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# CONGRUENCES FOR FRANEL NUMBERS 

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Abstract. The Franel numbers given by $f_{n}=\sum_{k=0}^{n}\binom{n}{k}^{3}(n=0,1,2, \ldots)$ play important roles in both combinatorics and number theory. In this paper we initiate the systematic investigation of fundamental congruences for the Franel numbers. We mainly establish for any prime $p>3$ the following congruences:

$$
\begin{gathered}
\sum_{k=0}^{p-1}(-1)^{k} f_{k} \equiv\left(\frac{p}{3}\right)\left(\bmod p^{2}\right), \sum_{k=0}^{p-1}(-1)^{k} k f_{k} \equiv-\frac{2}{3}\left(\frac{p}{3}\right)\left(\bmod p^{2}\right), \\
\sum_{k=1}^{p-1} \frac{(-1)^{k}}{k} f_{k} \equiv 0\left(\bmod p^{2}\right), \sum_{k=1}^{p-1} \frac{(-1)^{k}}{k^{2}} f_{k} \equiv 0(\bmod p) .
\end{gathered}
$$

## 1. Introduction

In 1894, Franel [F] noted that the numbers

$$
\begin{equation*}
f_{n}=\sum_{k=0}^{n}\binom{n}{k}^{3} \quad(n=0,1,2, \ldots) \tag{1.1}
\end{equation*}
$$

(cf. [Sl, A000172]) satisfy the recurrence relation:

$$
\begin{equation*}
(n+1)^{2} f_{n+1}=\left(7 n^{2}+7 n+2\right) f_{n}+8 n^{2} f_{n-1} \quad(n=1,2,3, \ldots) \tag{1.2}
\end{equation*}
$$

Such numbers are now called Franel numbers. For a combinatorial interpretation of the Franel numbers, see Callan [C]. Recall that the Apéry numbers given by

$$
A_{n}=\sum_{k=0}^{n}\binom{n}{k}^{2}\binom{n+k}{k}^{2}=\sum_{k=0}^{n}\binom{n+k}{2 k}^{2}\binom{2 k}{k}^{2}(n=0,1,2, \ldots)
$$

[^0]were introduced by Apéry [A], and they can be expressed in terms of Franel numbers as follows:
\[

$$
\begin{equation*}
A_{n}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k} f_{k} \tag{1.3}
\end{equation*}
$$

\]

(see Strehl [St92]). The Franel numbers are also related to the theory of modular forms, see, e.g., Zagier [Z].

In this paper we study congruences for the Franel numbers systematically. As usual, for any odd prime $p$ and integer $a,\left(\frac{a}{p}\right)$ denotes the Legendre symbol, and $q_{p}(a)$ stands for the Fermat quotient $\left(a^{p-1}-1\right) / p$ if $p \nmid a$.

Now we state our main result.
Theorem 1.1. Let $p>3$ be a prime. For any p-adic integer $r$ we have

$$
\begin{equation*}
\sum_{k=0}^{p-1}(-1)^{k}\binom{k+r}{k} f_{k} \equiv \sum_{k=0}^{p-1}\binom{2 k}{k}\binom{k+r}{k}^{2} \quad\left(\bmod p^{2}\right) \tag{1.4}
\end{equation*}
$$

In particular,

$$
\begin{align*}
\sum_{k=0}^{p-1}(-1)^{k} f_{k} & \equiv\left(\frac{p}{3}\right) \quad\left(\bmod p^{2}\right),  \tag{1.5}\\
\sum_{k=0}^{p-1}(-1)^{k} k f_{k} & \equiv-\frac{2}{3}\left(\frac{p}{3}\right) \quad\left(\bmod p^{2}\right),  \tag{1.6}\\
\sum_{k=0}^{p-1}(-1)^{k} k^{2} f_{k} & \equiv \frac{10}{27}\left(\frac{p}{3}\right) \quad\left(\bmod p^{2}\right), \tag{1.7}
\end{align*}
$$

and

$$
\begin{equation*}
\sum_{k=0}^{p-1} \frac{\binom{2 k}{k} f_{k}}{(-4)^{k}} \equiv \sum_{k=0}^{p-1} \frac{\binom{2 k}{k}^{3}}{16^{k}} \quad\left(\bmod p^{2}\right) \tag{1.8}
\end{equation*}
$$

We also have

$$
\begin{align*}
\sum_{k=1}^{p-1} \frac{(-1)^{k}}{k} f_{k} & \equiv 0\left(\bmod p^{2}\right)  \tag{1.9}\\
\sum_{k=1}^{p-1} \frac{(-1)^{k}}{k^{2}} f_{k} & \equiv 0(\bmod p)  \tag{1.10}\\
\sum_{k=1}^{p-1} \frac{(-1)^{k}}{k} f_{k-1} & \equiv 3 q_{p}(2)+3 p q_{p}(2)^{2}\left(\bmod p^{2}\right) \tag{1.11}
\end{align*}
$$

Remark 1.1. Fix a prime $p>3$. In contrast with (1.5), we conjecture that

$$
\sum_{n=0}^{p-1}(-1)^{n} \sum_{k=0}^{n}\binom{n}{k}^{3}(-8)^{k} \equiv \sum_{k=0}^{p-1} \frac{f_{k}}{8^{k}} \equiv\left(\frac{p}{3}\right) \quad\left(\bmod p^{2}\right)
$$

As $f_{k} \equiv(-8)^{k} f_{p-1-k}(\bmod p)$ for all $k=0, \ldots, p-1$ by [JV, Lemma 2.6], (1.11) implies that

$$
\sum_{k=1}^{p-1} \frac{f_{k}}{k 8^{k}} \equiv \sum_{k=1}^{p-1} \frac{(-1)^{k}}{k} f_{p-1-k}=\sum_{k=1}^{p-1} \frac{(-1)^{p-k}}{p-k} f_{k-1} \equiv 3 q_{p}(2) \quad(\bmod p)
$$

Motivated by (1.5) and (1.6), we conjecture that both $\left(\sum_{k=0}^{n-1}(-1)^{k} f_{k}\right) / n^{2}$ and $\left(\sum_{k=0}^{n-1}(-1)^{k} k f_{k}\right) / n^{2}$ are 3 -adic integers for any positive integer $n$. Concerning (1.8) the author [S11, Conj. 5.2(ii)] conjectured that

$$
\sum_{k=0}^{p-1} \frac{\binom{2 k}{k}^{3}}{16^{k}} \equiv \begin{cases}4 x^{2}-2 p\left(\bmod p^{2}\right) & \text { if } p=x^{2}+3 y^{2}(x, y \in \mathbb{Z}) \\ 0\left(\bmod p^{2}\right) & \text { if } p \equiv 2(\bmod 3)\end{cases}
$$

See also [S13] for other connections between $p=x^{2}+3 y^{2}$ and Franel numbers. (1.10) can be extended as

$$
\begin{equation*}
\sum_{k=1}^{p-1} \frac{(-1)^{k r}}{k^{r-1}} f_{k}^{(r)} \equiv 0 \quad(\bmod p) \tag{1.12}
\end{equation*}
$$

where $r$ is any positive integer and $f_{k}^{(r)}:=\sum_{j=0}^{k}\binom{k}{j}^{r}$. Note that $f_{k}^{(2)}=\binom{2 k}{k}$ and $\sum_{k=1}^{p-1}\binom{2 k}{k} / k \equiv 0\left(\bmod p^{2}\right)$ by [ST10].

Let $p>3$ be a prime. Similar to (1.5)-(1.7), we are also able to show that

$$
\sum_{k=0}^{p-1}(-1)^{k} k^{3} f_{k} \equiv-\frac{10}{81}\left(\frac{p}{3}\right)\left(\bmod p^{2}\right) \text { and } \sum_{k=0}^{p-1}(-1)^{k} k^{4} f_{k} \equiv-\frac{14}{243}\left(\frac{p}{3}\right)\left(\bmod p^{2}\right)
$$

In general, for any positive integer $r$ and prime $p>\max \{r, 3\}$ there should be an odd integer $a_{r}$ (not dependent on $p$ ) such that

$$
\sum_{k=0}^{p-1}(-1)^{k} k^{r} f_{k} \equiv \frac{2 a_{r}}{3^{2 r-1}}\left(\frac{p}{3}\right) \quad\left(\bmod p^{2}\right)
$$

## 2. Proof of Theorem 1.1

We first establish an auxiliary theorem on the polynomials

$$
f_{n}(x):=\sum_{k=0}^{n}\binom{n}{k}^{2}\binom{2 k}{n} x^{k}=\sum_{k=0}^{n}\binom{n}{k}\binom{k}{n-k}\binom{2 k}{k} x^{k} \quad(n=0,1,2, \ldots)
$$

Theorem 2.1. Let $p$ be an odd prime and let $r$ be any p-adic integer. Then

$$
\begin{equation*}
\sum_{l=0}^{p-1}(-1)^{l}\binom{l+r}{l} f_{l}(x) \equiv \sum_{k=0}^{p-1}\binom{2 k}{k} x^{k}\binom{k+r}{k}^{2} \quad\left(\bmod p^{2}\right) \tag{2.1}
\end{equation*}
$$

Proof. Observe that

$$
\begin{aligned}
\sum_{l=0}^{p-1}(-1)^{l}\binom{l+r}{l} f_{l}(x) & =\sum_{l=0}^{p-1}(-1)^{l}\binom{l+r}{l} \sum_{k=0}^{l}\binom{l}{k}\binom{k}{l-k}\binom{2 k}{k} x^{k} \\
& =\sum_{k=0}^{p-1}\binom{2 k}{k} x^{k} \sum_{l=k}^{\min \{2 k, p-1\}}(-1)^{l}\binom{l}{k}\binom{k}{l-k}\binom{l+r}{l}
\end{aligned}
$$

If $(p-1) / 2<k \leqslant p-1$ and $p \leqslant l \leqslant 2 k$, then

$$
\binom{2 k}{k}=\frac{(2 k)!}{(k!)^{2}} \equiv 0(\bmod p) \text { and }\binom{l}{k}=\frac{l!}{k!(l-k)!} \equiv 0(\bmod p)
$$

Thus
$\sum_{l=0}^{p-1}(-1)^{l}\binom{l+r}{l} f_{l}(x) \equiv \sum_{k=0}^{p-1}\binom{2 k}{k} x^{k} \sum_{l=k}^{2 k}(-1)^{l}\binom{l}{k}\binom{k}{l-k}\binom{l+r}{l} \quad\left(\bmod p^{2}\right)$, and hence it suffices to show the identity

$$
\begin{equation*}
\sum_{l=k}^{2 k}(-1)^{l}\binom{l}{k}\binom{k}{l-k}\binom{x+l}{l}=\binom{x+k}{k}^{2} \tag{2.2}
\end{equation*}
$$

By the well-known Chu-Vandermonde identity (cf. (3.1) of [G, p.22]),

$$
\sum_{j=0}^{k}\binom{y}{j}\binom{z}{k-j}=\binom{y+z}{k}
$$

Therefore

$$
\begin{aligned}
& \sum_{l=k}^{2 k}(-1)^{l}\binom{l}{k}\binom{k}{l-k}\binom{x+l}{l} \\
= & \sum_{l=k}^{2 k}\binom{l}{k}\binom{k}{l-k}\binom{-x-1}{l}=\binom{-x-1}{k} \sum_{l=k}^{2 k}\binom{-x-1-k}{l-k}\binom{k}{l-k} \\
= & \binom{-x-1}{k} \sum_{j=0}^{k}\binom{-x-1-k}{j}\binom{k}{k-j}=\binom{-x-1}{k}^{2}=\binom{x+k}{k}^{2} .
\end{aligned}
$$

This proves (2.2) and hence (2.1) follows.

Lemma 2.1. For any nonnegative integer $n$, the integer $f_{n}(1)$ coincides with the Franel number $f_{n}$.
Proof. The identity $\sum_{k=0}^{n}\binom{n}{k}^{2}\binom{2 k}{n}=f_{n}$ is a known result due to Strehl [St94].

Lemma 2.2. For each positive integer $m$ we have

$$
\sum_{k=0}^{n-1} P_{m}(k)\binom{2 k}{k}=n^{m}\binom{2 n}{n} \quad \text { for all } n=1,2,3, \ldots
$$

where $P_{m}(x):=2(2 x+1)(x+1)^{m-1}-x^{m}$.
Proof. The desired result follows immediately by induction on $n$.
Lemma 2.3. Let $m$ be a positive integer. For $n=0,1, \ldots, m$ we have

$$
\sum_{k=0}^{n}\binom{x}{k}\binom{-x}{m-k}=\frac{m-n}{m}\binom{x-1}{n}\binom{-x}{m-n}
$$

Remark 2.1. This is a known result due to Andersen, see, e.g., (3.14) of [G, p. 23].

Lemma 2.4 ([S11, Lemma 2.1]). Let $p$ be an odd prime. For any $k=1, \ldots, p-$ 1 we have

$$
k\binom{2 k}{k}\binom{2(p-k)}{p-k} \equiv(-1)^{\lfloor 2 k / p\rfloor-1} 2 p \quad\left(\bmod p^{2}\right)
$$

Recall that the harmonic numbers and the second-order harmonic numbers are given by

$$
H_{n}=\sum_{0<k \leqslant n} \frac{1}{k} \text { and } H_{n}^{(2)}=\sum_{0<k \leqslant n} \frac{1}{k^{2}} \quad(n=0,1,2, \ldots)
$$

respectively. Let $p>3$ be a prime. In 1862, Wolstenholme [W] proved that

$$
H_{p-1} \equiv 0\left(\bmod p^{2}\right) \quad \text { and } \quad H_{p-1}^{(2)} \equiv 0(\bmod p)
$$

Note that

$$
H_{(p-1) / 2}^{(2)} \equiv \frac{1}{2} \sum_{k=1}^{(p-1) / 2}\left(\frac{1}{k^{2}}+\frac{1}{(p-k)^{2}}\right)=\frac{1}{2} H_{p-1}^{(2)} \equiv 0 \quad(\bmod p)
$$

In 1938, Lehmer [L] showed that

$$
\begin{equation*}
H_{(p-1) / 2} \equiv-2 q_{p}(2)+p q_{p}(2)^{2} \quad\left(\bmod p^{2}\right) \tag{2.3}
\end{equation*}
$$

Lemma 2.5. Let $p>3$ be a prime. Then

$$
\begin{equation*}
f_{p-1} \equiv 1+3 p q_{p}(2)+3 p^{2} q_{p}(2)^{2} \quad\left(\bmod p^{3}\right) \tag{2.4}
\end{equation*}
$$

Proof. For any $k=1, \ldots, p-1$, we obviously have

$$
\begin{aligned}
& (-1)^{k}\binom{p-1}{k}=\prod_{j=1}^{k}\left(1-\frac{p}{j}\right) \\
\equiv & 1-p H_{k}+\frac{p^{2}}{2} \sum_{1 \leqslant i<j \leqslant k} \frac{2}{i j}=1-p H_{k}+\frac{p^{2}}{2}\left(H_{k}^{2}-H_{k}^{(2)}\right) \quad\left(\bmod p^{3}\right) .
\end{aligned}
$$

Thus

$$
\begin{aligned}
f_{p-1}-1 & =\sum_{k=1}^{p-1}\binom{p-1}{k}^{3} \equiv \sum_{k=1}^{p-1}(-1)^{k}\left(1-p H_{k}+\frac{p^{2}}{2}\left(H_{k}^{2}-H_{k}^{(2)}\right)\right)^{3} \\
& \equiv-3 p \sum_{k=1}^{p-1}(-1)^{k} H_{k}+\frac{9}{2} p^{2} \sum_{k=1}^{p-1}(-1)^{k} H_{k}^{2}-\frac{3}{2} p^{2} \sum_{k=1}^{p-1}(-1)^{k} H_{k}^{(2)}\left(\bmod p^{3}\right) .
\end{aligned}
$$

Clearly

$$
\begin{aligned}
\sum_{k=1}^{p-1}(-1)^{k} H_{k} & =\sum_{k=1}^{p-1} \sum_{j=1}^{k} \frac{(-1)^{k}}{j}=\sum_{j=1}^{p-1} \frac{\sum_{k=j}^{p-1}(-1)^{k}}{j}=\sum_{\substack{j=1 \\
2 \mid j}}^{p-1} \frac{1}{j} \\
& =\frac{1}{2} H_{(p-1) / 2} \equiv-q_{p}(2)+\frac{p}{2} q_{p}(2)^{2}\left(\bmod p^{2}\right) \quad(\text { by }(2.3))
\end{aligned}
$$

and

$$
\sum_{k=1}^{p-1}(-1)^{k} H_{k}^{(2)}=\sum_{j=1}^{p-1} \frac{\sum_{k=j}^{p-1}(-1)^{k}}{j^{2}}=\sum_{i=1}^{(p-1) / 2} \frac{1}{(2 i)^{2}}=\frac{H_{(p-1) / 2}^{(2)}}{4} \equiv 0 \quad(\bmod p)
$$

Observe that

$$
\begin{aligned}
\sum_{k=1}^{p-1}(-1)^{k} H_{k}^{2} & =\sum_{k=1}^{p-1}(-1)^{p-k} H_{p-k}^{2}=\sum_{k=1}^{p-1}(-1)^{k-1}\left(H_{p-1}-\sum_{0<j<k} \frac{1}{p-j}\right)^{2} \\
& \equiv-\sum_{k=1}^{p-1}(-1)^{k}\left(H_{k}-\frac{1}{k}\right)^{2} \\
& =-\sum_{k=1}^{p-1}(-1)^{k} H_{k}^{2}+2 \sum_{k=1}^{p-1} \frac{(-1)^{k}}{k} H_{k}-\sum_{k=1}^{p-1} \frac{(-1)^{k}}{k^{2}}(\bmod p)
\end{aligned}
$$

Clearly,

$$
\sum_{k=1}^{p-1} \frac{(-1)^{k}}{k^{2}} \equiv \sum_{k=1}^{p-1} \frac{1+(-1)^{k}}{k^{2}}=\sum_{j=1}^{(p-1) / 2} \frac{2}{(2 j)^{2}} \equiv 0 \quad(\bmod p)
$$

and

$$
\sum_{k=1}^{p-1} \frac{(-1)^{k}}{k} H_{k}=\sum_{\substack{k=1 \\ 2 \mid k}}^{p-1} \frac{H_{k}}{k}-\sum_{\substack{k=1 \\ 2 \nmid k}}^{p-1} \frac{H_{k}}{k} \equiv \frac{q_{p}(2)^{2}}{2}-\left(-\frac{q_{p}(2)^{2}}{2}\right) \quad(\bmod p)
$$

by [S12a, Lemma 2.3]. Therefore

$$
\sum_{k=1}^{p-1}(-1)^{k} H_{k}^{2} \equiv \sum_{k=1}^{p-1} \frac{(-1)^{k}}{k} H_{k} \equiv q_{p}(2)^{2} \quad(\bmod p)
$$

Combining the above, we finally obtain

$$
f_{p-1}-1 \equiv-3 p\left(-q_{p}(2)+\frac{p}{2} q_{p}(2)^{2}\right)+\frac{9}{2} p^{2} q_{p}(2)^{2}\left(\bmod p^{3}\right)
$$

and hence (2.4) holds.
Lemma 2.6. Let $p$ be any prime. Then

$$
\binom{p-1}{k}\binom{p+k}{k} \equiv(-1)^{k} \quad\left(\bmod p^{2}\right) \quad \text { for } k=0,1, \ldots, p-1
$$

and

$$
\binom{2 k}{k} \sum_{n=k}^{p-1}(2 n+1)\binom{n+k}{2 k} \equiv p^{2} \frac{(-1)^{k}}{k+1} \quad\left(\bmod p^{4}\right) \quad \text { for } k=0, \ldots, p-2
$$

Proof. Let $k \in\{0,1, \ldots, p-1\}$. Clearly

$$
\binom{p-1}{k}\binom{p+k}{k}=\prod_{0<j \leqslant k}\left(\frac{p-j}{j} \cdot \frac{p+j}{j}\right) \equiv(-1)^{k}\left(\bmod p^{2}\right) .
$$

In view of the known identity $\sum_{n=0}^{m}\binom{n}{l}=\binom{m+1}{l+1}(l, m=0,1, \ldots)$ (see, e.g., (1.52) of [G, p. 7]) which can be easily proved by induction, we have

$$
\begin{aligned}
\sum_{n=k}^{p-1} \frac{2 n+1}{2 k+1}\binom{n+k}{2 k} & =\sum_{n=k}^{p-1}\left(\frac{2(n+k+1)}{2 k+1}-1\right)\binom{n+k}{2 k} \\
& =2 \sum_{n=k}^{p-1}\binom{n+k+1}{2 k+1}-\sum_{n=k}^{p-1}\binom{n+k}{2 k} \\
& =2\binom{p+k+1}{2 k+2}-\binom{p+k}{2 k+1}=\frac{p}{k+1}\binom{p+k}{2 k+1}
\end{aligned}
$$

and hence

$$
\binom{2 k}{k} \sum_{n=k}^{p-1}(2 n+1)\binom{n+k}{2 k}=p \frac{2 k+1}{k+1}\binom{2 k}{k}\binom{p+k}{2 k+1}=\frac{p^{2}}{k+1}\binom{p-1}{k}\binom{p+k}{k}
$$

Thus, if $k<p-1$ then

$$
\binom{2 k}{k} \sum_{n=k}^{p-1}(2 n+1)\binom{n+k}{2 k} \equiv \frac{p^{2}}{k+1}(-1)^{k} \quad\left(\bmod p^{4}\right)
$$

as desired.
Proof of Theorem 1.1. In view of Lemma 2.1, (2.1) with $x=1$ gives (1.4).
(2.1) with $r=0$ yields the congruence

$$
\sum_{k=0}^{p-1}(-1)^{k} f_{k}(x) \equiv \sum_{k=0}^{p-1}\binom{2 k}{k} x^{k} \quad\left(\bmod p^{2}\right)
$$

In the case $x=1$, this gives (1.5) since $\sum_{k=0}^{p-1}\binom{2 k}{k} \equiv\left(\frac{p}{3}\right)\left(\bmod p^{2}\right)$ by [ST11, (1.9)].

By (2.1) with $r=0,1$,

$$
\begin{aligned}
& \sum_{k=0}^{p-1}(3(k+1)-1)(-1)^{k} f_{k}(x) \\
\equiv & \sum_{k=0}^{p-1}\binom{2 k}{k} x^{k}\left(3(k+1)^{2}-1\right)=\sum_{k=0}^{p-1} P_{2}(k)\binom{2 k}{k} x^{k}\left(\bmod p^{2}\right)
\end{aligned}
$$

where $P_{2}(x)=2(2 x+1)(x+1)-x^{2}=3 x^{2}+6 x+2$. Thus, with the help of Lemmas 2.1-2.2, we have

$$
\begin{equation*}
\sum_{k=0}^{p-1}(3 k+2)(-1)^{k} f_{k} \equiv 0 \quad\left(\bmod p^{2}\right) \tag{2.5}
\end{equation*}
$$

and hence (1.6) holds in view of (1.5).
Taking $r=2$ in (2.1) we get

$$
2 \sum_{k=0}^{p-1}\left(k^{2}+3 k+2\right)(-1)^{k} f_{k}(x) \equiv \sum_{k=0}^{p-1}\binom{2 k}{k} x^{k}((k+1)(k+2))^{2} \quad\left(\bmod p^{2}\right)
$$

In view of (2.5), this yields

$$
2 \sum_{k=0}^{p-1}(-1)^{k} k^{2} f_{k} \equiv \sum_{k=0}^{p-1}\binom{2 k}{k}\left(k^{2}+3 k+2\right)^{2} \quad\left(\bmod p^{2}\right)
$$

Note that

$$
27\left(k^{2}+3 k+2\right)^{2}=9 P_{4}(k)+12 P_{3}(k)+23 P_{2}(k)+20
$$

where $P_{m}(x)$ is given by Lemma 2.2. Therefore, with the help of Lemma 2.3 and $[S T 11,(1.9)]$, we have
$54 \sum_{k=0}^{p-1}(-1)^{k} k^{2} f_{k} \equiv \sum_{k=0}^{p-1}\left(9 P_{4}(k)+12 P_{3}(k)+23 P_{2}(k)+20\right)\binom{2 k}{k} \equiv 20\left(\frac{p}{3}\right)\left(\bmod p^{2}\right)$
and hence (1.7) follows.
Putting $r=-1 / 2$ in (2.1) and noting that $\binom{k-1 / 2}{k}=\binom{2 k}{k} / 4^{k}$, we then obtain

$$
\begin{equation*}
\sum_{k=0}^{p-1} \frac{\binom{2 k}{k} f_{k}(x)}{(-4)^{k}} \equiv \sum_{k=0}^{p-1} \frac{\binom{2 k}{k}^{3}}{16^{k}} x^{k} \quad\left(\bmod p^{2}\right) \tag{2.6}
\end{equation*}
$$

In the case $x=1$ this gives (1.8).
Now we prove (1.9). Observe that

$$
\sum_{l=1}^{p-1} \frac{(-1)^{l}}{l} \sum_{k=0}^{l}\binom{l}{k}\binom{k}{l-k}\binom{2 k}{k} x^{k}=\sum_{k=1}^{p-1} \frac{\binom{2 k}{k}}{k} x^{k} \sum_{l=k}^{p-1}(-1)^{l}\binom{l-1}{k-1}\binom{k}{l-k} .
$$

If $1 \leqslant k \leqslant(p-1) / 2$, then

$$
\begin{aligned}
\sum_{l=k}^{p-1}(-1)^{l}\binom{l-1}{k-1}\binom{k}{l-k} & =\sum_{l=k}^{2 k}(-1)^{l}\binom{l-1}{k-1}\binom{k}{l-k} \\
& =\sum_{j=0}^{k}(-1)^{k+j}\binom{k+j-1}{j}\binom{k}{j} \\
& =(-1)^{k} \sum_{j=0}^{k}\binom{-k}{j}\binom{k}{k-j}=(-1)^{k}\binom{0}{k}=0
\end{aligned}
$$

by the Chu-Vandermonde identity. If $(p+1) / 2 \leqslant k \leqslant p-1$, then

$$
\begin{aligned}
\sum_{l=k}^{p-1}(-1)^{l}\binom{l-1}{k-1}\binom{k}{l-k} & =\sum_{j=0}^{p-1-k}(-1)^{k+j}\binom{k+j-1}{j}\binom{k}{j} \\
& =(-1)^{k} \sum_{j=0}^{p-1-k}\binom{-k}{j}\binom{k}{k-j}
\end{aligned}
$$

and hence applying Lemma 2.3 we get

$$
\begin{aligned}
& \sum_{l=k}^{p-1}(-1)^{l}\binom{l-1}{k-1}\binom{k}{l-k} \\
= & (-1)^{k} \frac{k-(p-1-k)}{k}\binom{-k-1}{p-1-k}\binom{k}{k-(p-1-k)} \\
= & (-1)^{p-1}\left(\frac{p-k}{k}\right)^{2}\binom{p-1}{k-1}\binom{k}{p-k} \\
\equiv & (-1)^{k-1}\binom{k}{p-k}=\binom{p-2 k-1}{p-k} \\
\equiv & \binom{2(p-k)-1}{p-k}=\frac{1}{2}\binom{2(p-k)}{p-k}(\bmod p) .
\end{aligned}
$$

Note that $\binom{2 k}{k} \equiv 0(\bmod p)$ for $k=(p+1) / 2, \ldots, p-1$. By the above,

$$
\begin{equation*}
\sum_{l=1}^{p-1} \frac{(-1)^{l}}{l} f_{l}(x) \equiv \sum_{k=(p+1) / 2}^{p-1} \frac{\binom{2 k}{k}}{k} x^{k} \frac{\binom{2(p-k)}{p-k}}{2} \equiv p \sum_{k=(p+1) / 2}^{p-1} \frac{x^{k}}{k^{2}} \quad\left(\bmod p^{2}\right) \tag{2.7}
\end{equation*}
$$

with the help of Lemma 2.4. Hence (1.9) follows from (2.7) in the case $x=1$ since

$$
2 \sum_{k=(p+1) / 2}^{p-1} \frac{1}{k^{2}} \equiv \sum_{k=(p+1) / 2}^{p-1}\left(\frac{1}{k^{2}}+\frac{1}{(p-k)^{2}}\right)=H_{p-1}^{(2)} \equiv 0 \quad(\bmod p)
$$

Instead of proving (1.10) we show its extension (1.12). Clearly,

$$
\sum_{k=1}^{p-1} \frac{(-1)^{k r}}{k^{r-1}}=\sum_{k=1}^{(p-1) / 2}\left(\frac{(-1)^{k r}}{k^{r-1}}+\frac{(-1)^{(p-k) r}}{(p-k)^{r-1}}\right) \equiv 0 \quad(\bmod p)
$$

Thus

$$
\begin{aligned}
\sum_{l=1}^{p-1} \frac{(-1)^{l r}}{l^{r-1}} f_{l}^{(r)} & \equiv \sum_{l=1}^{p-1} \frac{(-1)^{l r}}{l^{r-1}} \sum_{k=1}^{l}\binom{l}{k}^{r}=\sum_{k=1}^{p-1} \frac{1}{k^{r-1}} \sum_{l=k}^{p-1}(-1)^{l r}\binom{l-1}{k-1}^{r-1}\binom{l}{k} \\
& =\sum_{k=1}^{p-1} \frac{1}{k^{r-1}} \sum_{j=0}^{p-1-k}(-1)^{(k+j) r}\binom{k+j-1}{j}^{r-1}\binom{k+j}{j} \\
& =\sum_{k=1}^{p-1} \frac{(-1)^{k r}}{k^{r-1}} \sum_{j=0}^{p-1-k}\binom{-k}{j}^{r-1}\binom{-k-1}{j} \\
& \equiv \sum_{k=1}^{p-1} \frac{(-1)^{k r}}{k^{r-1}} \sum_{j=0}^{p-k-1}\binom{p-k}{j}^{r-1}\binom{p-k-1}{j}(\bmod p) .
\end{aligned}
$$

For any positive integer $n$, we have

$$
f_{n}^{(r)}=\sum_{k=0}^{n}\left(\frac{k}{n}+\frac{n-k}{n}\right)\binom{n}{k}^{r}=2 \sum_{k=0}^{n} \frac{n-k}{n}\binom{n}{k}^{r}=2 \sum_{k=0}^{n-1}\binom{n}{k}^{r-1}\binom{n-1}{k} .
$$

Therefore,

$$
\begin{aligned}
\sum_{l=1}^{p-1} \frac{(-1)^{l r}}{l^{r-1}} f_{l}^{(r)} & \equiv \sum_{k=1}^{p-1} \frac{(-1)^{k r}}{k^{r-1}} \cdot \frac{f_{p-k}^{(r)}}{2}=\frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^{(p-k) r} f_{k}^{(r)}}{(p-k)^{r-1}} \\
& \equiv-\frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^{k r}}{k^{r-1}} f_{k}^{(r)}(\bmod p)
\end{aligned}
$$

and hence (1.12) follows.
Finally we show (1.11). By (1.3) and Lemma 2.6,

$$
\begin{aligned}
\frac{1}{p} \sum_{n=0}^{p-1}(2 n+1) A_{n} & =\frac{1}{p} \sum_{n=0}^{p-1}(2 n+1) \sum_{k=0}^{n}\binom{n+k}{2 k}\binom{2 k}{k} f_{k} \\
& =\frac{1}{p} \sum_{k=0}^{p-1}\binom{2 k}{k} f_{k} \sum_{n=k}^{p-1}(2 n+1)\binom{n+k}{2 k} \\
& \equiv \frac{f_{p-1}}{p}\binom{2 p-2}{p-1}(2 p-1)+p \sum_{k=0}^{p-2} \frac{(-1)^{k} f_{k}}{k+1} \\
& =\binom{2 p-1}{p-1} f_{p-1}-p \sum_{k=1}^{p-1} \frac{(-1)^{k}}{k} f_{k-1}\left(\bmod p^{3}\right)
\end{aligned}
$$

Combining this with Wolstenholme's congruence $\binom{2 p-1}{p-1} \equiv 1\left(\bmod p^{3}\right)(c f .[W])$ and [S12b, (1.6)] we obtain

$$
\sum_{k=1}^{p-1} \frac{(-1)^{k} f_{k-1}}{k} \equiv \frac{f_{p-1}-1}{p} \equiv 3 q_{p}(2)+3 p q_{p}(2)^{2} \quad\left(\bmod p^{2}\right)
$$

by Lemma 2.5 .
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