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SOME q-CONGRUENCES RELATED TO 3-ADIC VALUATIONS

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Abstract. In 1992 Strauss, Shallit and Zagier proved that for any positive integer a we have

$$\sum_{k=0}^{3^a-1}\binom{2k}{k}\equiv 0 \pmod{3^{2a}}$$

and furthermore

$$\frac{1}{3^{2a}}\sum_{k=0}^{3^a-1}\binom{2k}{k}\equiv 1 \pmod{3}.$$

Recently a q-analogue of the former congruence was conjectured by Guo and Zeng. In this paper we prove the conjecture of Guo and Zeng, and also give a q-analogue of the latter congruence.

1. Introduction

Partially motivated by the work of Pan and Sun [PS], Sun and Tauraso [ST1] proved that for any prime p and $a\in\mathbb{Z}^+=\{1,2,3,\ldots\}$ we have

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}\equiv\left(\frac{p^a}{3}\right) \pmod{p^2},$$

where $(-)$ denotes the Legendre symbol. (See also [ST2, ZPS, S09a, S09b, S09c] for related results.) When checking whether there are composite numbers n such that

$$\sum_{k=0}^{n-1}\binom{2k}{k}\equiv\left(\frac{n}{3}\right) \pmod{n^2},$$

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Sun and Tauraso found that

$$\nu_3\bigg(\sum_{k=0}^{3^a-1}\binom{2k}{k}\bigg)\geqslant 2a \quad \text{for } a=1,2,3,\ldots, \tag{1.1}$$

where $\nu_3(m)$ denotes the 3-adic valuation of an integer m (i.e., $\nu_3(m)=\sup\{a\in\mathbb{N}:\ 3^a\mid m\}$ with $\mathbb{N}=\{0,1,2,\ldots\}$). However, a refinement of this was proved earlier by Strauss, Shallit and Zagier [SSZ] in 1992.

Theorem 1.1 (Strauss, Shallit and Zagier [SSZ])**.** *For any $a\in\mathbb{Z}^+$ we have*

$$\sum_{k=0}^{3^a-1}\binom{2k}{k}\equiv 3^{2a}\ (\mathrm{mod}\ 3^{2a+1}). \tag{1.2}$$

Furthermore,

$$\frac{\sum_{k=0}^{n-1}\binom{2k}{k}}{n^2\binom{2n}{n}}\equiv -1\ (\mathrm{mod}\ 3)\quad \textit{for all } n\in\mathbb{Z}^+.$$

Recall that the usual q-analogue of $n\in\mathbb{N}$ is

$$[n]_q=\frac{1-q^n}{1-q}=\sum_{0\leqslant k<n}q^k$$

which tends to n as $q\to 1$. For $d\in\mathbb{Z}^+$ the d-th cyclotomic polynomial is given by

$$\Phi_d(q)=\prod_{\substack{r=1\\(r,d)=1}}^{d}\left(q-e^{2\pi ir/d}\right)\in\mathbb{Z}[q].$$

Given a positive integer $n>1$ we obviously have

$$[n]_q=\frac{q^n-1}{q-1}=\prod_{k=1}^{n-1}\left(q-e^{2\pi ik/n}\right)=\prod_{\substack{d\mid n\\d>1}}\Phi_d(x).$$

It is well known that if $d_1,d_2\in\mathbb{Z}^+$ are distinct then $\Phi_{d_1}(q)$ and $\Phi_{d_2}(q)$ are relatively prime in the polynomial ring $\mathbb{Z}[q]$. If p is a prime and a is a positive integer, then

$$\Phi_{p^a}(q)=\frac{q^{p^a}-1}{q^{p^{a-1}}-1}=[p]_{q^{p^{a-1}}}\quad\text{and}\quad [p^a]_q=\prod_{j=1}^{a}\Phi_{p^j}(q).$$

For $n,k\in\mathbb{N}$ the usual q-analogue of the binomial coefficient $\binom{n}{k}$ is the folloqing q-binomial coefficient:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} ([n]_q\cdots[n-k+1]_q)/([1]_q\cdots[k]_q) & \text{if } 0<k\leqslant n,\\ 1 & \text{if } k=0,\\ 0 & \text{if } k>n. \end{cases}$$

Note that $\begin{bmatrix} n \\ k \end{bmatrix}_q \to \binom{n}{k}$ as $q\to 1$. Many combinatorial identities and congruences involving binomial coefficients have their q-analogues (cf. [St]).

Recently Guo and Zeng [GZ] proposed a q-analogue of (1.1), namely they formulated the following conjecture.

Conjecture 1.2 (Guo and Zeng [GZ, Conjecture 3.5])**.** *Let a be a positive integer. Then*

$$\sum_{k=0}^{3^a m-1} q^k \begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv 0 \pmod{[3^a]_q^2} \quad \textit{for any } m\in\mathbb{Z}^+. \tag{1.3}$$

Concerning this conjecture, Guo and Zeng [GZ] were able to show (1.3) with the modulus $[3^a]_q^2$ replaced by $[3^a]_q$.

In this paper we confirm Conjecture 1.2 and give a q-analogue of (1.2).

Theorem 1.3. *Let $a\in\mathbb{Z}^+$. Then* (1.3) *holds. Furthermore, we have the following q-analogue of* (1.2)*:*

$$\frac{1}{[3^a]_q^2}\sum_{k=0}^{3^a-1} q^k \begin{bmatrix} 2k \\ k \end{bmatrix}_q \equiv 2R(a,q) \pmod{\Phi_{3^a}(q)} \tag{1.4}$$

where

$$R(a,q) := \sum_{\substack{k=1\\ 3\mid k-1}}^{3^a-1} q^{\frac{(k+2)(k-1)}{6}} \frac{(-1)^k}{[k]_q^2}\left(1+\left(\frac{k-1}{3}-\frac{3^{a-1}+1}{2}\right)(1-q^k)\right). \tag{1.5}$$

Remark 1.4. Let $a\in\mathbb{Z}^+$. Then $\lim_{q\to 1} R(a,q)\equiv -1 \pmod 3$ since

$$\sum_{\substack{k=1\\ 3\mid k-1}}^{3^a-1} \frac{(-1)^k}{k^2} = \sum_{j=0}^{3^{a-1}-1} \frac{(-1)^{3j+1}}{(3j+1)^2} \equiv -\sum_{j=0}^{3^{a-1}-1} (-1)^j = -1 \pmod 3.$$

Also, for $k\in\mathbb{Z}^+$ with $k\equiv 1 \pmod 3$, $[k]_q$ is relatively prime to $[3^a]_q$ since k is relatively prime to 3^a. Therefore (1.4) implies both (1.2) and (1.3) in the case $m=1$.

We are going to prove an auxiliary result in the next section and then show Theorem 1.3 in Section 3.

2. An auxiliary theorem

Theorem 2.1. *Let $a, m \in \mathbb{Z}^+$ and let $\psi : \mathbb{Z} \to \mathbb{Z}$ be a function such that for any $k \in \mathbb{Z}$ and $j = 1, \ldots, a$ we have*

$$\psi(k) \equiv \psi(-k) \pmod{3^a} \quad \text{and} \quad \psi(k + 3^j) \equiv \psi(k) \pmod{3^j}.$$

Then

$$\sum_{k=1}^{3^a m-1} q^{\psi(k)} \left(\frac{k}{3}\right) \begin{bmatrix} 2 \cdot 3^a m \\ k \end{bmatrix}_q \equiv 0 \pmod{[3^a]_q^2}. \tag{2.1}$$

In particular,

$$\sum_{k=1}^{3^a m-1} \left(\frac{k}{3}\right) \begin{bmatrix} 2 \cdot 3^a m \\ k \end{bmatrix}_q \equiv 0 \pmod{[3^a]_q^2}. \tag{2.2}$$

Proof. Clearly $[x]_q \equiv [y]_q \pmod{\Phi_d(q)}$ provided that $x \equiv y \pmod d$. By the q-Lucas congruence (cf. [Sa]),

$$\begin{bmatrix} x_1 d + y_1 \\ x_2 d + y_2 \end{bmatrix}_q \equiv \binom{x_1}{x_2} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_q \pmod{\Phi_d(q)}$$

for $x_1, x_2, y_1, y_2 \in \mathbb{N}$ with $0 \le y_1, y_2 \le d-1$. Recall that

$$[3^a]_q = \prod_{j=1}^{a} \Phi_{3^j}(q).$$

Since these $\Phi_{3^j}(q)$ are relatively prime and $[2 \cdot 3^a m]_q \equiv 0 \pmod{[3^a]_q}$, we only need to show that

$$\sum_{k=1}^{3^a m-1} \left(\frac{k}{3}\right) \frac{q^{\psi(k)}}{[k]_q} \begin{bmatrix} 2 \cdot 3^a m - 1 \\ k-1 \end{bmatrix}_q \equiv 0 \pmod{\Phi_{3^j}(q)}$$

for every $j = 1, \ldots, a$.

For any $1 \le j \le a$ and $1 \le k \le 3^a m - 1$ with $3 \nmid k$, write $k = 3^j s + t$ where $1 \le t \le 3^j - 1$. Then, by the q-Lucas congruence,

$$\begin{bmatrix} 2 \cdot 3^a m - 1 \\ k-1 \end{bmatrix}_q \equiv \binom{2 \cdot 3^{a-j} m - 1}{s} \begin{bmatrix} 3^j - 1 \\ t-1 \end{bmatrix}_q \pmod{\Phi_{3^j}(q)}.$$

And we have

$$\begin{bmatrix} 3^j - 1 \\ t-1 \end{bmatrix}_q = \prod_{j=1}^{t-1} \frac{[3^j - j]_q}{[j]_q}$$

$$= \prod_{j=1}^{t-1} \frac{q^{-j}([3^j]_q - [j]_q)}{[j]_q} \equiv (-1)^{t-1} q^{-\binom{t}{2}} \pmod{\Phi_{3^j}(q)}.$$

Hence

$$\begin{aligned}
&\sum_{k=1}^{3^a m-1}\left(\frac{k}{3}\right)\frac{q^{\psi(k)}}{[k]_q}{\begin{bmatrix}2\cdot 3^a m-1\\ k-1\end{bmatrix}}_q\\
=&\sum_{s=0}^{3^{a-j}m-1}\sum_{t=1}^{3^j-1}\left(\frac{3^j s+t}{3}\right)\frac{q^{\psi(3^j s+t)}}{[3^j s+t]_q}{\begin{bmatrix}2\cdot 3^a m-1\\ 3^j s+t-1\end{bmatrix}}_q\\
\equiv&\sum_{s=0}^{3^{a-j}m-1}\binom{2\cdot 3^{a-j}m-1}{s}\sum_{t=1}^{3^j-1}\left(\frac{t}{3}\right)\frac{(-1)^{t-1}q^{\psi(t)-\binom{t}{2}}}{[t]_q}\pmod{\Phi_{3^j}(q)}.
\end{aligned}$$

Clearly,

$$\begin{aligned}
&2\sum_{t=1}^{3^j-1}\left(\frac{t}{3}\right)\frac{(-1)^{t-1}q^{\psi(t)-\binom{t}{2}}}{[t]_q}\\
=&\sum_{t=1}^{3^j-1}\left(\left(\frac{t}{3}\right)\frac{(-1)^{t-1}q^{\psi(t)-\binom{t}{2}}}{[t]_q}+\left(\frac{3^j-t}{3}\right)\frac{(-1)^{3^j-t-1}q^{\psi(3^j-t)-\binom{3^j-t}{2}}}{[3^j-t]_q}\right)\\
\equiv&\sum_{\substack{t=1\\ 3\nmid t}}^{3^j-1}\left(\frac{t}{3}\right)\left(\frac{(-1)^{t-1}q^{\psi(t)-\binom{t}{2}}}{[t]_q}+\frac{(-1)^{t-1}q^{\psi(t)-\binom{-t}{2}}}{-q^{-t}[t]_q}\right)=0\pmod{\Phi_{3^j}(q)}.
\end{aligned}$$

So (2.1) holds.

Note that (2.2) is just (2.1) with ψ replaced by the zero function from $\mathbb{Z}\to\mathbb{Z}$. So (2.2) is also valid. This concludes the proof. □

3. Proof of Theorem 1.3

Lemma 3.1. *Suppose that* $k\equiv l\pmod{3^a}$ *where* $k,l\in\mathbb{Z}$ *and* $a\in\mathbb{Z}^+$. *Then*

$$2k^2-k\left(\frac{k}{3}\right)\equiv 2l^2-l\left(\frac{l}{3}\right)\pmod{3^{a+1}}.$$

Proof. Observe that

$$2(k+l)-\left(\frac{k}{3}\right)\equiv 4k-\left(\frac{k}{3}\right)\equiv 0\pmod 3.$$

Thus

$$\begin{aligned}
&2k^2-k\left(\frac{k}{3}\right)-\left(2l^2-l\left(\frac{l}{3}\right)\right)\\
=&(k-l)\left(2(k+l)-\left(\frac{k}{3}\right)\right)\equiv 0\pmod{3^{a+1}}.
\end{aligned}$$

We are done. □

Lemma 3.2. *Let $a \in \mathbb{Z}^+$ and let ψ be a function as in Theorem 2.1. Then*

$$\begin{aligned} &\frac{1}{2[3^a]_q^2}\sum_{k=1}^{3^a-1} q^{\psi(k)}\left(\frac{k}{3}\right)\begin{bmatrix}2\cdot 3^a\\ k\end{bmatrix}_q \\ \equiv&\sum_{\substack{k=1\\ 3\mid k-1}}^{3^a-1} q^{\psi(k)-\binom{k}{2}}\frac{(-1)^{k-1}}{[k]_q^2}(1+\Psi_a(k)(1-q^k)) \pmod{\Phi_{3^a}(q)}, \end{aligned} \tag{3.1}$$

where

$$\Psi_a(k) := \frac{\psi(3^a-k)-\psi(k)}{3^a}+\frac{3^a-1}{2}-k. \tag{3.2}$$

Proof. We have

$$\begin{aligned} &\sum_{k=1}^{3^a-1}\left(\frac{k}{3}\right)\frac{q^{\psi(k)}}{[k]_q}\begin{bmatrix}2\cdot 3^a-1\\ k-1\end{bmatrix}_q \\ =&\sum_{k=1}^{3^a-1}\left(\frac{k}{3}\right)\frac{q^{\psi(k)}}{[k]_q}\prod_{j=1}^{k-1}\frac{q^{-j}([2\cdot 3^a]_q-[j]_q)}{[j]_q} \\ \equiv&\sum_{k=1}^{3^a-1}\left(\frac{k}{3}\right)\frac{(-1)^{k-1}q^{\psi(k)-\binom{k}{2}}}{[k]_q}\left(1-2\sum_{j=1}^{k-1}\frac{[3^a]_q}{[j]_q}\right) \pmod{\Phi_{3^a}(q)^2}, \end{aligned}$$

since

$$[2\cdot 3^a]_q=[3^a]_q(1+q^{3^a})=[3^a]_q(2+q^{3^a}-1)\equiv 2[3^a]_q \pmod{[3^a]_q^2}.$$

Note that for $s=0,1,2,\dots$ we have

$$\begin{aligned} q^{3^a s}=&1+(q^{3^a}-1)\sum_{j=0}^{s-1}q^{3^a j}=1+(q^{3^a}-1)\left(s+\sum_{j=0}^{s-1}(q^{3^a j}-1)\right) \\ \equiv&1+s(q^{3^a}-1) \pmod{\Phi_{3^a}(q)^2} \end{aligned}$$

and

$$q^{-3^a s}\equiv\frac{1}{1+s(q^{3^a}-1)}=\frac{1-s(q^{3^a}-1)}{1-s^2(q^{3^a}-1)^2}\equiv 1-s(q^{3^a}-1) \pmod{\Phi_{3^a}(q)^2}.$$

Also, for each $1 \le k \le 3^a - 1$, we have

$$\begin{aligned}
&\frac{q^{\psi(3^a-k)-\binom{3^a-k}{2}}}{[3^a-k]_q}\left(1-2\sum_{j=1}^{3^a-k-1}\frac{[3^a]_q}{[j]_q}\right)\\
=&\frac{q^{\psi(3^a-k)-\binom{3^a}{2}+3^ak-\binom{k+1}{2}}([3^a]_q+[k]_q)}{q^{-k}([3^a]_q^2-[k]_q^2)}\left(1-2\sum_{j=k+1}^{3^a-1}\frac{[3^a]_q}{[3^a-j]_q}\right)\\
\equiv&\frac{q^{\psi(k)-\binom{k}{2}}(1+(\frac{\psi(3^a-k)-\psi(k)}{3^a}+k-\frac{3^a-1}{2})(q^{3^a}-1))([3^a]_q+[k]_q)}{-[k]_q^2}\\
&\times\left(1+2\sum_{j=k+1}^{3^a-1}\frac{q^j[3^a]_q}{[j]_q}\right)\\
\equiv&-\frac{q^{\psi(k)-\binom{k}{2}}}{[k]_q}\left(1+\left(\frac{\psi(3^a-k)-\psi(k)}{3^a}+k-\frac{3^a-1}{2}\right)(q^{3^a}-1)\right)\\
&-2\frac{q^{\psi(k)-\binom{k}{2}}}{[k]_q}\sum_{j=k+1}^{3^a-1}\frac{q^j[3^a]_q}{[j]_q}-\frac{q^{\psi(k)-\binom{k}{2}}[3^a]_q}{[k]_q^2}\pmod{\Phi_{3^a}(q)^2}.
\end{aligned}$$

Clearly,

$$\sum_{j=k+1}^{3^a-1}\frac{q^j}{[j]_q}=\sum_{j=k+1}^{3^a-1}\frac{1+q^j-1}{[j]_q}=-(3^a-1-k)(1-q)+\sum_{j=k+1}^{3^a-1}\frac{1}{[j]_q},$$

and

$$\begin{aligned}
\sum_{j=1}^{3^a-1}\frac{1}{[j]_q}=&\frac{1}{2}\sum_{j=1}^{3^a-1}\left(\frac{1}{[j]_q}+\frac{1}{[3^a-j]_q}\right)\\
\equiv&\frac{1}{2}\sum_{j=1}^{3^a-1}\left(\frac{1}{[j]_q}-\frac{q^j}{[j]_q}\right)=\frac{3^a-1}{2}(1-q)\pmod{\Phi_{3^a}(q)}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\frac{q^{\psi(k)-\binom{k}{2}}}{[k]_q}\left(1-\sum_{j=1}^{k-1}\frac{[2\cdot 3^a]_q}{[j]_q}\right)+\frac{q^{\psi(3^a-k)-\binom{3^a-k}{2}}}{[3^a-k]_q}\left(1-\sum_{j=1}^{3^a-k-1}\frac{[2\cdot 3^a]_q}{[j]_q}\right)\\
\equiv&-\frac{q^{\psi(k)-\binom{k}{2}}}{[k]_q}\left(\left(\frac{\psi(3^a-k)-\psi(k)}{3^a}+\frac{3}{2}(3^a-1)-k\right)(q^{3^a}-1)\right)\\
&-\frac{q^{\psi(k)-\binom{k}{2}}}{[k]_q}\left(2\sum_{j=1}^{k-1}\frac{[3^a]_q}{[j]_q}+2\sum_{j=k+1}^{3^a-1}\frac{[3^a]_q}{[j]_q}+\frac{q^{\psi(k)-\binom{k}{2}}[3^a]_q}{[k]_q^2}\right)\\
\equiv&\frac{q^{\psi(k)-\binom{k}{2}}[3^a]_q}{[k]_q^2}+\frac{q^{\psi(k)-\binom{k}{2}}}{[k]_q}\Psi_a(k)(1-q^{3^a})\pmod{\Phi_{3^a}(q)^2}.
\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{k=1}^{3^a-1}\left(\frac{k}{3}\right)\frac{q^{\psi(k)}}{[k]_q}\begin{bmatrix}2\cdot 3^a-1\\ k-1\end{bmatrix}_q\\ \equiv&\sum_{\substack{k=1\\ 3\mid k-1}}^{3^a-1}(-1)^{k-1}\frac{q^{\psi(k)-\binom{k}{2}}}{[k]_q}\left(\frac{[3^a]_q}{[k]_q}+\Psi_a(k)(1-q^{3^a})\right) \pmod{\Phi_{3^a}(q)^2}.\end{aligned}$$

Noting that $[3^a]_q$ divides both sides of the above congruence by Theorem 2.1 and $[2\cdot 3^a]\equiv 2[3^a] \pmod{[3^a]_q^2}$, we are done. □

Proof of Theorem 1.3. Let $m\in\mathbb{Z}^+$. By [T, (4.3)] in the case $d=0$, we have

$$\sum_{k=0}^{3^am-1}q^k\begin{bmatrix}2k\\ k\end{bmatrix}_q=-\sum_{k=1}^{3^am-1}q^{\psi_m(k)}\left(\frac{k}{3}\right)\begin{bmatrix}2\cdot 3^am\\ k\end{bmatrix}_q, \tag{3.3}$$

where

$$\psi_m(k)=\frac{2(3^am-k)^2-(3^am-k)\left(\frac{3^am-k}{3}\right)-1}{3}.$$

According to Lemma 3.1, the function $\psi=\psi_m$ has the property described in Theorem 2.1. Combining (2.1) with (3.3) we get (1.3).

Now it remains to prove (1.4). By (3.3) and Lemma 3.2, we finally obtain

$$\begin{aligned}&\sum_{k=0}^{3^a-1}q^k\begin{bmatrix}2k\\ k\end{bmatrix}_q=-\sum_{k=1}^{3^a-1}q^{\psi_1(k)}\left(\frac{k}{3}\right)\begin{bmatrix}2\cdot 3^a\\ k\end{bmatrix}_q\\ \equiv&2[3^a]_q^2\sum_{\substack{k=1\\ 3\mid k-1}}^{3^a-1}q^{\frac{(k+2)(k-1)}{6}}\frac{(-1)^k}{[k]_q^2}\left(1+\left(\frac{k-1}{3}-\frac{3^{a-1}+1}{2}\right)(1-q^k)\right)\\ &\pmod{\Phi_{3^a}(q)[3^a]_q^2}.\end{aligned}$$

This concludes our proof. □

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