A FILTRATION OF (Q,T)-CATALAN NUMBERS

N. BERGERON, F. DESCOUENS, AND M. ZABROCKI

Abstract. Using the operator ∇ of F. Bergeron, Garsia, Haiman and Tesler [2] acting on the k-Schur functions [15, 16, 17] indexed by a single column has a coefficient in the expansion which is an analogue of the (q,t)-Catalan number with a level k. When k divides n we conjecture a representation theoretical model in this case such that the graded dimensions of the module are the coefficients of the (q,t)-Catalan polynomials of level k. When the parameters t is set to 1, the Catalan numbers of level k are shown to count the number of Dyck paths that lie below a certain Dyck path with q counting the area of the path.

1. Introduction

In the study of the (q,t)-Catalan polynomials, Bergeron, Garsia, Haiman and Tesler [2] introduced a remarkable operator on symmetric functions, ∇, to help explain the conjectured graded Frobenius series of the space of the diagonal harmonic alternants. The operator ∇ has Macdonald's symmetric functions as eigenfunctions (see equation (4) for a definition) and it was a necessary tool for arriving at a combinatorial formula for the (q,t)-Catalan numbers [4]. The original definition of the (q,t)-Catalan numbers is equivalent to the coefficient of the symmetric function $s_{1^n}(X)$ in the expression $\nabla s_{1^n}(X)$.

The authors Lapointe, Lascoux, Morse [15] introduced and Lapointe, Morse [16], [17], [18], [19] further developed an analogue of the Schur basis of the space of symmetric functions that they called k-Schur functions. Here the parameter $k \geq 1$ indicates a level of a filtration of the space of symmetric functions and the parts of the partitions indexing the k-Schur functions are all less than or equal to k. In summary, the k-Schur functions $\{s^{(k)}_\lambda(X;t)\}_{\lambda_1 \leq k}$ are the 'fundamental' basis of the space linearly spanned by the elements $\{s_\lambda(X/(1-t))\}_{\lambda_1\leq k}$ where $f(X/(1-t))$ is the symmetric function $f(X)$ with the primitive power sum elements $p_k(X)$ replaced by $p_k(X)/(1-t^k)$. k-Schur functions are a remarkable analogue of the Schur basis and the Schur functions and the k-Schur functions are equal when $k \to \infty$. In special cases, k-Schur functions are equal to Hall-Littlewood symmetric functions, but in general there is currently no relatively simple definition of these symmetric functions.

Since the k-Schur functions are an analogue of the Schur functions, we decided to consider the action of the operator ∇ on these symmetric functions, in particular in the case when k-Schur functions are indexed by a single column. We found that when ∇ acts on the k-Schur function $s^{(k)}_{1^n}(X;1/t)$ then this expands positively again in the k-Schur functions $s^{(k)}_\lambda(X;1/t)$. In fact, our experiments suggest that ∇ acting on the k-Schur functions $s^{(k)}_\lambda(X;1/t)$ where $\lambda = (a^b)$ is a rectangle also expands positively in the k-Schur functions with inverted parameter, however it is not true for arbitrary λ that $\nabla s^{(k)}_\lambda(X;1/t)$ again lies in the space linearly spanned by the k-Schur functions with inverted parameter (the first failed example is $\nabla s^{(4)}_{2211}(X;1/t)$)

We define a version of the (q,t)-Catalan polynomials that includes a level k by setting $C^{(k)}_n(q,t)$ to be the coefficient of $s^{(k)}_{(1^n)}(X;1/t)$ in the expression $\nabla s^{(k)}_{1^n}(X;1/t)$ or more simply (or, more simply, their definition is $\langle s_{1^n}(X), \nabla s^{(k)}_{1^n}(X;1/t)\rangle$). Experimental evidence and special cases lead us to believe that these numbers form a filtration of the (q,t)-Catalan numbers (see Conjectures 12 and 14) and so we suspect that $C^{(k)}_n(q,t)$ is a (q,t)-counting of some subsets of Dyck paths. In certain cases, we can provided a combinatorial interpretation of these polynomials in terms of subsets of Dyck paths. Because the operator ∇ is an algebra

This work is supported in part by CRC and NSERC. It is a result of the Algebraic Combinatorics Seminar at the Fields Institute with the active participation of Anouk Bergeron-Brlek, Philippe Choquette, and Muge Taskin.

homomorphism at $t=1$, we are able to give an explicit interpretation of the polynomials $C_n^{(k)}(q,1)$ as the sum over q raised to the area statistic for each Dyck path which lies below the path which has k steps up followed by k steps over, followed by k steps up, followed by k steps over, etc., followed finally by $n\ mod\ k$ steps up and $n\ mod\ k$ steps over.

It is interesting to remark that $s^{(k)}_{(1^n)}(X;1/t)$ is equal to the modified Hall-Littlewood polynomials $t^{-n(\mu)}\omega Q'_\mu(X;t)$ where $\mu = (k^{n \operatorname{div} k}, n \operatorname{mod} k)$ [See Eq. (6)]. From this one can refine our filtration further and define for any partition $\mu \vdash n$ the μ-Catalan number $C_\mu(q,t)$ to be the coefficient of $s_{(1^n)}(X)$ in the expression $\nabla t^{-n(\mu)}\omega Q'_\mu(X;t)$. This would would give a filtration of the (q,t)-Catalan numbers compatible with the dominance order of partitions of n. All the results and conjecture presented here work in the same way. We do not consider that generality since J. Haglund and J. Morse [11] have comunicated to us an even more refined definition indexed by composition of n, see Remark 11.

The remainder of this paper is divided into 4 sections. In section 2, we discuss necessary definitions. In section 3, we introduce the Catalan numbers indexed by a level k and consider special cases and specializations. In section 4, we briefly consider the analogous filtrations of the Schröder paths and parking functions. Finally in the last section we define a filtration of the space of diagonal harmonic alternants. We conjecture based on experimental data that for k dividing n that the graded dimensions of this space are given by the polynomials $C_n^{(k)}(q,t)$.

2. Basic definitions

2.1. **Symmetric functions.** For symmetric functions, we mainly follow the notations of [23]. Let $X = \{x_1, x_2, \ldots\}$ be a sequence of variables. The complete homogeneous symmetric function of degree n in the variables X are defined by

$$h_n(X) = \sum_{i_1 \le i_2 \le \cdots \le i_n} x_{i_1} x_{i_2} \cdots x_{i_n} .$$

The space of symmetric functions Sym over a field F is the polynomial ring $F[h_1, h_2, \ldots]$, where $h_n = h_n(X)$. This is a graded ring where $\deg(h_n) = n$. It is convenient to index bases of Sym by partitions which are sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k > 0$. The sequence λ is a partition of n if $n = \lambda_1 + \cdots + \lambda_k$ and its length $\ell(\lambda)$ is k. The homogeneous basis can be defined by

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k} .$$

The elementary basis is defined by $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$, where e_n is defined by the recurrence

$$\begin{cases} e_{-k} &=& h_{-k} = 0 \text{ for } k > 0 , \\ e_0 &=& 1 , \\ 0 &=& \sum_{i+j=n} (-1)^i h_i e_j . \end{cases}$$

And for any partition λ of n, the Schur basis can be defined in an algebraic way by

$$s_\lambda = \det \left[h_{\lambda_i + i - j} \right]_{1 \le i,j \le n} .$$

The usual scalar product on the space Sym is defined on the Schur basis by

$$\langle s_\lambda \ , \ s_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu , \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

2.2. **Macdonald polynomials and Hall-Littlewood functions.** For any partition λ, we denote by λ' the conjugate partition of λ. The usual normalization constant $n(\lambda)$ is defined by

$$n(\lambda) = \sum_{i \ge 1} (i-1) \lambda_i . \tag{2}$$

Let us now recall some basic definitions on Macdonald polynomials. The modified Macdonald polynomials $\widetilde{H}_\lambda(X;q,t)$ are defined by

$$\widetilde{H}_\lambda(X;q,t) = t^{n(\lambda)} J_\lambda\left(X;q,\frac{1}{t}\right) , \tag{3}$$

where $J_\lambda(X;q,t)$ is the integral version of Macdonald polynomials defined in VI.8 of [23]. The modified Hall-Littlewood polynomials can be obtained as a specialization of the Macdonald polynomials

$$Q'_\lambda(X;t) = \widetilde{H}_\lambda(X;0,t) .$$

The linear operator ∇ introduced in [2] is defined by

$$\nabla \widetilde{H}_\lambda(X;q,t) = t^{n(\lambda)} q^{n(\lambda')} \widetilde{H}_\lambda(X;q,t) . \tag{4}$$

There exist a long list of conjectures about the action of ∇ on different bases of symmetric functions. For many of them, recent work in this area has developed combinatorial models (proved or conjectural [4, 5, 7, 8, 9, 10, 21, 22]) which explains the different properties.

2.3. **k-Schur functions.** The k-Schur functions $s_\lambda^{(k)}(X;t)$ of Lapointe, Lascoux, Morse (see [15, 16, 17, 18]) are the fundamental basis of the space $\mathcal{L}\{Q'_\lambda(X;t)$ with $\lambda_1 \leq k\}$, where $\mathcal{L}$ represents the vector space linear span of the elements.
We are interested in this short note only in the explicit definition of the elements $s_{1^n}^{(k)}(X;t)$. Let μ be the partition defined by

$$\mu = (k^{n \operatorname{div} k}, n \operatorname{mod} k) . \tag{5}$$

For these symmetric functions, we simply define them to be

$$s_{1^n}^{(k)}(X;t) = t^{n(\mu)} \omega\left(Q'_\mu\left(X;\frac{1}{t}\right)\right) . \tag{6}$$

This definition permits us to give the explicit expansion of $s_{1^n}^{(k)}(X)$, for the special cases $k > n/2$

$$s_{1^n}^{(k)}(X;t) = s_{1^n}(X) + t s_{21^{n-2}}(X) + \ldots + t^{n-k} s_{2^{n-k}1^{2k-n}}(X) . \tag{7}$$

For a complete definition of the k-Schur functions with the parameter t, we refer the reader to the references [15, 16]. Note that these two references provide two different definitions which are conjectured to be equivalent. In the case of the indexing partition equal to 1^n we can show that they are both equal to (7).

3. Generalizations of (q,t)-Catalan numbers

The (q,t)-Catalan numbers $C_n(q,t)$ defined in [6], are related to the operator ∇ applied to an elementary symmetric function $e_n(X)$. As defined in the previous section, the k-Schur functions indexed by column partitions are a generalization of these elementaries functions which are equal to $e_n(X)$, for $k \geq n$. Hence, a natural way to obtain filtrations of (q,t)-Catalan numbers is to replace in this picture, the functions $e_n(X)$ by the k-Schur functions. With this process, we obtain new polynomials in q and t with positive coefficients, which are smaller than the usual (q,t)-Catalan numbers. By specializing $q = 1$ and $t = 1$ in these filtrations, we obtain different generalizations of Catalan numbers than those given in [13].

3.1. **(q,t)-Catalan numbers.** Let first recall the definition and the combinatorial interpretation for the (q,t)-Catalan numbers in terms of Dyck paths [4, 5, 6, 7, 8].

Definition 1 ((q,t)-Catalan numbers)**.** *The (q,t)-Catalan numbers are the polynomials in the parameters q and t defined by*

$$C_n(q,t) = \langle \nabla e_n(X) \ , s_{1^n}(X) \rangle , \tag{8}$$

where $\langle \ , \ \rangle$ is the usual scalar product on symmetric functions.

These polynomials are in $\mathbb{N}[q,t]$. Their specialization at $t=1$ and $q=1$ gives the usual Catalan numbers C_n

$$C_n(1,1) = C_n \ .$$

The (q,t)-Catalan numbers are symmetric in the variables q and t, i.e. $C_n(q,t) = C_n(t,q)$. The maximum degree in these parameters are

$$\deg_q(C_n(q,t)) = \deg_t(C_n(q,t)) = \tbinom{n}{2} \ .$$

Example 2. *For $n=6$, the (q,t)-Catalan number $C_6(q,t)$ can be represented by an array, where the entry (i,j) corresponds to the coefficient of $q^i\ t^{\binom{n}{2}-j}$.*

q^i

															1
														1	
													1	1	
												1	1	1	
											1	1	2	1	
										1	1	2	2	1	
									1	1	2	3	2		
								1	1	2	3	3	1		
							1	1	2	3	4	2	1		
						1	1	2	3	4	3	2			
					1	1	2	3	4	3	2				
				1	1	2	3	4	3	2	1				
			1	1	2	3	3	2	2						
		1	1	2	2	2	1	1							
	1	1	1	1	1										
1															

$t^{\binom{n}{2}-j}$

In [7], the authors proved that these polynomials have positive coefficients by interpreting them as generating polynomials of Dyck paths with two statistics *area* for the t and *dinv* for the q. We should also mention that the original combinatorial interpretation, given in [5], uses the statistics of *area* and *bounce* with the two parameters interchanged.

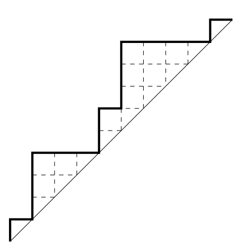
Definition 3. *A Dyck path of length n is a lattice path from the point $(0,0)$ to the point (n,n) consisting of n north steps and n east steps that never go below the line $y=x$.*

We denote by DP_n, the set of all the Dyck paths of length n. Dyck paths of length n are in bijection with sequences $(g_0, \ldots, g_{n-1})$ of n nonnegative integers satisfying the two conditions

$$\begin{cases} g_0 = 0 \ , \\ g_{i+1} \leq g_i + 1 \ , \quad \forall i < n-1 \ . \end{cases} \tag{9}$$

The i-th entry g_i of the sequence g corresponds to the number of complete lattice squares between the north step of the i-th row of the Dyck path and the diagonal $y=x$. Such sequences are called Dyck sequences. We denote by DS_n, the set of all the Dyck sequences of length n. From now, we use indifferently Dyck sequences or Dyck paths.

Example 4. *The Dyck sequence $g = (0,0,1,2,0,1,1,2,3,0)$ corresponds to the Dyck path*



Definition 5. *The statistic* area *associated to a Dyck sequence g is defined by*

$$\operatorname{area}(g) = \sum_{i=0}^{n-1} g_i \ . \tag{10}$$

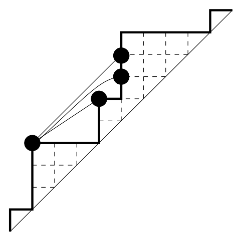
On the corresponding Dyck path, this statistic is the number of complete lattice squares between the path and the diagonal $y = x$.

Definition 6. *The statistic* dinv*, which is the number of inversions of a Dyck sequence g, is defined by*

$$\mathrm{dinv}(g) = \sum_{0 \le i < j < n} \chi(g_i - g_j \in \{0, 1\}) \; . \tag{11}$$

We recall a graphical interpretation of this statistic on Dyck path. Let us call a *north point*, a point where a north step arrives. Two north points give a contribution of 1 in *dinv*, if they are in the same diagonal or if the second point is in the diagonal just below the diagonal of the first one.

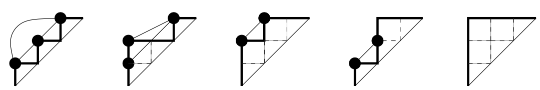
Example 7. *The fourth entry of the Dyck sequence $g = (0, 0, 1, 2, 0, 1, 1, 2, 3, 0)$ gives 3 inversions. One inversion comes from the same diagonal and the two others come from the diagonal just below.*



Theorem 8 ([5, 7])**.** *The (q, t)-Catalan numbers $C_n(q, t)$ are the generating polynomials of Dyck sequences of size n with the two statistics* area *and* dinv

$$C_n(q, t) = \sum_{g \in DS_n} t^{\mathrm{area}(g)} q^{\mathrm{dinv}(g)} \; . \tag{12}$$

Example 9. *The (q, t)-Catalan number $C_3(q, t) = q^3 + q^2 t + q t^2 + q t + t^3$ can be computing using the following five Dyck paths and the two previous statistics.*



The black linked points correspond to pairs of points which give a contribution of 1 in the statistic dinv*.*

3.2. **Definition of a filtration of Catalan numbers.** By definition, the k-Schur functions indexed by column partitions (1^n) are generalizations of the elementary functions $e_n(X)$ in the space of symmetric functions over $\mathbb{C}(t, q)$. Hence, we can replace elementary functions in Definition 1 by these k-Schur functions, in order to obtain a k-level version of (q, t)-Catalan numbers.

Definition 10. *Let k and n be two positive integers. The generalized (q, t)-Catalan numbers of level k are defined by*

$$C_n^{(k)}(q, t) = \left\langle \; \nabla s_{1^n}^{(k)}\left(X; \frac{1}{t}\right) \; , \; s_{1^n}(X) \right\rangle \; , \tag{13}$$

where $\langle \; , \; \rangle$ is the usual scalar product on symmetric functions.

Remark 11. In fact, k-Schur function $s_{1^n}^{(k)}\left(X; \frac{1}{t}\right)$ are special cases of Hall-Littlewood functions (see Equation 6). We can define more general extensions of the (q, t)-Catalan numbers indexed by partitions by

$$C_\lambda(q, t) = \left\langle \; \nabla \left(t^{n(\lambda)} \omega \, Q'_\lambda \left(X; \frac{1}{t}\right)\right) \; , \; s_{1^n}(X) \right\rangle \; . \tag{14}$$

Recently J. Haglund and J. Morse [11] have defined generalizations of (q, t)-Catalan numbers indexed by compositions using Jing operators on Hall-Littlewood functions and have found the corresponding combinatorial interpretation on Dyck paths. These generalizations are equivalent to our definition (14) in the case of partitions.

Conjecture 12 (Positivity)**.** *Let k and n be two positive integers. The polynomial $C_n^{(k)}(q,t)$ is in $\mathbb{N}[q,t]$.*

In [2], the authors make the conjecture (originally formulated by Lascoux) that the Schur expansions of the operator ∇ applied on the modified Hall-Littlewood functions $Q'_\lambda(X;t)$ are positive, up to a global sign. The positivity of our generalizations follows directly from this conjecture.

Remark 13. It is important to invert the parameter t inside the k-Schur functions in order to obtain polynomials in q and t with integral positive coefficients.

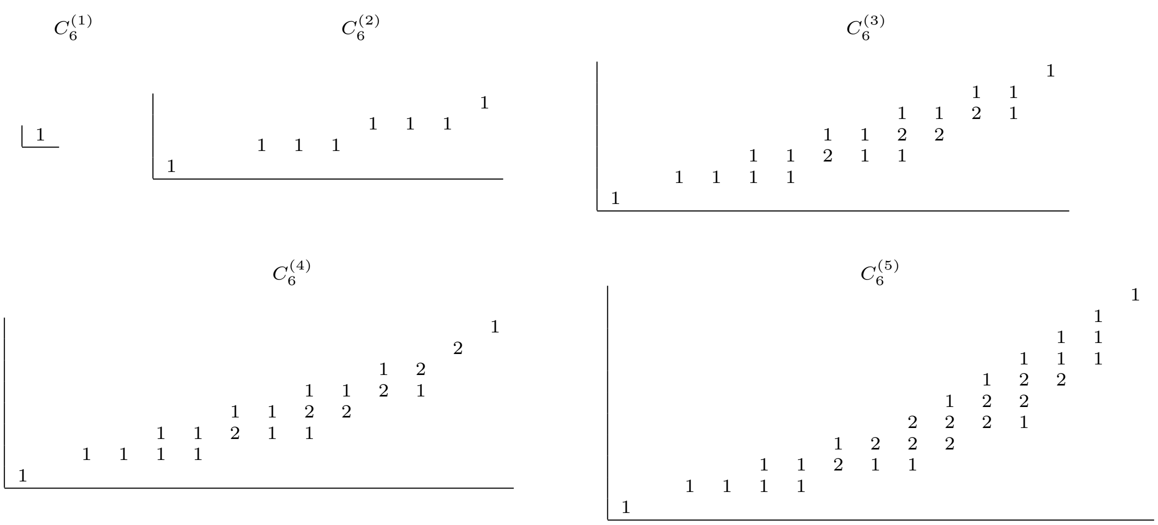
Conjecture 14 (Filtration)**.** *Let n be a positive integer. The family of polynomials $\left(C_n^{(k)}(q,t)\right)_{k\geq 1}$ is a filtration of the usual (q,t)-Catalan numbers $C_n(q,t)$. More precisely, we have*

$$\begin{cases} \forall k \geq 1, \quad C_n^{(k+1)}(q,t) - C_n^{(k)}(q,t) \in \mathbb{N}[t,q], \\ \forall k \geq n, \quad C_n^{(k)}(q,t) = C_n(q,t) \ . \end{cases} \tag{15}$$

Proof: The first statement is a consequence of Conjecture 3 of [2]. The second statement of the proposition follows immediately from the stability property

$$\forall k \geq n, \ s_{1^n}^{(k)}(X;t) = e_n(X) \ . \tag{16}$$

Example 15. *Using the same conventions as in Example 2, the generalized (q,t)-Catalan numbers $C_6^{(k)}(q,t)$ are given by the following matrices*



Conjecture 16. *At the level $k=2$, the generalized (q,t)-Catalan numbers statisfy the recursive formula for $n>1$,*

$$\begin{cases} C_{n+1}^{(2)}(q,t) & = & t^n C_n^{(2)}(q/t,t) & \textit{if } n \textit{ is even}, \\ C_{n+1}^{(2)}(q,t) & = & t^n C_n^{(2)}(q,t) + qt^{n-1}C_{n-1}^{(2)}(q,t) & \textit{if } n \textit{ is odd}. \end{cases}$$

Example 17. *In the even case, for $n=2$*

$$C_3^{(2)}(q,t) = t^2 C_2^{(2)}(q,t) = t^2(t+q/t) = t^3 + qt \ . \tag{17}$$

In the odd case, for $n=5$

$$\begin{aligned} C_6^{(2)}(q,t) &= t^5 C_5^{(2)}(q,t) + qt^4 C_4^{(2)}(q,t) \\ &= t^5(q^2t^4 + qt^7 + qt^6 + t^{10}) + qt^4(q^2t^2 + qt^4 + qt^3 + t^6) \\ &= q^3t^6 + q^2t^9 + q^2t^8 + q^2t^7 + qt^{12} + qt^{11} + qt^{10} + t^{15} \end{aligned}$$

Definition 18. *By specializing $q = 1$ and $t = 1$ in the generalized (q,t)-Catalan numbers $C_n^{(k)}(q,t)$, we define a new filtration $C_n^{(k)}$ of the usual Catalan numbers C_n*

$$C_n^{(k)} = C_n^{(k)}(1,1) \ .$$

Example 19. *The triangle of the specialization of the generalized (q,t)-Catalan at $q = 1$ and $t = 1$ is*

(18)

$n : k$	1	2	3	4	5	6
1	1					
2	1	2				
3	1	2	5			
4	1	4	5	14		
5	1	4	10	14	42	
6	1	8	25	28	42	132

The first diagonal below the main diagonal corresponds to the sequence C_{n-1} and the second diagonal below the main one corresponds to the sequence $2\,C_{n-2}$. The others diagonal sequences are unknown in Sloane's integer encyclopedia. But using the combinatorial interpretation at $t = 1$, we give an explicit expression for these numbers in the next section.

3.3. **Combinatorial interpretation at** $t = 1$. When the parameter t is specialized at 1 in the generalized (q,t)-Catalan numbers $C_n^{(k)}(q,t)$, we are able to give an explicit combinatorial interpretation of these polynomials. This interpretation is based on the fact that ∇ is multiplicative at $t = 1$ and was remarked in [2].

Proposition 20. *Let n and k be two positive integers. The generalized (q,t)-Catalan numbers satisfy the following factorisation formula at $t = 1$*

$$C_n^{(k)}(q,1) = C_n(q,1)^{n \text{ div } k}\ C_{(n \bmod k)}(q,1) \ . \tag{19}$$

Proof: By definition, we have

$$\begin{aligned} s_{1^n}^{(k)}(X;1) &= \omega\left(H_{(k^{n \text{ div } k},\ n \bmod k)}(X;1)\right) = \omega\left(h_{(k^{n \text{ div } k},\ n \bmod k)}(X)\right) \\ &= e_{(k^{n \text{ div } k},\ n \bmod k)}(X) \ . \end{aligned}$$

Now, since the operator ∇ at $t = 1$ is multiplicative, we can write

$$\begin{aligned} C_n^{(k)}(q,1) &= \langle \nabla_{t=1}\left(s_{1^n}^{(k)}(X;1)\right) \ , \ e_n(X)\rangle \\ &= \langle \nabla_{t=1}(e_k(X))^{n \text{ div } k}\ \nabla_{t=1}(e_{n \bmod k}(X)) \ , \ e_n(X)\rangle \ . \end{aligned}$$

By consideration of degree, the coefficient in $e_n(X)$ in the right part of the scalar product can only be obtained as the product of the coefficient of $e_n(X)$ in $\nabla_{t=1}(e_k(X))^{n \text{ div } k}$ and in $\nabla_{t=1}(e_{n \bmod k}(X))$. □

Remark 21. Using the previous proposition, we obtain an explicit expression for the generalized Catalan numbers $C_n^{(k)}$ in terms of usual Catalan numbers

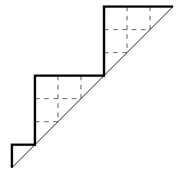
$$C_n^{(k)} = (C_k)^{n \,\mathrm{div}\, k}\ C_{n \,\mathrm{mod}\, k} \ . \tag{20}$$

Corollary 22. *Let n and k be two positive integers. The combinatorial interpretation of $C_n^{(k)}$ is given by*

$$C_n^{(k)}(q,1) = \sum_g q^{\mathrm{area(g)}} \ , \tag{21}$$

where the sum is taken over the Dyck paths g, which are below the Dyck path built with n div k blocks of k steps up and k steps right and a last block of n mod k steps up and n mod k step right.

Example 23. *The set of Dyck paths for the combinatorial interpretation of $C_7^{(3)}(q,1)$ are those under the following Dyck path.*

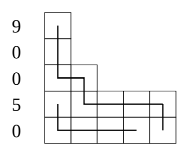


3.4. **Combinatorial interpretation of $C_n^{(k)}(q,t)$.** We find some conjectural combinatorial models for special cases of these generalizations of (q,t)-Catalan numbers. We use the combinatorics of configurations of Dyck paths which permits to give some conjectures on a combinatorial model for $\langle \nabla s_\lambda(X)\ ,\ s_{1^n}(X)\rangle$. It is known that the Schur expansion of $\nabla s_\lambda(X)$ on the Schur basis is always positive, up to a global sign. This sign is interpreted by M. Bousquet-Mélou using determinants of Catalan numbers. In the special case of $t = 1$, a proof using configurations of Dyck paths is given in [20]. More recently, an interpretation using nested Dyck paths is given in [22] and the following interpretation is mainly based on this work.

3.4.1. *A combinatorial interpretation of $\langle \nabla s_\lambda(X)\ ,\ s_{1^n}(X)\rangle$.* We recall the combinatorial interpretation of the scalar product $\langle \nabla s_\lambda(X)\ ,\ s_{1^n}(X)\rangle$ given in [22] in terms of nested Dyck paths. For a given partition, we can describe a set of configurations of Dyck paths with two statistics which permits to express the previous scalar product as a generating polynomial. The global sign of these expressions and the characterization of the corresponding configurations of Dyck paths can be computed directly from the partition λ.

For a given partition $\lambda = (\lambda_1, \ldots, \lambda_p)$, we will associate a sequence of λ_1 nonnegative integers $\tilde{n}(\lambda) = (\tilde{n}_0, \ldots, \tilde{n}_{\lambda_1})$, called the dissection sequence of λ. Define the maximal rim-hook of a partition μ as the skew diagram $\mu/(\mu_2 - 1, \mu_3 - 1, \ldots, \mu_{\ell(\mu)} - 1)$. We consider the tiling of the conjugate partition of λ obtained by removing successively the maximal rim-hooks. The entry $\tilde{n}_i$ of $\tilde{n}(\lambda)$ is the length of the maximal rim-hook of the tilling starting in the $(\lambda_1 - i)$-th row and is 0 if no rim-hook starts in the $(\lambda_1 - i)$-th row.

Example 24. *The dissection sequence corresponding to the partition $\lambda = (53222)$ is $\tilde{n}(53222) = (9, 0, 0, 5, 0)$ as described by the picture*



We define the spin of a partition λ by

$$\mathrm{sp}(\lambda) = \sum_R (h(R) - 1)\ , \tag{22}$$

where the sum is over all the border rim-hooks of λ' and $h(R)$ the height of these ribbons. The sign of a partition λ, corresponding to the global sign of $\langle \nabla s_\lambda(X)\ ,\ s_{1^n}(X)\rangle$, is defined by

$$\mathrm{sgn}(\lambda) = (-1)^{\mathrm{sp}(\lambda)}\ . \tag{23}$$

We also define the diagonal inversion adjustment by

$$\mathrm{adj}(\lambda) = \sum_{i=0}^{\lambda_1 - 1} (\lambda_1 - 1 - i)\chi(\tilde{n}_i > 0) = \sum_{i=0}^{\lambda_1 - 1} \lambda'_i\ \chi(\tilde{n}_i > 0)\ . \tag{24}$$

The adjustment is the sum of the row indices of λ' (starting from the top of the diagram of the partition) where a border rim-hook starts.

Example 25. *For the partition $\lambda = (53222)$, the spin and the sign are*

$$\mathrm{sp}(\lambda) = 4 + 1 = 5 \textit{ and consequently } \mathrm{sgn}(\lambda) = -1\ . \tag{25}$$

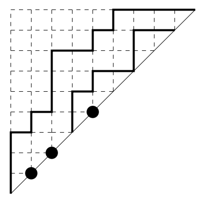
In this case, the adjustment is

$$\operatorname{adj}(\lambda) = 1 + 4 = 5 \ . \tag{26}$$

Definition 26. *Let $\lambda = (\lambda_1, \dots, \lambda_p)$ be a partition of dissection sequence $\tilde{n}(\lambda) = (\tilde{n}_0, \dots, \tilde{n}_{\lambda_1 - 1})$. Let $\Pi = (\pi_0, \dots, \pi_{\lambda_1 - 1})$ be a sequence of Dyck paths π_i of length $\tilde{n}_i$ from (i,i) to $(i+\tilde{n}_i, i+\tilde{n}_i)$. If $\tilde{n}_i$ is equal to 0, π_i is a degenerate Dyck path consisting in a single vertex at (i,i). The sequence Π is a nested Dyck path for the partition λ, if for all $i \neq j$, no edge or vertex of π_i coincides with any edge or vertex of π_j.*

We denote by NDP_λ the set of all the nested Dyck paths for the partition λ.

Example 27. *A nested Dyck path of $NDP_{(53222)}$ corresponding to the dissection sequence $\tilde{n}(53222) = (9, 0, 0, 5, 0)$.*



The encoding of Dyck paths using Dyck sequences can be extended to nested Dyck paths. Let $\Pi = (\pi_o, \dots, \pi_{l-1})$ a nested Dyck path. The nested Dyck configuration corresponding to Π is an l-tuple of words $G = (g^{(0)}, \dots, g^{(l-1)})$, where $g^{(i)}$ is the Dyck sequence encoding the Dyck path π_i. The indexing of the letters in these Dyck sequences are chosen to match the alignment of paths in the picture. In the following, we use indifferently nested Dyck paths and nested Dyck configurations.

Example 28. *The nested Dyck path of Example 27 corresponds to the following nested Dyck configuration*

$$G = \begin{pmatrix} g^{(0)}: & 0 & 1 & 2 & 2 & 2 & 3 & 4 & 3 & 3 \\ g^{(1)}: & \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ g^{(2)}: & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ g^{(3)}: & \cdot & \cdot & \cdot & 0 & 1 & 1 & 0 & 1 & \cdot \\ g^{(4)}: & \cdot & \cdot & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot \end{pmatrix} . \tag{27}$$

The statistic *area* and *dinv* can be extended to nested Dyck paths. The area of a nested Dyck path G, written $\overline{area}$, is the sum of the areas of the Dyck paths of G:

$$\overline{\operatorname{area}}(G) = \sum_{i=0}^{l-1} \operatorname{area}(g_i) = \sum_{i=0}^{l-1} \sum_{i \le j < i+n_i} g_j^{(i)} \ . \tag{28}$$

The diagonal inversion statistic for a nested Dyck path G, written $\overline{dinv}$, is defined by

$$\begin{aligned} \overline{\operatorname{dinv}}(G) \quad = \quad & \operatorname{adj}(\lambda) + \textstyle\sum_{a,b,u,v} \chi\left(g_a^{(u)} - g_b^{(v)} = 1\right) \chi(a \le b) \\ & + \textstyle\sum_{a,b,u,v} \chi\left(g_a^{(u)} - g_b^{(v)} = 0\right) \chi((a < b) \text{ or } (a = b \text{ and } u < v)) \ . \end{aligned} \tag{29}$$

The $\overline{dinv}$ of a nested Dyck path $G = (g^{(0)}, \dots, g^{(l)})$ corresponds to the sum of the *dinv* of each Dyck path $g^{(i)}$ plus the number of pairs of points coming from different $g^{(i)}$'s which form a inversion. A pair of points which form an inversion and which are in the same row are just counted one time.

Example 29. *The statistics $\overline{\operatorname{area}}(G)$ and $\overline{\operatorname{dinv}}(G)$ for the nested Dyck path G of Example 27 are*

$$\overline{\operatorname{area}}(G) = 24 \quad \textit{and} \quad \overline{\operatorname{dinv}}(G) = 37 \ . \tag{30}$$

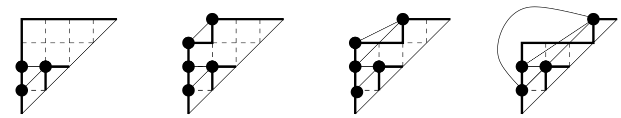
One of the main conjectures of [22] gives the following expression for the coefficient of $\nabla s_\lambda(X)$ on the Schur function $s_{1^n}(X)$

$$\langle \nabla s_\lambda(X) \ , \ s_{1^n}(X) \rangle = \operatorname{sgn}(\lambda) \sum_{G \in NDP_\lambda} q^{\overline{\operatorname{area}}(G)} t^{\overline{\operatorname{dinv}}(G)} \ . \tag{31}$$

Example 30. *For $\lambda = (221)$, we have*

$$\langle \nabla s_{221}(X)\ ,\ s_{1^5}(X)\rangle = -\left(q^6t^3 + q^5t^4 + q^4t^5 + q^3t^6\right)\ . \tag{32}$$

The dissection vector of the partition $\lambda = (221)$ is $n = (4,1)$ and $\mathrm{adj}(\lambda) = 1$. The combinatorial interpretation of (32) is given by the following four nested Dyck paths where we have linked the pairs of points which give a contribution of 1 in $\overline{\mathrm{dinv}}$.



3.4.2. *Combinatorial interpretation for the filtration in some special cases.* We give an explicit combinatorial interpretation of the generalizations of (q,t)-Catalan numbers in the cases of the level $k = n-1$ and $k = n-2$ using the combinatorial materials given in the previous section. The goal is to find bijections between sets of nested Dyck paths and usual Dyck paths. These bijections have to be compatible with the statistics $(area,\ dinv)$ and $(\overline{area},\ \overline{dinv})$ in order to explain why the terms are canceling in the right way, giving at the end, a polynomial with only positive coefficients.

For level $n-1$

For the level $k = n-1$, we have an explicit characterization of the Schur functions which appear in the k-Schur functions we are considering. Using Equation (7), we have that

$$s^{(n-1)}_{1^n}(X;t) = s_{1^n}(X) + t\ s_{21^{n-2}}(X)\ . \tag{33}$$

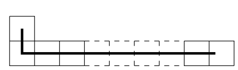
Conjecture 31 (Combinatorial interpretation for $k = n-1$)**.** *Let $DP_n^{(1,1)}$ denotes the set of Dyck paths which go through the lattice point $(1,1)$. The generalized (q,t)-Catalan numbers of level $(n-1)$ are given by*

$$C_n^{(n-1)}(q,t) = \sum_{g\in DP_n^{(1,1)}} q^{\mathrm{area}(g)}t^{\mathrm{dinv}(g)}\ . \tag{34}$$

Proof (based on Conjecture (31)): As $s_{1^n}(X) = e_n(X)$, we know that

$$\langle \nabla s_{1^n}(X)\ ,\ s_{1^n}(X)\rangle = \sum_{g\in DP_n} q^{\mathrm{area}(g)}t^{\mathrm{dinv}(g)}\ , \tag{35}$$

where the sum is over all the Dyck paths of length n.
Let now compute the combinatorial interpretation of $\langle \nabla s_{21^{n-2}}(X)\ ,\ s_{1^n}(X)\rangle$ in terms of nested Dyck paths. The dissection vector of the partition (21^{n-2}) is $\tilde{n}(21^{n-2}) = (n,0)$, as described by the following picture



This implies that the nested Dyck paths corresponding to the partition (21^{n-2}) are the sequences of two non intersecting Dyck paths $G = (g^{(0)}, g^{(1)})$, such that

- Dyck path $g^{(0)}$ is a Dyck path of length n avoiding the lattice point $(1,1)$,
- Dyck path $g^{(1)}$ is reduced to the degenerated Dyck path of size 0 at the lattice point $(1,1)$.

Hence, we have

$$\langle \nabla s_{21^{n-2}}(X)\ ,\ s_{1^n}(X)\rangle = -\sum_{G\in NDP_{21^{n-2}}} q^{\overline{\mathrm{area}}(G)}t^{\overline{\mathrm{dinv}}(G)}\ . \tag{36}$$

Let denote by $DP_n^{(1,1)^c}$ the set of all Dyck paths of size n avoiding the lattice point $(1,1)$. Let consider the following bijection Φ_n defined by

$$\begin{array}{cccc} \Phi_n & : & NDP_{21^{n-2}} & \longrightarrow & DP_n^{(1,1)^c} \\ & & (g^{(0)}, g^{(1)}) & \longmapsto & g^{(0)}\ . \end{array} \tag{37}$$

The compatibility of Φ_n with the statistics area and dinv is given by

$$\left\{\begin{array}{lcl} \mathrm{dinv}(\Phi_n(g^{(0)},g^{(1)})) & = & \overline{\mathrm{dinv}}(G)-1\ , \\ \mathrm{area}(\Phi_n(g^{(0)},g^{(1)})) & = & \overline{\mathrm{area}}(G)\ . \end{array}\right. \tag{38}$$

In order to prove this compatibility, let $G=(g^{(0)},g^{(1)})$ be a nested Dyck path of $NDP_{21^{n-2}}$. By definition of G, the corresponding Dyck configuration is of the form

$$\left(\begin{array}{cccccc} g^{(0)}: & 0 & 1 & g^{(0)}_2 & \cdots & g^{(0)}_{n-1} \\ g^{(1)}: & \cdot & \times & \cdot & \cdots & \cdot \end{array}\right)\ . \tag{39}$$

Hence, the Dyck path $g^{(1)}$ always give a contribution of 1 in $\overline{dinv}(G)$. Using the property of Φ_n given in (38), Equation (36) can be rewritten as

$$\langle s_{21^{n-2}}(X)\ ,\ s_{1^n}(X)\rangle = -\ \textstyle\sum_{G\in NDP_{(21^{n-2})}} q^{\overline{area}(G)}t^{\overline{dinv}(G)} \tag{40}$$

$$= -\ \textstyle\sum_{g\in DP_n^{(1,1)^c}} q^{\mathrm{area}(g)}t^{\mathrm{dinv}(g)+1}\ . \tag{41}$$

Hence, we have for generalized (q,t)-Catalan of level $(n-1)$

$$C_n^{(n-1)}(q.t) = \sum_{g\in DP_n} q^{\mathrm{area}(g)}t^{\mathrm{dinv}(g)} - \frac{1}{t}\sum_{g\in DP_n^{(1,1)^c}} q^{\mathrm{area}(g)}t^{\mathrm{dinv}(g)+1} \tag{42}$$

$$= \sum_{g\in DP_n^{(1,1)}} q^{\mathrm{area}(g)}t^{\mathrm{dinv}(g)}\ . \tag{43}$$

□

Corollary 32. *The generalized Catalan numbers of level $(n-1)$ are given by*

$$C_n^{(n-1)}(1,1) = C_{n-1}\ . \tag{44}$$

The proof is immediate using the combinatorial interpretation given in the previous theorem.

For level $n-2$

In order to give a combinatorial interpretation for generalized (q,t)-Catalan numbers of level $(n-2)$, we use the combinatorial interpretation for level $(n-1)$ combined with the combinatorial interpretation of $\langle\nabla s_{221^{n-4}}(X)\ ,\ s_{1^n}(X)\rangle$. Using Equation (7), we have

$$s_{1^n}^{(n-2)}(X;t) = s_{1^n}(X) + t\ s_{21^{n-2}}(X) + t^2\ s_{221^{n-4}}(X)\ . \tag{45}$$

Conjecture 33. *Let denote by $DP_n^{(1,1),(3,2)}$ the set of Dyck paths which go through the lattice points $(1,1)$ and $(3,2)$. The generalized (q,t)-Catalan numbers of level $(n-2)$ are given by*

$$C_n^{(n-2)}(q,t) = \sum_{g\in DP_n^{(1,1),(3,2)}} t^{\mathrm{dinv(g)}}q^{\mathrm{area}(g)}\ . \tag{46}$$

Proof (based on Conjecture (31)): Let us compute the combinatorial interpretation of $\langle\nabla s_{221^{n-4}}(X)\ ,\ s_{1^n}(X)\rangle$. The dissection vector of the partition (221^{n-4}) is $\tilde{n}(221^{n-4})=(n-1,1)$ and $adj(221^{n-4})=1$, as described in the following picture

Hence, a nested Dyck paths $G=(g^{(0)},g^{(1)})$ is a couple of non intersecting Dyck path $(g^{(0)},g^{(1)})$ satisfying

- $g^{(0)}$ is a Dyck path of length $n-1$ avoiding the lattice point $(2,1)$,
- $g^{(1)}$ is the unique Dyck path of length 1 starting from the lattice point $(1,1)$.

Hence, we have

$$\langle \nabla s_{221^{n-4}}(X)\ ,\ s_{1^n}(X)\rangle = \sum_{G\in NDP_{221^{n-4}}} q^{\overline{\mathrm{area}}(G)} t^{\overline{\mathrm{dinv}}(G)}\ . \tag{47}$$

Let denote by $DP_n^{(1,1),(3,2)^c}$ the set of Dyck paths which go through the lattice point $(1,1)$ and avoid the lattice point $(2,1)$. Let consider the following bijection Ψ_n defined by

$$\begin{array}{rccc} \Psi_n : & NDP_{221^{n-4}} & \longrightarrow & DP_n^{(1,1),(3,2)^c} \\ & G=(g^{(0)},g^{(1)}) & \longmapsto & g^{(1)}\cdot g^{(0)}\ , \end{array} \tag{48}$$

where $g^{(1)}\cdot g^{(0)}$ is the Dyck path of length n obtained by concatenation of $g^{(1)}$ and $g^{(0)}$. The compatibility of Ψ_n with the statistics area and dinv is given by

$$\left\{ \begin{array}{rcl} \mathrm{dinv}(\Psi_n(g^{(0)},g^{(1)})) & = & \overline{\mathrm{dinv}}(G)-2\ , \\ \mathrm{area}(\Psi_n(g^{(0)},g^{(1)})) & = & \overline{\mathrm{area}}(G)\ . \end{array}\right. \tag{49}$$

In order to prove this compatibility, let $G=(g^{(0)},g^{(1)})$ be a nested Dyck path in $NDP_{221^{n-4}}$. The corresponding Dyck configuration is of the form

$$G = \left(\begin{array}{ccccccc} g^{(0)}: & 0 & 1 & 2 & g_3^{(0)} & \cdots & g_{n-2}^{(0)} \\ g^{(1)}: & \cdot & 0 & \cdot & \cdot & \cdots & \cdot \end{array} \right) \tag{50}$$

The zero of $g^{(1)}$ give a contribution of 2 in $\overline{dinv}(G)$. By definition of the statistic *dinv* of a Dyck configuration, we have

$$\overline{\mathrm{dinv}}(G) = \mathrm{adj}(221^{n-4}) + 2 + \mathrm{dinv}\left(g^{(0)}\right) = 3 + \mathrm{dinv}\left(g^{(0)}\right)\ . \tag{51}$$

The concatenation of Dyck paths $g^{(1)}\cdot g^{(0)}$ corresponds to the following Dyck sequence

$$g^{(1)}\cdot g^{(0)} = \left(g_1^{(1)}=0,\ 0,\ 1,\ 2,\ g_3^{(0)},\ \cdots, g_{n-2}^{(0)}\right)\ . \tag{52}$$

The first 0 gives now a contribution of 1 to $dinv\left(g^{(1)}\cdot g^{(0)}\right)$. Hence,

$$\mathrm{dinv}\left(\Psi_n\left(g^{(0)},g^{(1)}\right)\right) = \mathrm{dinv}\left(g^{(1)}\cdot g^{(0)}\right) = 1+\mathrm{dinv}\left(g^{(0)}\right)\ . \tag{53}$$

Finally, by combining (51) and (53), we have $dinv\left(\Psi_n\left(g^{(0)},g^{(1)}\right)\right) = \overline{dinv}(G)-2$.
Using Equation (49), we have

$$\langle \nabla s_{221^{n-4}}(X)\ ,\ s_{1^n}(X)\rangle = -\sum_{g\in DP_n^{(1,1),(3,2)^c}} q^{\mathrm{area(g)}} t^{\mathrm{dinv(g)}+2}\ . \tag{54}$$

Hence, using Expression (45) of k-Schur functions when $k=n-2$ and the combinatorial interpretation for level $n-1$, we obtain

$$\begin{aligned} \left\langle \nabla s_{1^n}^{(n-2)}\left(X;\frac{1}{t}\right)\ ,\ s_{1^n}(X)\right\rangle &= \sum_{g\in DP_n^{(1,1)}} q^{\mathrm{area(g)}}t^{\mathrm{dinv(g)}} - \frac{1}{t^2}\sum_{g\in DP_n^{(1,1),(3,2)^c}} q^{\mathrm{area(g)}}t^{\mathrm{dinv(g)}+2} \\ &= \sum_{g\in DP_n^{(1,1),(3,2)}} t^{\mathrm{dinv(g)}}q^{\mathrm{area}(g)}\ . \end{aligned}$$

□

Corollary 34. *The generalized Catalan numbers of level $(n-2)$ are given by*

$$C_n^{(n-2)}(1,1) = 2\ C_{n-2}\ . \tag{55}$$

Proof: There are two configurations for the first two steps of Dyck paths in $DP_n^{(1,1),(3,2)}$ given in the following picture

And it is well known that the number of lattice paths of length $n-3$ starting at the lattice point $(3,2)$ is C_{n-2}. Hence the cardinality of $DP_n^{(1,1),(3,2)}$ is $2\ C_{n-2}$. $\square$

For the level 2

In the special case of $k=2$, we have a conjectural interpretation for the generalized (q,t)-Catalan numbers.

Conjecture 35. *Let $DP_n^{(2)}$ be the set of the Dyck paths which are under the Dyck path given by the sequence $\big((10)^{n/2-1}, 1, (10)^{n/2}, 0\big)$. The generalized (q,t)-Catalan number of level k=2 are given by*

$$C_n^{(2)}(q,t) = \sum_{g \in DP_n^{(2)}} q^{\mathrm{area}(g)} t^{\mathrm{dinv}(g)}\ . \tag{56}$$

For other levels $2 < k < n-2$

For the others levels the problem splits into two different cases. For the levels $n/2 < k < n-2$, it exists an algorithm which describe how cancellations are behaving correctly but the characterizations of the corresponding subsets of Dyck paths are not as nice as for the case of the level $n-1$ and $n-2$.
For the levels $2 < k < n/2$, the coefficients of the k-Schur functions indexed by column partitions on the Schur basis are not just monomials in t. Hence, the terms are more complicated and are not compatible with the combinatorial interpretation using the same process than before.

4. Filtration of parking functions and Schröder paths

There exist other interesting polynomials of $\mathbb{N}[q,t]$ computed using scalar products involving the operator ∇. For each of these polynomials, a combinatorial model is associated in order to interpret them as generating polynomials with respect with somes statistics. We are mainly interested in the following two examples

$$\begin{array}{ccc} \langle \nabla(e_n(X))\ ,\ h_{1^n}(X)\rangle & \longrightarrow & (q,t)\text{-parking functions}, \\ \langle \nabla(e_n(X))\ ,\ e_d h_{n-d}(X)\rangle & \longrightarrow & (q,t)\text{-Schröder paths}\ . \end{array}$$

In order to generalize these combinatorial models, we apply the same idea as in the previous sections, i.e. replacing the elementary functions in ∇ by the k-Schur functions indexed by column partitions.

The combinatorial model for the (q,t)-parking functions have been conjectured by Haglund and Loehr in [10, 21]. Using k-Schur functions, we define a new filtration of these polynomials.

Definition 36. *Let k and n be two positive integers. The generalized (q,t)-parking numbers of level k are defined by*

$$P_n^{(k)}(q,t) = \left\langle \nabla s_{1^n}^{(k)}\left(X;\frac{1}{t}\right)\ ,\ h_{1^n}(X)\right\rangle\ . \tag{57}$$

Example 37. *For $n=3$, the different levels of the filtration are given by*

$$\begin{array}{rc} P_3^{(1)}(q,t) = & t^3+2t^2+2t+1 \\ P_3^{(2)}(q,t) = & 2q+2t+2t^2+t^3+qt+1 \\ P_3^{(3)}(q,t) = & q^3+q^2t+2q^2+qt^2+3qt+2q+t^3+2t^2+2t+1\ . \end{array}$$

The specialization of $t=1$ and $q=1$ in $P_n^{(k)}(q,t)$ gives new sequences of numbers $P_n^{(k)} = (P_k)^{n \text{ div } k} P_{n \text{ mod } k}$, where $P_j = (j+1)^{j-1}$.

$n:k$	1	2	3	4	5	6
1	1					
2	2	3				
3	6	9	16			
4	24	54	64	125		
5	120	270	480	625	1296	
6	720	2430	5120	5625	7776	16807

The combinatorial model for the (q,t)-Schröder paths have been conjectured by Egge, Haglund, Kremer and Killpatrick in [1] and proved by Haglund in [9]. Using the same kind of idea, we can define filtration of (q,t)-Schröder paths.

Definition 38. *Let n, d, k be tree positive integers. The generalized (q,t)-Schröder numbers of level k are defined by*

$$\forall d > 0, \quad S_{n,d}^{(k)}(q,t) = \left\langle \nabla s_{1^n}^{(k)}\left(X; \frac{1}{t}\right)\ ,\ e_d(X) h_{n-d}(X) \right\rangle\ . \tag{58}$$

5. Representation theoretic interpretation of $C_n^{(k)}(q,t)$

The (q,t)-Catalan numbers $C_n(q,t)$ are related to the space of diagonal harmonics DH_n and the $n!$ conjecture on Macdonald polynomials. Using our generalized (q,t)-Catalan numbers $C_n^{(k)}(q,t)$, we define subspaces $DH_n^{(k)}$ for k dividing n of the space DH_n. In the special cases where k divides n, we give an explicit algebraic description of these spaces. We briefly recall some basic statement on the space of diagonal harmonics and the operator theorem of Haiman [12].

5.1. **A generalization of the space of diagonal harmonics DH_n.** Let n be a positive integer and $\mathbb{Q}[X_n, Y_n]$ the space of polynomials over $\mathbb{Q}$ in the two sets of variables $X_n = \{x_1, x_2, \ldots, x_n\}$ and $Y_n = \{y_1, y_2, \ldots, y_n\}$. We call bidegree of a polynomial $f(X_n, Y_n)$, the couple of non-negative integers (i,j) such that $\deg_X(f) = i$ and $\deg_Y(f) = j$.

The symmetric group S_n acts diagonaly on $\mathbb{Q}[X_n, Y_n]$ by

$$\forall\ f \in \mathbb{Q}[X_n, Y_n], \quad \sigma.f(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)})\ . \tag{59}$$

Let I be the ideal in $\mathbb{Q}[X_n, Y_n]$ generated by all the S_n-invariant polynomials without constant term. Define the quotient ring

$$R_n = \mathbb{Q}[X_n, Y_n]/I\ . \tag{60}$$

For each S_n-invariant polynomials $P(X_n, Y_n)$ of the ideal I, the component of P in I is bihomogeneous in X_n and Y_n. Thus, I is a bihomogeneous ideal. Consequently, R_n has a structure of a doubly graded ring, i.e.

$$R_n = \bigoplus_{i,j} (R_n)_{i,j}\ , \tag{61}$$

where the subspace $(R_n)_{i,j}$ consists of all images of homogeneous polynomials of bidegree (i,j).

Let us denote by ∂x_i (resp. ∂y_i) the partial derivative operator with respect to the variable x_i (resp. y_i). Define the scalar product $\langle\ ,\ \rangle_\partial$ on $\mathbb{Q}[X_n, Y_n]$ by

$$\forall f, g \in \mathbb{Q}[X_n, Y_n]\ , \quad \langle f,\ g\rangle_\partial = f\left(\partial x_1, \ldots, \partial x_n, \partial y_1, \ldots, y_n\right) g(x_1, \ldots, x_n, y_1, \ldots, \partial y_n)\ |_{X=Y=0}\ . \tag{62}$$

For this scalar product the multplication by x_i (resp. y_i) is the adjoint operator of ∂x_i (resp. ∂y_i).

Definition 39. *The space DH_n of the diagonal harmonics is defined by*

$$DH_n = I^{\perp} = \{h \in \mathbb{Q}[X_n, Y_n] \mid f(\partial x_1, \ldots, \partial x_n, \partial y_1, \ldots, \partial y_n)h = 0\} \ . \tag{63}$$

This definition of the diagonals harmonics is equivalent to the following caracterization

$$DH_n = \left\{ P(X,Y) \in \mathbb{Q}[X_n, Y_n] \ \text{ such that } \sum_{i=1}^{n} \partial x_i^h \partial y_i^k P \text{ with } h+k>0 \right\} \ . \tag{64}$$

The two rings DH_n and R_n are isomorphic and an explicit isomorphism $\phi : DH_n \longrightarrow R_n$ can be defined by

$$\begin{array}{llll} \phi : & DH_n & \longrightarrow & R_n \\ & h & \longmapsto & \text{the equivalent class of } h \text{ modulo } I \ . \end{array} \tag{65}$$

In the space DH_n, the subspace DHA_n of the alternating harmonics is defined as the diagonals harmonics which are alternating, i.e.

$$DHA_n = \{P(X_n, Y_n) \in DH_n \text{ such that } \sigma P(X_n, Y_n) = -P(X_n, Y_n) \ , \forall \sigma \in S_n\} \ . \tag{66}$$

Proposition 40 ([12])**.** *Let C_n be the n-th Catalan number. The dimension of the space DHA_n is given by*

$$\dim DHA_n = C_n \ . \tag{67}$$

The space DHA_n is a bigraded vector space which can be decomposed as

$$DHA_n = \bigoplus_{i=1}^{\binom{n}{2}} \bigoplus_{j=1}^{\binom{n}{2}} (DHA_n)_{i,j} \ , \tag{68}$$

where $(DHA_n)_{i,j}$ is the space of the polynomials in DHA_n of bidegree (i,j). The Hilbert series of DHA_n is defined by

$$\mathcal{F}_{DHA_n}(q,t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} t^i q^j \dim(DHA_n)_{i,j} \ . \tag{69}$$

Theorem 41 ([12])**.** *The (q,t)-Catalan numbers are defined by*

$$C_n(q,t) = \mathcal{F}_{DHA_n}(q,t) \ . \tag{70}$$

5.2. **The operator theorem.** The structure of the space DH_n can be more explicitly, but not entirely, described using the operator theorem given in [12]. The idea is to introduce differential operators E_k which generate the space DH_n only from the Vandermonde determinant of level n in variables X defined by

$$\Delta_n(X_n) = \prod_{1 \le i < j \le n} (x_i - x_j) \ . \tag{71}$$

These operators E_k are defined for all $k > 0$ by

$$E_k = \sum_{i=1}^{n} y_i \partial x_i^p \ . \tag{72}$$

Theorem 42 ([12])**.** *The space of diagonal harmonics H_n is the smallest space containing $\Delta_n(X_n)$ and closed under the action of the operators E_p for all $1 \le p \le n-1$ and the operators ∂x_i for all $1 \le i \le n$. We write this statement using the following notation*

$$DH_n = \mathcal{L}_{E_1, \ldots, E_n, \partial x_1, \ldots, \partial x_n}(\Delta_n(X_n)) \ . \tag{73}$$

If we consider the operators F_p obtained by interchanging X_n and Y_n in E_p, the space DH_n can also be described as the smallest space containing $\Delta_n(Y_n)$ and closed under the action of the operators F_p for all $1 \le p \le n-1$ and the operators ∂y_i for all $1 \le i \le n$. In that sense, the operator conjecture is symmetric. Our generalization of the operators conjecture is not symmetric because our generalization of (q,t)-Catalan numbers are not symmetric and the space $DH_n^{(k)}$ does not contain the Vandermonde determinant $\Delta_n(Y_n)$ in variables Y_n.

Corollary 43 ([12])**.** *The space of the alternants DHA_n is the smallest space containing $\Delta_n(X_n)$ and closed under the action of the operators E_p for all $1 \leq p \leq n-1$, i.e.*

$$DHA_n = \mathcal{L}_{E_1,\ldots,E_n}(\Delta_n(X_n)) \ . \tag{74}$$

5.3. **Special case when k divides n.**

Conjecture 44. *Let k and n be two integers such that k divides n and $d = n/k$. Let us define the space $DHA_n^{(k)}$ by*

$$DHA_n^{(k)} = \mathcal{L}_{E_d,E_{d+1},\ldots,E_n}(\Delta_n(X_n)) \ . \tag{75}$$

The Hilbert series of $DHA_n^{(k)}$ is given by $\mathcal{F}_{DHA_n^{(k)}}(q,t) = C_n^{(k)}(q,t)$.

Example 45. *For $n = 8$ and $k = 4$, we have the triangle corresponding to $C_8^{(4)}(q,t)$ is*

																								1
																						1	1	
																				1	1	2	1	
																		1	1	2	3	2	1	
																1	1	2	3	5	3	1		
														1	1	2	3	5	6	4	2			
q^i												1	1	2	3	5	6	$\boxed{7}$	3	1				
										1	1	2	3	5	6	6	5	2	1					
								1	1	2	3	5	5	6	4	3	1							
						1	1	2	3	4	4	4	3	1	1									
				1	1	2	2	3	2	2	1	1												
		1	1	1	1	1	1																	
	1																							

$t^{\binom{n}{2}-j}$

The boxed entry of coordinates $(19,7)$ corresponds to the subspace of $DHA_8^{(4)}$ of bidegree $t^{28-19+1}q^{7-1} = t^{10}q^6$ with dimension 7. And

$$\operatorname{rank}\left\{\begin{array}{llll} E_{732222}.\Delta_n(X_n), & E_{642222}.\Delta_n(X_n), & E_{633222}.\Delta_n(X_n), & E_{552222}.\Delta_n(X_n), \\ E_{543222}.\Delta_n(X_n), & E_{533322}.\Delta_n(X_n), & E_{444222}.\Delta_n(X_n), & E_{443322}.\Delta_n(X_n), \\ E_{433332}.\Delta_n(X_n), & E_{333333}.\Delta_n(X_n) & & \end{array}\right\} = 7 \ .$$

This conjecture has been verified up to $n = 8$.

References

[1] E.Egge, J. Haglund, D. Kremmer, K. Killpatrick, *A Schröder generalization of Haglund's statistic on Catalan paths*, Electronic Journal of Combinatorics, **10**, (2003), R16, 21p.

[2] F. Bergeron, A. Garsia, M. Haiman and G. Tesler, *Identities and Positivity Conjectures for Some Remarkable Operators in the Theory of Symmetric Functions*, Methods and Applications of Analysis, **6**, 3, (1999), 363-420.

[3] F. Descouens, *Making research on symmetric functions using MuPAD-Combinat*, *Lectures Notes in Computer Sciences*, Springer, **4151**, (2006).

[4] A. Garsia and J. Haglund, *A positivity result in the theory of Macdonald polynomials*, Proc. Natl. Acad. Sci, USA, **98**, 8, (2001), 4313-4316 (electronic).

[5] A. Garsia and J. Haglund, *A proof of the (q,t)-Catalan positivity conjecture*, Discrete Math. **256**, 3, (2002), 677-717.

[6] A. Garsia and M. Haiman, *A remarkable q,t-Catalan sequence and q-Lagrange inversion*, J. Algebraic Combinatorics **5**, (1996), 191-244.

[7] M. Haiman, J. Haglund, N. Loehr, J. Remmel and A. Ulyanov, *A combinatorial formula of the diagonal coinvariants*, Duke Math. J., **126** (2005), pp. 195-232.

[8] J. Haglund, *Conjectured statistics for the (q,t)-Catalan numbers*, Adv. Maths, **175**, 2, (2003), 319-334.

[9] J. Haglund, *A proof of the q,t-Schröder conjecture*, Intl. Math. Res. Notices, **11**, (2004), 525-560.

[10] J. Haglund and N. Loehr, *A conjectured combinatorial formula for the Hilbert series for diagonal harmonics*, Discrete Math. **298**, (2005), 189-204.

[11] J. Haglund and J. Morse, private communication.

[12] M. Haiman, *Vanishing theorem and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149**, 2, (2002), 371-407.

[13] P. Hilton and J. Pedersen, *Catalan numbers, their generalization, and their uses*, Math. Intelligencer, **13**, (1991), 64-75.

[14] F. Hivert and N. Thiéry, *MuPAD-Combinat, an open source package for research in algebraic combinatorics*, Séminaire Lotharingien de Combinatoire, **51**, (2003), 70p electronic.

[15] L. Lapointe, A. Lascoux and J. Morse, *Tableaux atoms and a new Macdonald positivity conjecture*, Duke Math. J., **116**, 1, (2003), 103-146.
[16] L. Lapointe and J. Morse, *Schur functions analogs for a filtration of the symmetric functions space*, J. Combin. Theory Ser. A, **101**, 2, (2003), 191-224.
[17] L. Lapointe and J. Morse, *Tableaux on $k+1$-cores, reduced words for affine permutations and k-Schur expansion*, J. Combin. Theory Ser. A, **112**, 1, (2005), 44-81.
[18] L. Lapointe and J. Morse, *a k-tableau characterization of k-Schur functions*, Preprint (2005) arXiv:math.CO/0505519.
[19] L. Lapointe and J. Morse, *Quantum cohomology and the k-Schur basis*, Trans. Amer. Math. Soc. to appear, arXiv:math.CO/0501529
[20] C. Lenart, *Lagrange inversion and Schur functions*, J. Algebraic Combinatorics, **11**, (2000), 69-78.
[21] N. Loehr, *Combinatorics of q,t-parking functions*, Adv. in Applied Maths, **34**, (2005), 408-425.
[22] N. Loehr and G. Warrington, *Nested quantum Dyck paths and* $\nabla(s_\lambda)$, Preprint (2007) arXiv:0705.4608.
[23] I.G. Macdonald, *Symmetric Functions and Hall-Polynomials*, Oxford Mathematical Monographs, Oxford Univ. Press, second edition, 1995.

(Nantel Bergeron) Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, CANADA
E-mail address: bergeron@mathstat.yorku.ca
URL: http://www.math.yorku.ca/bergeron

(F. Descouens) The Fields Institute, 222 College Street, Toronto, Ontario, M5T 3J1, CANADA
E-mail address: francois.descouens@utoronto.ca
URL: http://www.fields.utoronto.ca/~chohlweg

(Mike Zabrocki) Department of Mathematics and Statistics, York University, Toronto, Ontario M3J 1P3, CANADA
E-mail address: zabrocki@mathstat.yorku.ca
URL: http://www.math.yorku.ca/~zabrocki