

Asymptotic enumeration of sparse nonnegative integer matrices with specified row and column sums

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Abstract

Let $\mathbf{s} = (s_1, \dots, s_m)$ and $\mathbf{t} = (t_1, \dots, t_n)$ be vectors of nonnegative integer-valued functions of m, n with equal sum $S = \sum_{i=1}^m s_i = \sum_{j=1}^n t_j$. Let $M(\mathbf{s}, \mathbf{t})$ be the number of $m \times n$ matrices with nonnegative integer entries such that the i th row has row sum s_i and the j th column has column sum t_j for all i, j . Such matrices occur in many different settings, an important example being the contingency tables (also called frequency tables) important in statistics. Define $s = \max_i s_i$ and $t = \max_j t_j$. Previous work has established the asymptotic value of $M(\mathbf{s}, \mathbf{t})$ as $m, n \rightarrow \infty$ with s and t bounded (various authors independently, 1971–1974), and when all entries of \mathbf{s} equal s , all entries of \mathbf{t} equal t , and $m/n, n/m, s/n \geq c/\log n$ for sufficiently large c (Canfield and McKay, 2007). In this paper we extend the sparse range to the case $st = o(S^{2/3})$. The proof in part follows a previous asymptotic enumeration of 0-1 matrices under the same conditions (Greenhill, McKay and Wang, 2006). We also generalise the enumeration to matrices over any subset of the nonnegative integers that includes 0 and 1.

Note added in proof, 2011: This paper appeared in *Advances in Applied Mathematics* **41** (2008), 459–481. Here we fix a small gap in the proof of Lemma 5.1 and make some other minor corrections. We emphasise that the statements of our results have not changed.

1 Introduction

Let $\mathbf{s} = \mathbf{s}(m, n) = (s_1, \dots, s_m)$ and $\mathbf{t} = \mathbf{t}(m, n) = (t_1, \dots, t_n)$ be vectors of nonnegative integers with equal sum $S = \sum_{i=1}^m s_i = \sum_{j=1}^n t_j$. Let $\mathcal{M}(\mathbf{s}, \mathbf{t})$ be the set of all $m \times n$ matrices with nonnegative integer entries such that the i th row has row sum s_i and the j th column has column sum t_j for each i, j . Then define $M(\mathbf{s}, \mathbf{t}) = |\mathcal{M}(\mathbf{s}, \mathbf{t})|$ to be the number of such matrices.

Our task in this paper is to determine the asymptotic value of $M(\mathbf{s}, \mathbf{t})$ as $m, n \rightarrow \infty$ under suitable conditions on \mathbf{s} and \mathbf{t} .

The matrices $\mathcal{M}(\mathbf{s}, \mathbf{t})$ appear in many combinatorial contexts; see Stanley [16, Chapter 1] for a brief history. A large body of statistical literature is devoted to them under the name of *contingency tables* or *frequency tables*; see [6, 7] for a partial survey. In theoretical computer science there has been interest in efficient algorithms for the problem of generating contingency tables with prescribed margins at random, and for approximately counting these tables. See for example [1, 8, 14].

The history of the enumeration problem for nonnegative integer matrices is surveyed in [5], while a history for the corresponding problem for 0-1 matrices is given in [12]. Here we recall only the few previous exact results on asymptotic enumeration for nonnegative integer matrices. Define $s = \max_i s_i$ and $t = \max_j t_j$. The first non-trivial case $s_1 = \dots = s_m = t_1 = \dots = t_n = 3$ was solved by Read [15] in 1958. During the period 1971–74, this was generalised to bounded s, t by three independent groups: Békéssy, Békéssy and Komlós [2], Bender [3], and Everett and Stein [9], under slightly different conditions.

In the case of denser matrices, the only precise asymptotics were found by Canfield and McKay [5] in the case that the row sums are all the same and the column sums are all the same. Let $M(m, s; n, t) = M((s, s, \dots, s), (t, t, \dots, t))$, where the vectors have length m, n , respectively, and $ms = nt$.

Theorem 1.1 ([5, Theorem 1]). *Let $s = s(m, n), t = t(m, n)$ be positive integers satisfying $ms = nt$. Define $\lambda = s/n = t/m$. Let $a, b > 0$ be constants such that $a + b < \frac{1}{2}$. Suppose that $m, n \rightarrow \infty$ in such a way that*

$$\frac{(1 + 2\lambda)^2}{4\lambda(1 + \lambda)} \left(1 + \frac{5m}{6n} + \frac{5n}{6m} \right) \leq a \log n. \quad (1.1)$$

Define $\Delta(m, s; n, t)$ by

$$M(m, s; n, t) = \frac{\binom{n+s-1}{s}^m \binom{m+t-1}{t}^n}{\binom{mn + \lambda mn - 1}{\lambda mn}} \times \left(\frac{m+1}{m} \right)^{(m-1)/2} \left(\frac{n+1}{n} \right)^{(n-1)/2} \exp\left(-\frac{1}{2} + \frac{\Delta(m, s; n, t)}{m+n} \right).$$

Then $\Delta(m, s; n, t) = O(n^{-b})(m + n)$ as $m, n \rightarrow \infty$. \square

Canfield and McKay conjectured that in fact $0 < \Delta(m, s; n, t) < 2$ for all $s, t \geq 1$. The results in the present paper establish that conjecture for sufficiently large m, n in the case $st = o((mn)^{1/5})$. (See Corollary 4.2.)

The main result in this paper is the asymptotic value of $M(\mathbf{s}, \mathbf{t})$ for $st = o(S^{2/3})$. Our proof uses the method of switchings in a number of different ways. In several aspects our approach is parallel to that which provided our previous asymptotic estimate of $N(\mathbf{s}, \mathbf{t})$, the number of 0-1 matrices in the class $\mathcal{M}(\mathbf{s}, \mathbf{t})$. We now restate that result for convenience. For any x , define $[x]_0 = 1$ and for a positive integer k , $[x]_k = x(x-1) \cdots (x-k+1)$. Also define

$$S_k = \sum_{i=1}^m [s_i]_k, \quad T_k = \sum_{j=1}^n [t_j]_k$$

for $k \geq 1$. Note that $S_1 = T_1 = S$.

Theorem 1.2 ([12, Corollary 5.1]). *Let $\mathbf{s} = \mathbf{s}(m, n) = (s_1, \dots, s_m)$ and $\mathbf{t} = \mathbf{t}(m, n) = (t_1, \dots, t_n)$ be vectors of nonnegative integers with equal sum $S = \sum_{i=1}^m s_i = \sum_{j=1}^n t_j$. Suppose that $m, n \rightarrow \infty$, $S \rightarrow \infty$ and $1 \leq st = o(S^{2/3})$. Then*

$$N(\mathbf{s}, \mathbf{t}) = \frac{S!}{\prod_{i=1}^m s_i! \prod_{j=1}^n t_j!} \exp\left(-\frac{S_2 T_2}{2S^2} - \frac{S_2 T_2}{2S^3} + \frac{S_3 T_3}{3S^3} - \frac{S_2 T_2 (S_2 + T_2)}{4S^4} - \frac{S_2^2 T_3 + S_3 T_2^2}{2S^4} + \frac{S_2^2 T_2^2}{2S^5} + O\left(\frac{s^3 t^3}{S^2}\right)\right). \quad \square$$

We now state our main result, which is the asymptotic value of $M(\mathbf{s}, \mathbf{t})$ for sufficiently sparse matrices. Note that the answer is obtained by multiplying the expression for $N(\mathbf{s}, \mathbf{t})$ from Theorem 1.2 by a simple adjustment factor.

Theorem 1.3. *Let $\mathbf{s} = \mathbf{s}(m, n) = (s_1, \dots, s_m)$ and $\mathbf{t} = \mathbf{t}(m, n) = (t_1, \dots, t_n)$ be vectors of nonnegative integers with equal sum $S = \sum_{i=1}^m s_i = \sum_{j=1}^n t_j$. Suppose that $m, n \rightarrow \infty$, $S \rightarrow \infty$ and $1 \leq st = o(S^{2/3})$. Then*

$$\begin{aligned} M(\mathbf{s}, \mathbf{t}) &= N(\mathbf{s}, \mathbf{t}) \exp\left(\frac{S_2 T_2}{S^2} + \frac{S_2 T_2}{S^3} + O\left(\frac{s^3 t^3}{S^2}\right)\right) \\ &= \frac{S!}{\prod_{i=1}^m s_i! \prod_{j=1}^n t_j!} \exp\left(\frac{S_2 T_2}{2S^2} + \frac{S_2 T_2}{2S^3} + \frac{S_3 T_3}{3S^3} - \frac{S_2 T_2 (S_2 + T_2)}{4S^4} - \frac{S_2^2 T_3 + S_3 T_2^2}{2S^4} + \frac{S_2^2 T_2^2}{2S^5} + O\left(\frac{s^3 t^3}{S^2}\right)\right). \end{aligned}$$

Proof. The proof of this theorem is presented in Sections 2 and 3. First we show that the set of matrices in $\mathcal{M}(\mathbf{s}, \mathbf{t})$ with an entry greater than 3 forms a vanishingly small proportion of $\mathcal{M}(\mathbf{s}, \mathbf{t})$. We also show that it is very unusual for an element of $\mathcal{M}(\mathbf{s}, \mathbf{t})$ to have a “large” number of entries equal to 2 or a “large” number of entries equal to 3, where “largeness” is defined in Section 2. We establish these facts using switchings on the matrix entries. This allows us to concentrate on matrices in $\mathcal{M}(\mathbf{s}, \mathbf{t})$ with no entries greater than 3 and not very many entries equal to 2 or 3.

We then proceed in Section 3 to compare the number of these matrices with the number $N(\mathbf{s}, \mathbf{t})$ of $\{0, 1\}$ -matrices with row sums \mathbf{s} and column sums \mathbf{t} . We do this by adapting the results from [12] used to prove Theorem 1.2. These calculations are carried out in the pairing model, which is described in Section 3. Our theorem follows on combining Lemmas 3.1, 3.2 and Corollary 3.8. \square

In the semiregular case where $s_i = s$ for $1 \leq i \leq m$ and $t_j = t$ for $1 \leq j \leq n$, Theorem 1.3 says the following.

Corollary 1.4. *Suppose that $m, n \rightarrow \infty$ and that $sm = tn = S$ for nonnegative integer functions $s = s(m, n)$, $t = t(m, n)$ and $S = S(m, n)$. If $1 \leq st = o(S^{2/3})$ then*

$$M(m, s; n, t) = \frac{S!}{(s!)^m (t!)^n} \exp\left(\frac{(s-1)(t-1)}{2} - \frac{(s-1)(t-1)(2st - s - t - 10)}{12S} + O\left(\frac{s^3 t^3}{S^2}\right)\right). \quad \square$$

For some applications the statement of Theorem 1.3 is not very convenient. In Section 4 we will derive an alternative formulation, very similar to one given for $N(\mathbf{s}, \mathbf{t})$ in [12]. For $k = 2, 3$, define

$$\hat{\mu}_k = \frac{mn}{S(mn + S)} \sum_{i=1}^m (s_i - S/m)^k$$

$$\hat{\nu}_k = \frac{mn}{S(mn + S)} \sum_{j=1}^n (t_j - S/n)^k.$$

To motivate the definitions, recall that S/m is the mean value of s_i and S/n is the mean value of t_j , so these are scaled central moments. We will prove Corollary 4.1, stated in Section 4, which has the following special case.

Corollary 1.5. *Under the conditions of Theorem 1.3, if $(1 + \hat{\mu}_2)(1 + \hat{\nu}_2) = O(S^{1/3})$ then*

$$M(\mathbf{s}, \mathbf{t}) = \frac{\prod_{i=1}^m \binom{n+s_i-1}{s_i} \prod_{j=1}^n \binom{m+t_j-1}{t_j}}{\binom{mn+S-1}{S}} \exp\left(\frac{1}{2}(1 - \hat{\mu}_2)(1 - \hat{\nu}_2) + O\left(\frac{st}{S^{2/3}}\right)\right). \quad \square$$

Corollary 1.5 has an instructive interpretation. Following [5], we write $M(\mathbf{s}, \mathbf{t}) = MP_1P_2E$, where

$$M = \binom{mn+S-1}{S}, \quad P_1 = M^{-1} \prod_{i=1}^m \binom{n+s_i-1}{s_i}, \quad P_2 = M^{-1} \prod_{j=1}^n \binom{m+t_j-1}{t_j},$$

$$E = \exp\left(\frac{1}{2}(1 - \hat{\mu}_2)(1 - \hat{\nu}_2) + O\left(\frac{st}{S^{2/3}}\right)\right).$$

Clearly, M is the number of $m \times n$ nonnegative matrices whose entries sum to S . In the uniform probability space on these M matrices, P_1 is the probability of the event that the row sums are given by \mathbf{s} and P_2 is the probability of the event that the column sums are given by \mathbf{t} . The final quantity E is thus a correction to account for the non-independence of these two events.

Finally, in Section 5 we show how to generalise Theorem 1.3 to matrices whose entries are restricted to any subset of the natural numbers that includes 0 and 1.

A note on our usage of the $O(\cdot)$ notation in the following is in order. Given a fixed function $f(S) = o(S^{2/3})$, and any quantity ϕ that depends on any of our variables, $O(\phi)$ denotes any quantity whose absolute value is bounded above by $|c\phi|$ for some constant c that depends on f and *nothing else*, provided that $1 \leq st \leq f(S)$.

Note added in proof, 2011: This version of the paper the same as the journal version [11], except as follows:

- Theorem 2.1, a statement of a special case of a more general result from [10], was previously incomplete. The first inequality in (2.1) need only hold if v is a sink, but this condition was absent in [11].
- A note has been added at the end of the proofs of Lemmas 3.5 and 3.7, clarifying why it is valid to apply [12, Lemma 4.6] and [12, Lemma 4.8] with a possibly larger value of N_2, N_3 than used in [12].
- The proof of Lemma 5.1 has been changed to fix a small gap. The old proof did not guarantee that $n_1(Q) = S - o(S)$ when $Q \in \mathcal{M}^- \setminus \mathcal{M}^*$. The definition of \mathcal{M}^- has changed and a new switching argument is given to correct this.
- We added a reference to the journal version [11] of this paper.

Note that none of the statements of our own results from [11] have changed.

2 Switchings on matrices

In this section we will show that the condition $st = o(S^{2/3})$ implies that most matrices have no entries greater than 3. We also find bounds on the number of entries equal to 2

or 3. Our tool will be the method of switchings, which we will analyse using results of Fack and McKay [10] from which we will distill the following special case.

Theorem 2.1. *Let $G = (V, E)$ be a finite simple acyclic directed graph, with each $v \in V$ being associated with a finite set $C(v)$, these sets being disjoint. Suppose that \mathcal{S} is a multiset of ordered pairs such that for each $(Q, R) \in \mathcal{S}$ there is an edge $vw \in E$ with $Q \in C(v)$ and $R \in C(w)$. Further suppose that $a, b : V \rightarrow \mathbb{R}$ are positive functions such that, for each $v \in V$,*

$$\begin{aligned} |\{(Q, R) \in \mathcal{S} \mid Q \in C(v)\}| &\geq a(v) |C(v)| \quad \text{if } v \text{ is not a sink,} \\ |\{(Q, R) \in \mathcal{S} \mid R \in C(v)\}| &\leq b(v) |C(v)|, \end{aligned} \tag{2.1}$$

where the left hand sides are multiset cardinalities. Let $\emptyset \neq Y \subseteq V$. Then there is a directed path v_1, v_2, \dots, v_k in G , where $v_1 \in Y$ and v_k is a sink, such that

$$\frac{\sum_{v \in Y} |C(v)|}{\sum_{v \in V} |C(v)|} \leq \frac{\sum_{v_i \in Y} N(v_i)}{\sum_{1 \leq i \leq k} N(v_i)}, \tag{2.2}$$

where $N(v_i)$ is defined by

$$\begin{aligned} N(v_1) &= 1, \\ N(v_i) &= \frac{a(v_1) \cdots a(v_{i-1})}{b(v_2) \cdots b(v_i)} \quad (2 \leq i \leq k). \end{aligned}$$

Proof. This follows from Theorems 1 and 2 of [10]. □

For $D \geq 2$, a D -switching is described by the sequence

$$(Q; (i_0, j_0), (i_1, j_1), \dots, (i_D, j_D))$$

where Q is a matrix in $\mathcal{M}(\mathbf{s}, \mathbf{t})$ and $(i_0, j_0), (i_1, j_1), \dots, (i_D, j_D)$ is a $(D+1)$ -tuple of positions such that

- the rows i_0, \dots, i_D are all distinct and the columns j_0, \dots, j_D are all distinct;
- there is a D in position (i_0, j_0) of Q ;
- the entries in positions (i_ℓ, j_ℓ) of Q are not equal to 0 or $D + 1$, for $1 \leq \ell \leq D$;
- there is a 0 in position (i_ℓ, j_0) and position (i_0, j_ℓ) of Q for $1 \leq \ell \leq D$.

This D -switching transforms Q into a matrix $R \in \mathcal{M}(\mathbf{s}, \mathbf{t})$ by acting on the $(D+1) \times (D+1)$ submatrix consisting of rows (i_0, \dots, i_D) and columns (j_0, \dots, j_D) as follows:

$$Q = \begin{pmatrix} D & 0 & 0 & \cdots & 0 \\ 0 & q_1 & & & \\ 0 & & q_2 & & \\ \vdots & & & \ddots & \\ 0 & & & & q_D \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & q_1-1 & & & \\ 1 & & q_2-1 & & \\ \vdots & & & \ddots & \\ 1 & & & & q_D-1 \end{pmatrix} = R.$$

Matrix entries not shown can have any values and are unchanged by the switching operation. Notice that the D -switching preserves all row and column sums and reduces the number of entries equal to D by at least 1 and at most $D + 1$. The number of entries greater than D is unchanged.

A *reverse D -switching*, which undoes a D -switching (and vice-versa), is described by a sequence $(R; (i_0, j_0), \dots, (i_D, j_D))$ where $R \in \mathcal{M}(\mathbf{s}, \mathbf{t})$ and $(i_0, j_0), (i_1, j_1), \dots, (i_D, j_D)$ is a $(D+1)$ -tuple of positions such that

- the rows i_0, \dots, i_D are all distinct and the columns j_0, \dots, j_D are all distinct;
- there is a zero in position (i_0, j_0) of R ;
- the entries in positions (i_ℓ, j_ℓ) of R are not equal to D , for $1 \leq \ell \leq D$;
- there is a 1 in position (i_ℓ, j_0) and position (i_0, j_ℓ) of R for $1 \leq \ell \leq D$.

Lemma 2.2. *Let $D \geq 2$ and let $Q \in \mathcal{M}(\mathbf{s}, \mathbf{t})$ have at least $K \geq 2st$ non-zero entries that are not greater than D , and at least J entries equal to D . Then there are at least $J(K - 2st)^D$ D -switchings and at most $S_D T_D$ reverse D -switchings that apply to Q .*

Proof. First consider D -switchings. We want a lower bound on the number of $(D+1)$ -tuples $(i_0, j_0), \dots, (i_D, j_D)$ of indices where a D -switching may be performed. There are at least J ways to choose the position (i_0, j_0) . Then we can choose the remaining positions one at a time, avoiding choices which violate the rules. The choice of the last position (i_D, j_D) is the most restricted, so we bound that. By assumption, there are at least K nonzero entries in Q that are not greater than D . Of these we must exclude the entry in position (i_0, j_0) as well as entries in the same column as a nonzero entry in row i_0 other than column j_0 (at most $(s - D)t$ positions), entries in the same row as a nonzero entry in column j_0 other than row i_0 (at most $(t - D)s$ positions), and entries in row i_ℓ or column j_ℓ for $1 \leq \ell \leq D - 1$ (at most $(D - 1)(s + t - 2)$ positions). Overall, we can choose position (i_D, j_D) in at least

$$K - 1 - (s - D)t - (t - D)s - (D - 1)(s + t - 2) \geq K - 2st$$

ways, and as we noted this also applies to each of the less restricted positions (i_ℓ, j_ℓ) , where $1 \leq \ell < D$. Hence at most $J(K - 2st)^D$ D -switchings involve Q .

Next consider reverse D -switchings. An ordered sequence of D entries in the same row which equal 1 may be chosen in at most S_D ways, and an ordered sequence of D entries in the same column which equal 1 may be chosen in at most T_D ways. Some of these choices will not give a legal position for a reverse D -switching, but $S_D T_D$ is certainly an upper bound. \square

Our first application of switchings will be to show that only a vanishing fraction of our matrices have any entries greater than 3. For $j \geq 0$ and $D \geq 2$, let $\mathcal{M}_D(j)$ be the set of all matrices in $\mathcal{M}(\mathbf{s}, \mathbf{t})$ with exactly j entries equal to D and none greater than D . Define $\mathcal{M}_D(>0) = \bigcup_{j>0} \mathcal{M}_D(j)$, and note that $\mathcal{M}_{D+1}(0) = \mathcal{M}_D(0) \cup \mathcal{M}_D(>0)$.

Lemma 2.3. *Suppose that $1 \leq st = o(S^{2/3})$. Let $U_1 = U_1(\mathbf{s}, \mathbf{t})$ be the set of all matrices in $\mathcal{M}(\mathbf{s}, \mathbf{t})$ which contain an entry greater than 3. Then*

$$|U_1|/M(\mathbf{s}, \mathbf{t}) = O(s^3 t^3 / S^2).$$

Proof. The largest possible entry of a matrix in $\mathcal{M}(\mathbf{s}, \mathbf{t})$ is $\Delta = \min\{s, t\}$. We will apply Theorem 2.1 to successively bound the possibility that the maximum entry is D , for $D = \Delta, \Delta - 1, \dots, 4$.

Fix D with $4 \leq D \leq \Delta$. Define a directed graph $G = (V, E)$ with vertex set $V = \{v_0, v_1, v_2, \dots\}$ and edge set $E = \{v_j v_i \mid j - D - 1 \leq i \leq j - 1\}$. Associate each v_i with the set $C(v_i) = \mathcal{M}_D(i)$. Define \mathcal{S} to be the set of pairs (Q, R) related by a D -switching, where $Q \in v_j, R \in v_i$ for some $v_j v_i \in E$. Define $Y = \{v_1, v_2, \dots\} \subseteq V$. Note that $S_D T_D > 0$ since $D \leq \Delta$.

We can now use Theorem 2.1 to bound

$$\frac{|\mathcal{M}_D(>0)|}{|\mathcal{M}_{D+1}(0)|} = \frac{\sum_{v \in Y} |C(v)|}{\sum_{v \in V} |C(v)|},$$

once we have found positive functions $a, b : V \rightarrow \mathbb{R}$ satisfying (2.1). These are provided by Lemma 2.2 with $J = j$ and $K = S/D$, the latter being clear since there are no entries greater than D and the total of all the entries is S . We have $S/D > 2st$ since $D \leq \Delta \leq (st)^{1/2}$. Thus we can take $a(v_j) = j(S/D - 2st)^D$ and $b(v_j) = S_D T_D$.

Theorem 2.1 tells us that, unless $\mathcal{M}_D(>0) = \emptyset$, there is a directed path $v_{t_1}, v_{t_2}, \dots, v_{t_q}$, where $q > 1$ and $t_1 > t_2 > \dots > t_q = 0$ (since v_0 is the only sink) such that (2.2) holds.

Hence, using the values of N as given in Theorem 2.1 we have

$$\begin{aligned} \frac{|\mathcal{M}_D(>0)|}{|\mathcal{M}_{D+1}(0)|} &\leq \frac{N(v_{t_{q-1}}) + \dots + N(v_{t_1})}{N(v_{t_q}) + \dots + N(v_{t_2})} \\ &\leq \max_{1 \leq i \leq q} \frac{N(v_{t_{i-1}})}{N(v_{t_i})} \\ &= \max_{1 \leq i \leq q} \frac{b(\mathcal{M}_D(t_i))}{a(\mathcal{M}_D(t_{i-1}))} \\ &\leq \frac{S_D T_D}{(S/D - 2st)^D}. \end{aligned}$$

Let ξ_D denote this upper bound: that is, $\xi_D = S_D T_D / (S/D - 2st)^D$ for $4 \leq D \leq \Delta$. Note

that $\xi_4 = O(s^3 t^3 / S^2)$. For $4 \leq D < \Delta$, we have $\xi_D > 0$ and

$$\begin{aligned} \frac{\xi_{D+1}}{\xi_D} &\leq st \left(\frac{(D+1)^{D+1}}{D^D} \right) \frac{(S-2stD)^D}{(S-2st(D+1))^{D+1}} \\ &= O(1) \frac{Dst}{S-2st(D+1)} \left(1 - \frac{2st}{S-2stD} \right)^{-D} \\ &= o(1) \end{aligned}$$

uniformly over D , where the last step uses the observation that $\Delta \leq (st)^{1/2} = o(S^{1/3})$.

Since $U_1 = \mathcal{M}_4(>0) \cup \mathcal{M}_5(>0) \cup \dots \cup \mathcal{M}_\Delta(>0)$ and $\mathcal{M}_{D+1}(0) \subseteq \mathcal{M}(\mathbf{s}, \mathbf{t})$ for $4 \leq D \leq \Delta$, we have $|U_1|/M(\mathbf{s}, \mathbf{t}) \leq \xi_4 + \xi_5 + \dots + \xi_\Delta = O(s^3 t^3 / S^2)$ as required. \square

We may therefore restrict our attention to matrices with no entry greater than 3. Next we find upper bounds on the numbers of entries equal to 2 or 3 which hold with high probability.

Define

$$N_2 = \begin{cases} 22 & \text{if } S_2 T_2 < S^{7/4}, \\ \lceil \log S \rceil & \text{if } S^{7/4} \leq S_2 T_2 < \frac{1}{5600} S^2 \log S, \\ \lceil 5600 S_2 T_2 / S^2 \rceil & \text{if } \frac{1}{5600} S^2 \log S \leq S_2 T_2; \end{cases}$$

$$N_3 = \max(\lceil \log S \rceil, \lceil 230000 S_3 T_3 / S^3 \rceil).$$

(Here and throughout the paper we have not attempted to optimise constants.)

We will use the following lemma.

Lemma 2.4. *Let k be a positive integer and let q and n be positive real numbers such that $n \geq kq$. Then*

$$n(n-q) \cdots (n-(k-1)q) \geq (n/e)^k.$$

Proof. Dividing the left side by n^k gives, for $n > kq$,

$$\begin{aligned} \prod_{i=0}^{k-1} (1 - iq/n) &= \exp\left(\sum_{i=0}^{k-1} \log(1 - iq/n) \right) \\ &\geq \exp\left(\int_0^k \log(1 - xq/n) dx \right) \\ &= \exp(-k - (n/q - k) \log(1 - kq/n)) \\ &\geq \exp(-k). \end{aligned}$$

The second line holds because $\log(1 - xq/n)$ is a decreasing function for $x \in [0, k]$. The case $n = kq$ follows by continuity. \square

Lemma 2.5. *Let $1 \leq st = o(S^{2/3})$. Then, with probability $1 - O(s^3 t^3 / S^2)$, a random element of $\mathcal{M}(\mathbf{s}, \mathbf{t})$ has no entry greater than 3, at most N_3 entries equal to 3, and at most N_2 entries equal to 2.*

Proof. In view of Lemma 2.3, we may restrict our attention to the set $\mathcal{M}_4(0)$ of all matrices in $\mathcal{M}(\mathbf{s}, \mathbf{t})$ with maximum entry at most 3. We will start by applying 3-switchings as in Lemma 2.3 but the analysis will be more delicate.

In applying Theorem 2.1 we have $V = \{v_0, v_1, \dots\}$, with v_h associated with $\mathcal{M}_3(h)$, and $Y = \{v_h \mid h > N_3\}$. For sufficiently large S , we have from Lemma 2.2 that we can take $a(v_h) = \frac{1}{28}hS^3$ and $b(v_h) = S_3T_3$. If $S_3T_3 = 0$ then entries equal to 3 are impossible, so we assume that $S_3T_3 > 0$. Define $\varphi = 28S_3T_3/S^3$.

According to Theorem 2.1, there is a sequence

$$h_1 > h_2 > \dots > h_q = 0,$$

with $h_1 > N_3$ and $h_{i-1} - 4 \leq h_i < h_{i-1}$ for all i , such that

$$\begin{aligned} \frac{|\mathcal{M}_3(>N_3)|}{|\mathcal{M}(\mathbf{s}, \mathbf{t})|} &\leq \frac{|\mathcal{M}_3(>N_3)|}{|\mathcal{M}_4(0)|} \\ &\leq \frac{N(h_\ell) + N(h_{\ell-1}) + \dots + N(h_1)}{N(h_q) + N(h_{q-1}) + \dots + N(h_1)}, \end{aligned}$$

where ℓ is the largest index such that $h_\ell \geq N_3 + 1$ and $N(h_i) = h_1 \dots h_{i-1} \varphi^{-i+1}$ for all i .

Define $u = \lfloor \frac{1}{4} \log S \rfloor$. Since $N_3 \geq \lceil \log S \rceil$, we have $\ell + u \leq q$. Also, for $0 \leq i \leq \ell - 1$,

$$\frac{N(h_{\ell-i})}{N(h_{\ell+u-i})} \leq \frac{N(h_\ell)}{N(h_{\ell+u})}.$$

Therefore,

$$\begin{aligned} \frac{|\mathcal{M}_3(>N_3)|}{|\mathcal{M}(\mathbf{s}, \mathbf{t})|} &\leq \frac{N(h_\ell) + N(h_{\ell-1}) + \dots + N(h_1)}{N(h_{\ell+u}) + N(h_{\ell+u-1}) + \dots + N(h_{u+1})} \\ &\leq \frac{N(h_\ell)}{N(h_{\ell+u})} \\ &= \frac{\varphi^u}{h_\ell h_{\ell+1} \dots h_{\ell+u-1}} \\ &\leq \frac{\varphi^u}{(N_3 + 1)(N_3 - 3) \dots (N_3 - 4u + 5)}. \end{aligned}$$

Since $N_3 + 1 > 4u$ we can apply Lemma 2.4 to obtain the bound

$$\frac{|\mathcal{M}_3(>N_3)|}{|\mathcal{M}(\mathbf{s}, \mathbf{t})|} \leq \left(\frac{\varphi e}{N_3 + 1} \right)^u.$$

Now $N_3 \geq 230000S_3T_3/S^3 \geq 8214\varphi$, and $u \geq \frac{1}{4}\log S - 1$, so this upper bound is at most

$$\left(\frac{e}{8214}\right)^{\frac{1}{4}\log S - 1} = O(1)S^{\frac{1}{4}\log(e/8214)} = O(S^{-2}).$$

This shows that with probability $O(s^3t^3/S^2)$ there are at most N_3 entries equal to 3, as required.

To bound the number of entries equal to 2, we proceed in the same manner using 2-switchings, working under the assumption that there are at most N_3 entries equal to 3 and none greater than 3. In applying Lemma 2.2, we can take $K = \frac{1}{2}(S - 3N_3)$, so that $(K - 2st)^2 \geq \frac{1}{5}S^2$ for sufficiently large S . Define $\psi = 5S_2T_2/S^2$. Arguing as above we find a sequence

$$d_1 > d_2 > \cdots > d_r = 0,$$

with the following properties: (i) $d_1 > N_2$ and $d_{i-1} - 3 \leq d_i < d_{i-1}$ for all i , and (ii) if p is the greatest integer such that $d_p > N_2$ then, for any w with $0 < w \leq r - p$, the probability that there are more than N_2 entries equal to 2, subject to there being at most N_3 equal to 3, is bounded above by

$$\frac{\psi^w}{d_p d_{p+1} \cdots d_{p+w-1}}. \quad (2.3)$$

First suppose that $S_2T_2 < S^{7/4}$, so that $N_2 = 22$ and $\psi < 5S^{-1/4}$. Since $d_p \geq N_2 + 1 = 23$, it follows that $r - p \geq 8$. Taking $w = 8$ in (2.3) gives

$$\frac{|Y|}{|\mathcal{M}(\mathbf{s}, \mathbf{t})|} \leq \frac{\psi^8}{d_p d_{p+1} \cdots d_{p+7}} = O(S^{-2}).$$

Now suppose that $S_2T_2 \geq S^{7/4}$. Then $N_2 \geq \lceil \log S \rceil$ so we can take $w = \lfloor \frac{1}{3}\log S \rfloor$. Arguing as above by applying Lemma 2.4 to (2.3), we obtain the bound $O(S^{-2})$ again. This completes the proof. \square

From now on we proceed in two cases, as in [12]. Say that the pair (S_2, T_2) is *substantial* if the following conditions hold:

- $1 \leq st = o(S^{2/3})$,
- $S_2 \geq s \log^2 S$ and $T_2 \geq t \log^2 S$,
- $S_2T_2 \geq (st)^{3/2}S$.

Lemma 2.6. *If $1 \leq st = o(S^{2/3})$ and (S_2, T_2) is insubstantial, then with probability $1 - O(s^3t^3/S^2)$, a random element of $\mathcal{M}(\mathbf{s}, \mathbf{t})$ has no entry greater than 3, at most one entry equal to 3 and at most two entries equal to 2.*

Proof. The absence of entries greater than 3 follows from Lemma 2.3. We can also, by Lemma 2.5, assume that the number of entries equal to 2 or 3 is $o(S)$. Therefore, most of the matrix entries are 0 or 1. Let \mathcal{N} be the set of all matrices in $\mathcal{M}(\mathbf{s}, \mathbf{t})$ with no entries greater than 3, at most N_2 entries equal to 2 and at most N_3 entries equal to 3.

To bound the number of entries equal to 2 or 3 even more tightly, as this lemma requires, we employ D -switchings ($D = 2, 3$) with the additional restriction that $q_1 = \dots = q_D = 1$. This ensures that these *restricted* D -switchings reduce the number of entries equal to D by exactly one and do not create any new entries equal to 2 or 3.

Let $N''(h)$ be the number of matrices in \mathcal{N} with h entries equal to 3. If Q is such a matrix then the number of restricted 3-switchings applicable to Q is $hS^3(1+o(1))$ and the number of reverse restricted 3-switchings is at most S_3T_3 . (This follows using arguments similar to those in Lemma 2.2, since there are $S - o(S)$ entries equal to 1.) Therefore, if the denominator is nonzero,

$$\frac{N''(h)}{N''(h-1)} = O(1) \frac{S_3T_3}{hS^3}. \quad (2.4)$$

We can now easily check that each of the three causes of insubstantiality (namely, $S_2 < s \log^2 S$, $T_2 < t \log^2 S$, and $S_2T_2 < (st)^{3/2}S$) imply that

$$\frac{S_3T_3}{S^3} = O(s^{3/2}t^{3/2}/S) = o(1).$$

Hence (2.4) implies that

$$\frac{\sum_{h \geq 2} N''(h)}{N''(0)} = O(s^3t^3/S^2).$$

In precisely the same way, using restricted 2-switchings, we find that

$$\frac{\sum_{d \geq 3} N'(d)}{N'(0)} = O(s^3t^3/S^2),$$

where $N'(d)$ is the number of matrices in \mathcal{N} with d entries equal to 2 and at most one entry equal to 3. The lemma follows. \square

3 From pairings to matrices

The remainder of the paper will involve calculations in the *pairing model*, which we now describe. (This model is standard for working with random bipartite graphs of fixed degrees: see for example [13].) Consider a set of S points arranged in *cells* x_1, x_2, \dots, x_m , where cell x_i has size s_i for $1 \leq i \leq m$, and another set of S points arranged in cells y_1, y_2, \dots, y_n where cell y_j has size t_j for $1 \leq j \leq n$. Take a partition P (called a *pairing*) of the $2S$ points into S pairs with each pair having the form (x, y) where $x \in x_i$ and $y \in y_j$ for some i, j . The set of all such pairings, of which there are $S!$, will be denoted by $\mathcal{P}(\mathbf{s}, \mathbf{t})$. We work in the uniform probability space on $\mathcal{P}(\mathbf{s}, \mathbf{t})$.

Two pairs are called *parallel* if they involve the same two cells. A *parallel class* is a maximal set of mutually parallel pairs. The *multiplicity* of a parallel class (and of the pairs in the class) is the cardinality of the class. As important special cases, a *simple pair* is a parallel class of multiplicity one, a *double pair* is a parallel class of multiplicity two, while a *triple pair* is a parallel class of multiplicity three.

Each pairing $P \in \mathcal{P}(\mathbf{s}, \mathbf{t})$ gives rise to a matrix in $\mathcal{M}(\mathbf{s}, \mathbf{t})$ by letting the (i, j) -th entry of the matrix equal the multiplicity of the parallel class from x_i to y_j in P .

In [12] we noted that the number of pairings which gives rise to each 0-1 matrix in $\mathcal{M}(\mathbf{s}, \mathbf{t})$ depends only on \mathbf{s} and \mathbf{t} and is independent of the structure of the matrix. Hence the task of counting such matrices reduces to finding the fraction of pairings that have no multiplicities greater than 1.

More generally, matrices in $\mathcal{M}(\mathbf{s}, \mathbf{t})$ correspond to different numbers of pairings. For a pairing $P \in \mathcal{P}(\mathbf{s}, \mathbf{t})$, define the *multiplicity vector* of P to be $\mathbf{a}(P) = (a_2, a_3, \dots)$ where a_r is the number of parallel classes of multiplicity r . Also define the *weight* of P as

$$w(P) = (2!)^{a_2} (3!)^{a_3} (4!)^{a_4} \dots$$

For $Q \in \mathcal{M}(\mathbf{s}, \mathbf{t})$, define $w(Q)$ and $\mathbf{a}(Q)$ to be the common weight and multiplicity vectors of the pairings that yield Q .

By elementary counting, a matrix $Q \in \mathcal{M}(\mathbf{s}, \mathbf{t})$ corresponds to exactly

$$\frac{1}{w(Q)} \prod_{i=1}^m s_i! \prod_{j=1}^n t_j!$$

pairings in $\mathcal{P}(\mathbf{s}, \mathbf{t})$. Therefore, if A is a set of multiplicity vectors, $\mathcal{P}_A = \{P \in \mathcal{P}(\mathbf{s}, \mathbf{t}) \mid \mathbf{a}(P) \in A\}$, and $\mathcal{M}_A = \{Q \in \mathcal{M}(\mathbf{s}, \mathbf{t}) \mid \mathbf{a}(Q) \in A\}$, then

$$|\mathcal{M}_A| = \frac{\sum_{P \in \mathcal{P}_A} w(P)}{\prod_{i=1}^m s_i! \prod_{j=1}^n t_j!}. \quad (3.1)$$

This holds in particular if A is the set of all nonnegative integer sequences, in which case $\mathcal{P}_A = \mathcal{P}(\mathbf{s}, \mathbf{t})$ and $\mathcal{M}_A = \mathcal{M}(\mathbf{s}, \mathbf{t})$.

We first prove Theorem 1.3 in the case that (S_2, T_2) is insubstantial.

Lemma 3.1. *If $1 \leq st = o(S^{2/3})$ and (S_2, T_2) is insubstantial then Theorem 1.3 holds.*

Proof. Similarly to [12, Lemma 2.2], define a *doublet* to be to be an unordered set of 2 parallel pairs. A double pair provides one doublet, while a triple pair provides 3 doublets. For the uniform probability space over $\mathcal{P}(\mathbf{s}, \mathbf{t})$, let b_r be the expectation of the number of

sets of r doublets, for $r \geq 0$. In [12, Lemma 2.2] it is shown that

$$\begin{aligned} b_0 &= 1, \\ b_1 &= \frac{S_2 T_2}{2[S]_2}, \\ b_2 &= \frac{S_3 T_3}{2[S]_3} + \frac{(S_2^2 - 4S_3 - 2S_2)(T_2^2 - 4T_3 - 2T_2)}{8[S]_4}, \\ b_3 &= \frac{S_3 T_3}{6[S]_3} + O(s^3 t^3 / S^2), \\ b_4 &= O(s^3 t^3 / S^2). \end{aligned}$$

Let p_k denote the probability that a randomly chosen pairing contains exactly k doublets, for $k \geq 0$. Then

$$p_k = \sum_{r \geq k} (-1)^{r+k} \binom{r}{k} b_r$$

and the partial sums of this series alternate above and below p_k (see for example [4, Theorem 1.10]). Applying this, we find that

$$\begin{aligned} p_0 &= 1 - \frac{S_2 T_2}{2[S]_2} + \frac{S_3 T_3}{3[S]_3} + \frac{(S_2^2 - 4S_3 - 2S_2)(T_2^2 - 4T_3 - 2T_2)}{8[S]_4} + O(s^3 t^3 / S^2), \\ p_1 &= \frac{S_2 T_2}{2[S]_2} - \frac{S_3 T_3}{2[S]_3} - \frac{(S_2^2 - 4S_3 - 2S_2)(T_2^2 - 4T_3 - 2T_2)}{4[S]_4} + O(s^3 t^3 / S^2), \\ p_2 &= \frac{(S_2^2 - 4S_3 - 2S_2)(T_2^2 - 4T_3 - 2T_2)}{8[S]_4} + O(s^3 t^3 / S^2), \\ p_3 &= \frac{S_3 T_3}{6[S]_3} + O(s^3 t^3 / S^2). \end{aligned}$$

(The expression for p_0 was also derived in [12, Lemma 2.2].) The configurations defining these cases are, respectively, no parallel pairs, one double pair, two double pairs, and one triple pair.

Applying Lemma 2.6 and (3.1),

$$\begin{aligned} M(\mathbf{s}, \mathbf{t}) &= (1 + O(s^3 t^3 / S^2)) \frac{S!}{\prod_{i=1}^m s_i! \prod_{j=1}^n t_j!} (p_0 + 2p_1 + 4p_2 + 6p_3) \\ &= \frac{S!}{\prod_{i=1}^m s_i! \prod_{j=1}^n t_j!} (p_0 + 2p_1 + 4p_2 + 6p_3 + O(s^3 t^3 / S^2)) \\ &= \frac{S!}{\prod_{i=1}^m s_i! \prod_{j=1}^n t_j!} \\ &\quad \times \left(1 + \frac{S_2 T_2}{2[S]_2} + \frac{S_3 T_3}{3[S]_3} + \frac{(S_2^2 - 4S_3 - 2S_2)(T_2^2 - 4T_3 - 2T_2)}{8[S]_4} + O(s^3 t^3 / S^2) \right), \end{aligned}$$

where we have used the fact that $p_0 + 2p_1 + 4p_2 + 6p_3 = 1 + o(1)$ in the insubstantial case to get the second line.

This expression is equal to the expression in Theorem 1.3 under our present assumptions. (Note that since (S_2, T_2) is insubstantial, the term $S_2^2 T_2^2 / 2S^5$ which appears in the statement of Theorem 1.3 is absorbed into the error term.) \square

For nonnegative integers d, h , define $\mathcal{C}_{d,h} = \mathcal{C}_{d,h}(\mathbf{s}, \mathbf{t})$ to be the set of all pairings in $\mathcal{P}(\mathbf{s}, \mathbf{t})$ with exactly d double pairs and h triple pairs, but no parallel classes of multiplicity greater than 3. Also define

$$w(\mathcal{C}_{d,h}) = \sum_{P \in \mathcal{C}_{d,h}} w(P) = 2^d 6^h |\mathcal{C}_{d,h}|.$$

A special case of (3.1), used in [12], is that the number of 0-1 matrices in $\mathcal{P}(\mathbf{s}, \mathbf{t})$ is

$$N(\mathbf{s}, \mathbf{t}) = \frac{|\mathcal{C}_{0,0}|}{\prod_{i=1}^m s_i! \prod_{j=1}^n t_j!}.$$

We will proceed by writing $M(\mathbf{s}, \mathbf{t})$ in terms of $N(\mathbf{s}, \mathbf{t})$, as follows.

Lemma 3.2. *If (S_2, T_2) is substantial then*

$$M(\mathbf{s}, \mathbf{t}) = N(\mathbf{s}, \mathbf{t}) \sum_{d=0}^{N_2} \sum_{h=0}^{N_3} \frac{w(\mathcal{C}_{d,h})}{w(\mathcal{C}_{0,0})} (1 + O(s^3 t^3 / S^2)).$$

Proof. By Lemma 2.5 and (3.1),

$$\begin{aligned} M(\mathbf{s}, \mathbf{t}) &= \frac{1}{\prod_{i=1}^m s_i! \prod_{j=1}^n t_j!} \sum_{d=0}^{N_2} \sum_{h=0}^{N_3} w(\mathcal{C}_{d,h}) (1 + O(s^3 t^3 / S^2)) \\ &= N(\mathbf{s}, \mathbf{t}) \sum_{d=0}^{N_2} \sum_{h=0}^{N_3} \frac{w(\mathcal{C}_{d,h})}{w(\mathcal{C}_{0,0})} (1 + O(s^3 t^3 / S^2)). \end{aligned} \quad \square$$

We will evaluate the sum in Lemma 3.2 using two summation lemmas proved in [12] and restated below.

Lemma 3.3 ([12, Corollary 4.3]). *Let $0 \leq A_1 \leq A_2$ and $B_1 \leq B_2$ be real numbers. Suppose that there exist integers N, K with $N \geq 2$ and $0 \leq K \leq N$, and a real number $c > 2e$ such that $0 \leq Ac < N - K + 1$ and $|BN| < 1$ for all $A \in [A_1, A_2]$ and $B \in [B_1, B_2]$. Further suppose that there are real numbers δ_i , for $1 \leq i \leq N$, and $\gamma_i \geq 0$, for $0 \leq i \leq K$, such that $\sum_{j=1}^i |\delta_j| \leq \sum_{j=0}^K \gamma_j [i]_j < \frac{1}{5}$ for $1 \leq i \leq N$. Given $A(1), \dots, A(N) \in [A_1, A_2]$ and $B(1), \dots, B(N) \in [B_1, B_2]$, define n_0, n_1, \dots, n_N by $n_0 = 1$ and*

$$\frac{n_i}{n_{i-1}} = \frac{A(i)}{i} (1 - (i-1)B(i)) (1 + \delta_i)$$

for $1 \leq i \leq N$, with the following interpretation: if $A(i) = 0$ then $n_j = 0$ for $i \leq j \leq N$. Then

$$\Sigma_1 \leq \sum_{i=0}^N n_i \leq \Sigma_2,$$

where

$$\begin{aligned} \Sigma_1 &= \exp\left(A_1 - \frac{1}{2}A_1^2B_2 - 4 \sum_{j=0}^K \gamma_j(3A_1)^j\right) - \frac{1}{4}(2e/c)^N, \\ \Sigma_2 &= \exp\left(A_2 - \frac{1}{2}A_2^2B_1 + \frac{1}{2}A_2^3B_1^2 + 4 \sum_{j=0}^K \gamma_j(3A_2)^j\right) + \frac{1}{4}(2e/c)^N. \quad \square \end{aligned}$$

Lemma 3.4 ([12, Corollary 4.5]). *Let $N \geq 2$ be an integer and, for $1 \leq i \leq N$, let real numbers $A(i)$, $B(i)$ be given such that $A(i) \geq 0$ and $1 - (i-1)B(i) \geq 0$. Define $A_1 = \min_{i=1}^N A(i)$, $A_2 = \max_{i=1}^N A(i)$, $C_1 = \min_{i=1}^N A(i)B(i)$ and $C_2 = \max_{i=1}^N A(i)B(i)$. Suppose that there exists a real number \hat{c} with $0 < \hat{c} < \frac{1}{3}$ such that $\max\{A/N, |C|\} \leq \hat{c}$ for all $A \in [A_1, A_2]$, $C \in [C_1, C_2]$. Define n_0, \dots, n_N by $n_0 = 1$ and*

$$\frac{n_i}{n_{i-1}} = \frac{A(i)}{i}(1 - (i-1)B(i))$$

for $1 \leq i \leq N$, with the following interpretation: if $A(i) = 0$ or $1 - (i-1)B(i) = 0$, then $n_j = 0$ for $i \leq j \leq N$. Then

$$\Sigma_1 \leq \sum_{i=0}^N n_i \leq \Sigma_2$$

where

$$\begin{aligned} \Sigma_1 &= \exp\left(A_1 - \frac{1}{2}A_1C_2\right) - (2e\hat{c})^N, \\ \Sigma_2 &= \exp\left(A_2 - \frac{1}{2}A_2C_1 + \frac{1}{2}A_2C_1^2\right) + (2e\hat{c})^N. \quad \square \end{aligned}$$

We obtain bounds on the ratios we require by applying results from [12]. To begin with we focus on the effect of changing the number of triple pairs while keeping the number of double pairs fixed.

Lemma 3.5. *Suppose $0 \leq d \leq N_2$ and $1 < h \leq N_3$, with $\mathcal{C}_{d,h} \neq \emptyset$. If (S_2, T_2) is substantial then*

$$\frac{w(\mathcal{C}_{d,h})}{w(\mathcal{C}_{d,h-1})} = \frac{S_3T_3 + O(s^2t^2(st + d + h)S)}{hS^3}.$$

Proof. This follows from [12, Lemma 4.6] since, for $h \geq 1$,

$$\frac{w(\mathcal{C}_{d,h})}{w(\mathcal{C}_{d,h-1})} = \frac{6|\mathcal{C}_{d,h}|}{|\mathcal{C}_{d,h-1}|}.$$

Note that the values of N_2, N_3 used in this paper are no smaller than, and are at most a constant factor larger than, the values used in [12]. For example, we have $N_3 = \max(\lceil \log S \rceil, \lceil 230000S_3T_3/S^3 \rceil)$, while in [12] the value $\max(\lceil \log S \rceil, \lceil 7S_3T_3/S^3 \rceil)$ was used. Examination of the proof of [12, Lemma 4.6] shows that the bound given there for $|\mathcal{C}_{d,h}|/|\mathcal{C}_{d,h-1}|$ also holds for all $0 \leq d \leq N_2$ and $1 \leq h \leq N_3$. \square

Next, adapting the proof of [12, Corollary 4.7] gives:

Corollary 3.6. *Suppose $0 \leq d \leq N_2$ with $\mathcal{C}_{d,0} \neq \emptyset$. Further suppose that (S_2, T_2) is substantial. Then*

$$\sum_{h=0}^{N_3} \frac{w(\mathcal{C}_{d,h})}{w(\mathcal{C}_{d,0})} = \exp\left(\frac{S_3T_3}{S^3} + O(s^2t^2(st+d)/S^2)\right).$$

Proof. We will apply Lemma 3.4. Let h' be the first value of $h \leq N_3$ for which $\mathcal{C}_{d,h} = \emptyset$, or $h' = N_3 + 1$ if there is no such value. Define $\alpha_h, 1 \leq h < h'$, by

$$\frac{|\mathcal{C}_{d,h}|}{|\mathcal{C}_{d,h-1}|} = \frac{S_3T_3 - \alpha_h(s^2t^2(st+d+(h-1)S))}{hS^3}. \quad (3.2)$$

Lemma 3.5 says that α_h is bounded independently of h, d and S .

For $1 \leq h < h'$, define

$$A(h) = \frac{S_3T_3 - \alpha_h(s^2t^2(st+d)S)}{S^3}, \quad C(h) = \frac{\alpha_h s^2t^2}{S^2}.$$

If $\alpha_h \leq 0$ then by definition $A(h) \geq S_3T_3/S^3$, and $S_3T_3 > 0$ since $h < h'$. Therefore $A(h) > 0$ in this case. If $\alpha_h > 0$ then $C(h) > 0$, which implies that $A(h) > 0$ since the right side of (3.2) has the same sign as $A(h) - (h-1)C(h)$. Therefore $A(h) > 0$ whenever $h < h'$. Define $B(h) = C(h)/A(h)$ for $1 \leq h < h'$. Also define $A(h) = B(h) = 0$ for $h' \leq h \leq N_3$.

Define A_1, A_2, C_1, C_2 by taking the minimum and maximum of the $A(h)$ and $C(h)$ over $1 \leq h \leq N_3$, as in Lemma 3.4. Let $A \in [A_1, A_2]$ and $C \in [C_1, C_2]$, and set $\hat{c} = \frac{1}{41}$. Since $A = S_3T_3/S^3 + o(1)$ and $C = o(1)$, we have that $\max\{A/N_3, |C|\} < \hat{c}$ for S sufficiently large, by the definition of N_3 .

Therefore Lemma 3.4 applies and says that

$$\sum_{h=0}^{N_3} \frac{|\mathcal{C}_{d,h}|}{|\mathcal{C}_{d,0}|} = \exp\left(\frac{S_3T_3}{S^3} + O(s^2t^2(st+d)/S^2)\right) + O((2e/41)^{N_3}).$$

Finally, $(2e/41)^{N_3} \leq (2e/41)^{\log S} \leq S^{-2}$. Since the sum we are estimating is at least equal to one, this additive error term is covered by the error terms inside the exponential. This completes the proof. \square

Now we must sum over pairings with no triple pairs.

Lemma 3.7. *Suppose that (S_2, T_2) is substantial and that $1 \leq d \leq N_2$ with $\mathcal{C}_{d,0} \neq \emptyset$. Then*

$$\frac{w(\mathcal{C}_{d,0})}{w(\mathcal{C}_{d-1,0})} = \frac{2A(d)}{d} (1 - (d-1)B)(1 + \delta_d)$$

where

$$\begin{aligned} A(d) &= \frac{S_2 T_2}{2S^2} \left(1 + \frac{S_2}{S^2} + \frac{T_2}{S^2} + \frac{1}{S} + \frac{2S_3 T_2}{S_2 S^2} + \frac{2S_2 T_3}{S^2 T_2} - \frac{S_3 T_3}{S S_2 T_2} - \frac{2S_2 T_2}{S^3} \right) + O\left(\frac{s^3 t^3}{S^2}\right), \\ B &= \frac{2}{S_2} + \frac{2}{T_2} + \frac{4T_3}{T_2^2} + \frac{4S_3}{S_2^2} - \frac{4}{S}, \\ \delta_d &= O\left(\frac{(d-1)^2 s^2}{S_2^2} + \frac{(d-1)^2 t^2}{T_2^2} + \frac{dst(d+st)}{S_2 T_2}\right). \end{aligned}$$

Proof. This follows from [12, Lemma 4.8] since, for $d \geq 1$,

$$\frac{w(\mathcal{C}_{d,0})}{w(\mathcal{C}_{d-1,0})} = \frac{2|\mathcal{C}_{d,0}|}{|\mathcal{C}_{d-1,0}|}.$$

As in Lemma 3.5, our value of N_2 is no smaller than, and is at most a constant factor larger than, the value used in [12]. Examination of the proof of [12, Lemma 4.8] shows that the expression given there for $|\mathcal{C}_{d,0}|/|\mathcal{C}_{d-1,0}|$ also holds for $1 \leq d \leq N_2$. \square

Adapting the proof of [12, Corollary 4.9] gives the following:

Corollary 3.8. *If (S_2, T_2) is substantial then*

$$\sum_{d=0}^{N_2} \sum_{h=0}^{N_3} \frac{w(\mathcal{C}_{d,h})}{w(\mathcal{C}_{0,0})} = \exp\left(\frac{S_2 T_2}{S^2} + \frac{S_2 T_2}{S^3} + O\left(\frac{s^3 t^3}{S^2}\right)\right).$$

Proof. We need to apply Lemma 3.3 to the result of Lemma 3.7, and take into account the terms coming from the triple pairs (as given by Corollary 3.6).

Let d' be the first value of $d \leq N_2$ for which $|\mathcal{C}_{d,0}| = 0$, or $d' = N_2 + 1$ if no such value of d exists. Define m_0, m_1, \dots, m_{N_2} by

$$m_d = \frac{w(\mathcal{C}_{d,0})}{w(\mathcal{C}_{0,0})} \sum_{h=0}^{N_3} \frac{w(\mathcal{C}_{d,h})}{w(\mathcal{C}_{d,0})}$$

for $0 \leq d < d'$, and $m_d = 0$ for $d' \leq d \leq N_2$. Then clearly

$$\sum_{d=0}^{N_2} \sum_{h=0}^{N_3} \frac{w(\mathcal{C}_{d,h})}{w(\mathcal{C}_{0,0})} = \sum_{d=0}^{N_2} m_d.$$

Corollary 3.6 tells us that for $d < d'$ we have

$$m_d = \frac{w(\mathcal{C}_{d,0})}{w(\mathcal{C}_{0,0})} \exp\left(\frac{S_3 T_3}{S^3} + O(s^3 t^3 / S^2) + \xi_d s^2 t^2 / S^2\right) \quad (3.3)$$

where $\xi_0 = 0$ and in general $\xi_d = O(d)$. (Note that (3.3) is also true for $d' \leq d \leq N_2$, since both sides equal zero.) If α is a constant such that $|\xi_d| \leq \alpha d$ for $0 \leq d \leq d'$, then

$$\exp\left(\frac{S_3 T_3}{S^3} + O(s^3 t^3 / S^2)\right) \sum_{d=0}^{N_2} n_d(-1) \leq \sum_{d=0}^{N_2} m_d \leq \exp\left(\frac{S_3 T_3}{S^3} + O(s^3 t^3 / S^2)\right) \sum_{d=0}^{N_2} n_d(1) \quad (3.4)$$

where

$$n_d(x) = \frac{w(\mathcal{C}_{d,0})}{w(\mathcal{C}_{0,0})} \exp(x\alpha d s^2 t^2 / S^2).$$

Next we note that, for $x \in \{-1, 1\}$, $n_0(x) = 1$, and for $1 \leq d \leq d'$,

$$\frac{n_d(x)}{n_{d-1}(x)} = 2A(d)(1 - (d-1)B)(1 + \delta_d)$$

with $A(d)$, B , and δ_d satisfying the expressions given in the statement of Lemma 3.7. This follows since the factor $\exp(x\alpha d s^2 t^2 / S^2)$ is covered by the error term on $A(d)$. For $d' \leq d \leq N_2$ define $A(d) = 0$.

Now let $A_1 = A_1(x) = \min_d 2A(d)$, $A_2 = A_2(x) = \max_d 2A(d)$, where the maximum and minimum are taken over $1 \leq d \leq N_2$. Also let $B_1 = B_2 = B$, and $K = 3$, and define $c = S^{1/4}$ if $S_2 T_2 < S^{7/4}$ and $c = 41$ otherwise. The conditions of Lemma 3.3 now hold as we will show. Let $A \in [A_1, A_2]$ be arbitrary.

Clearly $c > 2e$. If $S_2 T_2 < S^{7/4}$ then $N_2 = 22$. Using the condition $S_2 T_2 \geq (st)^{3/2} S$ implied by the substantiality of (S_2, T_2) , we find that $Ac = 1 + o(1)$. For $S_2 T_2 \geq S^{7/4}$, $Ac = 41S_2 T_2 / S^2 (1 + o(1))$. It is also easy to check that $BN_2 = o(1)$. Thus, in all cases we have that $Ac < N_2 - 2$ and $|BN_2| < 1$ for sufficiently large S .

If $d = O(S_2 T_2 / S^2)$ then

$$\sum_{d=1}^{N_2} |\delta_d| = O\left(\frac{s^2 S_2 T_2^3}{S^6} + \frac{t^2 S_2^3 T_2}{S^6} + \frac{st S_2^2 T_2^2}{S^6} + \frac{s^2 t^2 S_2 T_2}{S^4}\right) = O\left(\frac{s^3 t^3}{S^2}\right) = o(1),$$

while if $d \leq \lceil \log S \rceil$ then

$$\sum_{d=1}^{N_2} |\delta_d| = O\left(\frac{s^2 \log^3 S}{S_2^2} + \frac{t^2 \log^3 S}{T_2^2} + \frac{st \log^3 S}{S_2 T_2} + \frac{s^2 t^2 \log^2 S}{S_2 T_2}\right) = o(1).$$

Finally, for $1 \leq k \leq N_2$, we have

$$\begin{aligned} \sum_{d=1}^k |\delta_d| &= O\left(\sum_{d=1}^k (d-1)^2 \left(\frac{s^2}{S_2^2} + \frac{t^2}{T_2^2}\right)\right) + O\left(\sum_{d=1}^k \frac{d^2 st}{S_2 T_2}\right) + O\left(\sum_{d=1}^k \frac{ds^2 t^2}{S_2 T_2}\right) \\ &= O\left(k(k-1)(2k-1) \left(\frac{s^2}{S_2^2} + \frac{t^2}{T_2^2}\right) + \frac{k(k+1)(2k+1)st}{S_2 T_2} + \frac{k(k+1)s^2 t^2}{S_2 T_2}\right) \\ &\leq \sum_{j=0}^K \gamma_j [k]_j, \end{aligned}$$

where

$$\gamma_0 = 0, \quad \gamma_1 = O\left(\frac{s^2 t^2}{S_2 T_2}\right), \quad \gamma_2 = O\left(\frac{s^2}{S_2^2} + \frac{t^2}{T_2^2} + \frac{s^2 t^2}{S_2 T_2}\right), \quad \gamma_3 = O\left(\frac{s^2}{S_2^2} + \frac{t^2}{T_2^2} + \frac{st}{S_2 T_2}\right).$$

Since $N_2^3(s^2/S_2^2 + t^2/T_2^2 + st/S_2 T_2) = o(1)$, which is easily checked, it follows that $\sum_{j=0}^K \gamma_j [k]_j < 1/5$ for $1 \leq k \leq N_2$, when S is large enough.

Therefore the conditions of Lemma 3.3 hold, and we conclude that each of the bounds given by that lemma for $\sum_{d=0}^{N_2} n_d(x)$ has the form

$$\exp\left(A - \frac{1}{2} A^2 B + O\left(A^3 B^2 + \sum_{j=0}^3 \gamma_j (3A)^j\right)\right) + O((2e/c)^{N_2}),$$

where A is either A_1 or A_2 . A somewhat tedious check shows that

$$O(A^3 B^2) + \sum_{j=0}^3 \gamma_j (3A)^j = O(s^3 t^3 / S^2).$$

Next consider the error term $O((2e/c)^{N_2})$. If $N_2 = 22$ then $(2e/c)^{N_2} = (2eS^{-1/4})^{22} = O(S^{-2})$, while in the other cases we have $(2e/c)^{N_2} = (2e/41)^{N_2} \leq (2e/41)^{\log S} = O(S^{-2})$. Since $n_0 = 1$, this additive error term is covered by a relative error of the same form. Therefore, each of the bounds on $\sum_{d=0}^{N_2} n_d(x)$ has the form

$$\exp\left(A - \frac{1}{2} A^2 B + O\left(\frac{s^3 t^3}{S^2}\right)\right) = \exp\left(\frac{S_2 T_2}{S^2} + \frac{S_2 T_2}{S^3} - \frac{S_3 T_3}{S^3} + O\left(\frac{s^3 t^3}{S^2}\right)\right).$$

Modulo the given error terms, the final expression does not depend on x , nor on whether we are taking a lower bound or upper bound in Lemma 3.3. To complete the proof, just apply (3.4). \square

Corollary 3.8 and Lemma 3.2 together prove Theorem 1.3 in the substantial case. The insubstantial case was already proved in Lemma 3.1.

4 Alternative formulation

We now derive an alternative formulation of Theorem 1.3. Recall the definition of $\hat{\mu}_k$ and $\hat{\nu}_k$ given in the Introduction.

Corollary 4.1. *Under the conditions of Theorem 1.3,*

$$\begin{aligned}
 M(\mathbf{s}, \mathbf{t}) &= \frac{\prod_{i=1}^m \binom{n+s_i-1}{s_i} \prod_{j=1}^n \binom{m+t_j-1}{t_j}}{\binom{mn+S-1}{S}} \\
 &\times \exp\left((1-\hat{\mu}_2)(1-\hat{\nu}_2) \left(\frac{1}{2} + \frac{3-\hat{\mu}_2\hat{\nu}_2}{4S} \right) \right. \\
 &\quad - \frac{(1-\hat{\mu}_2)(3+\hat{\mu}_2-2\hat{\mu}_2\hat{\nu}_2)}{4n} - \frac{(1-\hat{\nu}_2)(3+\hat{\nu}_2-2\hat{\mu}_2\hat{\nu}_2)}{4m} \\
 &\quad \left. + \frac{(1-3\hat{\mu}_2^2+2\hat{\mu}_3)(1-3\hat{\nu}_2^2+2\hat{\nu}_3)}{12S} + O\left(\frac{s^3t^3}{S^2}\right) \right).
 \end{aligned}$$

Proof. By Stirling's formula or otherwise,

$$\binom{N+x-1}{x} = \frac{N^x}{x!} \exp\left(\frac{[x]_2}{2N} - \frac{[x]_3}{6N^2} - \frac{[x]_2}{4N^2} + O(x^4/N^3) \right)$$

as $N \rightarrow \infty$, provided that the error term is bounded. This gives us the approximations

$$\begin{aligned}
 \prod_{i=1}^m \binom{n+s_i-1}{s_i} &= \frac{n^S}{\prod_i s_i!} \exp\left(\frac{S_2}{2n} - \frac{S_2}{4n^2} - \frac{S_3}{6n^2} + O\left(\frac{s^3t^3}{S^2}\right) \right) \\
 \prod_{j=1}^n \binom{m+t_j-1}{t_j} &= \frac{m^S}{\prod_j t_j!} \exp\left(\frac{T_2}{2m} - \frac{T_2}{4m^2} - \frac{T_3}{6m^2} + O\left(\frac{s^3t^3}{S^2}\right) \right) \\
 \binom{mn+S-1}{S} &= \frac{(mn)^S}{S!} \exp\left(\frac{S^2}{2mn} - \frac{S}{2mn} - \frac{S^3}{6m^2n^2} + O\left(\frac{s^3t^3}{S^2}\right) \right).
 \end{aligned}$$

Substitute these expressions into Theorem 1.3 and replace S_2, S_3, T_2, T_3 by their equivalents in terms of $\hat{\mu}_2, \hat{\mu}_3, \hat{\nu}_2, \hat{\nu}_3$. The desired result is obtained. \square

As noted in the Introduction, Theorem 1.3 establishes the conjecture recalled after Theorem 1.1 in some cases. Using Corollary 4.1, the following is easily seen. (Note that $\hat{\mu}_2 = \hat{\mu}_3 = \hat{\nu}_2 = \hat{\nu}_3 = 0$ in the semiregular case.)

Corollary 4.2. *If $s = s(m, n)$ and $t = t(m, n)$ satisfy $ms = nt$ and $st = o((mn)^{1/5})$, then*

$$\Delta(m, s; n, t) = \frac{5(s+t)}{6st} (1 + o(1)). \quad \square$$

Most of the terms inside the exponential of Corollary 4.1 are tiny unless at least one of $\hat{\mu}_2, \hat{\nu}_2$ is quite large (that is, the graph is very far from semiregular). In particular we can now prove Corollary 1.5 which was stated in the Introduction.

Proof of Corollary 1.5. It is only necessary to check that the additional terms in Corollary 4.1 have the required size. It helps to realise that $\hat{\mu}_2 \leq s$, $|\hat{\mu}_3| \leq s\hat{\mu}_2$, $\hat{\nu}_2 \leq t$ and $|\hat{\nu}_3| \leq t\hat{\nu}_2$. \square

A random nonnegative $m \times n$ matrix with entries summing to S is just a random composition of S into mn parts. (A composition is an ordered sum of nonnegative numbers.) In particular, for $1 \leq i \leq m$ the row sum s_i satisfies

$$\Pr(s_i = k) = \binom{k+n-1}{k} \binom{S-k+(m-1)n-1}{S-k} / \binom{S+mn-1}{S} \quad (0 \leq k \leq S).$$

From this we can compute the following expected values.

$$\begin{aligned} \mathbb{E} \hat{\mu}_2 &= \frac{n(m-1)}{mn+1}, & \mathbb{E} \hat{\nu}_2 &= \frac{m(n-1)}{mn+1}, \\ \mathbb{E} \hat{\mu}_3 &= \frac{n(m-1)(m-2)(mn+2S)}{m(mn+1)(mn+2)}, & \mathbb{E} \hat{\nu}_3 &= \frac{m(n-1)(n-2)(mn+2S)}{n(mn+1)(mn+2)}. \end{aligned}$$

The first two expectations suggest that the argument of the exponential in Corollary 1.5 is close to 0 with high probability for such a random matrix. We will prove this in a future paper, and note that the result gives a model for the row and column sums of random matrices.

5 Restricted sets of allowed entries

Given a subset \mathcal{J} of the nonnegative integers, let $\mathcal{M}(\mathbf{s}, \mathbf{t}, \mathcal{J})$ denote the set of matrices in $\mathcal{M}(\mathbf{s}, \mathbf{t})$ with all entries in the set \mathcal{J} . Let $M(\mathbf{s}, \mathbf{t}, \mathcal{J}) = |\mathcal{M}(\mathbf{s}, \mathbf{t}, \mathcal{J})|$. By generalising the techniques of the preceding sections, we can find an asymptotic expression for $M(\mathbf{s}, \mathbf{t}, \mathcal{J})$ whenever $0, 1 \in \mathcal{J}$.

Lemma 5.1. *Let $\mathcal{J} \subseteq \mathbb{N}$ with $0, 1 \in \mathcal{J}$. Define $\chi_2 = 0$ if $2 \notin \mathcal{J}$, $\chi_2 = 1$ if $2 \in \mathcal{J}$, and similarly χ_3 . Then*

$$\begin{aligned} M(\mathbf{s}, \mathbf{t}, \mathcal{J}) &= N(\mathbf{s}, \mathbf{t}) \exp\left(\chi_2 \frac{S_2 T_2}{S^2} + \chi_2 \frac{S_2 T_2}{S^3} + (\chi_3 - \chi_2) \frac{S_3 T_3}{S^3} + O\left(\frac{s^3 t^3}{S^2}\right)\right) \\ &= \frac{S!}{\prod_{i=1}^m s_i! \prod_{j=1}^n t_j!} \exp\left(\left(\chi_2 - \frac{1}{2}\right) \frac{S_2 T_2}{S^2} + \left(\chi_2 - \frac{1}{2}\right) \frac{S_2 T_2}{S^3} + \left(\chi_3 - \chi_2 + \frac{1}{3}\right) \frac{S_3 T_3}{S^3} \right. \\ &\quad \left. - \frac{S_2 T_2 (S_2 + T_2)}{4S^4} - \frac{S_2^2 T_3 + S_3 T_2^2}{2S^4} + \frac{S_2^2 T_2^2}{2S^5} + O\left(\frac{s^3 t^3}{S^2}\right)\right). \end{aligned}$$

Proof. Our general approach will be similar to that we used for Theorem 1.3, but the methods of Section 2 will need significant modification. The source of the problem is that a D -switching may introduce an entry that is not in \mathcal{J} .

For $Q \in \mathcal{M}(\mathbf{s}, \mathbf{t})$ and $i \geq 1$, let $n_i(Q)$ be the number of entries of Q equal to i . Also let $n_{\geq 5}(Q) = \sum_{i \geq 5} n_i(Q)$. Define N_2 and N_3 as in Section 2 when (S_2, T_2) is substantial, and $N_2 = 2$ and $N_3 = 1$ when (S_2, T_2) is insubstantial. For $Q \in \mathcal{M}^+$, let

$$E^+(Q) = \sum_{D > \lceil (st)^{1/4} \rceil} n_D(Q), \quad E^-(Q) = \sum_{D=5}^{\lceil (st)^{1/4} \rceil} n_D(Q).$$

Consider the following subsets of $\mathcal{M}(\mathbf{s}, \mathbf{t})$:

$$\begin{aligned} \mathcal{M}^+ &= \mathcal{M}(\mathbf{s}, \mathbf{t}, \mathcal{J} \cup \{4, 5, 6, \dots\}), \\ \mathcal{M} &= \mathcal{M}(\mathbf{s}, \mathbf{t}, \mathcal{J}), \\ \mathcal{M}^- &= \{Q \in \mathcal{M}^+ \mid n_2(Q), n_3(Q), n_4(Q) \leq S^{5/6}, \\ &\quad E^-(Q) \leq \lceil 2(stS)^{1/2} \rceil, E^+(Q) \leq \lceil 2(st)^{1/4} S^{1/2} \rceil\}, \\ \mathcal{M}^* &= \{Q \in \mathcal{M}(\mathbf{s}, \mathbf{t}, \mathcal{J} \cap \{0, 1, 2, 3\}) \mid n_2(Q) \leq N_2, n_3(Q) \leq N_3\}. \end{aligned}$$

Also define the cardinalities M^+, M, M^-, M^* , respectively. By monotonicity, we have $M^* \leq M \leq M^+$ and $M^* \leq M^- \leq M^+$.

We now employ switchings to establish that $M^+ - M^- < M^- - M^*$ and $M^- - M^* = O(s^3 t^3 / S^2) M^*$, from which it follows that $M = (1 + O(s^3 t^3 / S^2)) M^*$. **(OK?)**

We start with the claim that $M^+ - M^- < M^- - M^*$. Let $Q \in \mathcal{M}^+ - \mathcal{M}^-$ such that $E^+(Q) > \lceil 2(st)^{1/4} S^{1/2} \rceil$. We will use the following switching, illustrated by this operation performed on submatrices:

$$\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \mapsto \begin{pmatrix} D_1 - 1 & 1 \\ 1 & D_2 - 1 \end{pmatrix} \quad (5.1)$$

where $D_1, D_2 \geq (st)^{1/4}$. The number of forward switchings is bounded below by

$$E^+(Q)^2 - O(stE^+(Q)) = E^+(Q)(1 - o(1)),$$

and the number of reverse switchings is bounded above by

$$\frac{2stS}{(st)^{1/2}} = 2\sqrt{st}S.$$

Hence the number of reverse switchings divided by the number of forward switchings is bounded above by

$$\frac{2(1 + o(1))\sqrt{st}S}{E^+(Q)^2} \leq \frac{1 + o(1)}{2} < \frac{2}{3},$$

using the assumed lower bound on $E^+(Q)$. After repeatedly applying this switching, we reach a matrix Q which satisfies

$$E^+(Q) \leq \lceil 2(st)^{1/4} S^{1/2} \rceil. \quad (5.2)$$

The next switching is applied to matrices $Q \in \mathcal{M}^+$ for which (5.2) holds but $E^-(Q) > \lceil 2(stS)^{1/2} \rceil$. The switching that we used is the same as that shown in (5.1) except that now $D_1, D_2 \in \{5, \dots, \lceil (st)^{1/4} \rceil\}$. The number of forward switchings is bounded below by

$$E^-(Q)^2 - O(stE^-(Q)) = E^-(Q)^2(1 - o(1)),$$

and the number of reverse switchings is bounded above by

$$2stS.$$

Hence the number of reverse switchings divided by the number of forward switchings is bounded above by

$$\frac{2(1 + o(1))stS}{E^-(Q)^2} \leq \frac{1 + o(1)}{2} < \frac{2}{3}.$$

We apply this switching until we reach a matrix Q which satisfies both (5.2) and

$$E^-(Q) \leq \lceil 2(stS)^{1/2} \rceil. \quad (5.3)$$

To analyse these two switchings using Theorem 2.1, we can define the sets

$$C(i) = \{Q \in \mathcal{M}^+ - \mathcal{M}^- \mid \sum_{D \geq 2} Dn_D(Q) = i\}.$$

If $Q \in C(i)$ and R can be obtained from Q using one of the switchings described above, then $R \in C(i - 2)$. This leads to an acyclic directed graph in each case, and we have shown above that all the ratios $b(v_i)/a(v_{i-1})$ in Theorem 2.1 are at most $2/3$.

Next we need to reduce each of $n_2(Q), n_3(Q), n_4(Q)$ to below $S^{5/6}$. We achieve this using a succession of three types of switchings, illustrated by the following operations on submatrices: for example, the switching

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

will be used to reduce $n_4(Q)$ (with analogous operations for $D = 2, 3$). First we apply the switching for $D = 4$ until $n_4(Q) \leq S^{5/6}$, then the switching for $D = 3$ until $n_3(Q) \leq S^{5/6}$, finally applying the switching for $D = 2$ until $n_2(Q) \leq S^{5/6}$. As a representative example,

take the switching for $D = 4$. By counting similarly to Lemma 2.2, this switching can be applied to Q in at least $(n_4(Q) - O(st))^4$ ways, and the inverse can be applied in at most Ss^3t^3 ways. For $n_4(Q) > S^{5/6}$, the condition $s^3t^3 = o(S^2)$ implies that $Ss^3t^3 = o((n_4(Q) - O(st))^4)$, so the ratios denoted by $b(v_i)/a(v_{i-1})$ in Theorem 2.1 are all $o(1)$.

Since none of the switchings can undo the work of a previous switching, the end result is a matrix $\mathcal{M}^- \setminus \mathcal{M}^*$. (Note that in the resulting matrix R , at least one of $E^+(R)$, $E^-(R)$, $n_2(R)$, $n_3(R)$ or $n_4(R)$ will be just under the threshold value. This implies that $R \notin \mathcal{M}^*$.) This establishes the bound $M^+ - M^- < M^- - M^*$.

For any matrix $Q \in \mathcal{M}^- \setminus \mathcal{M}^*$ we have

$$\sum_{D \geq \lceil (st)^{1/4} \rceil} Dn_D(Q) \leq \min\{s, t\}E^+(Q) \leq 3(st)^{3/4}S^{1/2},$$

since using (5.2) and since $\min\{s, t\} \leq (st)^{1/2}$. Similarly, (5.3) implies that

$$\sum_{D=5}^{\lceil (st)^{1/4} \rceil} Dn_D(Q) \leq \lceil (st)^{1/4} \rceil E^-(Q) \leq 3(st)^{3/4}S^{1/2},$$

which leads to

$$\sum_{D \geq 2} Dn_D(Q) \leq 6(st)^{3/4}S^{1/2} + 3S^{5/6} = o(S).$$

Hence when $Q \in \mathcal{M}^- \setminus \mathcal{M}^*$, we know that $n_1(Q) = S - o(S)$. We can now continue precisely as in Lemmas 2.3, 2.5, using D -switchings restricted to $q_1 = \dots = q_D = 1$. This restriction ensures that D -switchings only create entries with value equal to 0 or 1. The various switching counts can be taken as essentially the same as before, since all but a vanishing fraction of the non-zero entries are 1. We conclude that $M^- - M^* = O(s^3t^3/S^2)M^*$ which, as noted above, implies that $M = (1 + O(s^3t^3/S^2))M^*$.

Having now reduced the task to evaluation of M^* , we can complete the proof following Lemma 3.1 in the insubstantial case, and Section 4 in the substantial case. In Lemma 3.1 the only modification is to replace the expression $p_0 + 2p_1 + 4p_2 + 6p_3$ by $p_0 + 2\chi_2 p_1 + 4\chi_2 p_2 + 6\chi_3 p_3$.

Now suppose that (S_2, T_2) is substantial. If $\chi_2 = \chi_3$ then the result is given by either Theorem 1.2 or Theorem 1.3. If $\chi_2 = 0$ and $\chi_3 = 1$ then the result follows from applying Corollary 3.6 with $d = 0$, since arguing as in Lemma 3.2 shows that

$$M(\mathbf{s}, \mathbf{t}, \mathcal{J}) = N(\mathbf{s}, \mathbf{t}) \sum_{h=0}^{N_3} \frac{w(\mathcal{C}_{0,h})}{w(\mathcal{C}_{0,0})} (1 + O(s^3t^3/S^2))$$

in this case. Finally, if $\chi_2 = 1$ and $\chi_3 = 0$ then

$$M(\mathbf{s}, \mathbf{t}, \mathcal{J}) = N(\mathbf{s}, \mathbf{t}) \sum_{d=0}^{N_2} \frac{w(\mathcal{C}_{d,0})}{w(\mathcal{C}_{0,0})} (1 + O(s^3t^3/S^2))$$

so in place of (3.3) we simply have $m_d = w(\mathcal{C}_{d,0})/w(\mathcal{C}_{0,0}) = n_d(0)$ for $0 \leq d \leq N_2$. The remainder of the proof is identical except that there is no need to apply (3.4) at the end. \square

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