

Counting polygon dissections in the projective plane

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Abstract

For each value of $k \geq 2$, we determine the number p_n of ways of dissecting a polygon in the projective plane into n subpolygons with $k + 1$ sides each. In particular, if $k = 2$ we recover a result of Edelman and Reiner (1997) on the number of triangulations of the Möbius band having n labelled points on its boundary. We also solve the problem when the polygon is dissected into subpolygons of arbitrary size. In each case, the associated generating function $\sum p_n z^n$ is a rational function in z and the corresponding generating function of plane polygon dissections. Finally, we obtain asymptotic estimates for the number of dissections of various kinds, and determine probability limit laws for natural parameters associated to triangulations and dissections.

MSC Classification. 05A, 05C30.

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1 Introduction

Counting polygon dissections in the plane is a well studied problem, beginning with the classical result of Euler that the number of triangulations of an $(n + 2)$ -gon whose vertices are labelled $1, 2, \dots, n$ is the Catalan number C_n . The result can also be phrased in terms of triangulations of a polygon in the sphere in which the diagonals are all of them either inside or outside the polygon. The aim of this paper is to start a similar study in surfaces other than the sphere.

Let \mathcal{P} denote the real projective plane, obtained by adding a cross-cap to the sphere. We fix a polygon Q in \mathcal{P} , that is, a simple contractible closed curve, in which n points are labelled $1, 2, \dots, n$ circularly. By a triangulation of Q we mean a 2-cell decomposition of the outside of Q into triangles using as vertices only the n labelled points, such that two intersecting triangles meet only in a common vertex or in a common edge. In the sequel by a triangulation we mean a triangulation of Q .

There are two possible ways of drawing Q and the cross-cap: either Q is drawn inside the cross-cap and the edges of a triangulation are outside Q , or the cross-cap is drawn inside Q and the edges of a triangulation are inside Q (see Figure 1). Throughout this paper we stick to the second representation, that is, the non-boundary edges of a decomposition of Q are drawn inside. For such an edge we have three possibilities: either it crosses the cross-cap, or it leaves the cross-cap to the right or to the left. In principle there are many more possibilities, since an edge can reenter the cross-cap several times, but from a combinatorial point of view there are

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only these three cases: the proof of this fact can be found in [2] and is based on the fact that the fundamental group of the projective plane is cyclic of order two.

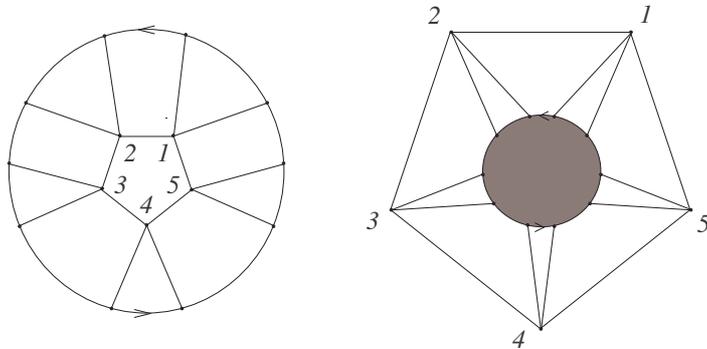


Figure 1: Two representations of the unique triangulation of a pentagon in the projective plane.

More generally, we are interested in dissections into quadrangles, pentagons, and so on, and in dissections into cells of any size. In each case we require that two cells of a dissection intersect only at a vertex or at an edge. Since removing the interior of a simple closed contractible curve in \mathcal{P} we obtain a Möbius band, our problem is equivalent to that of dissecting a Möbius band, where n points labelled $1, 2, \dots, n$ are placed in the boundary. The number of triangulations of the Möbius band in this sense was first determined in [2]. We reprove the same result with a different approach, using the symbolic method for handling generating functions [4]. We believe our proof is more transparent and moreover this approach allows us to solve other related problems which appear difficult to obtain using recurrence equations as in [2].

In Section 2 we present our derivation for computing the number of triangulations. In Section 3 we obtain the number of dissections into $(k + 1)$ -gons for each value of $k \geq 2$; the cases $k = 3$ and $k = 4$ are somehow exceptional and need a special treatment. And in Section 4 we compute the number of dissections into arbitrary cells. In each case our result gives a closed form for the corresponding generating function (GF for short), which is always a rational function of the independent variable z and the corresponding GF of plane polygon dissections.

In Section 5 we obtain precise asymptotic estimates for the numbers of polygon dissections of various kinds. Finally, in Section 6 we derive limit laws for two parameters of interest: the number of cyclic triangles (to be defined later on) in triangulations; and the number of cells in arbitrary polygon dissections. In the second case we obtain a classical normal law, whereas in the first case the limit law is the *absolute value* of a normal law.

For more general surfaces it is possible to obtain the asymptotic number of simplicial triangulations depending on the genus and the number of components of the boundary [1]. The case of arbitrary maps was treated in [5], which contains many interesting results.

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Note added. After this paper was submitted for publication, we have learned of an unpublished preprint [6] which contains a proof of Theorem 2.1 very similar to ours. It also contains

an estimate for the expected value of the random variable discussed in Section 6.1 in agreement with ours.

1.1 Preliminaries

First notice that in a triangulation τ the number of triangles equals the number of vertices. This follows from Euler's formula

$$v - e + f = 1$$

for a 2-cell decomposition of the projective plane and double counting incidences between edges and faces. More generally, in a dissection into $(k+1)$ -gons, the number of vertices is $(k-1)n$, where n is the number of $(k+1)$ -gons. Finally, in an arbitrary dissection, the number of cells is equal to the number of internal edges.

The generating function $C(z)$ of plane triangulations of a polygon whose vertices are labelled $1, 2, \dots, n$ circularly, where z marks a triangle, is the GF of Catalan numbers and satisfies

$$C = 1 + zC^2.$$

Similarly, if z marks a $(k+1)$ -gon, the generating function $L(z)$ of plane dissections into $(k+1)$ -gons satisfies

$$L = 1 + zL^k$$

and

$$[z^n]L(z) = \frac{1}{(k-1)n+1} \binom{kn}{n}$$

Finally, the bivariate generating function $D(z)$ of plane dissections of a polygon, where z now marks a vertex of the polygon to be dissected and u marks a region, satisfies

$$(1+u)D^2 - z(1+z)D + z^3 = 0. \tag{1}$$

See, for instance, [3] for a simple proof.

2 Triangulations

In this section we recover a result from [2]. Our approach is similar but we deal directly with the generating functions involved, thus avoiding working with recurrence relations with one and two indices. The purpose of including a new proof is to exemplify in a simple case the tools we use in later sections.

The expression for the GF $P(z)$ obtained in [2] is written differently from ours but it can be checked they are in fact the same. The sequence $1, 14, 113, 720, \dots$ counting triangulations is A007817 in the On-Line Encyclopedia of Integer Sequences.

Theorem 2.1 *Let p_n be the number of triangulations of a polygon with n vertices in the projective plane, and let $C(z)$ be the generating function for the Catalan numbers. Then*

$$\begin{aligned} P(z) = \sum_{n \geq 5} p_n z^n &= \frac{(2 - 9z + 6z^2 + 7z^3 - 2z^4)C(z) - (2 - 7z + z^2 + 5z^3)}{z(1 - 4z)} \\ &= z^5 + 14z^6 + 113z^7 + 720z^8 + 4033z^9 + \dots \end{aligned}$$

Proof. Let τ be a triangulation of an n -gon, and let $12x$ be the unique triangle of τ that contains the side 12 of the polygon. In principle this triangle can appear as a very involved closed curve in the projective plane, but in fact there are only three possibilities from a combinatorial point of view, as shown in Figure 2; the proof that this is indeed the case is again based on the fact that the fundamental group of the projective plane is cyclic of order two. This implies the following equation

$$P(z) = z(2C(z)P(z) + T(z)),$$

where $T(z)$ is the GF associated to triangulations of the region T indicated in Figure 2, $C(z)$ is the GF of plane triangulations, and the factor z takes account of the root polygon. Thus we have

$$P(z) = \frac{zT(z)}{1 - 2zC(z)}, \quad (2)$$

and it only remains to compute $T(z)$.

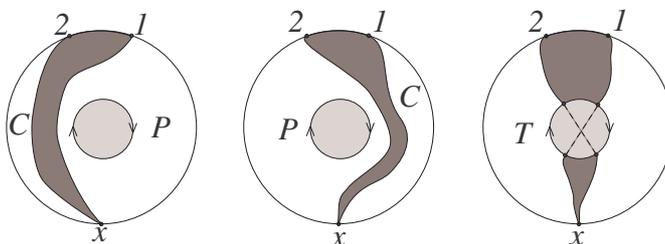


Figure 2: Three cases for the root polygon $12x$, shown shaded.

By a “cut and paste” argument, it is clear that region T is homeomorphic to a polygon with $n + 1$ vertices (if there are n vertices in region T) in which two vertices labelled x are identified. For triangulating T it is necessary that in the arc between vertex 1 and the non-adjacent copy of vertex x there is at least one vertex, and similarly in the arc between vertex 2 and the other copy of x (see Figure 3), since otherwise we cannot triangulate the polygon in a way compatible with the root triangle $12x$. It follows that region T must contain at least 6 vertices or, equivalently, 4 triangles are needed for triangulating it.

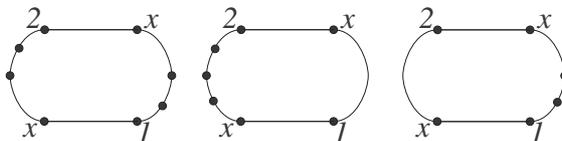


Figure 3: The first case can be extended to a valid triangulation; in the two other cases, any triangulation would not be compatible with the root triangle $12x$.

We must find the number of plane triangulations of an $(n + 1)$ -gon in which two points are identified that give rise to compatible triangulations with the root triangle $12x$ in the projective plane. To this end, we compute the total number of triangulations and subtract, using the principle of inclusion-exclusion, a complete set of forbidden configurations.

Let $C_r = C - \sum_{i=0}^{r-1} c_i z^i$ be the GF of plane triangulations with at least r triangles; observe that C_r is just a truncation of the Catalan series $C(z)$. The total number of triangulations of region T is counted by C_4 (as noticed before, we need at least four triangles), modified in order to mark the double vertex x . Another way to put it is the following: we have to split $n - 3 = (n + 1) - 4$ points between the arcs $\widehat{1x}$ and $\widehat{2x}$ in such a way that there is at least one point in each arc, and this can be done in $n - 3$ different ways. Hence the associated generating function is $\Theta(C_4) - 3C_4$, where $\Theta(f(z)) = zf'(z)$ is the “pointing” operator.

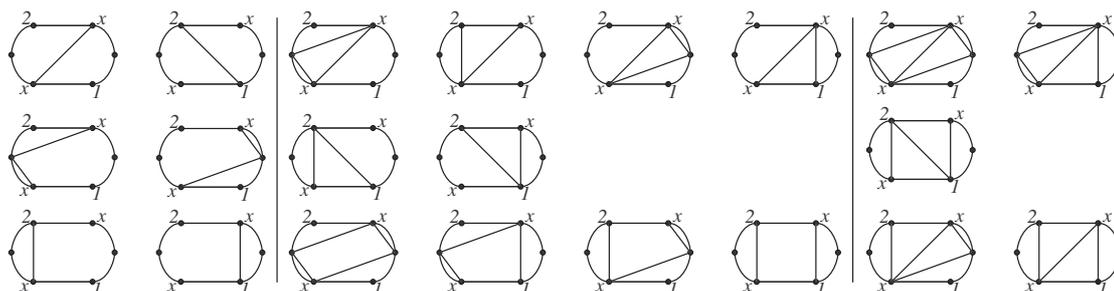


Figure 4: From left to right: forbidden configurations, pairwise intersections and triple-wise intersections.

On the other hand, the forbidden configurations are those shown on the left of Figure 4: edges $12, 1x, 2x$ and xx cannot be used, since either of them gives rise to a triangle sharing exactly two vertices with the root polygon; and no point y can be joined to both copies of x , since we would have two triangles sharing only vertices x and y .

The pairwise intersections of these configurations are shown in the middle of the figure, and finally the triple-wise intersections are on the right of the figure. The GFs associated to each configuration are computed easily: for instance, the presence of edge xx (first configuration) leaves us with two triangulations with at least two triangles each, hence the term C_2^2 . Similarly, when there is a triangle we add a factor z , and where there is a quadrangle we add a factor $2z^2$ (there are two ways of triangulating a quadrangle). Applying symbolically the principle of inclusion-exclusion, we arrive at

$$\begin{aligned}
 T &= \Theta(C_4) - 3C_4 \\
 &\quad - (2C_2^2 + 2(1 + C_0)C_1C_3) \\
 &\quad + (2z(2 + C_0)C_1C_2 + 2z^2C_1^2(1 + C_0)^2) \\
 &\quad - (z^2C_1^2 + z^2C_1^2(1 + C_0)^2).
 \end{aligned}$$

A simple calculation, together with Equation (2) and the fact that $C'(z) = C(z)^2(1 - 2zC(z))^{-1}$, gives the result as claimed. \square

3 Dissections into r -gons

In this section we study cellular dissections (or decompositions) of a polygon in the projective plane where all the cells are r -polygons, i.e., polygons with r edges; we call such a decomposition simply an r -dissection. The case of triangulations (3-polygons) has been treated in the previous

section. For r -dissections the essence is the same: we find a combinatorial encoding of the cellular decompositions, we compute generating functions associated to dissections of some particular regions, we single out forbidden configurations, and we put everything together to obtain the desired generating function.

Let $k > 2$ be an arbitrary but fixed integer. We use the same notation as in the case of triangulations: P is the generating function of projective $(k + 1)$ -dissections, $L = \sum_{n \geq 0} l_n z^n$ the GF of planar $(k + 1)$ -dissections and $L_s = \sum_{n \geq s} l_n z^n$, the truncation of L of order s . Recall that L satisfies $L = 1 + zL^k$, where z marks a cell in the dissection. As opposed to the case of triangulations, in a $(k + 1)$ -dissection the number of vertices cannot be arbitrary; it must be of the form $(k - 1)n$, where n is the number of cells in the dissection. It is necessary also that $n \geq 3$.

Theorem 3.1 *Let $k > 4$ be fixed. Let p_n be the number of dissections into n $(k + 1)$ -gons of a polygon with $(k - 1)n$ vertices in the projective plane, and let $L(z)$ be the generating function for plane dissections into $(k + 1)$ -gons, which satisfies $L = 1 + zL^k$. Then*

$$P(z) = \sum_{n \geq 3} p_n z^n = \frac{(k - 1)(L - 1)\alpha}{2L^3(L - kL + k)^2},$$

where

$$\begin{aligned} \alpha = & (4k^3 - 12k^2 + 12k - 4)L^6 + (-14k^3 + 46k^2 - 38k + 6)L^5 + (18k^3 - 72k^2 + 42k)L^4 \\ & + ((k^3 - 6k^2 + 5k)z + (-10k^3 + 68k^2 - 25k - 4))L^3 \\ & + ((-k^3 + 5k^2)z + (2k^3 - 47k^2 + 9k + 4))L^2 + (21k^2 + 2k - 2)L - (4k^2 + 2k). \end{aligned}$$

For $k = 3$ and $L = 1 + zL^3$ we have

$$\begin{aligned} P(z) &= \frac{(128z^2 - 216z + 32)L^2 + (-105z^2 + 263z - 32)L + (18z^2 - 79z)}{4 - 27z} \\ &= z^3 + 25z^4 + 348z^5 + 3703z^6 + 34240z^7 + 291485z^8 + 2353422z^9 + \dots \end{aligned}$$

and for $k = 4$ and $L = 1 + zL^4$ we have

$$\begin{aligned} P(z) &= 3 \frac{(316z^2 - 84z)L^3 + (80z^2 + 1454z - 162)L^2 + (-64z^2 - 2574z + 261)L + (1267z - 99)}{256z - 27} \\ &= 12z^3 + 336z^4 + 5499z^5 + 73302z^6 + 880548z^7 + 9951336z^8 + 108136104z^9 + \dots \end{aligned}$$

Proof. Let γ be a dissection into $(k + 1)$ -gons. As in the case of triangulations, we fix the $(k + 1)$ -polygon in γ which has the edge 12 as the root, and consider the possible ways in which this polygon crosses the cross-cap. In Figure 5 we show the basic cases for 4-dissections: for instance, in the first picture, the cross-cap could be instead on the left or right white region; and in the third picture, the “leg” to the right of the cross-cap could be instead to the left. The combinatorial definition of region T is the same as in the case of triangulations. Region S is similar but now there are no repeated points.

This gives the equation

$$P = z \left(kPL^{k-1} + (k - 1)L^{k-2}T + \binom{k-1}{2}SL^{k-2} \right),$$

where S and T stand for the GFs associated to the number of compatible dissections of regions S and T , respectively. The term kPL^{k-1} arises because there are k slots for placing the cross-cap;

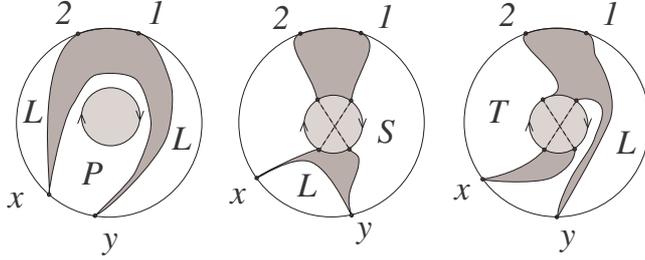


Figure 5: Different combinatorial cases for 4-dissections.

the term $(k-1)L^{k-2}T$ since there are $k-1$ choices for x ; and the last term because there are $\binom{k-1}{2}$ choices for x and y . As before, the factor z indicates the root polygon. Solving for P we have

$$P = \frac{zL^{k-2}(2(k-1)T + (k^2 - 3k + 2)S)}{2(1 - zL^{k-1})}.$$

Hence in order to find P we must compute S and T . Denote by E the GF of planar $(k+1)$ -dissections incompatible with the root polygon (we call them *externally incompatible*); and by I the GF of planar $(k+1)$ -dissections which are compatible with the root polygon, but *internally incompatible* because of the existence of a repeated point. Both E and I consist of forbidden configurations.

The total number of possible $(k+1)$ -dissections of S and T is counted by $(k-1)\Theta(L_2) - 3L_2$, where $\Theta(f(z)) = zf'(z)$. The argument is the same as in the case of triangulations, with two differences:

1. We take L_2 instead of C_4 because now the minimum number of polygons needed to dissect the projective plane is 3 instead of 5.
2. We consider the planar relation $(k-1)n + 2 = v$, where v denotes the number of vertices instead of $n + 2 = v$, the latter being a particular case when $k = 2$ (triangulations).

By construction, a forbidden dissection in S comes from an externally incompatible dissection, so that

$$S = (k-1)\Theta(L_2) - 3L_2 - E.$$

In the case of T , a forbidden dissection comes from either an externally incompatible dissection or an internally incompatible dissection, so that

$$T = (k-1)\Theta(L_2) - 3L_2 - E - I = S - I.$$

We first find the generating function for E . The possible forbidden configurations are those shown in Figure 6, but there is an essential difference with Figure 4. The incompatibility with the root polygon is produced if any of $2x$, $1y$, 12 or xy are either edges or diagonals of some cell. A solid line indicates that it is an edge of some $(k+1)$ -polygon, and a dashed edge that it is a diagonal; and otherwise it is neither an edge nor a diagonal. This means that the configurations are mutually exclusive and there is no need in this case to apply inclusion-exclusion.

Some of the labels correspond in fact to several configurations. For instance, there are four possibilities for U_2 , depending on whether the solid edge is $2x$ or $1y$ and whether the diagonal

dashed edge is 12 or xy . The associated GFs are shown in Table 1, together with their multiplicities, that is, the number of times we have to consider the configuration. For instance, the GF for U_2 is $zL_1^2L^{k-2}$: z marks the cell c containing 12 as a diagonal; one factor L_1 is for the cell containing edge $2x$ and the point to the left; and the remaining factor L_1L^{k-2} accounts for the cells arising from the remaining edges of cell c . The multiplicity is 4 since, as we have discussed, there are four different possibilities.

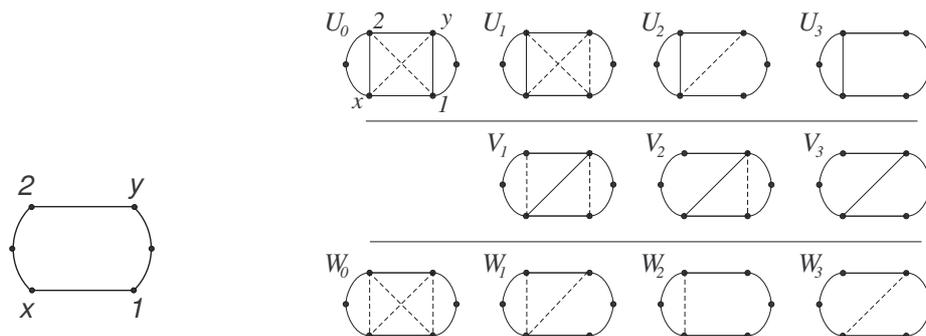


Figure 6: Forbidden configurations for an external incompatible planar k -dissection. The labels of the four marked points are shown on the left.

Configuration	GF	Multiplicity
U_0	zL_1^2	1
U_1	zL_1L^{k-2}	2
U_2	$zL_1^2L^{k-2}$	4
U_3	$zL_1^3L^{k-2}$	2
V_1	z^2L^{2k-2}	2
V_2	$z^2L_1L^{2k-2}$	4
V_3	$z^2L_1^2L^{2k-2}$	2
W_0	$zL^{k-1} - z$	1
W_1	zL_1L^{k-1}	4
W_2	$zL_1^2L^{k-1}$	2
W_3	$zL_1^2L^{k-1}$	2

Table 1: Generating functions and multiplicities of forbidden configurations.

In order to obtain E we only need to sum the corresponding terms. There are however some exceptional cases depending on the value of k .

- $k = 3$, that is, we are considering dissections into quadrangles. In this case configurations U_1, W_0, W_1 cannot occur, since they imply the existence of a polygon with more than four sides. Hence

$$E = U_0 + 4U_2 + 2U_3 + 2V_1 + 4V_2 + 2V_3 + 2W_2 + 2W_3.$$

- $k = 4$, In this case configurations U_0 and W_0 cannot occur, hence

$$E = 2U_1 + 4U_2 + 2U_3 + 2V_1 + 4V_2 + 2V_3 + 4W_1 + 2W_2 + 2W_3.$$

- $k > 4$. Configuration U_0 does not occur, since it implies the existence of a quadrangle. Hence

$$E = 2U_1 + 4U_2 + 2U_3 + 2V_1 + 4V_2 + 2V_3 + W_0 + 4W_1 + 2W_2 + 2W_3.$$

In all cases, the series S is a polynomial in z and the corresponding planar generating function L .

To obtain the GF associated to T , we need only to compute I . It corresponds to those “decompositions” which are internally non-compatible; we use quotes to denote that in a planar sense it is a cellular decomposition into $(k+1)$ -polygons, but when we introduce a repeated point it is not a cellular decomposition in the projective plane. A decomposition which is internally non-compatible is a planar decomposition in which either:

- There are two polygons whose intersection is exactly the repeated point and an edge which does not contain the repeated point. We denote them by X_1, X_2, X_3 .
- There are two polygons whose intersection consists of the repeated point and a second point. We denote them by Y_1, Y_2 .

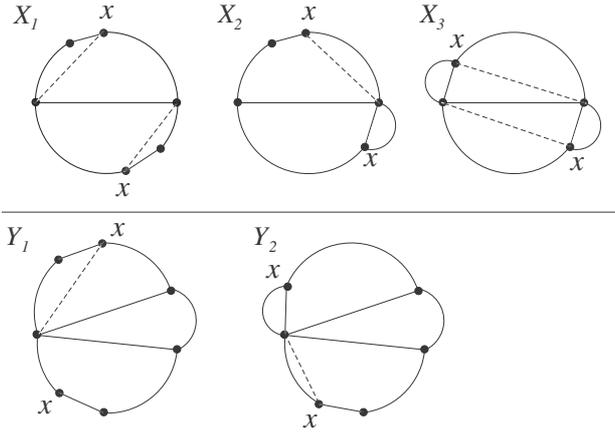


Figure 7: Configurations internally incompatible for a planar k -decomposition with marked points.

Configuration	GF	Multiplicity
X_1	$(k-2)^2 z^2 L^{2k}$	1
X_2	$(k-2) z^2 L^{2k-1} L_1$	2
X_3	$z^2 L^{2k-2} L_1^2$	1
Y_1	$(k-2)(k-1) z^2 L^{2k} L_1$	2
Y_2	$(k-1) z^2 L^{2k-1} L_1^2$	2

Table 2: Forbidden configurations for a compatible planar k -decomposition.

These restrictions are summarized in Figure 7, and their respective GFs appear in Table 2, together with their multiplicities. These configurations do not depend on the value of k , and we

obtain

$$I = X_1 + X_2 + X_3 + Y_1 + Y_2,$$

which is also a polynomial in z and L . Observe that the configurations in Figure 7 are disjoint from the ones in Figure 6, so that there is no need for inclusion-exclusion.

Having computed both S and T , we can write down in an explicit way the generating function $P(z)$, and a routine computation using $L = 1 + zL^k$ for simplifying the final expressions proves the claim. \square

4 Unrestricted dissections

In this section we consider dissections of a polygon in the projective plane into cells of any size at least three. We count them according to the number of vertices and to the number of cells in the dissection. Again we demand that two intersecting cells meet only in a common vertex or in a common edge.

Theorem 4.1 *Let $p_{n,q}$ be the number of dissections into m cells of a polygon with n vertices in the projective plane, and let $D(z, u)$ be the generating function for plane dissections, which satisfies $(1 + u)D^2 - z(1 + z)D + z^3 = 0$. Then*

$$\begin{aligned} P(z, u) &= \sum_{n \geq 5} p_{n,m} u^m z^n = \frac{\alpha + \beta D(z, u)}{\gamma} \\ &= u^5 z^5 + (u^3 + 6u^4 + 18u^5 + 14u^6)z^6 + (7u^3 + 56u^4 + 182u^5 + 245u^6 + 113u^7)z^7 + \dots \end{aligned}$$

where

$$\begin{aligned} \alpha &= -z^9 + (u^2 + 9u + 7)z^8 + (-5u^3 - 29u^2 - 43u - 19)z^7 + (5u^4 + 32u^3 + 73u^2 + 70u + 24)z^6 \\ &\quad + (-5u^4 - 25u^3 - 46u^2 - 37u - 11)z^5 + (-u^3 - 7u^2 - 11u - 5)z^4 + (7u^2 + 14u + 7)z^3 \\ &\quad + (-2u - 2)z^2 \\ \beta &= (u + 1)z^7 + (-u^3 - 9u^2 - 15u - 7)z^6 + (4u^4 + 26u^3 + 59u^2 + 56u + 19)z^5 \\ &\quad + (-2u^5 - 19u^4 - 65u^3 - 105u^2 - 81u - 24)z^4 + (7u^4 + 29u^3 + 48u^2 + 37u + 11)z^3 \\ &\quad + (6u^3 + 17u^2 + 16u + 5)z^2 + (-9u^2 - 16u - 7)z + 2u + 2 \\ \gamma &= u^2 z^3 (-4zu + z^2 - 2z + 1) \end{aligned}$$

Proof. Let $D(z, u) = \sum_n d_n(u)z^n$, where $d_n(u)$ is a polynomial of degree $n - 1$. As usual we set $D_r = \sum_{i \geq r} d_i(u)z^i$. As a general rule it is convenient to work with D/z , since we work with sequences of consecutive planar dissections, and each vertex is counted twice in the sequence. Hence let $C = D/z$. As before S and T are the GFs of marked planar dissections without and with a repeated point, respectively.

We fix the root polygon containing edge 12 and we argue as follows. Assume first the root polygon r does not cross the cross-cap. If r has k edges, then it determines $(k - 2)$ planar dissections and one projective dissection, and there are $k - 1$ ways of choosing the region containing the cross-cap. This contributes a term $u(k - 1)PC^{k-2}$, where u marks the root polygon. Summing up we obtain

$$uP(2C + 3C^2 + 4C^4 + \dots) = uP \frac{2C - C^2}{(1 - C)^2}.$$

If the cross-cap is crossed then we distinguish three kinds of edges in r : those found before crossing the cross-cap, those found after first crossing the cross-cap, and finally those found after crossing the cross-cap a second time. This decomposition gives rise to three sequences of planar dissections, drawn as grey zones in Figure 8. Each grey zone must contain at least one vertex, and those at the bottom of the first three pictures contain at least two vertices.

The final equation is

$$P = uP \frac{2C - C^2}{(1 - C)^2} + \frac{u}{z} \left(\frac{C}{1 - C} + 2 \left(\frac{C}{1 - C} \right)^2 + \left(\frac{C}{1 - C} \right)^3 \right) S + \frac{u}{z} \frac{1}{(1 - C)^2} T. \quad (3)$$

The first term is the one obtained above. The second term corresponds to the three cases where we have region S ; and the final term to the case where we have region T . The series $C/(1 - C)$ is the GF for a non-empty sequence of planar dissections.

Solving for P , and recalling that $C = D/z$, we obtain

$$P = \frac{uz(DS + zT - DT)}{(D - z)(-(1 + u)D^2 + 2(1 + u)zD - z^2)}.$$

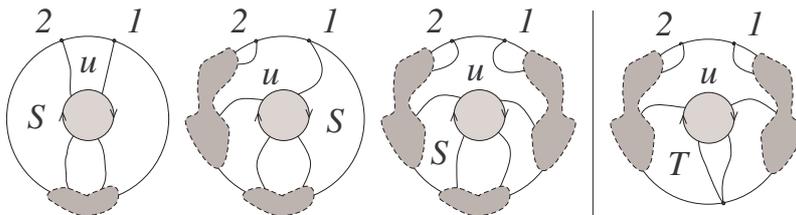


Figure 8: Possible cases where the polygon crosses the cross-cap. The GF associated to the grey zones is always $(1 - D/z)^{-1}$.

In order to compute S and T , we proceed as in the case of k -dissections. Define E and I as in the previous section. We count the number of marked planar dissections and we subtract E to obtain S or $E + I$ to obtain T .

To count marked planar dissections we proceed as follows. If n is the number of vertices, we have to distribute $n - 4$ points into two non-empty sets. This can be done in $n - 5$ ways; the minimum number of vertices is 6, hence we must consider D_6 . As a conclusion, the corresponding GF is $\Theta(D_6) - 5D_6$. Then

$$S = \Theta(D_6) - 5D_6 - E, \quad T = \Theta(D_6) - 5D_6 - E - I = S - I.$$

We consider the computation of S and T separately.

Computing S . In this case, we count marked planar dissections and subtract those which are externally incompatible, that is, we only consider E .

To compute E we use again exclusion-inclusion. Basic configurations are shown in Figure 9, and their combinatorial specification is shown in Table 3; the difference now is that a diagonal can be either an edge of a face (which introduces the factor u) or a diagonal in a face (which introduces the factor $1/u$).

For instance, the GF associated to V_0 must have four terms. In the case with edges bd and bc (first row, first column in V_0 , Figure 9) the four possible configurations are those showed in

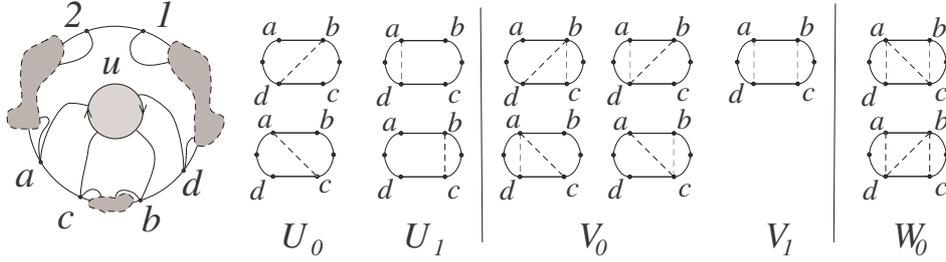


Figure 9: Forbidden dissections for an external incompatible planar dissection.

Configuration	GF	Multiplicity
U_0	$\frac{1}{z^2} D_3 D_5 (1 + \frac{1}{u})$	2
U_1	$\frac{1}{z^2} D_4^2 (1 + \frac{1}{u})$	2
V_0	$\frac{u}{z} (1 + \frac{1}{u})^2 D_3 D_4$	4
V_1	$u (1 + \frac{1}{u})^2 (u + 2u^2) D_3^2$	1
W_0	$u^2 (1 + \frac{1}{u})^3 D_3^2$	2

Table 3: Forbidden configurations for external incompatible planar dissections.

Figure 10. The corresponding GFs are, from left to right, $u/z(D_4 D_3)$, $1/z(D_4 D_3)$, $1/z(D_4 D_3)$ and $1/(zu)(D_4 D_3)$. In all cases we consider two planar dissections (D_4 on the left and D_3 on the right) which can be pasted together using the diagonals bd and bc . There is always a term z in the denominator because in all cases we are counting the point b twice. Adding these four terms we obtain the expression in Table 3.

In conclusion, we obtain

$$S = \Theta(D_6) - 5D_6 - E = \Theta(D_6) - 5D_6 - 2U_0 - 2U_1 + 4V_0 + V_1 - 2W_0. \quad (4)$$

Computing T. In this case we have identified points, hence we count marked planar dissections and we subtract those which are externally and internally incompatible. The ones which are externally incompatible where counted by S , and this has been done already. We must compute now the number of internally incompatible dissections which are externally compatible

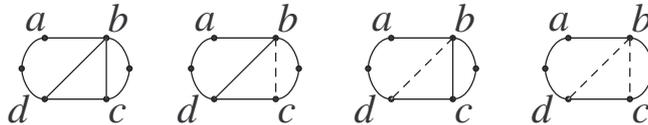


Figure 10: A particular case of external incompatible dissection. A continuous line denotes an edge, and a dashed line denotes a diagonal.

with the root polygon. We have

$$T = \Theta(D_6) - 5D_6 - E - I = S - I.$$

To describe the possible internal incompatible configurations, we define three fundamental blocks and show how to build from them all possible forbidden configurations. See Figure 11, where dashed lines must be taken either as edges or diagonals.

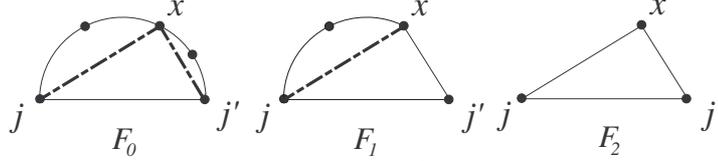


Figure 11: Blocks used to construct all possible forbidden configurations.

Configuration	GF
F_0	$\frac{1}{z}(u + 2 + \frac{1}{u})D_3^2$
F_1	$z(1 + u)D_3$
F_2	uz^3

Table 4: Translation of the previous configurations into generating functions.

Dissections that are internally incompatible appear because there is one point (denoted by x) that is a vertex of two polygons in the dissection. Denote these two polygons by P_1 and P_2 . Because we are dealing with cellular decompositions, $P_1 \cap P_2$ must be a cell. The existence of the double point x produces internally incompatible configurations in two different ways:

1. The ones such that $P_1 \cap P_2$ is $\{x, j\}$, where j is another vertex.
2. The ones such that $P_1 \cup P_2 = \{x\} \cup \{jj'\}$.

That is, $P_1 \cap P_2$ can be the union of two points or the union of an edge and a third point, and in either case it is not a cell. The two cases are shown in Figure 12, and the corresponding GFs are shown in Table 5. In G_0 and G_1 a factor 2 appears because we can choose the point j in the two sides of the polygon.

Configuration	GF	Multiplicity
F	$\frac{1}{z^2}(F_0 + F_1)^2$	1
G_0	$(F_0 + F_1)^2 D_3 \frac{1}{z^4}$	2
G_1	$(F_0 + F_1)(F_1 + F_2) D_3 \frac{1}{z^4}$	2

Table 5: Translation of the previous configurations into GFs.

The final expression for S is

$$T = S - F - G_0 - G_1. \quad (5)$$

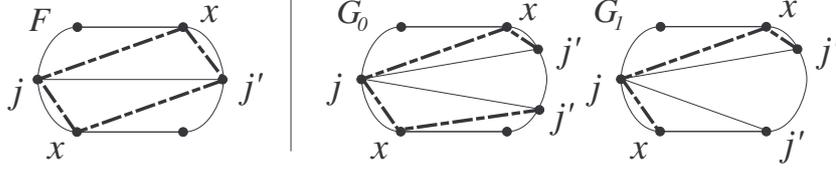


Figure 12: Blocks used to construct all possible forbidden configurations.

It only remains to substitute (5) and (4) into (1), and a routine computation give the result as claimed. \square

Notice that the sequence $[u^k z^k]P(z, u)$ is precisely the sequence of triangulations obtained before. This is because, for a fixed number of vertices, the dissections having a larger number of cells are triangulations.

5 Asymptotic estimates

In this section we obtain precise asymptotic expressions for the number of polygon dissections studied before. It turns out that they are invariably of the form

$$p_n \sim c \cdot \rho^{-n},$$

where c is a constant, and ρ is the radius of convergence of the corresponding planar generating function. In the planar case, it was shown in [3] that the estimates were always of the form $c \cdot n^{-3/2} \rho^{-n}$. Thus we can say that 0 is the universal exponent for dissections in the projective plane, whereas in the plane it is $-3/2$.

In order to obtain estimates for coefficients of generating functions defined implicitly, we follow [4, Ch. 5]. Let $L(z) = z\phi(L(z))$, and assume that:

1. $\phi(0) \neq 0$, $\phi''(z) \neq 0$.
2. $\phi(z)$ is analytic at $z = 0$, and has an expansion with positive coefficients.
3. Let R be the radius of convergence of ϕ . There exists a unique positive real solution $0 < \tau < R$ of the equation $\phi(\tau) - \tau\phi'(\tau) = 0$.

Then the radius of convergence of $L(z)$ is equal to $\rho = \tau/\phi(\tau)$, and the singular expansion of $L(z)$ at $z = \rho$ is of the form

$$L(z) \sim \tau - \sqrt{\frac{2\phi(\tau)}{\phi''(\tau)}} (1 - z/\rho)^{1/2} + O\left((1 - z/\rho)^{1/2}\right).$$

The generating functions we have obtained in the previous sections are all of the form

$$P(z) = \frac{1}{Q(z)} \sum_{j=0}^r P_j(z) L(z)^j,$$

where $P_j(z)$ and $Q(z)$ are polynomials, $L(z)$ is an algebraic function that verifies a relation of the form $L(z) = z\phi(L(z))$, and the smallest real root of $Q(z)$ is precisely equal to the radius of convergence of $L(z)$. Hence, the singular expansion of $P(z)$ at $z = \rho$ is of the form

$$P(z) = \sum_{j=r}^{\infty} a_j (1 - z/\rho)^{j/2},$$

where the integer r (possibly negative) depends on the given generating function.

To make calculations easier, we set $Z = \sqrt{1 - z/\rho}$ and make a change of variables. Then we expand the function that is obtained at $Z = 0$, obtaining a development of the form

$$P(Z) = \sum_{j=s}^{\infty} a_j Z^j,$$

where the integer s can be a negative. In our case s is always equal to -2 . Performing this calculations for all types of dissections we have studied, we obtain the following singular expansions:

- Triangulations: $P(Z) \sim \frac{1}{4}Z^{-2} + O(Z^{-1})$
- Quadrangulations: $P(Z) \sim \frac{2}{4}Z^{-2} + O(Z^{-1})$
- Decomposition into 5-gons: $P(Z) \sim \frac{3}{4}Z^{-2} + O(Z^{-1})$
- Decomposition into k -agons, $k > 5$: $P(Z) \sim \frac{k-1}{4}Z^{-2} + O(Z^{-1})$
- Dissections: $P(Z) \sim \frac{1}{4}Z^{-2} + O(Z^{-1})$

Observe that the constant term in the first four cases is $\frac{k-1}{4}$. The corresponding result is summarized in the following theorem.

Theorem 5.1 *For fixed $k \geq 2$, the number of dissections of a polygon in the projective plane with $(k-1)n$ vertices into $(k+1)$ -gons is asymptotically*

$$\frac{k-1}{4} \left(\frac{k^k}{(k-1)^{k-1}} \right)^n.$$

Proof. Let $L(z)$ be the solution of $L = 1 + zL^k$. The radius of convergence of $L(z)$ is $(k-1)^{k-1}/k^k$. This is a well-known result and is obtained by solving

$$\Phi(y, z) = 0, \quad \frac{\partial}{\partial y} \Phi(y, z) = 0,$$

where $\Phi = y - zy^k$ is the polynomial equation satisfied by L . The transfer theorems of singularity analysis apply and we obtain the estimate as claimed. \square

Theorem 5.2 *The number of dissections of a polygon in the projective plane with n vertices is asymptotically*

$$\frac{1}{4} \left(3 + 2\sqrt{2} \right)^n.$$

Proof. The equation satisfied by the generating function $D(z)$ of plane dissections is (just set $u = 1$ in Equation (1))

$$2D^2 - z(1+z)D + z^3.$$

The radius of convergence is $3 - 2\sqrt{2}$, and the result follows again by singularity analysis. \square

Table 6 summarizes the results in this section, together with the corresponding results for plane dissections taken from [3].

Class	In the plane	In the projective plane
Dissections into $(k + 1)$ -gons	$\sqrt{\frac{k}{2\pi(k-1)}} n^{-3/2} \left(\frac{(k-1)^{k-1}}{k^k}\right)^n$	$\frac{k-1}{4} \left(\frac{(k-1)^{k-1}}{k^k}\right)^n$
Unrestricted dissections	$\frac{\sqrt{-140+99\sqrt{2}}}{4\sqrt{\pi}} n^{-3/2} (3 + 2\sqrt{2})^n$	$\frac{1}{4} (3 + 2\sqrt{2})^n$

Table 6: Asymptotic estimates for planar and projective-planar dissections.

6 Limit laws

There are two statistical parameters we study in this section: the number of “cyclic” triangles in a triangulation (defined below), and the number of cells in an arbitrary polygon dissection. In the first case we obtain as a limit the absolute value of a normal law with expected value of order \sqrt{n} and variance of order n . In the second case we obtain a normal limit law with linear expected value and variance.

6.1 Cyclic triangles in triangulations

Let τ be a polygon triangulation in the projective plane, and let τ^* be the dual graph, whose vertices are the triangles of τ and edges are pairs of triangles sharing a diagonal. Then τ^* is a connected unicyclic graph, that is a graph with a unique cycle. This is most easily seen by considering τ as a triangulation of the Möbius band where the vertices are on the boundary: the triangles corresponding to the unique cycle of τ^* are those whose deletion disconnect the Möbius band.

We say that a triangulation is *cyclic* if its dual graph is a cycle.

Lemma 6.1 *The number s_n of cyclic triangulations of a polygon with n vertices is equal to $2^{n-1} - n^2 + 2n$.*

Proof. Let us analyze the proof of Theorem 2.1 and see how we can obtain a cyclic triangulation. First of all, the unique triangle $12x$ containing edge 12 has to cross the cross-cap, otherwise there would be non cyclic triangles. In the sequel we refer to the left picture in Figure 3, corresponding to a valid triangulation. Notice that x varies between 4 and $n - 1$.

For a triangulation to be cyclic, all diagonals must join a point on the left with a point on the right, forming a zigzag pattern. This can be done in $\binom{n-1}{x-2}$ ways. However, we have to subtract the configurations giving raise to a non valid triangulation. These are those containing one of the edges $12, xx, (x-1)x$ or $x(x+1)$. A simple computation shows that for each x there exactly n forbidden configurations. The total number of cyclic triangulations is thus

$$s_n = \sum_{x=4}^{n-3} \binom{n-1}{x-2} - n(n-4) = 2^{n-1} - n^2 + 2n.$$

□

From the previous lemma it follows that the GF of cyclic triangulations is equal to

$$S(z) = \sum_{n \geq 5} s_n z^n = \frac{z^5 (1 + 3z - 2z^2)}{(1 - 2z)(1 - z)^3}.$$

Given a triangulation τ , let $\alpha(\tau)$ be the length of the unique cycle in τ^* . We are interested in the distribution of the parameter α among all triangulations of size n . Let $p_{n,k}$ be the number

of triangulations with n vertices and $\alpha = k$, and let

$$P(z, u) = \sum p_{n,k} u^k z^n.$$

Lemma 6.2 *The generating functions $P(z, u)$ and $S(z)$ are related through the equation*

$$u \frac{\partial}{\partial u} P(z, u) = z \frac{\partial}{\partial z} S(zuC(z)). \quad (6)$$

Proof. Let τ be a triangulation and let σ be the union of all triangles in τ that belong to the unique cycle of τ^* . Then σ is a cyclic triangulation of a polygon Q with $k = \alpha(\tau)$ vertices, and τ is obtained from σ by gluing planar triangulations to the boundary edges of Q .

Equation (6) expresses in two different ways the GF of triangulations with vertices labeled $1, 2, \dots, n$ in circular order and one cyclic triangle marked. In the left term the triangle is marked by means of the $\partial/\partial u$ operator. In the right term, the triangle is marked using the $\partial/\partial z$ operator. The substitution $z \rightarrow zuC(z)$ means that to the outer edge of each cyclic triangle we glue a plane triangulation. \square

From this equation we obtain an alternative proof for the enumeration of triangulations of a polygon in the projective plane. We must integrate the previous expression, taking care of the initial conditions. The result is

$$P(z, u) = \left(1 + \frac{zC'(z)}{C(z)}\right) S(zuC(z)). \quad (7)$$

The first terms are

$$P(z, u) = u^5 z^5 + (6u^5 + 8u^6)z^6 + (28u^5 + 56u^6 + 29u^7)z^7 + O(z^8).$$

Setting $u = 1$, we recover the series for polygon triangulations in the projective plane.

Let X_n be the discrete random variable on the set of all triangulations of a polygon with n vertices in the projective plane, defined by $X_n(\tau) = \alpha(\tau)$. Our next result gives the limit law for the normalized variable X_n/\sqrt{n} . We denote by $\mathcal{N}(0, 1)$ the standard normal law with zero mean and unit variance.

Theorem 6.3 *Let $Y = \sqrt{2}|Z|$, where $Z \sim \mathcal{N}(0, 1)$. Then $X_n/\sqrt{n} \rightarrow Y$ in distribution.*

Proof. The proof is based on the method of moments. Let $Y_n = X_n/\sqrt{n}$. We show that the k -moments $\mathbb{E}(Y_n^k)$ converge to $\mathbb{E}(Y^k)$ as $n \rightarrow \infty$ for each k , and that Y is characterized by its moments. It follows then that Y_n converges in distribution to Y .

In order to compute moments from the generating function $P(z, u)$, it is convenient to work with the factorial moments $\mathbb{E}((X_n)_k) = \mathbb{E}(X_n(X_n - 1)(X_n - 2) \dots (X_n - k + 1))$. Using the identity $x^k = \sum_{j=0}^k S(j, k)(x)_j$, where the $S(j, k)$ are Stirling numbers of the second kind and $(x)_j = x(x - 1) \dots (x - j + 1)$, it follows easily that $\mathbb{E}(X_n^k) \sim \mathbb{E}((X_n)_k)$. Hence, it is enough to compute the factorial moments.

The density probability function of Y is

$$f(u) = \frac{1}{\sqrt{\pi}} e^{-u^2/4}, \quad u \geq 0.$$

Hence the moments are

$$\mathbb{E}(Y^k) = \int_0^\infty f(u) u^k du = \frac{2^k}{\sqrt{\pi}} \Gamma\left(\frac{k+1}{2}\right) < 2^k k!.$$

The latter inequality implies that the moment generating function $\mathbb{E}(e^{tY})$ has positive radius of convergence, so that the distribution Y is determined by its moments (see, for instance, Theorem 5.7.12 in [7]).

Disregarding the terms in (7) which are analytic, we obtain that the singular expansion of $P(z, u)$ at $(z, u) = (1/4, 1)$ is

$$P(z, u) \sim \frac{1}{4} \frac{1}{\sqrt{1-4z}} \frac{1}{1-u(1-\sqrt{1-4z})}$$

and from this it follows that

$$\sum_{n \geq 0} p_n \mathbb{E}((X_n)_k) = \frac{\partial^k}{\partial u^k} P(z, u) \Big|_{u=1} \sim \frac{1}{4} \frac{k!}{(1-4z)^{1+k/2}}.$$

Extracting coefficients and using singularity analysis we arrive at

$$p_n \mathbb{E}((X_n)_k) \sim \frac{1}{4} \frac{k! n^{k/2}}{\Gamma(1 + \frac{k}{2})} 4^n.$$

Using the estimate $p_n \sim 4^{n-1}$ from Theorem 5.1, we obtain

$$\mathbb{E}((X_n)_k) \sim \frac{k! n^{k/2}}{\Gamma(1 + \frac{k}{2})}.$$

The same estimate holds for the ordinary moment $\mathbb{E}(X_n^k)$, since $\mathbb{E}(X_n^k) \sim \mathbb{E}((X_n)_k)$. Finally, using the multiplication formula $\Gamma(k)\Gamma(k+1/2) = \sqrt{2\pi} 2^{1/2-2k} \Gamma(2k)$, the result follows. \square

A direct consequence of the previous result is the following.

Corollary 6.4 *The first two moments of X_n are, asymptotically,*

$$\begin{aligned} \mathbb{E}(X_n) &\sim \frac{2}{\sqrt{\pi}} \sqrt{n}, \\ \sigma^2(X_n) &\sim \left(2 - \frac{4}{\pi}\right) n. \end{aligned}$$

Proof. These are cases $k = 1$ and $k = 2$ in the previous proof. \square

6.2 Cells in dissections

Let X_n be the number of cells in a dissection of a polygon in the projective plane with n vertices. Intuitively, one should expect that X_n behaves very much like in the planar case, and this is indeed true. The first term asymptotic term for the mean and variance is the same as for planar dissections (see [3]). However, there is a difference in the second term. Taking additional terms in the computations from the planar case in [3], the expected value is shown to be asymptotically

$$\frac{\sqrt{2}}{2} n - \left(\frac{3\sqrt{2}}{4} - \frac{1}{8} \right) + O(n^{-1})$$

where one should notice that $\frac{3\sqrt{2}}{4} - \frac{1}{8}$ is positive. If we compare it with the result in the next theorem, we see that the expected number of cells in projective dissections is larger than in plane projections just by an additive constant.

Theorem 6.5 X_n is asymptotically normal and

$$E(X_n) \sim \frac{\sqrt{2}}{2}n + A\frac{1}{\sqrt{n}} + O(n^{-1}), \quad \sigma^2(X_n) \sim \frac{\sqrt{2}}{8}n + B\sqrt{n} + O(1),$$

where $A = \frac{1}{16\sqrt{\pi}}\sqrt{4 + 3\sqrt{2}}(1089\sqrt{2} - 1536) \approx 0.4129$,

and $B = \frac{1}{128\sqrt{\pi}}\sqrt{4 + 3\sqrt{2}}(3820\sqrt{2} - 6015) \approx -7.7535$.

Proof. From Theorem 4.1, the bivariate GF for dissections is

$$P(z, u) = \frac{\alpha + \beta D(z, u)}{\gamma},$$

where $D(z, u)$ satisfies (1). Hence $P(z, u)$ is an algebraic function and its defining equation can be computed directly using resultants. It follows that P satisfies the quadratic equation

$$aP^2 + bP + c = 0$$

where

$$\begin{aligned} a &= u^2 z^6 - (8u^3 + 4u^2)z^5 + (16u^4 + 16u^3 + 6u^2) \\ &\quad z^4 - (8u^3 + 4u^2)z^3 + u^2 z^2 \\ b &= z^{10} - (u^2 + 14u + 10)z^9 + (10u^3 + 79u^2 + 109u + 43)z^8 + \\ &\quad - (32u^4 + 211u^3 + 423u^2 + 348u + 103)z^7 + \\ &\quad (32u^5 + 232u^4 + 648u^3 + 875u^2 + 574u + 147)z^6 + \\ &\quad - (48u^5 + 272u^4 + 648u^3 + 790u^2 + 485u + 119)z^5 + \\ &\quad (32u^4 + 126u^3 + 186u^2 + 129u + 35)z^4 + (75u^3 + 146u^2 + 106u + 27)z^3 \\ &\quad - (72u^2 + 92u + 32)z^2 + (21u + 13)z - 2 \\ c &= (u^4 + u^3)z^9 - (u^6 + 9u^5 + 13u^4 + 3u^3)z^8 \\ &\quad + (4u^7 + 22u^6 + 36u^5 + 13u^4 + u^3)z^7 + (7u^6 + 23u^5 + 12u^4 + 2u^3)z^6 + 2z^5 u^5 \end{aligned}$$

In order to compute the expected value and the variance it is enough (see [4]) to estimate $\partial P(z, 1)/\partial u$ and $\partial^2 P(z, 1)/\partial u^2$. Since we have an explicit expression for $P(z, u)$, this can be achieved by singularity analysis as in the previous section. The necessary singular expansions are

$$\left. \frac{\partial P(z, u)}{\partial u} \right|_{u=1} = \frac{\sqrt{2}}{8}Z^{-4} + \sqrt{4 + 3\sqrt{2}} \left(-\frac{23}{8}\sqrt{2} + \frac{61}{16} \right) Z^{-3} - \frac{\sqrt{2}}{8}Z^{-2} + O(Z^{-1})$$

and

$$\begin{aligned} \left. \frac{\partial^2 P(z, u)}{\partial u^2} \right|_{u=1} &= \frac{1}{4}Z^{-6} + \sqrt{4 + 3\sqrt{2}} \left(\frac{183}{64}\sqrt{2} - \frac{69}{16} \right) Z^{-5} + \\ &\quad \left(-\frac{3}{8} - \frac{3}{32}\sqrt{2} \right) Z^{-4} + \sqrt{4 + 3\sqrt{2}} \left(-\frac{587}{128}\sqrt{2} + \frac{4063}{512} \right) Z^{-3} + O(Z^{-2}) \end{aligned}$$

where $Z = \sqrt{1 - z/\rho}$ and $\rho = 3 - 2\sqrt{2}$. Using singularity analysis we estimate the coefficient of z^n in the two series above, and find the asymptotics for $n \rightarrow \infty$. The computations are routine using Maple. Together with Theorem 5.2, we obtain the result as claimed. \square

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