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CONGRUENCES FOR FRANEL NUMBERS

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ABSTRACT. The Franel numbers given by $f_n = \sum_{k=0}^n \binom{n}{k}^3$ $(n=0,1,2,\ldots)$ play important roles in both combinatorics and number theory. In this paper we initiate the systematic investigation of fundamental congruences for the Franel numbers. We mainly establish for any prime p>3 the following congruences:

$$\sum_{k=0}^{p-1} (-1)^k f_k \equiv \left(\frac{p}{3}\right) \pmod{p^2}, \quad \sum_{k=0}^{p-1} (-1)^k k f_k \equiv -\frac{2}{3} \left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} f_k \equiv 0 \pmod{p^2}, \quad \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} f_k \equiv 0 \pmod{p}.$$

1. Introduction

In 1894, Franel [F] noted that the numbers

$$f_n = \sum_{k=0}^n \binom{n}{k}^3 \quad (n = 0, 1, 2, \dots)$$
 (1.1)

(cf. [Sl, A000172]) satisfy the recurrence relation:

$$(n+1)^2 f_{n+1} = (7n^2 + 7n + 2)f_n + 8n^2 f_{n-1} \quad (n=1,2,3,\ldots).$$
 (1.2)

Such numbers are now called Franel numbers. For a combinatorial interpretation of the Franel numbers, see Callan [C]. Recall that the Apéry numbers given by

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^n \binom{n+k}{2k}^2 \binom{2k}{k}^2 \ (n=0,1,2,\dots)$$

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were introduced by Apéry [A], and they can be expressed in terms of Franel numbers as follows:

$$A_n = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} f_k \tag{1.3}$$

(see Strehl [St92]). The Franel numbers are also related to the theory of modular forms, see, e.g., Zagier [Z].

In this paper we study congruences for the Franel numbers systematically. As usual, for any odd prime p and integer a, $(\frac{a}{p})$ denotes the Legendre symbol, and $q_p(a)$ stands for the Fermat quotient $(a^{p-1}-1)/p$ if $p \nmid a$.

Now we state our main result.

Theorem 1.1. Let p > 3 be a prime. For any p-adic integer r we have

$$\sum_{k=0}^{p-1} (-1)^k \binom{k+r}{k} f_k \equiv \sum_{k=0}^{p-1} \binom{2k}{k} \binom{k+r}{k}^2 \pmod{p^2}.$$
 (1.4)

In particular,

$$\sum_{k=0}^{p-1} (-1)^k f_k \equiv \left(\frac{p}{3}\right) \pmod{p^2},\tag{1.5}$$

$$\sum_{k=0}^{p-1} (-1)^k k f_k \equiv -\frac{2}{3} \left(\frac{p}{3}\right) \pmod{p^2},\tag{1.6}$$

$$\sum_{k=0}^{p-1} (-1)^k k^2 f_k \equiv \frac{10}{27} \left(\frac{p}{3}\right) \pmod{p^2},\tag{1.7}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} f_k}{(-4)^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} \pmod{p^2}.$$
 (1.8)

We also have

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} f_k \equiv 0 \pmod{p^2},\tag{1.9}$$

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} f_k \equiv 0 \pmod{p}, \tag{1.10}$$

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} f_{k-1} \equiv 3q_p(2) + 3p \, q_p(2)^2 \pmod{p^2},\tag{1.11}$$

Remark 1.1. Fix a prime p > 3. In contrast with (1.5), we conjecture that

$$\sum_{n=0}^{p-1} (-1)^n \sum_{k=0}^n \binom{n}{k}^3 (-8)^k \equiv \sum_{k=0}^{p-1} \frac{f_k}{8^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}.$$

As $f_k \equiv (-8)^k f_{p-1-k} \pmod{p}$ for all $k = 0, \ldots, p-1$ by [JV, Lemma 2.6], (1.11) implies that

$$\sum_{k=1}^{p-1} \frac{f_k}{k8^k} \equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{k} f_{p-1-k} = \sum_{k=1}^{p-1} \frac{(-1)^{p-k}}{p-k} f_{k-1} \equiv 3q_p(2) \pmod{p}.$$

Motivated by (1.5) and (1.6), we conjecture that both $(\sum_{k=0}^{n-1} (-1)^k f_k)/n^2$ and $(\sum_{k=0}^{n-1} (-1)^k k f_k)/n^2$ are 3-adic integers for any positive integer n. Concerning (1.8) the author [S11, Conj. 5.2(ii)] conjectured that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 3y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

See also [S13] for other connections between $p = x^2 + 3y^2$ and Franel numbers. (1.10) can be extended as

$$\sum_{k=1}^{p-1} \frac{(-1)^{kr}}{k^{r-1}} f_k^{(r)} \equiv 0 \pmod{p}, \tag{1.12}$$

where r is any positive integer and $f_k^{(r)} := \sum_{j=0}^k {k \choose j}^r$. Note that $f_k^{(2)} = {2k \choose k}$ and $\sum_{k=1}^{p-1} {2k \choose k}/k \equiv 0 \pmod{p^2}$ by [ST10].

Let p > 3 be a prime. Similar to (1.5)-(1.7), we are also able to show that

$$\sum_{k=0}^{p-1} (-1)^k k^3 f_k \equiv -\frac{10}{81} \left(\frac{p}{3} \right) \pmod{p^2} \text{ and } \sum_{k=0}^{p-1} (-1)^k k^4 f_k \equiv -\frac{14}{243} \left(\frac{p}{3} \right) \pmod{p^2}.$$

In general, for any positive integer r and prime $p > \max\{r, 3\}$ there should be an odd integer a_r (not dependent on p) such that

$$\sum_{k=0}^{p-1} (-1)^k k^r f_k \equiv \frac{2a_r}{3^{2r-1}} \left(\frac{p}{3}\right) \pmod{p^2}.$$

2. Proof of Theorem 1.1

We first establish an auxiliary theorem on the polynomials

$$f_n(x) := \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{n} x^k = \sum_{k=0}^n \binom{n}{k} \binom{k}{n-k} \binom{2k}{k} x^k \quad (n = 0, 1, 2, \dots).$$

Theorem 2.1. Let p be an odd prime and let r be any p-adic integer. Then

$$\sum_{l=0}^{p-1} (-1)^l \binom{l+r}{l} f_l(x) \equiv \sum_{k=0}^{p-1} \binom{2k}{k} x^k \binom{k+r}{k}^2 \pmod{p^2}. \tag{2.1}$$

Proof. Observe that

$$\sum_{l=0}^{p-1} (-1)^l \binom{l+r}{l} f_l(x) = \sum_{l=0}^{p-1} (-1)^l \binom{l+r}{l} \sum_{k=0}^l \binom{l}{k} \binom{k}{l-k} \binom{2k}{k} x^k$$
$$= \sum_{k=0}^{p-1} \binom{2k}{k} x^k \sum_{l=k}^{\min\{2k,p-1\}} (-1)^l \binom{l}{k} \binom{k}{l-k} \binom{l+r}{l}.$$

If $(p-1)/2 < k \le p-1$ and $p \le l \le 2k$, then

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} \equiv 0 \pmod{p} \text{ and } \binom{l}{k} = \frac{l!}{k!(l-k)!} \equiv 0 \pmod{p}.$$

Thus

$$\sum_{l=0}^{p-1} (-1)^l \binom{l+r}{l} f_l(x) \equiv \sum_{k=0}^{p-1} \binom{2k}{k} x^k \sum_{l=k}^{2k} (-1)^l \binom{l}{k} \binom{k}{l-k} \binom{l+r}{l} \pmod{p^2}$$

and hence it suffices to show the identity

$$\sum_{l=k}^{2k} (-1)^l \binom{l}{k} \binom{k}{l-k} \binom{x+l}{l} = \binom{x+k}{k}^2. \tag{2.2}$$

By the well-known Chu-Vandermonde identity (cf. (3.1) of [G, p.22]),

$$\sum_{j=0}^{k} {y \choose j} {z \choose k-j} = {y+z \choose k}.$$

Therefore

$$\sum_{l=k}^{2k} (-1)^l \binom{l}{k} \binom{k}{l-k} \binom{x+l}{l}$$

$$= \sum_{l=k}^{2k} \binom{l}{k} \binom{k}{l-k} \binom{-x-1}{l} = \binom{-x-1}{k} \sum_{l=k}^{2k} \binom{-x-1-k}{l-k} \binom{k}{l-k}$$

$$= \binom{-x-1}{k} \sum_{j=0}^{k} \binom{-x-1-k}{j} \binom{k}{k-j} = \binom{-x-1}{k}^2 = \binom{x+k}{k}^2.$$

This proves (2.2) and hence (2.1) follows. \square

Lemma 2.1. For any nonnegative integer n, the integer $f_n(1)$ coincides with the Franel number f_n .

Proof. The identity $\sum_{k=0}^{n} {n \choose k}^2 {2k \choose n} = f_n$ is a known result due to Strehl [St94]. \square

Lemma 2.2. For each positive integer m we have

$$\sum_{k=0}^{n-1} P_m(k) \binom{2k}{k} = n^m \binom{2n}{n} \quad \text{for all } n = 1, 2, 3, \dots,$$

where $P_m(x) := 2(2x+1)(x+1)^{m-1} - x^m$.

Proof. The desired result follows immediately by induction on n. \square

Lemma 2.3. Let m be a positive integer. For n = 0, 1, ..., m we have

$$\sum_{k=0}^{n} {x \choose k} {-x \choose m-k} = \frac{m-n}{m} {x-1 \choose n} {-x \choose m-n}.$$

Remark 2.1. This is a known result due to Andersen, see, e.g., (3.14) of [G, p. 23].

Lemma 2.4 ([S11, Lemma 2.1]). Let p be an odd prime. For any $k = 1, \ldots, p-1$ we have

$$k \binom{2k}{k} \binom{2(p-k)}{p-k} \equiv (-1)^{\lfloor 2k/p \rfloor - 1} 2p \pmod{p^2}.$$

Recall that the harmonic numbers and the second-order harmonic numbers are given by

$$H_n = \sum_{0 < k \le n} \frac{1}{k}$$
 and $H_n^{(2)} = \sum_{0 < k \le n} \frac{1}{k^2}$ $(n = 0, 1, 2, ...)$

respectively. Let p > 3 be a prime. In 1862, Wolstenholme [W] proved that

$$H_{p-1} \equiv 0 \pmod{p^2}$$
 and $H_{p-1}^{(2)} \equiv 0 \pmod{p}$.

Note that

$$H_{(p-1)/2}^{(2)} \equiv \frac{1}{2} \sum_{k=1}^{(p-1)/2} \left(\frac{1}{k^2} + \frac{1}{(p-k)^2} \right) = \frac{1}{2} H_{p-1}^{(2)} \equiv 0 \pmod{p}.$$

In 1938, Lehmer [L] showed that

$$H_{(p-1)/2} \equiv -2q_p(2) + p q_p(2)^2 \pmod{p^2}.$$
 (2.3)

Lemma 2.5. Let p > 3 be a prime. Then

$$f_{p-1} \equiv 1 + 3p q_p(2) + 3p^2 q_p(2)^2 \pmod{p^3}.$$
 (2.4)

Proof. For any $k = 1, \ldots, p - 1$, we obviously have

$$(-1)^k \binom{p-1}{k} = \prod_{j=1}^k \left(1 - \frac{p}{j}\right)$$

$$\equiv 1 - pH_k + \frac{p^2}{2} \sum_{1 \le i < j \le k} \frac{2}{ij} = 1 - pH_k + \frac{p^2}{2} \left(H_k^2 - H_k^{(2)}\right) \pmod{p^3}.$$

Thus

$$f_{p-1} - 1 = \sum_{k=1}^{p-1} {p-1 \choose k}^3 \equiv \sum_{k=1}^{p-1} (-1)^k \left(1 - pH_k + \frac{p^2}{2} \left(H_k^2 - H_k^{(2)} \right) \right)^3$$

$$\equiv -3p \sum_{k=1}^{p-1} (-1)^k H_k + \frac{9}{2} p^2 \sum_{k=1}^{p-1} (-1)^k H_k^2 - \frac{3}{2} p^2 \sum_{k=1}^{p-1} (-1)^k H_k^{(2)} \pmod{p^3}.$$

Clearly

$$\sum_{k=1}^{p-1} (-1)^k H_k = \sum_{k=1}^{p-1} \sum_{j=1}^k \frac{(-1)^k}{j} = \sum_{j=1}^{p-1} \frac{\sum_{k=j}^{p-1} (-1)^k}{j} = \sum_{j=1}^{p-1} \frac{1}{j}$$
$$= \frac{1}{2} H_{(p-1)/2} \equiv -q_p(2) + \frac{p}{2} q_p(2)^2 \pmod{p^2} \quad \text{(by (2.3))}$$

and

$$\sum_{k=1}^{p-1} (-1)^k H_k^{(2)} = \sum_{j=1}^{p-1} \frac{\sum_{k=j}^{p-1} (-1)^k}{j^2} = \sum_{i=1}^{(p-1)/2} \frac{1}{(2i)^2} = \frac{H_{(p-1)/2}^{(2)}}{4} \equiv 0 \pmod{p}.$$

Observe that

$$\begin{split} \sum_{k=1}^{p-1} (-1)^k H_k^2 &= \sum_{k=1}^{p-1} (-1)^{p-k} H_{p-k}^2 = \sum_{k=1}^{p-1} (-1)^{k-1} \left(H_{p-1} - \sum_{0 < j < k} \frac{1}{p-j} \right)^2 \\ &\equiv -\sum_{k=1}^{p-1} (-1)^k \left(H_k - \frac{1}{k} \right)^2 \\ &= -\sum_{k=1}^{p-1} (-1)^k H_k^2 + 2\sum_{k=1}^{p-1} \frac{(-1)^k}{k} H_k - \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \pmod{p}. \end{split}$$

Clearly,

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \equiv \sum_{k=1}^{p-1} \frac{1 + (-1)^k}{k^2} = \sum_{j=1}^{(p-1)/2} \frac{2}{(2j)^2} \equiv 0 \pmod{p},$$

and

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} H_k = \sum_{\substack{k=1\\2|k}}^{p-1} \frac{H_k}{k} - \sum_{\substack{k=1\\2\nmid k}}^{p-1} \frac{H_k}{k} \equiv \frac{q_p(2)^2}{2} - \left(-\frac{q_p(2)^2}{2}\right) \pmod{p}$$

by [S12a, Lemma 2.3]. Therefore

$$\sum_{k=1}^{p-1} (-1)^k H_k^2 \equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{k} H_k \equiv q_p(2)^2 \pmod{p}.$$

Combining the above, we finally obtain

$$f_{p-1} - 1 \equiv -3p\left(-q_p(2) + \frac{p}{2}q_p(2)^2\right) + \frac{9}{2}p^2q_p(2)^2 \pmod{p^3}$$

and hence (2.4) holds. \square

Lemma 2.6. Let p be any prime. Then

$$\binom{p-1}{k} \binom{p+k}{k} \equiv (-1)^k \pmod{p^2} \quad \text{for } k = 0, 1, \dots, p-1,$$

and

$$\binom{2k}{k} \sum_{n=k}^{p-1} (2n+1) \binom{n+k}{2k} \equiv p^2 \frac{(-1)^k}{k+1} \pmod{p^4} \quad \text{for } k = 0, \dots, p-2.$$

Proof. Let $k \in \{0, 1, \dots, p-1\}$. Clearly

$$\binom{p-1}{k} \binom{p+k}{k} = \prod_{0 \le j \le k} \left(\frac{p-j}{j} \cdot \frac{p+j}{j} \right) \equiv (-1)^k \pmod{p^2}.$$

In view of the known identity $\sum_{n=0}^{m} {n \choose l} = {m+1 \choose l+1}$ $(l, m=0,1,\ldots)$ (see, e.g., (1.52) of [G, p. 7]) which can be easily proved by induction, we have

$$\sum_{n=k}^{p-1} \frac{2n+1}{2k+1} \binom{n+k}{2k} = \sum_{n=k}^{p-1} \left(\frac{2(n+k+1)}{2k+1} - 1\right) \binom{n+k}{2k}$$

$$= 2\sum_{n=k}^{p-1} \binom{n+k+1}{2k+1} - \sum_{n=k}^{p-1} \binom{n+k}{2k}$$

$$= 2\binom{p+k+1}{2k+2} - \binom{p+k}{2k+1} = \frac{p}{k+1} \binom{p+k}{2k+1}$$

and hence

$$\binom{2k}{k} \sum_{n=k}^{p-1} (2n+1) \binom{n+k}{2k} = p \frac{2k+1}{k+1} \binom{2k}{k} \binom{p+k}{2k+1} = \frac{p^2}{k+1} \binom{p-1}{k} \binom{p+k}{k}.$$

Thus, if k then

$$\binom{2k}{k} \sum_{n=k}^{p-1} (2n+1) \binom{n+k}{2k} \equiv \frac{p^2}{k+1} (-1)^k \pmod{p^4}$$

as desired. \square

Proof of Theorem 1.1. In view of Lemma 2.1, (2.1) with x = 1 gives (1.4). (2.1) with r = 0 yields the congruence

$$\sum_{k=0}^{p-1} (-1)^k f_k(x) \equiv \sum_{k=0}^{p-1} {2k \choose k} x^k \pmod{p^2}.$$

In the case x = 1, this gives (1.5) since $\sum_{k=0}^{p-1} {2k \choose k} \equiv {p \choose 3} \pmod{p^2}$ by [ST11, (1.9)].

By (2.1) with r = 0, 1,

$$\sum_{k=0}^{p-1} (3(k+1)-1)(-1)^k f_k(x)$$

$$\equiv \sum_{k=0}^{p-1} {2k \choose k} x^k \left(3(k+1)^2 - 1\right) = \sum_{k=0}^{p-1} P_2(k) {2k \choose k} x^k \pmod{p^2}$$

where $P_2(x) = 2(2x+1)(x+1) - x^2 = 3x^2 + 6x + 2$. Thus, with the help of Lemmas 2.1-2.2, we have

$$\sum_{k=0}^{p-1} (3k+2)(-1)^k f_k \equiv 0 \pmod{p^2}$$
 (2.5)

and hence (1.6) holds in view of (1.5).

Taking r = 2 in (2.1) we get

$$2\sum_{k=0}^{p-1} (k^2 + 3k + 2)(-1)^k f_k(x) \equiv \sum_{k=0}^{p-1} {2k \choose k} x^k ((k+1)(k+2))^2 \pmod{p^2}.$$

In view of (2.5), this yields

$$2\sum_{k=0}^{p-1} (-1)^k k^2 f_k \equiv \sum_{k=0}^{p-1} {2k \choose k} (k^2 + 3k + 2)^2 \pmod{p^2}.$$

Note that

$$27(k^2 + 3k + 2)^2 = 9P_4(k) + 12P_3(k) + 23P_2(k) + 20$$

where $P_m(x)$ is given by Lemma 2.2. Therefore, with the help of Lemma 2.3 and [ST11, (1.9)], we have

$$54\sum_{k=0}^{p-1} (-1)^k k^2 f_k \equiv \sum_{k=0}^{p-1} (9P_4(k) + 12P_3(k) + 23P_2(k) + 20) \binom{2k}{k} \equiv 20 \left(\frac{p}{3}\right) \pmod{p^2}$$

and hence (1.7) follows.

Putting r = -1/2 in (2.1) and noting that $\binom{k-1/2}{k} = \binom{2k}{k}/4^k$, we then obtain

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} f_k(x)}{(-4)^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} x^k \pmod{p^2}.$$
 (2.6)

In the case x = 1 this gives (1.8).

Now we prove (1.9). Observe that

$$\sum_{l=1}^{p-1} \frac{(-1)^l}{l} \sum_{k=0}^l \binom{l}{k} \binom{k}{l-k} \binom{2k}{k} x^k = \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} x^k \sum_{l=k}^{p-1} (-1)^l \binom{l-1}{k-1} \binom{k}{l-k}.$$

If $1 \leqslant k \leqslant (p-1)/2$, then

$$\sum_{l=k}^{p-1} (-1)^l \binom{l-1}{k-1} \binom{k}{l-k} = \sum_{l=k}^{2k} (-1)^l \binom{l-1}{k-1} \binom{k}{l-k}$$

$$= \sum_{j=0}^k (-1)^{k+j} \binom{k+j-1}{j} \binom{k}{j}$$

$$= (-1)^k \sum_{j=0}^k \binom{-k}{j} \binom{k}{k-j} = (-1)^k \binom{0}{k} = 0$$

by the Chu-Vandermonde identity. If $(p+1)/2 \le k \le p-1$, then

$$\sum_{l=k}^{p-1} (-1)^l \binom{l-1}{k-1} \binom{k}{l-k} = \sum_{j=0}^{p-1-k} (-1)^{k+j} \binom{k+j-1}{j} \binom{k}{j}$$
$$= (-1)^k \sum_{j=0}^{p-1-k} \binom{-k}{j} \binom{k}{k-j}$$

and hence applying Lemma 2.3 we get

$$\sum_{l=k}^{p-1} (-1)^l \binom{l-1}{k-1} \binom{k}{l-k}$$

$$= (-1)^k \frac{k - (p-1-k)}{k} \binom{-k-1}{p-1-k} \binom{k}{k - (p-1-k)}$$

$$= (-1)^{p-1} \left(\frac{p-k}{k}\right)^2 \binom{p-1}{k-1} \binom{k}{p-k}$$

$$= (-1)^{k-1} \binom{k}{p-k} = \binom{p-2k-1}{p-k}$$

$$= \binom{2(p-k)-1}{p-k} = \frac{1}{2} \binom{2(p-k)}{p-k} \pmod{p}.$$

Note that $\binom{2k}{k} \equiv 0 \pmod{p}$ for $k = (p+1)/2, \dots, p-1$. By the above,

$$\sum_{l=1}^{p-1} \frac{(-1)^l}{l} f_l(x) \equiv \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}}{k} x^k \frac{\binom{2(p-k)}{p-k}}{2} \equiv p \sum_{k=(p+1)/2}^{p-1} \frac{x^k}{k^2} \pmod{p^2}$$
 (2.7)

with the help of Lemma 2.4. Hence (1.9) follows from (2.7) in the case x = 1 since

$$2\sum_{k=(p+1)/2}^{p-1} \frac{1}{k^2} \equiv \sum_{k=(p+1)/2}^{p-1} \left(\frac{1}{k^2} + \frac{1}{(p-k)^2}\right) = H_{p-1}^{(2)} \equiv 0 \pmod{p}.$$

Instead of proving (1.10) we show its extension (1.12). Clearly,

$$\sum_{k=1}^{p-1} \frac{(-1)^{kr}}{k^{r-1}} = \sum_{k=1}^{(p-1)/2} \left(\frac{(-1)^{kr}}{k^{r-1}} + \frac{(-1)^{(p-k)r}}{(p-k)^{r-1}} \right) \equiv 0 \pmod{p}.$$

Thus

$$\sum_{l=1}^{p-1} \frac{(-1)^{lr}}{l^{r-1}} f_l^{(r)} \equiv \sum_{l=1}^{p-1} \frac{(-1)^{lr}}{l^{r-1}} \sum_{k=1}^{l} \binom{l}{k}^r = \sum_{k=1}^{p-1} \frac{1}{k^{r-1}} \sum_{l=k}^{p-1} (-1)^{lr} \binom{l-1}{k-1}^{r-1} \binom{l}{k}$$

$$= \sum_{k=1}^{p-1} \frac{1}{k^{r-1}} \sum_{j=0}^{p-1-k} (-1)^{(k+j)r} \binom{k+j-1}{j}^{r-1} \binom{k+j}{j}$$

$$= \sum_{k=1}^{p-1} \frac{(-1)^{kr}}{k^{r-1}} \sum_{j=0}^{p-1-k} \binom{-k}{j}^{r-1} \binom{-k-1}{j}$$

$$\equiv \sum_{k=1}^{p-1} \frac{(-1)^{kr}}{k^{r-1}} \sum_{j=0}^{p-k-1} \binom{p-k}{j}^{r-1} \binom{p-k-1}{j} \pmod{p}.$$

For any positive integer n, we have

$$f_n^{(r)} = \sum_{k=0}^n \left(\frac{k}{n} + \frac{n-k}{n}\right) \binom{n}{k}^r = 2\sum_{k=0}^n \frac{n-k}{n} \binom{n}{k}^r = 2\sum_{k=0}^{n-1} \binom{n}{k}^{r-1} \binom{n-1}{k}.$$

Therefore.

$$\sum_{l=1}^{p-1} \frac{(-1)^{lr}}{l^{r-1}} f_l^{(r)} \equiv \sum_{k=1}^{p-1} \frac{(-1)^{kr}}{k^{r-1}} \cdot \frac{f_{p-k}^{(r)}}{2} = \frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^{(p-k)r} f_k^{(r)}}{(p-k)^{r-1}}$$
$$\equiv -\frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^{kr}}{k^{r-1}} f_k^{(r)} \pmod{p}$$

and hence (1.12) follows.

Finally we show (1.11). By (1.3) and Lemma 2.6,

$$\frac{1}{p} \sum_{n=0}^{p-1} (2n+1)A_n = \frac{1}{p} \sum_{n=0}^{p-1} (2n+1) \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k} f_k$$

$$= \frac{1}{p} \sum_{k=0}^{p-1} \binom{2k}{k} f_k \sum_{n=k}^{p-1} (2n+1) \binom{n+k}{2k}$$

$$\equiv \frac{f_{p-1}}{p} \binom{2p-2}{p-1} (2p-1) + p \sum_{k=0}^{p-2} \frac{(-1)^k f_k}{k+1}$$

$$= \binom{2p-1}{p-1} f_{p-1} - p \sum_{k=1}^{p-1} \frac{(-1)^k}{k} f_{k-1} \pmod{p^3}.$$

Combining this with Wolstenholme's congruence $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$ (cf. [W]) and [S12b, (1.6)] we obtain

$$\sum_{k=1}^{p-1} \frac{(-1)^k f_{k-1}}{k} \equiv \frac{f_{p-1} - 1}{p} \equiv 3q_p(2) + 3p \, q_p(2)^2 \pmod{p^2}$$

by Lemma 2.5. \square

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