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{(q^{1-c+s},q^{1-d+s},q^{e+f+s};q)_\infty}{(q^{a+s},q^{b+s},q^{f+s};q)_\infty}\,\frac{\pi q^s ds}{\sin\pi(c-s)\sin\pi(d-s)}$$

$$=\frac{q^c}{\sin\pi(c-d)}\,\frac{(q,q^{1+c-d},q^{d-c},q^{a+b+c+d},q^{c+e+f};q)_\infty}{(q^{a+c},q^{a+d},q^{b+c},q^{b+d},q^{c+f};q)_\infty}$$

$$\times{}_3\phi_2\left[\begin{matrix}q^e,&q^{a+c},&q^{b+c}\\&q^{c+e+f},&q^{a+b+c+d}\end{matrix};q,q^{d+f}\right],\qquad(7.3)$$

where $\max\{|q^f|,|q^{c+f}|,|q^{d+f}|\}<1$.

Proof. Applying the operator $T(q^e,q^f;D_q)$ to (7.1) with respect to the parameter q^a, we arrive at (7.3). ∎

Let us consider the special case when $e=a+b+c+d$. The ${}_3\phi_2$ sum on the right hand side of (7.3) turns out to be a ${}_2\phi_1$ sum and can be summed by the q-Gauss formula (3.14). Hence we get the following formula derived by Liu [22], which is also an extension of q-Barnes' first Lemma.

Corollary 7.2 *We have*

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{(q^{1-c+s},q^{1-d+s},q^{a+b+c+d+f+s};q)_\infty}{(q^{a+s},q^{b+s},q^{f+s};q)_\infty}\,\frac{\pi q^s ds}{\sin\pi(c-s)\sin\pi(d-s)}$$

$$=\frac{q^c}{\sin\pi(c-d)}\,\frac{(q,q^{1+c-d},q^{d-c},q^{a+b+c+d},q^{a+c+d+f},q^{b+c+d+f};q)_\infty}{(q^{a+c},q^{a+d},q^{b+c},q^{b+d},q^{c+f},q^{d+f};q)_\infty},\qquad(7.4)$$

where $\max\{|q^f|,|q^{c+f}|,|q^{d+f}|\}<1$.

Clearly, (7.4) becomes q-Barnes' first Lemma (7.1) for $f \to \infty$. Based on Corollary 7.2, employing the Cauchy operator again, we derive the following further extension of q-Barnes' first Lemma.

Theorem 7.3 *We have*

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(q^{1-c+s}, q^{1-d+s}, q^{a+b+c+d+f+s}, q^{e+g+s}; q)_\infty}{(q^{a+s}, q^{b+s}, q^{f+s}, q^{g+s}; q)_\infty} \frac{\pi q^s}{\sin\pi(c-s)\sin\pi(d-s)}$$

$$\times {}_3\phi_2 \left[\begin{array}{ccc} q^e, & q^{a+s}, & q^{b+s} \\ & q^{e+g+s}, & q^{a+b+c+d+f+s} \end{array} ; q, q^{c+d+f+g} \right] ds$$

$$= \frac{q^c}{\sin\pi(c-d)} \frac{(q, q^{1+c-d}, q^{d-c}, q^{a+b+c+d}, q^{a+c+d+f}, q^{b+c+d+f}, q^{c+e+g}; q)_\infty}{(q^{a+c}, q^{a+d}, q^{b+c}, q^{b+d}, q^{c+f}, q^{c+g}, q^{d+f}; q)_\infty}$$

$$\times {}_3\phi_2 \left[\begin{array}{ccc} q^e, & q^{a+c}, & q^{b+c} \\ & q^{c+e+g}, & q^{a+b+c+d} \end{array} ; q, q^{d+g} \right], \tag{7.5}$$

where $\max\{|q^f|, |q^g|, |q^{c+f}|, |q^{c+g}|, |q^{d+f}|, |q^{d+g}|, |q^{c+d+f+g}|\} < 1$.

We conclude this paper with the following extension of Agarwal's q-analogue of Barnes' second lemma. The proof is omitted.

Theorem 7.4 *We have*

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(q^{1+s}, q^{d+s}, q^{1+a+b+c+s-d}, q^{e+f+s}; q)_\infty}{(q^{a+s}, q^{b+s}, q^{c+s}, q^{f+s}; q)_\infty} \frac{\pi q^s}{\sin\pi s \sin\pi(d+s)}$$

$$\times {}_3\phi_2 \left[\begin{array}{ccc} q^e, & q^{a+s}, & q^{b+s} \\ & q^{e+f+s}, & q^{1+a+b+c+s-d} \end{array} ; q, q^{1+c+f-d} \right] ds$$

$$= \csc\pi d \frac{(q, q^d, q^{1-d}, q^{1+b+c-d}, q^{1+a+c-d}, q^{1+a+b-d}, q^{e+f}; q)_\infty}{(q^a, q^b, q^c, q^f, q^{1+a-d}, q^{1+b-d}, q^{1+c-d}; q)_\infty}$$

$$\times {}_3\phi_2 \left[\begin{array}{ccc} q^a, & q^b, & q^e \\ & q^{e+f}, & q^{1+a+b-d} \end{array} ; q, q^{1+f-d} \right], \tag{7.6}$$

where $\max\{|q^f|, |q^{1+f-d}|, |q^{1+c+f-d}|\} < 1$ *and* $Re\{s\log q - \log(\sin\pi s \sin\pi(d+s))\} < 0$ *for large* $|s|$.

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