THE STRUCTURE OF SYMMETRIC N-PLAYER GAMES WHEN INFLUENCE AND INDEPENDENCE COLLIDE

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Abstract. We study the mathematical properties of probabilistic processes in which the independent actions of n players ('causes') can influence the outcome of each player ('effects'). In such a setting, each pair of outcomes will generally be statistically correlated, even if the actions of all the players provide a complete causal description of the players' outcomes, and even if we condition on the outcome of any one player's action. This correlation always holds when $n = 2$, but when $n = 3$ there exists a highly symmetric process, recently studied, in which each cause can influence each effect, and yet each pair of effects is probabilistically independent (even upon conditioning on any one cause). We study such symmetric processes in more detail, obtaining a complete classification for all $n \geq 3$. Using a variety of mathematical techniques, we describe the geometry and topology of the underlying probability space that allows independence and influence to coexist.

1. Introduction

The study of causality is a long-standing topic at the interface of statistics and the philosophy of science. It is also an area where the mathematical analysis of graphical models has led to some important recent advances (see e.g. [2, 5]). In this paper, we investigate a particular class of symmetric causal processes which achieves two apparently conflicting requirements: 'independence' and 'influence' which we define shortly.

In Section 2, we provide formal definitions, but give the main ideas here to facilitate the discussion. Let $E_1, \ldots, E_n$ be n dichotomous (two states) random variables with the same state spaces, which we call 'effects' and let $C_1, \ldots, C_n$ be n independent dichotomous random variables, also with the same state spaces, which we call 'causes'.

We say that an effect E_i is 'influenced by' C_i if there exists at least one assignment of states for the remaining causes such that a change in the state of C_j changes the (conditional) probability of at least one state of E_i [8]. 'Independence' refers to pairwise probabilistic independence of the effects either absolutely, or conditional on knowing the state of any one cause.

We explore a symmetric system because it is applicable to any scenario in which the probability of E_i depends only on how many causes take the same value as C_i. We can view this process as a game where we identify C_i with the action of some player i and the outcome, E_i, for each player i then depends solely on how many of the other players chose the same action.

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For example, suppose there are n flowering plants in an area of study. For plant i, the cause C_i might describe whether the plant flowers early or late. The corresponding effect E_i could denote whether or not a plant is pollinated. For example, flowering early with many other flowers might be advantageous because such a mass flowering attracts more bees and increases the probability the plant is pollinated. On the other hand, there may be a limit in the number of bees, so flowering early with many other flowers may instead be a disadvantage. Either way, the probability of an effect (pollination of plant i) depends on the number of causes which match the cause of that particular effect (i.e. how many other plants flower at the same time as plant i).

Recently, such processes have been studied in the philosophy of science literature as they provide insights into the extent to which subsets of causes can render effects independent (Theorem 5b of [8]). The authors of [8] illustrated such a process with an entertaining application involving n people playing a tequila drinking game. In [8] they consider just the case $n = 3$. In the game, the n people simultaneously and independently reveal a clenched fist or an open hand (with equal probability), and the states of the n hands are regarded as the n causes. The event that person i drinks tequila is E_i, for $1 \leq i \leq n$. The rules for determining if person E_i drinks when $n = 3$ are that if a player's hand position is unique then they drink with probability $p_1 = 1$. For the ties (e.g. a tie of two or three), those in the tie drink independently with probability $p_2 = \frac{1}{2}$ when there are two people in the tie and probability $p_3 = \frac{1}{3}$ when there are three people in the tie (see Fig. 1). The probabilities used here are quite special when we consider influence and independence in relation to each other and the effect on the system. We study what is special and how it can generalize. We call this extension of this game to n players the 'extended symmetric tequila problem' (EST) but, as noted in the previous paragraph, the relevance of such processes extends well beyond bar drinking games.

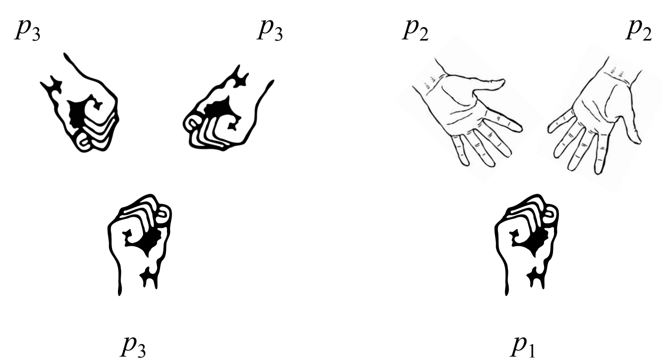


FIGURE 1. A simple three-player game exhibiting independence and influence, for various values of (p_1, p_2, p_3); including $(1, \frac{1}{2}, \frac{1}{3})$ from [8], and $(3^{-1}, 3^{-2}, 3^{-3})$ from Section 4.1.

Our main results assume the system has some symmetry, as explained at the beginning of Section 3 and we define three spaces in this context: Inf_n, Ind_n and $\mathrm{EST}_n = \mathrm{Inf}_n \cap \mathrm{Ind}_n$.

These spaces are formally defined in Section 3 but, in short, are the set of probabilities for the fully symmetric system which lead to influence, independence and both, respectively.

We fully analyze the case $n = 3$ (Section 3), we establish a useful equivalence relation on Ind_n (Section 6), we show that Ind_n is contractible (Section 7) but not convex (Section 6), and that EST_n is neither.

We establish a characterization (Proposition 3.1) for the system to be in Inf_n. We show, via a quadratic form and its Hessian matrix, that EST_n contains infinitely many points for any $n \geq 3$. We use this structure to investigate the topology and geometry of the space EST_n, with a main objective being to determine whether or not it is connected. We show that EST_n is disconnected for $n = 3, 4$ and connected when $n \geq 5$ in Theorem 7.2.

Our results involve an interplay of linear algebra, analysis, combinatorics and topology, including some classical results in these fields, such as Sylvester's Inertia Theorem, Alexander Duality and Smith's theorem on periodic maps.

2. Formal Setup

We begin by giving the formal set-up of the system of causes and effects, and proceed to provide formal definitions of influence, and conditional independence.

Let $E_1, \ldots, E_n$ and $C_1, \ldots, C_n$ be random variables with two possible states (also called 'dichotomous'), labeled throughout this paper as 0 and 1. We assume that the C_i are (mutually) independent, and each event E_j depends on the outcome of the events C_i; accordingly we call the C_i *causes* and the E_j *effects*. To simplify notation, we write conditional probabilities of the form $\mathbb{P}(E_i = 1|*)$ more simply as $\mathbb{P}(E_i|*)$ (i.e. $E_i = 1$ is the event that E_i 'occurs'). The model we study makes the following assumptions:

(A1) The causes are (mutually) independent, with $\mathbb{P}(C_i = 1) = r$ for some $0 < r < 1$.
(A2) The effects are conditionally independent, given the joint outcome of the causes.

$$\mathbb{P}(E_i|\bigwedge_{j=1}^{n} C_j = x_j) = \begin{cases} p_k, & x_i = 0, \text{ and } k \text{ total causes are in state } 0; \\ q_k, & x_i = 1, \text{ and } k \text{ total causes are in state } 1. \end{cases}$$

Property (A2) states that the probability of E_i depends on the state of C_i and the number of causes in that same state. If we assume $p_k = q_k$, then $\mathbb{P}(E_i)$ depends only on the number of causes in the same state as C_i. In our examples, flowers often seem to flower with some dependence on the number of other flowers which have also flowered and in the tequila example, $p_1 = q_1 = 1$, $p_2 = q_2 = \frac{1}{2}$ and $p_3 = q_3 = \frac{1}{3}$.

In this paper we will mostly deal with the case where $p_k = q_k$ for all k, and $r = \frac{1}{2}$ (the fully-symmetric (or EST) model), but it is helpful to pose the problem more generally.

2.1. Influence and Independence.

While the set-up we explore has the same number of causes as effects, we give the definitions here for arbitrary numbers of causes and effects.

Definition 2.1. *(Influence)*

- *Given a set of s causes $C_1, \ldots, C_s$ of an effect E we say that E is* influenced by *cause C_j if there exists at least one assignment of states for the remaining $s-1$ causes, so that some change in the state of C_j alters the probability of at least one state of E.*

- *A set of s causes of t effects satisfies the* influence property *if each effect is influenced by each cause.*

The influence property (called 'weak influence' in [8]) is equivalent to the requirement that none of the causes can be eliminated for any effect – that is, for each i, there is no proper subset J of $\{1, \dots, s\}$ for which $\mathbb{P}(E_i | \bigwedge_{j=1}^{s} C_j = x_j)$ can be written as a function of $(x_j : j \in J)$, for all $(x_1, \dots, x_s)$.

We also study probabilistic independence. Recall that two events X and Y are independent with respect to a third event Z if and only if $\mathbb{P}(X \wedge Y | Z) = \mathbb{P}(X|Z)\mathbb{P}(Y|Z)$. In the language of causality and graphical models we would say that Z *screens off* X from Y. We use the standard probabilistic language of independence throughout the paper. The *independence condition* is that any two events are independent with respect to any cause.

For example, in the tequila drinking game, any pair of effects are independent with respect to any cause C_k as $\mathbb{P}(E_i \wedge E_j \mid C_k = x_k) = \mathbb{P}(E_i \mid C_k = x_k)\mathbb{P}(E_j \mid C_k = x_k)$ for $x_k = 0$ and $x_k = 1$ (so the game has the independence condition). However, the reason this example is of interest in [8] is because any pair of effects E_i and E_j are not independent with respect to any pair of causes (C_{k_1}, C_{k_2}) and yet they are independent with respect to the set of all three causes. This provides a contrast to what happens when $n = 2$. In that case, Theorem 2 of [8] shows the independence condition fails whenever

(a) the causes have non-zero joint probability for any combination of states,
(b) both E_1 and E_2 are independent with respect to the pair of causes.
(c) the causes each influence E_1 and E_2.

3. The fully symmetric (EST) model: structure of the probabilities

We call the model where $p_k = q_k$ and $r_k = \frac{1}{2}$ the *extended symmetric tequila* (EST) setting, as it generalizes the tequila example in [8], where $n = 3$.[1] The EST setting is of particular interest, as it is tractable and leads to interesting results when we couple influence with independence.

We explore the case $n = 3$ further to characterize all the solutions satisfying influence and independence, before turning to general values of n as it serves to further understand the example in [8]; it also serves as a 'boundary' example for larger n, and we return to this example throughout the text.

Firstly, notice that in the EST setting, $\mathbb{P}(E_i | C_j = x)$ takes the same value for each choice of i, j and x (this probability is given formally in the proof of Proposition 3.2). In particular, E_i and C_j are (pairwise) independent, for any pair i, j (including $i = j$). If influence applies then E_i 'depends on' C_j (and the other causes) but this does not translate through to probabilistic independence.

In the EST setting, the conditions (A1) and (A2), coupled with influence and independence, can be stated more succinctly as:

(i) The causes represent independent tosses of a fair coin;

[1] We note that taking $r_k = \frac{1}{2}$ is the natural choice for symmetric games where it is beneficial to each player to play a minority action (for example, if $p_k = q_k$ is decreasing with k), as this provides a Nash equilibrium strategy.

(ii) The effects are mutually (probabilistically) independent once we specify the states of all the causes;
(iii) The probability of E_i depends (exactly) on the number of causes that take the same value as C_i;
(iv) Each pair of effects is (probabilistically) independent;
(v) Each cause can influence each effect.

3.1. **The cases $n = 2$ and $n = 3$.** In the case where $n = 2$, it is easy to verify that any process that satisfies properties (i)–(iv) must have $p_1 = p_2$ and so must fail to satisfy the influence condition (v).

The case where $n = 3$ is more interesting. We study independence by studying the following equation, which follows from direct computation.

$$\begin{aligned} 0 &= \mathbb{P}(E_i \mid C_j = 0)^2 - \mathbb{P}(E_i, E_j \mid C_j = 0) \\ &= \left(\frac{1}{16}\right)(p_3 + 2p_2 + p_1)^2 - \left(\frac{1}{4}\right)(p_3^2 + p_2^2 + 2p_2p_1) \\ &= \frac{1}{16}(p_1 - p_3)(p_1 - 4p_2 + 3p_3) \end{aligned} \tag{1}$$

Notice that $p_1 = 1$, $p_2 = \frac{1}{2}$, $p_3 = \frac{1}{3}$ is a solution to the equation which corresponds to the solution presented for the original tequila game in [8].

Observe that the space of probabilities leading to independence consists of two planes. Further, any solution with $p_1 = p_3$ corresponding to the vanishing of the first term $(p_1 - p_3)$ in Eqn. (1) fails to satisfy the influence property. The intersection of the two planes is $p_1 = p_2 = p_3$, where influence clearly fails. For the remaining points on the plane $p_1 - 4p_2 + 3p_3 = 0$, $p_1 \neq p_2 \neq p_3$ which implies influence. Therefore the space of probabilities satisfying both influence and independence for $n = 3$ consists of two connected pieces formed by removing the line $p_1 = p_2 = p_3$ from the plane $p_1 - 4p_2 + 3p_3 = 0$.

3.2. **Characterizing influence.** For the fully symmetric model we can characterize when the system satisfies the influence property.

Proposition 3.1. *Assume the EST setting, so $r = \frac{1}{2}$ and $p_i = q_i$. Then the following are equivalent:*

(i) *The system satisfies the influence property.*
(ii) *There exists $s \in [n]$ such that $p_s \neq p_{n-s+1}$.*

Proof. ((i) $\Rightarrow$ (ii)) We prove the contrapositive. Assume that $p_s = p_{n-s+1}$ for all $1 \leq s \leq n$. Then

$$\mathbb{P}(E_i \mid C_i = 0 \bigwedge_{j \neq i} C_j = x_j) = p_{k+1} = p_{n-k} = \mathbb{P}(E_i \mid C_i = 1 \bigwedge_{j \neq i} C_j = x_j),$$

where k is the number of zeros occurring in the sequence $(x_j : j \neq i)$. Therefore E_i is not influenced by C_i, and so the system fails to satisfy the influence property.

((ii) $\Rightarrow$ (i)) Suppose that $p_s \neq p_{n-(s+1)}$ for some $s \in [n]$. As above, since

$$\mathbb{P}(E_i \mid C_i = 0 \bigwedge_{j \neq i} C_j = x_j) = p_{k+1} \neq p_{n-k} = \mathbb{P}(E_i \mid C_i = 1 \bigwedge_{j \neq i} C_j = x_j),$$

where k is the number of zeros occurring in the sequence $(x_j : j \neq i)$, E_i is influenced by C_i. We must also show that E_i is influenced by C_j for each $j \neq i$. To this end, observe that if $p_s \neq p_{n-(s+1)}$ for some $s \in [n]$, there must exist some $t \in [n]$ such that $p_t \neq p_{t+1}$. Let $j \neq i \in [n]$. Set $x_k = 0$ for any $t-1$ values of $k \neq i, j$, and $x_k = 1$ for the remaining values of $k \neq i, j$. Then

$$\mathbb{P}(E_i \mid C_i = 0, C_j = 0, \bigwedge_{k \neq i,j} C_k = x_k) = p_{t+1} \neq p_t = \mathbb{P}(E_i \mid C_i = 0, C_j = 1, \bigwedge_{k \neq i,j} C_k = x_k).$$

Therefore each E_i is influenced by C_j for all $i, j \in [n]$ and so the system satisfies the influence property.

$\square$

To aid in further discussions, set Inf_n to be the set of points $\mathbf{p} \in [0,1]^n$ such that the system has influence.

3.3. **Characterizing independence.** We continue to assume the EST setting, that is $r = \frac{1}{2}$ and $p_i = q_i$. For the vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, let

$$\psi(\mathbf{p}) = \left(\frac{1}{2^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} p_{k+1} \right)^2 - \frac{1}{2^{n-1}} \sum_{k=0}^{n-2} \binom{n-2}{k} (p_{k+2}^2 + p_{k+1} p_{n-(k+1)}). \tag{2}$$

The function ψ allows us to characterize independence as follows.

Proposition 3.2. *The effects are pairwise independent if and only if $\psi(\mathbf{p}) = 0$.*

Proof. The symmetry in the EST model implies that for all $i, j \in \{1, \ldots, n\}$

$$\mathbb{P}(E_i) = \mathbb{P}(E_i \mid C_j = x) = \mathbb{P}(E_1 \mid C_1 = 0) = \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} p_{k+1}.$$

This last expression comes from summing the binomial probability $(\binom{n-1}{k} 2^{-(n-1)})$ that k of the causes $C_2, \ldots, C_n$ are also in state 0, times the probability (p_{k+1}) of E_1 given that k causes are also in state 0 and that $C_1 = 0$. This gives the first term in $\psi(\mathbf{p})$.

Similarly, for any $i \neq j$

$$\mathbb{P}(E_i \wedge E_j) = \mathbb{P}(E_1 \wedge E_2 | C_1 = 0) = \frac{1}{2^{n-1}} \sum_{k=0}^{n-2} \binom{n-2}{k} (p_{k+2}^2 + p_{k+1} p_{n-(k+1)})$$

The last equality follows from considering the two cases $C_2 = 0$ or $C_2 = 1$, each of which has probability $\frac{1}{2}$. The binomial probabilities arise as above and in the case $C_2 = 0$ we use that the probability of E_1 and E_2 given that k of the causes are also in state zero and given that $C_1 = 0$ and $C_2 = 0$ is p_{k+2}^2. Similarly when $C_2 = 1$ the probability of E_1 and E_2 in this context is $p_{k+1} p_{n-(k+1)}$. $\square$

As with influence, to aid our discussion set

$$\mathrm{Ind}_n := \{\mathbf{p} \in [0,1]^n \mid \psi(\mathbf{p}) = 0\};$$

that is Ind_n is the set of all points so that the system has independence. Finally, we set

$$\mathrm{EST}_n := \mathrm{Ind}_n \cap \mathrm{Inf}_n.$$

While our discussion is entirely in the "EST setting," meaning that we assume $r = \frac{1}{2}$ and $p_k = q_k$, we will only use the notation EST_n when talking about subsets of the probability space $[0,1]^n$ consisting of points which give a system exhibiting both influence and independence.

4. Some special points in EST_n

Before we dig deep into the geometric and topological structure of EST_n, we show the space is non-empty by explicitly establishing a few useful points in the space. We start with Ind_n and move on to points that are in EST_n.

The quadratic form discussed in the next section gives us an easy way, from details in the proof of Theorem 7.2, to show that there are infinitely many points in EST_n. However, we found the following explicit points useful for proving that both EST_n and Ind_n are not convex. These examples also illustrate the challenge of trying to write down explicit points and are interesting because "natural" points like $p_i = p$ for all $1 \le i \le n$ are in Ind_n but not Inf_n and $p_i = \frac{1}{i}$ for $1 \le i \le n$ (which naturally generalizes the tequila example) are in Inf_n but not Ind_n.

4.1. **Explicit points in EST_n with all coordinates non-zero.** For the first set of points set $p_k = \theta^k$ for some $0 < \theta < 1$. Then $p_i \neq p_j$ for all $i \neq j$, which implies influence. We claim there exists at least one θ that implies independence of effects. Since we are in the EST setting we use Eqn. (2) and substitute θ^k for p_k to obtain:

$$\begin{aligned}
\psi(\mathbf{p}) &= \left(\frac{1}{2^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}\theta^{k+1}\right)^2 - \frac{1}{2^{n-1}}\sum_{k=0}^{n-2}\binom{n-2}{k}((\theta^{k+2})^2 + \theta^{k+1}\theta^{n-(k+1)}) \\
&= \left(\frac{1}{2^{n-1}}\theta(1+\theta)^{n-1}\right)^2 - \frac{1}{2^{n-1}}(\theta^4(1+\theta^2)^{n-2} + 2^{n-2}\theta^n) \qquad (3)\\
&= \frac{1}{2^{2n-2}}\theta^2\left((1+\theta)^{2n-2} - 2^{n-1}\theta^2(1+\theta^2)^{n-2} - 2^{2n-3}\theta^{n-2}\right).
\end{aligned}$$

To determine θ such that two events are independent, given a cause, we need to determine when Eqn. (3) is equal to zero. Of course, $\theta = 0$ is a solution but it fails to satisfy influence, by Proposition 3.1. So we study the equation

$$(1+\theta)^{2n-2} - 2^{n-1}\theta^2(1+\theta^2)^{n-2} - 2^{2n-3}\theta^{n-2} = 0. \qquad (4)$$

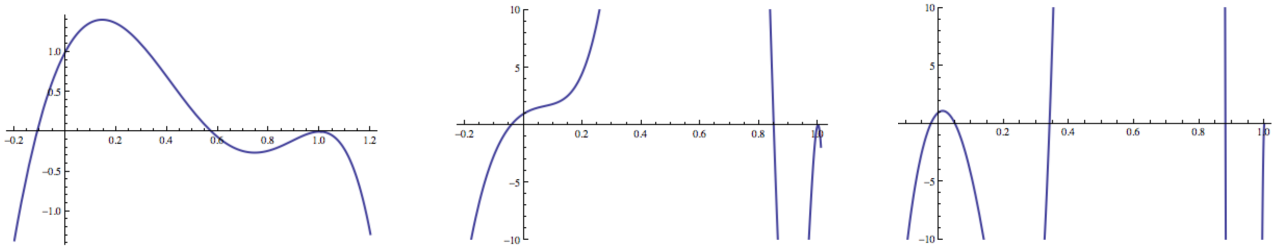
When $n = 3$ Eqn. (4). factors as

$$(1-\theta^2)(1-4\theta+3\theta^2) = 0,$$

The solution $\theta = 1$ corresponds to no influence by Proposition 3.1, and $\theta = -1$ is not stochastic. That leaves $1 - 4\theta + 3\theta^2 = (1-3\theta)(1-\theta) = 0$, showing two solutions: $\theta = 1$ and $\theta = \frac{1}{3}$. Therefore, for $n = 3$, there is one value of θ which is stochastic and all the probabilities involved are distinct, so the causes influence the effects (i.e. the system satisfies influence). Note that $\theta = \frac{1}{3}$ provides a different point in EST_n than that used in [8].

Set $f(\theta) = (1+\theta)^{2n-2} - 2^{n-1}\theta^2(1+\theta^2)^{n-2} - 2^{2n-3}\theta^{n-2}$. Notice that

$$\begin{aligned}
f(0) &= 1, \\
f(1) &= 2^{2n-2} - 2^{2n-3} - 2^{2n-3} = 0.
\end{aligned}$$



(A) $n = 4$ (B) $n = 9$ (C) $n = 11$

FIGURE 2. Graphs of $f(\theta)$.

Further, straightforward computation of f' and f'' show that $f'(1) = 0$ and $f''(1) < 0$ for all $n \geq 3$. Therefore, since f is 0 at $x = 1$, is positive at $x = 0$ and has a local maximum at $x = 1$, it must be 0 for some x in $(0, 1)$.

The few graphs of $f(\theta)$, given in Fig. 2, are instructive. We observe that for $n = 4, 9$ there is only the root guaranteed by the argument above, but starting with $n = 11$, f has three roots strictly between zero and one.

4.2. **Explicit points in** EST_n **with many zero coordinates.** A second way to construct explicit elements of EST_n is to look at 'boundary points'.

Proposition 4.1. *For any $n \geq 4$, there is exactly one value of p_n such that the point $\mathbf{p} = (1, 0, 0, 0, 0, \ldots, 0, p_n)$ lies in EST_n.*

Proof. To simplify initial computations, we let $N = 2^{n-1}$ to obtain:

$$\begin{aligned}\psi(\mathbf{p}) =& \frac{1}{N^2}\left(\sum_{k=0}^{n-1}\binom{n-1}{k}p_{k+1}\right)^2 - \frac{1}{N}\sum_{k=0}^{n-2}\binom{n-2}{k}p_{k+2}^2 - \frac{1}{N}\sum_{k=0}^{n-2}\binom{n-2}{k}p_{k+1}p_{n-(k+1)} \\ =& \frac{1}{N^2}(1+p_n)^2 - \frac{1}{N}(p_n^2).\end{aligned}$$

Thus the quadratic formula gives

$$p_n = \frac{-2 \pm \sqrt{4 - 4(1-N)}}{2(1-N)} = \frac{-1 \pm \sqrt{N}}{1-N}.$$

Then for any $N > 1$, one root lies between 0 and 1, namely $\frac{-1-\sqrt{N}}{1-N} = \frac{1}{\sqrt{N}-1}$. The point $\mathbf{p} = (1, 0, 0, 0, 0, \ldots, 0, \frac{1}{\sqrt{N}-1})$ also satisfies influence as $1 \neq \frac{1}{\sqrt{2^{n-1}}-1}$ for any $n \geq 4$ and so is in EST_n. $\square$

The computations in the proof above work for $n = 3$, but when $n = 3$, $\frac{1}{\sqrt{2^{3-1}}-1} = 1$. Therefore, the point we get, using this approach is $(1, 0, 1)$, which satisfies independence, but not influence. Similar computations (or Remark 6.1 below) show that $\mathbf{1} - \mathbf{p} = (0, 1, \ldots, 1, \frac{\sqrt{N}-2}{\sqrt{N}-1})$ is an element of EST_n as well.

5. The quadratic form ψ

To understand EST_n, we use the structure of ψ given in Eqn. (2). Since ψ is a quadratic form, the Hessian matrix, denoted H_n, seems to be most helpful in our study of the geometry and topology of EST_n and we explore the structure of H_n in this section. However, there are other helpful facts about ψ, like the fact that the first partial derivatives of ψ are zero at $\mathbf{p} = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, that we will pick up over the course of the next three sections. This turns out to be one piece of evidence that this point is special; another is that there are lots of lines, which are mostly in EST_n, passing through this point, as we show and use in Section 7.

To compute the Hessian matrix we begin with the first derivative. Throughout this section we use $N = 2^{n-1}$ to simplify expressions. For all $i \neq 1, n$,

$$\frac{\partial \psi}{\partial p_i} = \frac{2}{N^2}\binom{n-1}{i-1}\left(\sum_{k=0}^{n-1}\binom{n-1}{k}p_{k+1}\right) - \frac{2}{N}\left[\binom{n-2}{i-2}p_i + \binom{n-2}{i-1}p_{n-i}\right]. \tag{5}$$

When $i = 1$ simply remove the term $\frac{2}{N}[\binom{n-2}{i-2}p_i]$ and when $i = n$ remove the term $\frac{2}{N}[\binom{n-2}{i-1}p_{n-i}]$. From this the second partial derivatives are easy to compute.

$$\frac{\partial^2 \psi}{\partial p_i \partial p_j} = \frac{2}{N^2}\binom{n-1}{i-1}\binom{n-1}{j-1} - \begin{cases} \frac{2}{N}\binom{n-2}{i-2} & i = j \neq 1, \frac{n}{2}; \\ \frac{2}{N}\binom{n-2}{i-1} & j = n-i, j \neq \frac{n}{2}; \\ \frac{2}{N}\binom{n-2}{i-2} + \frac{2}{N}\binom{n-2}{i-1} & i = j = \frac{n}{2}; \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Since ψ is a quadratic polynomial, the Hessian matrix is constant, as expected. Furthermore, since ψ is a quadratic form corresponding to a symmetric matrix we label Q_n, $H_n = Q_n + Q_n^T = 2Q_n$. Therefore, knowing H_n gives us Q_n as well.

To determine for which values of n the space EST_n is connected – our main goal – we need several results regarding the eigenvalues and eigenspaces of the Hessian matrix H_n, which we collect here.

Proposition 5.1. *For all $n \geq 3$, the Hessian matrix H_n has 0 as an eigenvalue with associated eigenvector $\mathbf{1}$.*

Proof. The vector $\mathbf{1}$ is an eigenvector for the eigenvalue 0 if and only if the row sums are 0. The sum of the the entries in the i^{th} row of H_n, for $i \neq 1, n$, using Eqn. (6), is

$$\sum_{j=1}^{n}\frac{2}{N^2}\binom{n-1}{i-1}\binom{n-1}{j-1} - \frac{2}{N}\binom{n-2}{i-2} - \frac{2}{N}\binom{n-2}{i-1} = \frac{2}{N}\binom{n-1}{i-1} - \frac{2}{N}\binom{n-1}{i-1} = 0.$$

This uses $\sum_{j=1}^{n}\binom{n-1}{j-1} = 2^{n-1} = N$ and $\binom{n-2}{i-2} + \binom{n-2}{i-1} = \binom{n-1}{i-1}$. The arguments for $i = 1, n$ are similar, with simpler computations. $\square$

Remark 5.2. Observe from Eqn. (6) that the Hessian matrix $H_n = \mathbf{v}\mathbf{v}^T - X$, where $\mathbf{v}$ is the vector with i^{th} entry equal to $\frac{\sqrt{2}}{N}\binom{n-1}{i-1}$. The matrix X has non-zero entries on the diagonal, except for the $(1,1)$ location, which is 0, and there are non-zero entries on the opposite diagonal given by $i + j = n$. For example, below are the matrices X for $n = 4$ and

$n = 5$, in both cases scaled by multiplying by $N/2 = 2^{n-2}$. These two cases also illustrate the differences in X for odd vs. even values of n. Finally, it is helpful to keep the shape of the matrix X in mind for many of the following arguments.

$$\begin{bmatrix} 0 & 0 & \binom{2}{0} & 0 \\ 0 & \binom{2}{0} & 0 & 0 \\ \binom{2}{2} & 0 & \binom{2}{1} & 0 \\ 0 & 0 & 0 & \binom{2}{2} \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & \binom{3}{0} & 0 \\ 0 & \binom{3}{0} & \binom{3}{1} & 0 & 0 \\ 0 & \binom{3}{1} & \binom{3}{1} & 0 & 0 \\ \binom{3}{3} & 0 & 0 & \binom{3}{2} & 0 \\ 0 & 0 & 0 & 0 & \binom{3}{3} \end{bmatrix}$$

Lemma 5.3. *The matrix X has rank n.*

Proof. Observing that rows 1, n, and, when n is even, row $\frac{n}{2}$ each have only one non-zero entry, and that rows i and $n-i$ for all other $i \neq n-1$ have two entries in the same columns, which are i and $n-i$ we see that it is easy to use elementary row operations to convert X into an upper triangular matrix with all non-zero entries on the diagonal. $\square$

Proposition 5.4. *For all $n \geq 4$, the eigenspace of H_n corresponding to the eigenvalue* 0 *has dimension 1.*

Proof. It is enough to prove that $\mathrm{rk}(H_n) = n-1$. Since $H_n = \mathbf{v}\mathbf{v}^T - X$, the subaddativity of matrix rank applied to $-X = H_n - \mathbf{v}\mathbf{v}^T$ gives $\mathrm{rk}(X) \leq \mathrm{rk}(H_n) + \mathrm{rk}(\mathbf{v}\mathbf{v}^T)$. Since $\mathrm{rk}(X) = n$ by Lemma 5.3 and $\mathrm{rk}(\mathbf{v}\mathbf{v}^T) = 1$, $n-1 \leq \mathrm{rk}(H_n)$. Since 0 is an eigenvector, $n-1 = \mathrm{rk}(H_n)$. $\square$

Remark 5.5. Since $\psi(\mathbf{x}) = \mathbf{x}^T Q_n \mathbf{x}$ is a quadratic form, we can diagonalize Q_n using an orthogonal matrix P, that is $P^T Q_n P = D$, where D is a diagonal matrix of real eigenvalues of Q_n. Since $H_n = 2Q_n$, we could equivalently write $\psi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T H_n \mathbf{x}$ and diagonalize H_n instead. Furthermore, all the results in this section apply equally to Q_n, but are easier to prove and think about in terms of H_n. However, in later arguments, we use Q_n instead of H_n to avoid having to keep track of the factor $\frac{1}{2}$.

We prove in Theorem 7.2 that the connectedness of EST_n depends on the number of strictly positive and strictly negative eigenvalues of H_n. We establish here that H_n has "enough" of each type of eigenvalue for $n \geq 6$. For ease of notation, we use $H = H_n$ in the following discussion.

Theorem 5.6. *For all $n \geq 6$, H (equivalently, Q_n) has at least two strictly positive and at least two strictly negative eigenvalues.*

Proof. Let $A = H + \epsilon B$ where $\epsilon > 0$ and

$$B_{ij} = \begin{cases} 1, & \text{if } i + j = n+1; \\ 0, & \text{otherwise.} \end{cases}.$$

Let A_k denote the submatrix of A consisting of the first k rows and columns of A so that $\det(A_k)$ is the k^{th} leading principal minor of A. Then $A_k = H_k$ for all $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. Therefore, for all $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, $A_k = (\mathbf{v}\mathbf{v}^T)_k - X_k$ for a vector $\mathbf{v}$ and a matrix X, where X_k is diagonal and its first entry is 0 (see Remark 5.2). Hence elementary row operations on A_k transform it into an upper triangular matrix T such that $T_{11} = A_{11} = \frac{2}{N^2}$ and $T_{ii} = X_{ii} = \frac{2}{N}\binom{n-2}{i-2} \neq 0$ for all $2 \leq i \leq k$. Thus $\det(A_k) \neq 0$ for all $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

If $k \geq \lfloor \frac{n}{2} \rfloor + 1$, then $\det(A_k)$ is a polynomial in ϵ (for example, when $k = \lfloor \frac{n}{2} \rfloor + 1$, and n is odd, ϵ appears in the $(\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 1)$ entry). Set $p_k(\epsilon) = \det(A_k)$ for $\lfloor \frac{n}{2} \rfloor + 1 \leq k \leq n$. This is a finite set of polynomials, each with a finite number of zeros. Call that set of zeros Z, and let

$$\epsilon_Z = \min(\{|z| : z \in Z\} - \{0\}), \tag{7}$$

which is strictly positive (since Z is finite). Then for any $\epsilon \in (0, \epsilon_Z)$ we have that $\det(A_k) \neq 0$ for all $\lfloor \frac{n}{2} \rfloor + 1 \leq k \leq n$. Therefore all the leading principal minors of A are non-zero (including $\det(A) = \det(A_n)$).

Since all of the leading principal minors of A are non-zero, A has a unique LU-decomposition [6, Theorem 2.13]. Since A is symmetric, the LU-decomposition can be transformed into an LDL^T-decomposition where L is lower triangular and D is diagonal [6, Theorem 2.14 and discussion]. Furthermore, simply writing this expression out gives the following recursive formulae for the entries of D and L, assuming $i > j$:

$$D_j = A_{jj} - \sum_{k=1}^{j-1} L_{jk}^2 D_k \tag{8}$$

$$L_{ij} = \frac{1}{D_j}\left(A_{ij} - \sum_{k=1}^{j-1} L_{ik} L_{jk} D_k\right). \tag{9}$$

We show that $D_1 > 0$, $D_i < 0$ for $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $D_{\lfloor \frac{n}{2} \rfloor + 1} > 0$. Therefore D has at least two strictly negative eigenvalues and two strictly positive eigenvalues for $n \geq 6$. By Sylvester's Theorem [10], A and D have the same index (or inertia) and hence A also has at least two strictly negative eigenvalues and two strictly positive eigenvalues for $n \geq 6$. Before digging into computing D_i we argue that H must also have at least two strictly negative eigenvalues and two strictly positive eigenvalues for $n \geq 6$.

Over the complex numbers, roots of a polynomial are continuous functions of the coefficients of the polynomial [3, Theorem (1,4)] which implies that each eigenvalue of A corresponds to an eigenvalue of H. More formally, let $p_A(x) = x^n + c_1 x^{n-1} + \cdots + c_n$ denote the characteristic polynomial of A and $p_H(x) = x^n + d_1 x^{n-1} + \cdots + d_n$ be the characteristic polynomial of H. By construction, $d_i = c_i + \epsilon_i$ for $1 \leq i \leq n$ and each ϵ_i approaches 0 as ϵ (in the definition of A) goes to 0. Suppose that:

$$p_A(x) = \Pi_{k=1}^{q} (x - a_i)^{m_i}$$

with the distinct $a_i \in \mathbb{R}$, since A is symmetric. Then for any

$$0 < r_k < \min\{|a_k - a_i|,\ i = 1, 2, \cdots, k-1, k+1, \cdots q\},$$

there exists a δ such that if $|c_j - d_j| < \delta$ for all $1 \leq j \leq n$, then $p_H(x)$ has m_k roots in a circle of radius r_k centered at a_k. Since H is also symmetric, its roots are also real and if a_k is positive (resp. negative), then for small enough values of r_k, the corresponding roots of $p_H(x)$ are also positive (resp. negative). Let ϵ (in the definition of A), be less than ϵ_Z from

(7), and also small enough so that if A has at least two strictly positive eigenvalues and at least two strictly negative eigenvalues for $n \geq 6$, then H does also.

We finish by showing that $D_1 > 0$, $D_i < 0$ for $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $D_{\lfloor \frac{n}{2} \rfloor + 1} > 0$ for A. Throughout this discussion, we assume $i > j$ and use Eqns. (8) and (9). For all $i \neq n-j+1$, $A_{ij} = H_{ij}$. Thus $D_1 = H_{11} = \frac{2}{N^2} > 0$. Furthermore,

$$L_{i1} = \frac{1}{D_1}\left(D_1 \binom{n-1}{i-1}\binom{n-1}{0}\right) = \binom{n-1}{i-1}, \text{ for } 1 \leq i \leq n-1.$$

Therefore

$$A_{ij} = H_{ij} = D_1 L_{i1} L_{j1}, \text{ for all } i \neq n-j,\ n-j+1.$$

We use this fact repeatedly throughout the remaining discussion. Also note that $i \neq n - j,\ n-j+1$ for all $1 \leq i, j \leq \lfloor \frac{n}{2} \rfloor$. Hence,

$$L_{ij} = -\frac{1}{D_j}\left(\sum_{k=2}^{j-1} L_{ik} L_{jk} D_k\right) \text{ for all } i \neq n-j,\ n-j+1.$$

By induction on j, $L_{ij} = 0$ for all $1 < i, j \leq \lfloor \frac{n}{2} \rfloor$ since L_{i2} is trivially zero. Therefore the sum for L_{ij} only includes expressions where the second index is strictly less than j. Hence

$$D_i = H_{ii} - \sum_{k=1}^{j-1} L_{jk}^2 D_k = -\frac{2}{N}\binom{n-2}{i-2} < 0, \text{ for all } 1 < i \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Thus we have $D_1 > 0$ and, for $n \geq 6$, at least two strictly negative eigenvalues for D.

Finally, we need to argue that $D_{\lfloor \frac{n}{2} \rfloor + 1} > 0$. While the arguments are similar, they differ slightly for even and odd values of n and are somewhat technical so we placed them in the appendix. When $n \geq 6$ is odd, we get

$$D_{\lfloor \frac{n}{2} \rfloor + 1} = \frac{2}{N}\binom{n-2}{\lfloor \frac{n}{2} \rfloor}\left(\frac{2}{\lfloor \frac{n}{2} \rfloor - 1}\right) + \epsilon > 0,$$

and when $n \geq 6$ is even, $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$, so that

$$D_{\frac{n}{2}+1} = \frac{2}{N}\binom{n-2}{t-1}\left(\frac{n-1}{\frac{n}{2}(\frac{n}{2}-2)}\right) + \frac{\epsilon^2 N}{\binom{n-2}{t-2}} > 0.$$

□

6. The Geometry of EST_n

The space EST_n is a bounded (but not closed) subspace of $\mathbb{R}^n$. Recall that computations from Section 3 show that when $n = 3$, this space consists of a pair of two-dimensional components, each of which is convex. Many of the ideas we develop in this section are useful in our discussion of connectivity in Section 7.

Remark 6.1. For any n, the symmetry of the states 0 and 1 in the EST problem implies if $\mathbf{p} \in \mathrm{EST}_n$ then $\mathbf{1} - \mathbf{p} = (1 - p_1, 1 - p_2, \ldots, 1 - p_n) \in \mathrm{EST}_n$. Therefore the map $\mathbf{p} \mapsto \mathbf{1} - \mathbf{p}$ is a involution from the solution space to itself; in the case $n = 3$, this maps each connected component onto the other. This involution also moves every point, since the unique fixed point has $p_i = \frac{1}{2}$ for all i and this point fails influence.

Furthermore, if $\mathbf{p} \in \mathrm{EST}_n$ lies in the EST solution space then for any constant $0 < c \leq 1$, the scaled vector $c \cdot \mathbf{p} \in \mathrm{EST}_n$, since ψ is a homogeneous quadratic in the coordinates of $\mathbf{p}$.

These observations are part of the following more general result.

Proposition 6.2.

(i) *For any real values x and y and real vector $\mathbf{p} = (p_1, \dots, p_n)$,*

$$\psi(x\mathbf{p} + y\mathbf{1}) = x^2\psi(\mathbf{p}).$$

(ii) *In particular, if $\mathbf{p} \in [0,1]^n$ satisfies independence then $x\mathbf{p} + y\mathbf{1}$ does also, provided this vector also lies in $[0,1]^n$.*

Proof. Part (i) holds for $y = 0$, since ψ is a homogeneous quadric polynomial, so it suffices to establish part (i) when $x = 1$. In that case, if we replace p_i by $p_i + y$ in ψ, we see that the coefficient of y^2 is $\psi(y\mathbf{1}) = 0$, and the coefficient of y^0 is $\psi(\mathbf{p})$. The remaining terms correspond to the coefficient of y^1. Checking that this coefficient is equal to 0 requires more careful algebraic analysis (and the use of the combinatorial identity: $\binom{n-2}{k-1} + \binom{n-2}{k} = \binom{n-1}{k}$), but the computation is straightforward. This establishes part (i). Part (ii) now follows from Proposition 3.2. $\square$

This proposition has a few consequences of note. First, it provides an alternative argument for the point made in Remark 6.1. However, it proves further that if $\mathbf{p} \in \mathrm{Ind}_n$ then the entire line between $\mathbf{p}$ and $\mathbf{1} - \mathbf{p}$ also lies in Ind_n. Note that any such line must pass through the 'middle point' of $[0,1]^n$, namely

$$\mathbf{m} = \left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right),$$

and this point will play an important role in forthcoming arguments.

Furthermore, if we want to explore points near $\mathbf{m} \in \mathrm{Ind}_n$ (which is helpful for the proof of Theorem 7.2) – say, points of the form $\mathbf{p} = (\frac{1}{2} + x_1, \dots, \frac{1}{2} + x_n)$ where $-\frac{1}{2} < x_i < \frac{1}{2}$ – then $\mathbf{p} \in \mathrm{Ind}_n$ if and only if $\psi(x_1, \dots, x_n) = 0$. Note that $(x_1, \dots, x_n)$ may or may not be in Ind_n since the coordinates may or may not all be non-negative. The question of which of these points are in EST_n is a bit more subtle but, generally, they will be so if $\mathbf{p} \in \mathrm{EST}_n$ to start with.

Remark 6.3. Let $\mathbf{p}, \mathbf{q} \in \mathrm{Ind}_n$. We note that Proposition 6.2 gives an equivalence relation on Ind_n. We say $\mathbf{p} \sim \mathbf{q}$ if and only if $\mathbf{p} = a\mathbf{q} + b\mathbf{1}$ for some $a, b \in \mathbb{R}$ with $a \neq 0$. For example, the two points given in Section 4.2 are equivalent, as are the two solutions to EST_3 shown in Fig. 1 (use $a = \frac{9}{4}$ and $b = \frac{1}{4}$). Also note that if $\mathbf{p}, \mathbf{q} \in [0,1]^n$ and $\mathbf{p} \sim \mathbf{q}$ then $\mathbf{p} \in \mathrm{EST}_n$ if and only if $\mathbf{q} \in \mathrm{EST}_n$.

The more general expression $\psi(x\mathbf{p} + y\mathbf{q})$ for two points $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^n$ is helpful for investigating the convexity of Ind_n and EST_n, and is useful for our next result regarding the equivalence relation $\sim$ which we use in our discussion of convexity in the next section.

$$\begin{aligned}\psi(x\mathbf{p} + y\mathbf{q}) =& (x\mathbf{p} + y\mathbf{q})^T Q_n (x\mathbf{p} + y\mathbf{q}) \\ =& x^2\psi(\mathbf{p}) + y^2\psi(\mathbf{q}) + xyCT(\mathbf{p}, \mathbf{q})\end{aligned}$$

where the 'cross term' CT is given by

$$CT(\mathbf{p},\mathbf{q}) = 2\mathbf{p}^T Q_n \mathbf{q}. \tag{10}$$

Proposition 6.4. *For any $n \geq 3$, a point $\mathbf{x} \in \mathrm{Ind}_n$ has the property that for all $\mathbf{p} \in \mathrm{Ind}_n$ the line segment from $\mathbf{p}$ to $\mathbf{x}$ lies in Ind_n if and only if $\mathbf{x} \sim \mathbf{1}$.*

Proof. The 'if' direction is readily established. If $\mathbf{x} \sim \mathbf{1}$ and $\mathbf{p} \in \mathrm{Ind}_n$ then Eqn. (10) and the identity $Q_n\mathbf{1} = \mathbf{0}$, imply that $CT(\mathbf{p},\mathbf{x}) = 0$. Thus, $\psi(t\mathbf{p} + (1-t)\mathbf{x}) = 0$ for all $t \in [0,1]$, and thus each point on this line lies in Ind_n.

For the 'only if' part, suppose that $\mathbf{x} \in [0,1]^n$ satisfies the property described (we will say that $\mathbf{x}$ is *permissible*). For all $\mathbf{q} \in [-\frac{1}{3},\frac{1}{3}]^n$ for which $\psi(\mathbf{q}) = 0$ we have $\mathbf{m}+\mathbf{q} \in \mathrm{Ind}_n$ by Proposition 6.2(ii). Thus, since $\mathbf{x} \in \mathrm{Ind}_n$ and by the special assumption concerning this point, we have:

$$0 = CT(\mathbf{x},\mathbf{m}+\mathbf{q}) = CT(\mathbf{x},\mathbf{m}) + CT(\mathbf{x},\mathbf{q}) = 0 + CT(\mathbf{x},\mathbf{q}),$$

which gives

$$CT(\mathbf{x},\mathbf{q}) = 0 \tag{11}$$

for all $\mathbf{q} \in [-\frac{1}{3},\frac{1}{3}]^n$ for which $\psi(\mathbf{q}) = 0$. Let P and D be as given in Remark 5.5. If we let (fixed) $\mathbf{y} = P^T\mathbf{x}$ and (variable) $\mathbf{z} = P^T\mathbf{q}$, then for all $\mathbf{z} \in B = P^T[-\frac{1}{3},\frac{1}{3}]^n$ for which$\mathbf{z}^T D\mathbf{z} = 0$ (i.e. $\psi(\mathbf{q}) = 0$) we have (from (11)):

$$2\mathbf{y}^T D\mathbf{z} = 0. \tag{12}$$

By Proposition 5.4, we can order the diagonal entries D as $d_1,\dots,d_n$ so that $d_1 = 0$, and $d_j \neq 0$ for $j > 1$. Set $c_i = d_i y_i$ for each i. Then for all $\mathbf{z}$ in B for which

$$\sum_{i=2}^{n} d_i z_i^2 = 0, \tag{13}$$

we must also have (from Eqn. (12)):

$$\sum_{i=2}^{n} c_i z_i = 0.$$

Now, D not only has $n-1$ non-zero eigenvalues, but at least one is strictly positive and at least one is strictly negative. This is readily verified for $3 \leq n \leq 5$, and for $n \geq 6$ it is an immediate consequence of the stronger result stated in Proposition 5.6. Consequently, for any $j > 1$, the equation $\sum_{i=2}^n d_i z_i^2 = 0$ has a solution for $\mathbf{z} \in B$ with $z_j \neq 0$.

Now, suppose that $c_j \neq 0$ for some value of j. Let $\mathbf{z}$ be a vector in B that satisfies Eqn. (13) and has $z_j \neq 0$, and let $\mathbf{z}'$ be the vector obtained from $\mathbf{z}$ by flipping the sign of z_j while leaving the z_i values unchanged for all $i \neq j$. Then $\mathbf{z}'$ still lies in B and satisfies Eqn. (13) but $\sum_{i=2}^n c_i z_i$ and $\sum_{i=2}^n c_i z_i'$ cannot both be zero, since they differ by a term of magnitude $2|c_i z_i| \neq 0$. Thus if $\mathbf{x}$ is permissible then c_i must be zero for all $i > 1$ and since $d_i \neq 0$ for all $i > 1$, we must have:

$$y_2 = y_3 = \dots y_n = 0.$$

Thus, the set of possible values of $\mathbf{y}$ for which $\mathbf{x}$ is permissible is precisely the set

$$\{\mathbf{y} = (y, 0, 0, \ldots, 0) : P\mathbf{y} \in [0,1]^n\},$$

and this is simply $\{p \cdot \mathbf{1} : p \in [0,1]\}$, since $(1, 1, \ldots, 1)$ is the eigenvector of H_n corresponding to 0.

□

6.1. **Convexity.** As previously noted, Proposition 6.2 shows that if $\mathbf{p} \in \mathrm{EST}_n$ then $\mathbf{1} - \mathbf{p}$ and the line segment $(1-t)\mathbf{p} + t(\mathbf{1} - \mathbf{p})$, for $0 \leq t \leq 1$, between them are all in Ind_n. Easy computations show that the point $\mathbf{m} = (\frac{1}{2}, \ldots, \frac{1}{2})$ lies on the line $(1-t)\mathbf{p} + t(\mathbf{1} - \mathbf{p})$ for any point $\mathbf{p}$ but $\mathbf{m}$ fails influence and hence is not in EST_n. Therefore EST_n is not convex. However, in this example, all the points still lie in independence space and so it might still seem possible that Ind_n is convex. Using the cross term given in Eqn. (10) and the points from Section 4, we see that there are points points in EST_n where the line between them does not lie in Ind_n and hence independence space is not convex either.

If we take the point $(1, 0, \ldots, 0, \frac{1}{\sqrt{N-1}})$ and a point $(\theta, \theta^2, \ldots, \theta^n)$ where θ is a solution to $f(\theta) = 0$, then a bit of computation and proceeding by contradiction shows that $CT(\mathbf{p}, \mathbf{q}) \neq 0$ and every point on the line $t\mathbf{p} + (1-t)\mathbf{q}$, except for $\mathbf{p}$ and $\mathbf{q}$, is outside independence space and hence outside EST_n. For example, if $n = 11$ and we use $\theta = .340336$, then $CT(\mathbf{p}, \mathbf{q}) = 14.3457$.

7. The Topology of EST_n

As noted previously, the space EST_n is a bounded (but not closed) subspace of $\mathbb{R}^n$. The discussion in Section 3 shows that when $n = 3$, this space consists of a pair of two-dimensional components, each of which is contractible.

7.1. **Contractible.** Recall that a space is *contractible* if it can be continuously shrunk to a point (i.e. if the identity map is homotopic to the constant map).

Proposition 7.1. *For each $n \geq 3$,* Ind_n *is contractible, but* EST_n *is not.*

Proof. For Ind_n, select any point $\mathbf{x} \in \mathrm{Ind}_n$ for which $\mathbf{x} \sim \mathbf{1}$ (e.g. $\mathbf{x} = \mathbf{0}$ or $\mathbf{m} = (\frac{1}{2}, \ldots, \frac{1}{2})$). Then we have the homotopy:

$$F : \mathrm{Ind}_n \times [0,1] \to \mathrm{Ind}_n$$

$$(\mathbf{p}, t) \mapsto (1-t)\mathbf{p} + t\mathbf{x},$$

for which $F(\cdot, 0)$ is the identity map, $F(\cdot, 1)$ maps Ind_n to $\mathbf{x}$, and $F(\mathbf{p}, t) \in \mathrm{Ind}_n$ for all $t \in [0,1]$ by Proposition 6.2.

An early classical topological result of Smith [7] implies that any subset S of Euclidean space is not contractible if there is a continuous function $f : S \to S$ that has period two (i.e. $f \circ f$ is the identity map) and which has no fixed point. For EST_n, the map $\mathbf{p} \mapsto \mathbf{1} - \mathbf{p}$ is such a function, and since EST_n is a subset of Euclidean space it follows that EST_n is not contractible. □

7.2. **Connectedness of EST_n.** Since Ind_n is contractible, it is connected. The connectedness of EST_n is much more subtle and depends on the eigenvalues of the Hessian matrix H_n of ψ. Consider any two points $\mathbf{p}, \mathbf{q} \in \mathrm{EST}_n$. By Proposition 6.4, there are straight-line-paths from $\mathbf{p}$ to $\mathbf{m} = (\frac{1}{2}, \dots, \frac{1}{2})$, and from $\mathbf{m}$ to $\mathbf{q}$ and the concatenation of these two paths lies entirely in Ind_n. However, exactly one point on this concatenated path, namely $\mathbf{m}$, fails to lie in Inf_n. It is not enough to show there is a 'perturbed' path within Ind_n from $\mathbf{p}$ to $\mathbf{q}$ that avoids $\mathbf{m}$; we must also avoid all points not in Inf_n.

Theorem 7.2. *If $n = 3, 4$, then EST_n is disconnected and consists of exactly two connected components. If $n \geq 5$ then EST_n is connected.*

The case $n = 3$ was covered in Section 3. We give rather different proofs for the cases $n = 4, 5, 6, 7$ as opposed to the case $n \geq 8$. We use some computation for the small dimensions that does not generalize easily to the larger dimensions and we use cohomology theory for the larger dimensions that requires $n \geq 8$. However, some of the argument applies to all dimensions, so we begin with that.

All Dimensions. Let Inf_n^c denote the complement of influence space, which is the linear subspace of $\mathbb{R}^n$ of dimension $\lceil n/2 \rceil$ defined by:

$$x_i - x_{n-i+1} = 0 \text{ for all } i \in [n].$$

Since Q_n is a matrix corresponding to a quadratic form there exist matrices P, a real orthogonal matrix, and D, the diagonal matrix of real eigenvalues of Q_n (Remark 5.5). Let $\mathbf{y} = P^T\mathbf{x}$ (so $\mathbf{x} = P\mathbf{y}$).

By Proposition 5.4, D has zero as an eigenvalue with geometric multiplicity one. Suppose that D has k strictly positive eigenvalues, and l strictly negative eigenvalues, so that $k+l+1 = n$. By Theorem 5.6, and direct computation for $n = 4, 5$, we have that $k > 0$ and $l > 0$. We may assume that the first eigenvalue is 0 and that the next k eigenvalues $\lambda_1, \dots, \lambda_k$ are all strictly positive, while the final l eigenvalues, $\mu_1, \dots, \mu_l$ are all strictly negative. Then for any $s > 0$ and $t \geq 0$, the set

$$S_{s,t} := \{\mathbf{y} \in \mathbb{R}^n : -s < y_1 < s, \sum_{i=1}^{k} \lambda_i y_{i+1}^2 = t \text{ and } \sum_{j=1}^{l} (-\mu_j) y_{k+j+1}^2 = t\} \tag{14}$$

is a set of solutions to the equation

$$\mathbf{y}^T D \mathbf{y} = 0.$$

Observe that $S_{s,t} \cong I_s \times S^k \times S^l$, where I_s is an open interval of length s.

Let $\mathcal{L}$ be the image of Inf_n^c under the transformation P^T, that is

$$\mathcal{L} = \{P^T\mathbf{x} : \mathbf{x} \in \mathrm{Inf}_n^c\}.$$

Since P has full rank, it follows that $\mathcal{L}$ is a linear subspace of $\mathbb{R}^n$ of dimension $\lceil n/2 \rceil$. We recall that P^T is a homeomorphism since it is orthogonal and transforms $\mathrm{EST}_n = \mathrm{Ind}_n \cap \mathrm{Inf}_n$ into $(\bigcup_{s,t \geq 0} S_{s,t}) - \mathcal{L}$ where studying the connectivity of the space is much easier. We use the following lemma in arguments for all dimensions.

Remark 7.3. We note that for all s, points in $S_{s,0}$ are in the vectorspace spanned by $(1,0,\ldots,0)$ which is isomorphic to the vectorspace spanned by $(1,1,\ldots,1)$ under the transformation given by P. Therefore $S_{s,0} \subseteq \mathcal{L}$ and therefore none of the corresponding $\mathbf{x}$ satisfy influence and hence are not in EST_n. We use this fact repeatedly in what follows.

Lemma 7.4. *If $S_{s,t} - \mathcal{L}$ is connected for all $s \geq 0$ and $t > 0$, then for $n \geq 3$, EST_n is connected also.*

Proof. Let $\mathbf{m} = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$. Let $s, t > 0$ be sufficiently small so that $\mathbf{m} + P\mathbf{x} \in [0,1]^n$ for all $\mathbf{x} \in S_{s,t}$. Consider any points $\mathbf{p}$ and $\mathbf{q} \in \mathrm{EST}_n$. Recall that for any point $\mathbf{x}$ in EST_n, $\mathbf{y} = P^T\mathbf{x}$ satisfies

$$\sum_{i=1}^{k} \lambda_i y_{i+1}^2 = M \text{ and } \sum_{j=1}^{l} (-\mu_j) y_{k+j+1}^2 = M, \text{ for some } M.$$

Denote the value of M corresponding to $\mathbf{p}$ and $\mathbf{q}$ within this probability space as M_p and M_q respectively. Since $\mathbf{p}, \mathbf{q} \in \mathrm{EST}_n$, Remark 7.3 implies $M_p, M_q > 0$ and therefore we can choose $c_1 = \frac{t'}{M_p}$ and $c_2 = \frac{t'}{M_q}$ for some $t' \in (0,t]$. Then for $\mathbf{y}_p = c_1 P^T \mathbf{p}$ and $\mathbf{y}_q = c_2 P^T \mathbf{q}$, we have $\mathbf{y}_p, \mathbf{y}_q \in S_{s,t'}$. Since $S_{s,t} - \mathcal{L}$ is connected for all $s, t > 0$, there exists a path from $\mathbf{y}_p$ to $\mathbf{y}_q$ in $S_{s,t'} - \mathcal{L}$. By the fact that P^T is a homeomorphism, there also exists a continuous path from $P\mathbf{y}_p$ to $P\mathbf{y}_q$ satisfying Ind_n and Inf_n, but not necessarily within the probability space $[0,1]^n$. In order to ensure that there is a path within this probability space, we scale the path from $P\mathbf{y}_p$ to $P\mathbf{y}_q$ by adding $\mathbf{m} = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$ to the entire path. For small enough t and t', this path from $\mathbf{m} + P\mathbf{y}_p$ to $\mathbf{m} + P\mathbf{y}_q$ remains in $[0,1]^n$ and hence is in EST_n. Note that $P\mathbf{y}_p = P(c_1 P^T \mathbf{p})$. Using Proposition 6.2 and Remark 6.3, we know that the straight-line paths from $P\mathbf{y}_p$ to $\mathbf{m} + P\mathbf{y}_p$ and $P\mathbf{y}_q$ to $\mathbf{m} + P\mathbf{y}_q$, remain in $[0,1]^n$ and are in EST_n as well. Therefore, if $S_{s,t} - \mathcal{L}$ is connected for all $t > 0$, then EST_n is connected also. □

Dimension $n = 4$. We examine the space $\mathrm{Ind}_4 \cap \mathrm{Inf}_4^c$ by setting $\Psi(\mathbf{x}) = 0$, $x_1 = x_4$ and $x_2 = x_3$ getting $0 = -\frac{1}{16}(x_1 - x_2)^2$. Since $-\frac{1}{16}(x_1 - x_2)^2 \leq 0$ for all real x_1, x_2, the only solution to this equation is $x_1 = x_2$. Therefore, if $\mathbf{x} \in \mathrm{Ind}_4 \cap \mathrm{Inf}_4^c$, then $x_1 = x_2 = x_3 = x_4$. Hence $\mathrm{Ind}_4 \cap \mathrm{Inf}_4^c$ forms a one-dimensional linear space with basis vector $(1,1,1,1)$. The result of applying P^T to this space gives the one-dimensional space with basis $(1,0,0,0)$. By Remark 7.3, $P^T(\mathrm{Ind}_4 \cap \mathrm{Inf}_4^c) = S_{s,0} \subseteq \mathcal{L}$ and hence for all $t > 0$ and for all $s \geq 0$, $\mathcal{L} \cap S_{s,t} = \emptyset$. Furthermore, by Lemma 7.4, it is enough to prove that $S_{s,t} - \mathcal{L}$ is connected for all $t > 0$ and hence we proceed to argue $S_{s,t}$ is connected for all $t > 0$ and all $s \geq 0$.

Fix $t > 0$ and $s \geq 0$ arbitrary. Using Mathematica we verify that H_4 has exactly one positive eigenvalue. Then recalling Eqn. (14), for any $\mathbf{y} \in S_{s,t}$, $y_2 = \pm\sqrt{\frac{t}{\lambda_2}}$. Let $\mathbf{p} \in S_{s,t}$, be any point where $p_2 < 0$ and $\mathbf{q} \in S_{s,t}$ be any point where $q_2 > 0$, then the Intermediate Value Theorem implies that any continuous path between these two points must contain a point $\mathbf{y}$ with $y_2 = 0$. However, $y_2 = 0$ implies $\mathbf{y} \in S_{s,0} = \mathcal{L}$ and hence there does not exist a continuous path from $\mathbf{p}$ to $\mathbf{q}$ contained in $S_{s,t}$. Therefore EST_4 consists of at least two components. In fact each $S_{s,t}$ is homeomorphic to the union of the disjoint cylinders $A_1 = I_s \times \sqrt{\frac{t}{\lambda_2}} \times S^1$ and $A_2 = I_s \times -\sqrt{\frac{t}{\lambda_2}} \times S^1$ and each cylinder is connected. Observe that the argument used in Lemma 7.4 implies that $\bigcup_{s,t} A_1$ and $\bigcup_{s,t} A_2$ are connected and therefore EST_4 consists of two connected components.

Remark 7.5. The only role that $n = 4$ plays in this argument is that H_4 has exactly 1 positive eigenvalue and two negative eigenvalues.

Dimensions $n = 5, 6, 7$. Just as in the case of $n = 4$ we consider the set of equations consisting of setting $\Psi(\mathbf{x}) = 0$ and the linear equations that specify Inf_n^c. We get

$$\begin{aligned}
0 &= -\frac{3}{64}(x_1 - x_3)^2 \text{ for } n = 5, \\
0 &= \frac{1}{2^8}(-7x_1^2 - 15x_2^2 - 28x_3^2 - 6x_1x_2 + 20x_1x_3 + 36x_2x_3) \text{ for } n = 6, \text{ and} \\
0 &= \frac{1}{2^{10}}(-15x_1^2 - 60x_2^2 - 15x_3^2 - 60x_4^2 - 20x_1x_2 + 30x_1x_3 \\
&\qquad + 20x_1x_4 + 20x_2x_3 + 120x_2x_4 - 20x_3x_4) \text{ for } n = 7.
\end{aligned}$$

For $n = 5$, this implies $x_1 = x_3$ and hence $x_1 = x_3 = x_5$. For $n = 6$ solving for x_1 gives

$$x_1 = \frac{1}{7}\left(-3x_2 + 10x_3 \pm \sqrt{-(x_2 - x_3)^2} \right)$$

and hence real solutions require $x_2 = x_3$. Substituting this back into the equation $\Psi(\mathbf{x}) = 0$ we get that $x_1 = x_2$ and hence $x_1 = x_2 = x_3 = x_4 = x_5 = x_6$. Similarly for $n = 7$, solving for x_4 gives

$$x_4 = \frac{1}{6}\left(x_1 + 6x_2 - x_3 - 2\sqrt{2}\sqrt{-(x_1 - x_3)^2} \right)$$

and so real solutions require $x_1 = x_3$. Making this substitution into the equation $\Psi(\mathbf{x}) = 0$ we get $x_4 = x_2$, so $x_1 = x_3 = x_5 = x_7$ and $x_2 = x_4 = x_6$.

In the case $n = 6$, we see that $\mathrm{Ind}_n \cap \mathrm{Inf}_6^c$ forms a one-dimensional linear space with basis vector $(1, 1, 1, 1, 1, 1)$. The result of applying P^T to this space gives the one-dimensional space with basis $(1, 0, 0, 0, 0, 0)$. So just as for the case $n = 4$, $\mathcal{L} \subseteq S_{s,0}$. Thus for all $t > 0$, $S_{s,t} - \mathcal{L} = S_{s,t}$ when $n = 6$. Recalling Eqn. (14) and using Mathematica to get the eigenvalues for H_6, $S_{s,t}$ for $n = 6$ is homeomorphic to $I_s \times S^1 \times S^2$, a path connected space. Therefore by Lemma 7.4, $\bigcup_{s \geq 0, t > 0} S_{s,t}$ is connected and hence EST_6 is as well.

In the cases $n = 5$ and $n = 7$, $\mathrm{Ind}_n \cap \mathrm{Inf}_n^c$ is a two-dimensional linear space that can be written in terms of x_1 and x_2. The argument for $n = 5$ is a simplified version of that for $n = 7$, so for ease of reading, we give only the argument for $n = 7$ here.

Applying the matrix P^T to the two dimensional linear space $\mathrm{Ind}_7 \cap \mathrm{Inf}_7^c$ results in a two-dimensional space with basis vectors $\mathbf{b_1} = (1, 0, 0, 0, 0, 0)$ and $\mathbf{b_2} = (0, b_{22}, b_{23}, b_{24}, b_{25}, b_{26}, b_{27})$. Recall that for $t = 0$, $S_{s,0} \subseteq \mathcal{L}$ (Remark 7.3) and so fix an $s \geq 0$ and $t > 0$. Now, from Mathematica, H_7 has exactly three positive and three negative eigenvalues. Then using Eqn. (14) again, $S_{s,t}$ is homeomorphic to $I_s \times S^2 \times S^2$ and further for any $\mathbf{y} \in S_{s,t}$,

$$\lambda_2 y_2^2 + \lambda_3 y_3^2 + \lambda_4 y_4^2 = t.$$

Substituting cb_{ij} in for y_j for $j \in \{2,3,4\}$ and solving for c we get

$$c = \pm\sqrt{\frac{t}{\lambda_2 b_{22}^2 + \lambda_3 b_{23}^2 + \lambda_4 b_{24}^2}}. \tag{15}$$

Therefore we can characterize $\mathcal{L} \cap S_{s,t}$ as the set of line segments of the form,

$$\mathcal{L} \cap S_{s,t>0} = \{\mathbf{y} \in \mathbb{R}^5 | \mathbf{y} = (y_1, \pm c \cdot b_{22}, \pm c \cdot b_{23}, \pm c \cdot b_{24}, \pm c \cdot b_{25}, \pm c \cdot b_{26}, \pm c \cdot b_{27}), -s < y_1 < s\}.$$

We first observe that if $\mathbf{p}$ is any point in $S_{s,t} - \mathcal{L}$ then for some p_i with $2 \leq i \leq 7$, $p_i \neq \pm cb_{2i}$, and hence $p_j \neq \pm cb_{2j}$ or $p_k \neq \pm cb_{2k}$ for $i,j,k \in \{2,3,4\}$ or $i,j,k \in \{5,6,7\}$ and $i \neq j \neq k$. Let $\mathbf{q}$ be any point in $S_{s,t}$ such that $q_i = p_i$, $q_j = p_j$, and $q_k = p_k$. Then $\mathbf{q} \in S_{s,t} - \mathcal{L}$ as well. Since S^2 is path-connected, there is a path in $S_{s,t}$ from $\mathbf{p}$ to $\mathbf{q}$ where every point on the path has p_i as the i^{th} coordinate. Hence this entire path is in $S_{s,t} - \mathcal{L}$.

Now let $\mathbf{p}$ and $\mathbf{q}$ be any two points in $S_{s,t} - \mathcal{L}$. As before, for some $2 \leq i \leq 7$, $p_i \neq \pm cb_{2i}$. Without loss of generallity suppose $p_2 \neq \pm cb_{22}$. We split the argument into two cases:

Case 1: Assume $q_5 = \pm cb_{25}$, $q_6 = \pm cb_{26}$ and $q_7 = \pm cb_{27}$. Then $q_2 \neq \pm cb_{22}$, $q_3 \neq \pm cb_{23}$ or $q_4 \neq \pm cb_{24}$. Without loss of generality, suppose $q_2 \neq \pm cb_{22}$. Since S^2 is continuous and $S_{s,t>0} \cap \mathcal{L}$ is discrete, there exist s_5, s_6, s_7 such that $\lambda_5 s_5^2 + \lambda_6 s_6^2 + \lambda_7 s_7^2 = t$ and $s_5 \neq \pm cb_{25}$, $s_6 \neq \pm cb_{26}$ or $s_7 \neq \pm cb_{27}$. Then, by the argument above, there is a path in $S_{s,t>0} - \mathcal{L}$ from $\mathbf{p}$ to $(p_1, p_2, p_3, p_4, s_5, s_6, s_7)$ and from $(p_1, p_2, p_3, p_4, s_5, s_6, s_7)$ to $(q_1, q_2, q_3, q_4, s_5, s_6, s_7)$. Since $q_2 \neq \pm cb_{22}$, there exists a path from $(q_1, q_2, q_3, q_4, s_5, s_6, s_7)$ to $\mathbf{q}$. These paths combine to give a path from $\mathbf{p}$ to $\mathbf{q}$ in $S_{s,t>0} - \mathcal{L}$.

Case 2: Assume $q_5 \neq \pm cb_{25}$, $q_6 \neq \pm cb_{26}$ or $q_7 \neq \pm cb_{27}$. Using the argument above, there is a path from $\mathbf{p}$ to $(p_1, p_2, p_3, p_4, q_5, q_6, q_7)$ and a path from $(p_1, p_2, p_3, p_4, q_5, q_6, q_7)$ to $\mathbf{q}$ both of which are in $S_{s,t} - \mathcal{L}$.

Therefore $S_{s,t} - \mathcal{L}$ is path connected for all s and all $t > 0$ and hence EST_7 is connected by Lemma 7.4.

Dimensions $n \geq 8$. We begin this proof with one more topological lemma.

Lemma 7.6. *Let M be a compact manifold and I an open interval. Let $\mathbf{p} = (\mathbf{x}, t) \in M \times I$ and $\mathbf{q} = (\mathbf{y}, s) \in M \times I$. Then there exists $\phi : M \to M \times I$ such that M is homeomorphic to $\mathrm{im}(\phi)$ and $\mathbf{p}, \mathbf{q} \in \mathrm{im}(\phi)$.*

Proof. Let $f : M \to I$ be any continuous function such that $f(\mathbf{x}) = t$ and $f(\mathbf{y}) = s$. Set $\phi : M \to M \times I$ to be $\phi(\mathbf{v}) = (\mathbf{v}, f(\mathbf{v}))$ for any $\mathbf{v} \in M$. By construction, ϕ is continuous, since f is continuous. It is one-to-one, since it is the identity on the first coordinate of the image. Since M is compact, $M \times I$ is Hausdorff and ϕ is continuous and one-to-one, ϕ^{-1} is also continuous [9, Corollary 5.9.2]. Hence M is homeomorphic to the image of ϕ. □

We use Lemma 7.4 yet again and so let $\mathbf{p}, \mathbf{q}$ be any two points in $S_{s,t} - \mathcal{L}$ for some fixed $s, t > 0$. Recall that we have the homeomorphism $S_{s,t} \cong I_s \times S^{k-1} \times S^{l-1}$ where I_s is an open interval. If $\min\{k, l\} > 1$, then $S^{k-1} \times S^{l-1}$ is a compact orientable $m = (n-3)$-manifold which we denote by M_t.

Set $A = M_t \cap \mathcal{L}$. Thus A is a closed and bounded subspace of $\mathbb{R}^{\lceil n/2 \rceil}$. Therefore, A is a proper closed subset of M_t as long as $m = n - 3 > \lceil n/2 \rceil$, which is true for $n \geq 8$. In addition, A is locally contractible (it is a CW-complex).

In the following discussion, we compute all homology modules over $\mathbb{Z}$. Consider the terminal end of the long exact sequence relating homology to relative homology:

$$\cdots \to H_1(M_t, M_t - A) \to H_0(M_t - A) \to H_0(M_t) \to H_0(M_t, M_t - A) \to 0. \tag{16}$$

By Alexander Duality [1, Proposition 3.46] we have:

$$H_i(M_t, M_t - A) \cong H^{m-i}(A).$$

Therefore,

$$H_1(M_t, M_t - A) \cong H^{m-1}(A) \text{ and } H_0(M_t, M_t - A) \cong H^m(A).$$

For $t > 0$, $\mathbf{0} \notin M_t$ and therefore $\mathbf{0} \notin A$. However, $\mathbf{0} \in \mathbb{R}^{\lceil n/2 \rceil}$, so A is a proper closed subset of $\mathbb{R}^{\lceil n/2 \rceil}$ and hence it is a proper closed subspace of a compact manifold (sphere) of dimension $\lceil n/2 \rceil$ as well. Since $\lceil n/2 \rceil \leq m - 1$ for $n \geq 8$, by [4, Proposition 6.5], $H^{m-1}(A) = H^m(A) = 0$ (we are using that A is a CW-complex so Čech cohomology coincides with singular cohomology). Hence the exactness of the sequence in (16) implies

$$H_0(M_t - A) \cong H_0(M_t) \cong \mathbb{Z}.$$

Therefore, $M_t - A$ is connected.

The connectivity of $M_t - A$ and the fact that ϕ is a homeomorphism, imply $S_{s,t} - \mathcal{L}$ is connected and hence by Lemma 7.4, EST_n is connected.

8. Concluding comments

Our proof of Theorem 7.2 treats the dimensions $n < 8$ different from those for $n \geq 8$. Our proof for $n \geq 8$ requires larger dimensions to apply the cohomology theorems we use. We have good evidence that our argument for $n < 8$ extends to all n. Such a proof requires proving our conjecture that $\dim(\mathrm{Ind}_n \cap \mathrm{Inf}_n^c)$ has dimension 1 for n even or dimension 2 for n odd.

Further exploration of the topology of EST_n may be of interest, for example classification up to homotopy or homeomorphism.

We gave a thorough analysis of the EST set-up where $r = \frac{1}{2}$ and $p_k = q_k$. One possible approach to the study of the probabilities where influence and independence collide for more general values of r, p_k, and q_k might be to treat r, p_k, q_k as variables in a polynomial ring $R = K[r, p_1, \ldots, p_n, q_1, \ldots, q_n]$ over a field K and use polynomial ring theory.

From a practical point of view, the flexibility to allow r to vary seems interesting. For independence conditioned on a single cause, we verified computationally that for $n = 3, 4$, the only value of r that allows influence and independence to collide is $r = \frac{1}{2}$. For independence without conditioning we have duplicated all of the results in Section 5 (much more technical than the arguments here), except the fact that $D_{\lfloor \frac{n}{2} \rfloor + 1} > 0$ — the last step of the proof of Theorem 5.6.

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10. Appendix

We include here some details for the computations of $D_{\lfloor \frac{n}{2} \rfloor +1}$ from the end of Section 5. As in that section, we set $H = H_n$ to clean up the notation.

We need to argue that $D_{\lfloor \frac{n}{2} \rfloor +1} > 0$. We recall a few of the formulae found in the proof of Theorem 5.1 since we use them all:

$$D_1 = H_{11} = \frac{2}{N^2}, \qquad A_{ij} = H_{ij} = D_1 L_{i1} L_{j1}, \text{ for all } i \neq n-j,\ n-j+1,$$

$$L_{i1} = \binom{n-1}{i-1}, \text{ for } 1 \leq i \leq n-1, \qquad L_{ij} = -\frac{1}{D_j}\left(\sum_{k=2}^{j-1} L_{ik} L_{jk} D_k\right) \text{ for all } i \neq n-j,\ n-j+1,$$

$$L_{ij} = 0 \text{ for all } 1 < i, j \leq \lfloor \frac{n}{2} \rfloor, \qquad D_i = -\frac{2}{N}\binom{n-2}{i-2} < 0, \text{ for all } 1 < i \leq \lfloor \frac{1}{2} \rfloor.$$

We first assume that n is odd, so that $\lfloor \frac{n}{2} \rfloor + 1 + \lfloor \frac{n}{2} \rfloor = n$. For ease of notation, let $t = \lfloor \frac{n}{2} \rfloor + 1$. Then:

$$L_{tt-1} = \frac{1}{D_{t-1}}\left(H_{tt-1} - \sum_{k=1}^{t-2} L_{tk} L_{t-1k} D_k\right).$$

However, $L_{t-1k} = 0$ for $2 \leq k \leq t-2 < \lfloor \frac{n}{2} \rfloor$ since $t-1 = \lfloor \frac{n}{2} \rfloor$. Using that $D_{t-1} = -\frac{2}{N}\binom{n-2}{t-3}$, we have:

(17)
$$L_{tt-1} = -\frac{1}{\frac{2}{N}\binom{n-2}{t-3}}\left(\frac{2}{N^2}\binom{n-1}{t-1}\binom{n-1}{t-2} - \frac{2}{N}\binom{n-2}{t-1} - \binom{n-1}{t-1}\binom{n-1}{t-2}\frac{2}{N^2}\right) = \frac{\binom{n-2}{t-1}}{\binom{n-2}{t-3}}$$

Therefore:

$$\begin{aligned} D_t =& H_{tt} - \sum_{k=1}^{t-1} L_{tk}^2 D_k \\ =& \frac{2}{N^2}\binom{n-1}{t-1}^2 - \frac{2}{N}\binom{n-2}{t-2} + \epsilon - \binom{n-1}{t-1}^2 \frac{2}{N^2} - L_{tt-1}^2 D_{t-1} \\ =& -\frac{2}{N}\binom{n-2}{t-2} + \epsilon - \left(\frac{\binom{n-2}{t-1}}{\binom{n-2}{t-3}}\right)^2\left(-\frac{2}{N}\binom{n-2}{t-3}\right) \\ =& \frac{2}{N}\binom{n-2}{\lfloor \frac{n}{2} \rfloor}\left(\frac{2}{\lfloor \frac{n}{2} \rfloor - 1}\right) + \epsilon > 0. \qquad (18) \end{aligned}$$

where (18) uses the symmetry of the binomial.

Now assume n is even, so that $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$. This time, let $t = \frac{n}{2}$. Then the entries of L we need to be concerned with are $L_{t+1,t-1}$ and $L_{t+1,t}$. In both cases, as in Eqn. (17), the sum has all terms zero, except for the first one. We note that ϵ potentially appears in L_{t+1k}, but L_{tk} or L_{t-1k} are still zero and hence the full sum is zero. Therefore:

$$L_{t+1,t} = \frac{\epsilon}{D_t} = -\frac{\epsilon N}{2\binom{n-2}{t-2}}, \text{ and } L_{t+1,t-1} = \frac{\binom{n-2}{t}}{\binom{n-2}{t-3}}.$$

We are now ready to compute D_{t+1}.

$$\begin{aligned} D_{t+1} =& A_{t+1,t+1} - \sum_{k=1}^{t} L_{t+1k}^2 D_k \\ =& -\frac{2}{N}\binom{n-2}{t-1} - L_{t+1t-1}^2 D_{t-1} - L_{t+1t} D_t \\ =& -\frac{2}{N}\binom{n-2}{t-1} - \left(\frac{\binom{n-2}{t}}{\binom{n-2}{t-3}}\right)^2\left(-\frac{2}{N}\binom{n-2}{t-3}\right) - \left(-\frac{\epsilon N}{2\binom{n-2}{t-2}}\right)^2\left(-\frac{2}{N}\binom{n-2}{t-2}\right) \\ =& \frac{2}{N}\binom{n-2}{t-1}\left(\frac{n-1}{\frac{n}{2}(\frac{n}{2}-2)}\right) + \frac{\epsilon^2 N}{2\binom{n-2}{t-2}} > 0. \end{aligned}$$