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Decomposing Tensors into Frames

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Abstract

A symmetric tensor of small rank decomposes into a configuration of only few vectors. We study the variety of tensors for which this configuration is a unit norm tight frame.

1 Introduction

A fundamental problem in computational algebraic geometry, with a wide range of applications, is the low rank decomposition of symmetric tensors; see e.g. [1, 3, 10, 20, 21]. If $T = (t_{i_1 i_2 \cdots i_d})$ is a symmetric tensor in $\mathrm{Sym}_d(\mathbb{C}^n)$, then such a decomposition takes the form

$$T \;=\; \sum_{j=1}^{r} \lambda_j \mathbf{v}_j^{\otimes d}. \tag{1}$$

Here $\lambda_j \in \mathbb{C}$ and $\mathbf{v}_j = (v_{1j}, v_{2j}, \ldots, v_{nj}) \in \mathbb{C}^n$ for $j = 1, 2, \ldots, r$. The smallest r for which a representation (1) exists is the *rank* of T. In particular, each $\mathbf{v}_j^{\otimes d}$ is a tensor of rank 1.

An equivalent way to represent a symmetric tensor T is as the homogeneous polynomial

$$T \;=\; \sum_{i_1,\ldots,i_d=1}^{n} t_{i_1 i_2 \cdots i_d} \cdot x_{i_1} x_{i_2} \cdots x_{i_d}. \tag{2}$$

If $d = 2$, then (2) is the identification of symmetric matrices with quadratic forms. Written as a polynomial, the right hand side of (1) is a linear combination of powers of linear forms:

$$T \;=\; \sum_{j=1}^{r} \lambda_j (v_{1j}x_1 + v_{2j}x_2 + \cdots + v_{nj}x_n)^d. \tag{3}$$

The decomposition in (1) and (3) is called *Waring decomposition*. When $d = 2$, it corresponds to orthogonal diagonalization of symmetric matrices. We could subsume the constants λ_i into the vectors $\mathbf{v}_i$ but we prefer to leave (1) and (3) as is, for reasons to be seen shortly. The (projective) variety of all such symmetric tensors is the *r-th secant variety of the Veronese variety*. The vast literature on the geometry and equations of this variety (cf. [18]) forms the mathematical foundation for low rank decomposition algorithms for symmetric tensors.

In many situations one places further restrictions on the summands in (1) and (3), such as being real and nonnegative. Applications to machine learning in [1] concern the case when $r = n$ and the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form an orthonormal basis of $\mathbb{R}^n$. The article [21] characterizes the *odeco variety* of all tensors that admit such an orthogonal decomposition.

The present paper takes this one step further by connecting tensors to *frame theory* [5, 6, 9, 12, 26]. We examine the scenario when the $\mathbf{v}_j$ form a **f**inite **u**nit **n**orm **t**ight **f**rame (or funtf) of $\mathbb{R}^n$, an object of recent interest at the interface of applied functional analysis and algebraic geometry. Consider a configuration $V = (\mathbf{v}_1, \ldots, \mathbf{v}_r) \in (\mathbb{R}^n)^r$ of r labeled vectors in $\mathbb{R}^n$. We also regard this as an $n \times r$-matrix $V = (v_{ij})$. We call V a *funtf* if

$$V \cdot V^T = \frac{r}{n} \cdot \mathrm{Id}_n \qquad \text{and} \qquad \sum_{j=1}^{n} v_{ij}^2 = 1 \quad \text{for } i = 1, 2, \ldots, r. \tag{4}$$

This is an inhomogeneous system of $n^2 + r$ quadratic equations in nr unknowns. The *funtf variety*, denoted $\mathcal{F}_{r,n}$ as in [6], is the subvariety of complex affine space $\mathbb{C}^{n \times r}$ defined by (4). For the state of the art we refer to the article [6] by Cahill, Mixon and Strawn, and the references therein. A detailed review, with some new perspectives, will be given in Section 2.

We homogenize the funtf variety by attaching a scalar λ_i to each vector $\mathbf{v}_i$. The result maps into the projective space $\mathbb{P}(\mathrm{Sym}_d(\mathbb{C}^n)) = \mathbb{P}^{\binom{n-1+d}{d}-1}$ of symmetric tensors, via the formulas (1) and (3). Our aim is to study the closure of the image of that map. This is denoted $\mathcal{T}_{r,n,d}$. We call it the *variety of* **fra***me* **deco***mposable tensors*, or the *fradeco variety*. Here r, n, d are positive integers with $r \geq n$. For $r = n$, $\mathcal{T}_{n,n,d}$ is the odeco variety of [21].

Example 1.1. Let $n = 3, d = 4$, and consider the symmetric $3{\times}3{\times}3{\times}3$-tensor

$$\begin{aligned} T \;\; = \;\; & 59(x_1^4 + x_2^4 + x_3^4) \,-\, 16(x_1^3x_2 + x_1x_2^3 + x_1^3x_3 + x_2^3x_3 + x_1x_3^3 + x_2x_3^3) \\ & + 66(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) \,+\, 96(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2). \end{aligned} \tag{5}$$

This ternary quartic lies in $\mathcal{T}_{4,3,4}$, i.e. this tensor has *fradeco rank* $r = 4$. To see this, note that

$$T \;=\; \frac{1}{12}(-5x_1{+}x_2{+}x_3)^4 + \frac{1}{12}(x_1{-}5x_2{+}x_3)^4 + \frac{1}{12}(x_1{+}x_2{-}5x_3)^4 + \frac{1}{12}(3x_1{+}3x_2{+}3x_3)^4. \tag{6}$$

The corresponding four vectors, appropriately scaled, form a finite unit norm tight frame:

$$V \quad = \quad \frac{1}{3\sqrt{3}} \begin{pmatrix} -5 & 1 & 1 & 3 \\ 1 & -5 & 1 & 3 \\ 1 & 1 & -5 & 3 \end{pmatrix} \;\in\; \mathcal{F}_{4,3}. \tag{7}$$

The fradeco variety $\mathcal{T}_{4,3,4}$ is a projective variety of dimension 6 and degree 74 in $\mathbb{P}^{14}$. It is parametrized by applying rotation matrices $\rho \in \mathrm{SO}_3$ to all ternary quartics of the form

$$T \;=\; \lambda_1(-5x_1{+}x_2{+}x_3)^4 + \lambda_2(x_1{-}5x_2{+}x_3)^4 + \lambda_3(x_1{+}x_2{-}5x_3)^4 + \lambda_4(3x_1{+}3x_2{+}3x_3)^4. \tag{8}$$

The title of our paper refers to the task of finding the output (6) from the input (5). In this particular case, the decomposition can be found easily using Sylvester's classical *Catalecticant Algorithm*, as explained in [20, Section 2.2]. In general, this will be more difficult to do. $\diamondsuit$

The *fradeco rank* of a symmetric tensor $T \in \mathrm{Sym}_d(\mathbb{R}^n)$ is defined as the smallest r such that $T \in \mathcal{T}_{r,n,d}$. This property does not imply that T also has a frame decomposition (1) of length $r+1$. Indeed, we often have $\mathcal{T}_{r,n,d} \not\subset \mathcal{T}_{r+1,n,d}$. For instance, the odeco quartic $x_1^4 + x_2^4 + x_3^4$ lies in $\mathcal{T}_{3,3,4} \backslash \mathcal{T}_{4,3,4}$, by the constraint in Example 4.5. See also Example 3.5.

This paper is organized as follows. In Section 2 we give an introduction to the algebraic geometry of the funtf variety $\mathcal{F}_{r,n}$. This lays the foundation for the subsequent study of fradeco tensors. Section 3 is concerned with the case of symmetric $2{\times}2{\times}\cdots{\times}2$-tensors T. These correspond to binary forms ($n = 2$). We characterize frame decomposable tensors in terms of rank conditions on matrices. In Section 4 we investigate the general case $n \geq 3$, and we present what we know about the fradeco varieties $\mathcal{T}_{r,n,d}$. Section 5 is devoted to numerical algorithms for studying $\mathcal{T}_{r,n,d}$ and for decomposing its elements into frames.

2 Finite unit norm tight frames

In this section we discuss various representations of the funtf variety $\mathcal{F}_{r,n}$. This may serve as an invitation to the emerging interaction between algebraic geometry and frame theory.

Each variety studied in this paper is defined over the real field $\mathbb{R}$ and is the Zariski closure of its set of real points. This Zariski closure lives in affine or projective space over $\mathbb{C}$. For instance, SO_n is the group of $n{\times}n$ rotation matrices ρ, and such matrices have entries in $\mathbb{R}$. However, when referring to SO_n as an algebraic variety we mean the irreducible subvariety of $\mathbb{C}^{n\times n}$ defined by the polynomial equations $\rho \cdot \rho^T = \mathrm{Id}_n$ and $\det(\rho) = 1$. Likewise, a funtf V is a real $n \times r$ matrix, but the funtf variety $\mathcal{F}_{r,n}$ lives in $\mathbb{C}^{n\times r}$. It consists of all complex solutions to the quadratic equations (4). In the frame theory literature [5, 6, 12, 26] there is also a complex Hermitian version of $\mathcal{F}_{r,n}$, but it will not be considered in this paper.

It is important to distinguish $\mathcal{F}_{r,n}$ from the variety of *Parseval frames*, here denoted $\mathcal{P}_{r,n}$. The latter is much easier than the former. The variety $\mathcal{P}_{r,n}$ is defined by the matrix equation

$$V \cdot V^T = \mathrm{Id}_n.$$

The real points on $\mathcal{P}_{r,n}$ are smooth and Zariski dense, and they form the Stiefel manifold of all orthogonal projections $\mathbb{R}^r \to \mathbb{R}^n$. Hence $\mathcal{P}_{r,n}$ is irreducible of dimension $nr - \binom{n+1}{2}$.

One feature that distinguishes $\mathcal{P}_{r,n}$ from $\mathcal{F}_{r,n}$ is the existence of a canonical map $\mathcal{P}_{r,n+1} \to \mathcal{P}_{r,n}$. Indeed, by Naimark's Theorem [8], every Parseval frame is the orthogonal projection of an orthonormal basis of $\mathbb{R}^r$, so we can add a row to $V \in \mathcal{P}_{r,n}$ and get a matrix in $\mathcal{P}_{r,n+1}$. There is no analogous statement for the variety $\mathcal{F}_{r,n}$. We begin with the following result.

Theorem 2.1. *The dimension of the funtf variety $\mathcal{F}_{r,n}$ is*

$$\dim(\mathcal{F}_{r,n}) \;=\; (n-1)\cdot\left(r - \frac{n}{2} - 1\right) \qquad \textit{provided } r > n \geq 2. \tag{9}$$

It is irreducible when $r \geq n+2 > 4$.

Proof. Cahill, Mixon and Strawn [6, Theorem 1.4] proved that $\mathcal{F}_{r,n}$ is irreducible when $r \geq n+2 > 4$. The dimension formula comes from two articles: one by Dykema and Strawn

r	n	$\dim \mathcal{F}_{r,n}$	$\deg \mathcal{F}_{r,n}$	# components & degrees
3	2	1	$8 \cdot 2$	8 components, each degree 2
4	2	2	$12 \cdot 4$	12 components, each degree 4
5	2	3	112	irreducible
6	2	4	240	irreducible
7	2	5	496	irreducible
4	3	3	$16 \cdot 8$	16 components, each degree 8
5	3	5	1024	irreducible
6	3	7	2048	irreducible
7	3	9	4096	irreducible
5	4	6	$32 \cdot 40$	32 components, each degree 40
6	4	9	20800	irreducible
7	4	12	65536	irreducible

Table 1: Dimension and degree of the funtf variety in some small cases

[12, Theorem 4.3(ii)] regarding the case when r and n are relatively prime, and one by Strawn [26, Corollary 3.5] which studies the local geometry for all r, n. In these articles it is shown that the real points in $\mathcal{F}_{r,n}$ have a dense open subset that forms a manifold of dimension $(n-1) \cdot (r - \frac{n}{2} - 1)$. The arguments in [6] show that the real points are Zariski dense in the complex variety $\mathcal{F}_{r,n}$. Hence (9) is the correct formula for the dimension of $\mathcal{F}_{r,n}$. $\square$

Next to the dimension, the most important invariant of an algebraic variety is its *degree*. By this we mean the degree of its projective closure [11, §8.4]. This can be computed using symbolic software for Gröbner bases, or using numerical algebraic geometry software. The dimension and degree of $\mathcal{F}_{r,n}$ for small r, n in Table 1 were computed using `Bertini` [2].

The case $r = n + 1$ is special. Here, the funtf variety decomposes into 2^{n+1} irreducible components, each of which is affinely isomorphic to the $\binom{n}{2}$-dimensional variety SO_n. This will be explained in Corollary 2.10. The next example discusses one other exceptional case.

Example 2.2 ($r = 4, n = 2$)**.** Following (4), the defining ideal of the funtf variety $\mathcal{F}_{4,2}$ equals

$$\langle\, v_{11}^2 + v_{12}^2 + v_{13}^2 + v_{14}^2 - 2,\ v_{11}v_{21} + v_{12}v_{22} + v_{13}v_{23} + v_{14}v_{24} \rangle \ + \ \langle\, v_{i1}^2 + v_{i2}^2 - 1 \ : \ i = 1, 2, 3, 4 \,\rangle.$$

Note that this contains $v_{21}^2 + v_{22}^2 + v_{23}^2 + v_{24}^2 - 2$. Using Gröbner basis software, such as `Macaulay2` [13], one checks that this ideal is the complete intersection of the six given quadrics, it is radical, and its degree is 48. Primary decomposition reveals that this ideal is the intersection of 12 prime ideals, each of degree 4. One of these associated primes is

$$\langle\, v_{11} - v_{22}, v_{12} + v_{21},\ v_{31} - v_{42}, v_{32} + v_{41} \,\rangle \ + \ \langle\, v_{i1}^2 + v_{i2}^2 - 1 \ : \ i = 1, 2, 3, 4 \,\rangle.$$

The irreducible variety of this particular prime ideal consists of the 2×4-matrices

$$V \quad = \quad \big(R_1 \mid R_2 \big)\,,$$

where R_1 and R_2 are rotation matrices of format 2×2. The other 11 components are obtained by replacing R_i by $-R_i$ and permuting columns. The image of V under the map to binary forms is a linear combination of two odeco forms, one given by R_1 and the other by R_2. $\diamondsuit$

The real points of $\mathcal{F}_{r,n}$ live in $(\mathbb{S}^{n-1})^r$ where $\mathbb{S}^{n-1} = \{u \in \mathbb{R}^n : \sum_{i=1}^n u_i^2 = 1\}$ denotes the unit sphere. However, the vectors on these spheres will get scaled by the multipliers $\lambda_i^{1/d}$ in (3) when we pass to the fradeco variety $\mathcal{T}_{r,n,d}$. To achieve better geometric properties and computational speed, we map each real sphere $\mathbb{S}^{n-1}$ to complex projective $(n-1)$-space $\mathbb{P}^{n-1}$.

The *projective funtf variety* $\mathcal{G}_{r,n}$ is the image of $\mathcal{F}_{r,n}$ in $(\mathbb{P}^{n-1})^r$. To describe its equations, we use an $n \times r$-matrix $V = (v_{ij})$ of unknowns as before, but now the i-th column of V represents coordinates on the i-th factor of $(\mathbb{P}^{n-1})^r$. We introduce the $r \times r$ diagonal matrix

$$D \quad = \quad \mathrm{diag}\left(\sum_{i=1}^n v_{i1}^2\,,\, \sum_{i=1}^n v_{i2}^2\,,\, \ldots,\, \sum_{i=1}^n v_{ir}^2\right). \tag{10}$$

The variety $\mathcal{G}_{r,n}$ is defined by the following matrix equation:

$$V \cdot D^{-1} \cdot V^T \;=\; \frac{r}{n} \cdot \mathrm{Id}_n. \tag{11}$$

Each entry on the left hand side is a homogeneous rational function of degree 0. In fact, these functions are multihomogeneous: they define rational functions on $(\mathbb{P}^{n-1})^r$.

The challenge is to clear denominators in (11), so as to obtain a system of polynomial equations that defines $\mathcal{G}_{r,n}$ as a subvariety of $(\mathbb{P}^{n-1})^r$. Next we solve this problem for $n = 2$.

For planar frames, equation (11) translates into the vanishing of the two rational functions

$$P \;=\; \sum_{j=1}^r \frac{2v_{1j}^2}{v_{1j}^2 + v_{2j}^2} \,-\, r \qquad \text{and} \qquad Q \;=\; \sum_{j=1}^r \frac{2v_{1j}v_{2j}}{v_{1j}^2 + v_{2j}^2}. \tag{12}$$

Consider the numerator of the rational function

$$P - iQ \quad = \quad \sum_{j=1}^r \frac{v_{1j}^2 - 2iv_{1j}v_{2j} - v_{2j}^2}{v_{1j}^2 + v_{2j}^2} \quad = \quad \sum_{j=1}^r \frac{v_{1j} - v_{2j}i}{v_{1j} + v_{2j}i}, \qquad \text{where } \; i = \sqrt{-1}.$$

Let $\widetilde{P}$ and $\widetilde{Q}$ denote the real part and the imaginary part of that numerator. These are two multilinear polynomials of degree r with integer coefficients in $v_{11}, v_{12}, \ldots, v_{2r}$. They define a complete intersection, and, by construction, this is precisely our funtf variety in $(\mathbb{P}^1)^r$:

Lemma 2.3. *The projective funtf variety $\mathcal{G}_{r,2}$ is a complete intersection of codimension 2 in $(\mathbb{P}^1)^r$, namely, it is the zero set of the two multilinear forms $\widetilde{P}$ and $\widetilde{Q}$.*

Here are explicit formulas for the multilinear forms that define $\mathcal{G}_{r,2}$ when $r \leq 5$:

Example 2.4. If $r = 3$, then $\widetilde{P} = 3v_{11}v_{12}v_{13} + v_{11}v_{22}v_{23} + v_{21}v_{12}v_{23} + v_{21}v_{22}v_{13}$ and $\widetilde{Q} = v_{11}v_{12}v_{23} + v_{11}v_{22}v_{13} + v_{21}v_{12}v_{13} + 3v_{21}v_{22}v_{23}$. If $r{=}4$, then $\widetilde{P} = 4(v_{11}v_{12}v_{13}v_{14} - v_{21}v_{22}v_{23}v_{24})$ and

$$\begin{aligned}\widetilde{Q} \quad = \quad & 2v_{11}v_{12}v_{13}v_{24} + 2v_{11}v_{12}v_{23}v_{14} + 2v_{11}v_{22}v_{13}v_{14} + 2v_{11}v_{22}v_{23}v_{24} + \\ & 2v_{21}v_{12}v_{13}v_{14} + 2v_{21}v_{12}v_{23}v_{24} + 2v_{21}v_{22}v_{13}v_{24} + 2v_{21}v_{22}v_{23}v_{14}.\end{aligned}$$

If $r = 5$, then

$$
\begin{aligned}
\widetilde{P} \;=\; & \;\; 5v_{11}v_{12}v_{13}v_{14}v_{15} - v_{11}v_{12}v_{13}v_{24}v_{25} - v_{11}v_{12}v_{23}v_{14}v_{25} - v_{11}v_{12}v_{23}v_{24}v_{15} \\
& -v_{11}v_{22}v_{13}v_{14}v_{25} - v_{11}v_{22}v_{13}v_{24}v_{15} - v_{11}v_{22}v_{23}v_{14}v_{15} - 3v_{11}v_{22}v_{23}v_{24}v_{25} \\
& -v_{21}v_{12}v_{13}v_{14}v_{25} - v_{21}v_{12}v_{13}v_{24}v_{15} - v_{21}v_{12}v_{23}v_{14}v_{15} - 3v_{21}v_{12}v_{23}v_{24}v_{25} \\
& -v_{21}v_{22}v_{13}v_{14}v_{15} - 3v_{21}v_{22}v_{13}v_{24}v_{25} - 3v_{21}v_{22}v_{23}v_{14}v_{25} - 3v_{21}v_{22}v_{23}v_{24}v_{15},
\end{aligned}
$$

and $\widetilde{Q}$ is obtained from $\widetilde{P}$ by switching the two rows of V. $\diamondsuit$

Such formulas are useful for parametrizing frames. We write the equations for $\mathcal{G}_{r,2}$ as

$$
\begin{pmatrix} \widetilde{P} \\ \widetilde{Q} \end{pmatrix} \;=\; \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \cdot \begin{pmatrix} v_{1r} \\ v_{2r} \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

The matrix entries m_{ij} are multilinear forms in $(v_{11} : v_{21})$, $(v_{12} : v_{22})$, $\ldots$, $(v_{1,r-1} : v_{2,r-1})$. Using the quadratic formula, we solve the following equation for one of its unknowns:

$$
m_{11}m_{22} \;=\; m_{12}m_{21}. \tag{13}
$$

This defines a hypersurface in $(\mathbb{P}^1)^{r-1}$, from which we can now easily sample points. The point in the remaining rth factor $\mathbb{P}^1$ is then recovered by setting $v_{1r} = m_{12},\; v_{2r} = -m_{11}$.

For $n \geq 3$, we do not know the generators of the multihomogeneous prime ideal of $\mathcal{G}_{r,n}$. Here are two instances where `Macaulay2` [13] succeeded in computing these ideals:

Example 2.5. The variety $\mathcal{G}_{4,3}$ is a threefold in $(\mathbb{P}^2)^4$. Its ideal is generated by 34 quartics. Among them are the equations that define the six coordinate projections into $(\mathbb{P}^2)^2$, like

$$
\begin{aligned}
8(v_{11}^2v_{12}^2 + v_{21}^2v_{22}^2 + v_{31}^2v_{32}^2) + 18(v_{11}v_{21}v_{12}v_{22} + v_{11}v_{31}v_{12}v_{32} + v_{21}v_{31}v_{22}v_{32}) \\
-v_{11}^2v_{22}^2 - v_{11}^2v_{32}^2 - v_{21}^2v_{12}^2 - v_{21}^2v_{32}^2 - v_{31}^2v_{12}^2 - v_{31}^2v_{22}^2.
\end{aligned}
$$

Example 2.6. Let $r = 5$ and $n = 3$. By saturating the denominators in (11), we found that the ideal of $\mathcal{G}_{5,3}$ is generated by a 120-dimensional SO_3-invariant space of sextics. The following polynomial (with 60 terms of $\mathbb{Z}^5$-degree $(2,2,2,0,0)$) is a highest weight vector:

$$
\begin{gathered}
50v_{11}^2v_{12}^2v_{13}^2 + 5v_{11}^2v_{12}^2v_{23}^2 + 5v_{11}^2v_{12}^2v_{33}^2 + 45v_{11}^2v_{12}v_{22}v_{13}v_{23} + 45v_{11}^2v_{12}v_{32}v_{13}v_{33} + 5v_{11}^2v_{22}^2v_{13}^2 + 5v_{11}^2v_{22}^2v_{23}^2 - 4v_{11}^2v_{22}^2v_{33}^2 \\
+18v_{11}^2v_{22}v_{32}v_{23}v_{33} + 5v_{11}^2v_{32}^2v_{13}^2 - 4v_{11}^2v_{32}^2v_{23}^2 + 5v_{11}^2v_{32}^2v_{33}^2 + 45v_{11}v_{21}v_{12}^2v_{13}v_{23} + 45v_{11}v_{21}v_{12}v_{22}v_{13}^2 + 18v_{11}v_{21}v_{32}^2v_{13}v_{23} \\
+45v_{11}v_{21}v_{12}v_{22}v_{23}^2 + 18v_{11}v_{21}v_{12}v_{22}v_{33}^2 + 27v_{11}v_{21}v_{12}v_{32}v_{23}v_{33} + 45v_{11}v_{21}v_{22}^2v_{13}v_{23} + 27v_{11}v_{21}v_{22}v_{32}v_{13}v_{33} \\
+45v_{11}v_{31}v_{12}^2v_{13}v_{33} + 27v_{11}v_{31}v_{12}v_{22}v_{23}v_{33} + 45v_{11}v_{31}v_{12}v_{32}v_{13}^2 + 18v_{11}v_{31}v_{12}v_{32}v_{23}^2 + 45v_{11}v_{31}v_{12}v_{32}v_{33}^2 - 4v_{21}^2v_{12}^2v_{33}^2 \\
+18v_{11}v_{31}v_{22}^2v_{13}v_{33} + 27v_{11}v_{31}v_{22}v_{32}v_{13}v_{23} + 45v_{11}v_{31}v_{32}^2v_{13}v_{33} + 5v_{21}^2v_{12}^2v_{13}^2 + 5v_{21}^2v_{12}^2v_{23}^2 + 45v_{21}^2v_{12}v_{22}v_{13}v_{23} \\
+18v_{21}^2v_{12}v_{32}v_{13}v_{33} + 5v_{21}^2v_{22}^2v_{13}^2 + 50v_{21}^2v_{22}^2v_{23}^2 + 5v_{21}^2v_{22}^2v_{33}^2 + 45v_{21}^2v_{22}v_{32}v_{23}v_{33} - 4v_{21}^2v_{32}^2v_{13}^2 + 5v_{21}^2v_{32}^2v_{23}^2 + 5v_{21}^2v_{32}^2v_{33}^2 \\
+18v_{21}v_{31}v_{12}^2v_{23}v_{33} + 27v_{21}v_{31}v_{12}v_{22}v_{13}v_{33} + 27v_{21}v_{31}v_{12}v_{32}v_{13}v_{23} + 45v_{21}v_{31}v_{22}^2v_{23}v_{33} + 18v_{21}v_{31}v_{22}v_{32}v_{13}^2 \\
+45v_{21}v_{31}v_{22}v_{32}v_{23}^2 + 45v_{21}v_{31}v_{22}v_{32}v_{33}^2 + 45v_{21}v_{31}v_{32}^2v_{23}v_{33} + 5v_{31}^2v_{12}^2v_{13}^2 - 4v_{31}^2v_{12}^2v_{23}^2 + 5v_{31}^2v_{12}^2v_{33}^2 + 18v_{31}^2v_{12}v_{22}v_{13}v_{23} \\
+45v_{31}^2v_{12}v_{32}v_{13}v_{33} - 4v_{31}^2v_{22}^2v_{13}^2 + 5v_{31}^2v_{22}^2v_{23}^2 + 5v_{31}^2v_{22}^2v_{33}^2 + 45v_{31}^2v_{22}v_{32}v_{23}v_{33} + 5v_{31}^2v_{32}^2v_{13}^2 + 5v_{31}^2v_{32}^2v_{23}^2 + 50v_{31}^2v_{32}^2v_{33}^2.
\end{gathered}
$$

The ideal of $\mathcal{G}_{5,3} \subset (\mathbb{P}^2)^5$ has 10 generators like this, each spanning a one-dimensional graded component. It has 30 components of degrees like $(2,2,1,1,0)$, each generated by a polynomial with 78 terms. Finally, it has five 16-dimensional components of degrees like $(2,1,1,1,1)$. $\diamondsuit$

In order to sample points from the funtf variety $\mathcal{F}_{r,n}$, we can also use the following parametrization found in [5, 26]. We write $V = (U', W)$, where U' is an $n \times n$-matrix and W is an $(r-n) \times n$-matrix. For the columns of W we take arbitrary points on the unit sphere $\mathbb{S}^{n-1}$. In practice, it is convenient to fix a rational parametrization of $\mathbb{S}^{n-1}$, so as to ensure that W has rational entries w_{ij}. For instance, for $n = 3$ we might use the following formulas:

$$w_{1j} = \frac{2\lambda_j \mu_j}{\lambda_j^2 + \mu_j^2 + \nu_j^2}, \; w_{2j} = \frac{2\lambda_j \nu_j}{\lambda_j^2 + \mu_j^2 + \nu_j^2}, \; w_{3j} = \frac{\lambda_j^2 - \mu_j^2 - \nu_j^2}{\lambda_j^2 + \mu_j^2 + \nu_j^2}, \quad \text{where } \lambda_j, \mu_j, \nu_j \in \mathbb{Z}. \tag{14}$$

After these choices have been made, we fix the following $n \times n$-matrix with entries in $\mathbb{Q}$:

$$S \;=\; \frac{r}{n} \cdot \mathrm{Id}_n \;-\; W \cdot W^T. \tag{15}$$

It now remains to study all $n \times n$-matrices $U = (u_{ij})$ that satisfy

$$U \cdot D^{-1} \cdot U^T \;=\; S, \text{ where } \; D \;=\; \mathrm{diag}\Big(\sum_{i=1}^{n} u_{i1}^2 \,, \, \ldots, \, \sum_{i=1}^{n} u_{in}^2 \Big).$$

For any such U we get a funtf $V = (U', W) \in \mathcal{F}_{r,n}$ by setting $U' = U \cdot D^{-1/2}$. For random choices in (14), the matrix S is invertible, and the previous equation is equivalent to

$$D \;=\; U^T \cdot S^{-1} \cdot U. \tag{16}$$

This identity of symmetric matrices defines $\binom{n+1}{2}$ equations in the entries u_{ij} of U. The equation in position (i,j) is bilinear in $(u_{1i}, u_{2i}, \ldots, u_{ni})$ and $(u_{1j}, u_{2j}, \ldots, u_{nj})$. We solve the system (16) iteratively for the columns of U. We begin with the (1,1) entry of (16). There are $n-1$ degrees of freedom to fill in the first column of U, then $n-2$ degrees of freedom to fill in the second column, etc. This involves repeatedly solving quadratic equations in one variable, so each solution lives in a tower of quadratic extensions over $\mathbb{Q}$. In summary:

Proposition 2.7. *The equations (14), (15), (16) represent a parametrization of $\mathcal{F}_{r,n}$.*

The rotation group SO_n acts by left multiplication on the funtf variety $\mathcal{F}_{r,n}$. There is a natural way to construct the quotient $\mathcal{F}_{r,n}/\mathrm{SO}_n$ as an algebraic variety, namely by mapping it into the Grassmannian $\mathrm{Gr}(n,r)$ of n-dimensional subspaces of $\mathbb{C}^r$. This is described by Cahill and Strawn in [5, Section 3.1], and we briefly develop some basic algebraic properties.

We here define $\mathrm{Gr}(n,r)$ to be the image of the *Plücker map* $\mathbb{C}^{n\times r} \to \mathbb{C}^{\binom{r}{n}}$ that takes an $n \times r$-matrix V to its vector $p = p(V)$ of $n \times n$-minors. The coordinates p_I of p are indexed by the set $\binom{[r]}{n}$ of n-element subsets of $[r] = \{1, 2, \ldots, r\}$. With this definition, $\mathrm{Gr}(n,r)$ is the affine subvariety of $\mathbb{C}^{\binom{r}{n}}$ defined by the *quadratic Plücker relations*, such as $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$ for $n = 2, r = 4$. The dimension of $\mathrm{Gr}(n,r)$ is $(r-n)n+1$. Note that if $VV^T = (r/n) \cdot \mathrm{Id}_n$, then the *Cauchy-Binet formula* (cf. [5, Prop. 6]) implies

$$\sum_{I \in \binom{[r]}{n}} p_I^2 \;=\; \Big(\frac{r}{n}\Big)^n. \tag{17}$$

The real points in $\mathrm{Gr}(n,r)$, up to scaling, correspond to n-dimensional subspaces of $\mathbb{R}^r$.

Proposition 2.8. *The image of $\mathcal{F}_{r,n}$ under the Plücker map is an affine variety of dimension $(r-n)n - r + 2$ in the Grassmannian $\mathrm{Gr}(n,r) \subset \mathbb{C}^{\binom{r}{n}}$. It is defined by the equations*

$$\sum_{I: i\in I} p_I^2 \;=\; \left(\frac{r}{n}\right)^{n-1} \qquad \textit{for } i = 1,2,\ldots,r. \tag{18}$$

The real points in this image correspond to SO_n-orbits of n-dimensional frames in $\mathcal{F}_{r,n}$.

Note that adding up the r relations in (18) and dividing by n gives precisely (17).

Proof. Both $\mathcal{F}_{r,n}$ and the constraints (18) are invariant under SO_n. Suppose that $V \in \mathbb{C}^{n\times r}$ satisfies $VV^T = (r/n)\cdot \mathrm{Id}_n$. We may assume (modulo SO_n) that the i-th column of V is $(\alpha, 0, \ldots, 0)^T$ for some $\alpha \in \mathbb{C}$. Let $\tilde{V}$ be the matrix obtained from V by deleting the first row and i-th column. Then $\tilde{V}\cdot\tilde{V}^T = (r/n)\cdot \mathrm{Id}_{n-1}$. Any p_I with $i \in I$ equals α times the maximal minor of $\tilde{V}$ indexed by $I\backslash\{i\}$. Applying (17) to $\tilde{V}$, this gives

$$\sum_{I: i\in I} p_I^2 \;=\; \alpha^2\cdot\left(\frac{r}{n}\right)^{n-1}.$$

Hence (18) holds if and only if $\alpha = \pm 1$, and this holds for all i if and only if V lies in $\mathcal{F}_{r,n}$. The dimension formula follows from Theorem 2.1 because SO_n acts faithfully on $\mathcal{F}_{r,n}$. □

Example 2.9. Let $n = 2$. If $r = 5$, then our construction realizes $\mathcal{F}_{5,2}/\mathrm{SO}_2$ as an irreducible surface of degree 80 in $\mathbb{C}^{10}$. Its prime ideal is generated by the ten quadratic polynomials

$$p_{14}p_{23} - p_{13}p_{24} + p_{12}p_{34},\; p_{15}p_{23} - p_{13}p_{25} + p_{12}p_{35},\; p_{15}p_{24} - p_{14}p_{25} + p_{12}p_{45},\; p_{15}p_{34} - p_{14}p_{35} \\ +p_{13}p_{45},\; p_{25}p_{34} - p_{24}p_{35} + p_{23}p_{45},\; p_{12}^2 + p_{13}^2 + p_{14}^2 + p_{15}^2 - 5/2,\; p_{12}^2 + p_{23}^2 + p_{24}^2 + p_{25}^2 - 5/2, \\ p_{13}^2 + p_{23}^2 + p_{34}^2 + p_{35}^2 - 5/2,\; p_{14}^2 + p_{24}^2 + p_{34}^2 + p_{45}^2 - 5/2,\; p_{15}^2 + p_{25}^2 + p_{35}^2 + p_{45}^2 - 5/2.$$

If $r = 4$, then $\mathcal{F}_{4,2}/\mathrm{SO}_2$ is a reducible curve of degree 24 in $\mathbb{C}^6$. Its defining equations are

$$p_{14}p_{23} - p_{13}p_{24} + p_{12}p_{34} = 0,\;\; p_{12}^2 + p_{13}^2 + p_{14}^2 = p_{12}^2 + p_{23}^2 + p_{24}^2 = p_{13}^2 + p_{23}^2 + p_{34}^2 = p_{14}^2 + p_{24}^2 + p_{34}^2 = 2.$$

As in Example 2.2, this curve breaks into 12 components. One of these 12 irreducible curves is $\{\, p \in \mathbb{C}^6 \,:\, p_{12} = p_{34} = 1,\; p_{13} = p_{24},\; p_{14} = -p_{23},\; p_{23}^2 + p_{24}^2 = 1 \,\}$. ♢

The analogous decomposition is found easily for the case $r = n+1$. Here, there are no Plücker relations, so $\mathrm{Gr}(n, n+1) \simeq \mathbb{S}^n$. For convenience of notation, we set $q_i = p_{[n+1]\backslash\{i\}}$ in (18). The quotient space $\mathcal{F}_{n+1,n}/\mathrm{SO}_n$ is the subvariety of $\mathbb{C}^{n+1}$ defined by the equations

$$q_1^2 + q_2^2 + \cdots + q_n^2 + q_{n+1}^2 \;=\; (n+1)^{n-1}/n^n + \; q_i^2 \qquad \text{for } i = 1,2,\ldots,n+1.$$

These are equivalent to the following equations, which imply Corollary 2.10:

$$q_1^2 \;=\; q_2^2 \;=\; q_3^2 \;=\; \cdots \;=\; q_{n+1}^2 \;=\; (n+1)^{n-1}/n^{n+1}.$$

Corollary 2.10. *The quotient space $\mathcal{F}_{n+1,n}/\mathrm{SO}_n$ is a variety consisting of 2^{n+1} isolated points in $\mathbb{R}^{n+1} = \mathrm{Gr}(n, n+1)$, namely those points with coordinates $\pm(n+1)^{(n-1)/2}/n^{(n+1)/2}$.*

Any of the 2^{n+1} components of $\mathcal{F}_{n+1,n}$ can be used to parametrize our variety $\mathcal{T}_{n+1,n,d}$.

Example 2.11. Let $n = 3$. The point $p = \sqrt{3}(\frac{4}{9}, \frac{4}{9}, \frac{4}{9}, \frac{4}{9})$ in $\mathrm{Gr}(3,4)$ corresponds to the SO_3-orbit of the frame V in Example 1.1. The variety $\mathcal{G}_{4,3}$ can be parametrized as follows:

$$V = (v_{ij}) \;=\; \begin{bmatrix} 1-2y^2-2z^2 & 2xy-2zw & 2xz+2yw \\ 2xy+2zw & 1-2x^2-2z^2 & 2yz-2xw \\ 2xz-2yw & 2yz+2xw & 1-2x^2-2y^2 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \\ 3 & -5 & 1 & 1 \end{bmatrix} \begin{bmatrix} \nu_1 & 0 & 0 & 0 \\ 0 & \nu_2 & 0 & 0 \\ 0 & 0 & \nu_3 & 0 \\ 0 & 0 & 0 & \nu_4 \end{bmatrix}.$$

The 3×3-matrix on the left is the familiar parametrization of SO_3 via unit quaternions. This gives the parametrization of the fradeco variety $\mathcal{T}_{4,3,d}$ seen in (8). ♢

The embedding of $\mathcal{F}_{r,n}/\mathrm{SO}_n$ into $\mathrm{Gr}(r,n)$ via (18) connects frame theory with *matroid theory*. The matroid of V is given by the set of Plücker coordinates p_I that are zero. If all Plücker coordinates are nonzero, then the matroid is uniform. It is a natural to ask which matroids are realizable over $\mathbb{R}$ when the additional constraints (18) are imposed.

The discussion in [5, Section 3.2] relates frame theory to the study of *orbitopes* [22]. Cahill and Strawn set up an optimization problem for computing Parseval frames that are most uniform. Their formulation in [5, p. 24] is a linear program over the *Grassmann orbitope*, which is the convex hull of $\mathrm{Gr}(n,r)$ intersected with (17). The same optimization problem makes sense with $\mathrm{Gr}(n,r)$ replaced by $\mathcal{F}_{r,n}/\mathrm{SO}_n$, or, algebraically, with (17) replaced by (18). If $n = 2$, then the former problem is a *semidefinite program*. This is the content of [22, Theorem 7.3]. For $n \geq 3$, the situation is more complicated, but the considerable body of results coming from calibrated manifolds, such as [22, Theorem 7.5], should still be helpful.

3 Binary forms

We now commence our study of the fradeco variety $\mathcal{T}_{r,n,d}$. In this section we focus on the case $n = 2$ of binary forms that are decomposable into small frames. The case $r = 2$ is the odeco surface known from [21, §3]. Proposition 3.6 in [21] gives an explicit list of quadrics that forms a Gröbner basis for the prime ideal of $\mathcal{T}_{2,2,d}$, and these are here expressed as the 2×2-minors of a certain $3 \times (d-3)$-matrix $\mathcal{M}_4$. What follows is our main result in Section 3. We are using coordinates $(t_0 : \cdots : t_d)$ for the space $\mathbb{P}^d = \mathbb{P}(\mathrm{Sym}_d(\mathbb{C}^2))$ of binary forms. In the notation of (2), the coordinate t_i would be $t_{111\cdots1222\cdots2}$ with i indices 1 and $d-i$ indices 2.

Theorem 3.1. *Fix $r \in \{3, 4, \ldots, 9\}$ and $d \geq 2r-2$. There exists a matrix $\mathcal{M}_r$ such that:*

(a) Its maximal minors form a Gröbner basis for the prime ideal of $\mathcal{T}_{r,2,d}$.

(b) It has $r-1$ rows and $d-r+1$ columns, and the entries are linear forms in $t_0, \ldots, t_d$.

(c) Each column involves r of the unknowns t_i, and they are identical up to index shifts.

These matrices can be chosen as follows:

$$\mathcal{M}_3 \quad = \quad \begin{pmatrix} t_0 - 3t_2 & t_1 - 3t_3 & t_2 - 3t_4 & t_3 - 3t_5 & \cdots & t_{d-3} - 3t_{d-1} \\ 3t_1 - t_3 & 3t_2 - t_4 & 3t_3 - t_5 & 3t_4 - t_6 & \cdots & 3t_{d-2} - t_d \end{pmatrix} \tag{19}$$

$$\mathcal{M}_4 \quad = \quad \begin{pmatrix} t_0+t_4 & t_1+t_5 & t_2+t_6 & t_3+t_7 & \cdots & t_{d-4}+t_d \\ t_1-t_3 & t_2-t_4 & t_3-t_5 & t_4-t_6 & \cdots & t_{d-3}+t_{d-1} \\ t_2 & t_3 & t_4 & t_5 & \cdots & t_{d-2} \end{pmatrix} \tag{20}$$

$$\mathcal{M}_5 \quad = \quad \begin{pmatrix} t_0+5t_2 & t_1+5t_3 & t_2+5t_4 & t_3+5t_5 & \cdots & t_{d-5}+5t_{d-3} \\ t_1-3t_3 & t_2-3t_4 & t_3-3t_5 & t_4-3t_6 & \cdots & t_{d-4}-3t_{d-2} \\ 3t_2-t_4 & 3t_3-t_5 & 3t_4-t_6 & 3t_5-t_7 & \cdots & 3t_{d-3}-t_{d-1} \\ 5t_3+t_5 & 5t_4+t_6 & 5t_5+t_7 & 5t_6+t_8 & \cdots & 5t_{d-2}+t_d \end{pmatrix} \tag{21}$$

$$\mathcal{M}_6 \quad = \quad \begin{pmatrix} t_0+3t_2 & t_1+3t_3 & t_2+3t_4 & t_3+3t_5 & \cdots & t_{d-6}+3t_{d-4} \\ t_1+t_5 & t_2+t_6 & t_3+t_7 & t_4+t_8 & \cdots & t_{d-5}+t_{d-1} \\ t_2-t_4 & t_3-t_5 & t_4-t_6 & t_5-t_7 & \cdots & t_{d-4}-t_{d-2} \\ t_3 & t_4 & t_5 & t_6 & \cdots & t_{d-3} \\ 3t_4+t_6 & 3t_5+t_7 & 3t_6+t_8 & 3t_7+t_9 & \cdots & 3t_{d-2}+t_d \end{pmatrix} \tag{22}$$

The first column of $\mathcal{M}_7$ is $(3t_0+7t_2, t_1+5t_3, t_2-3t_4, 3t_3-t_5, 5t_4+t_6, 7t_5+3t_7)^T$, the first column of $\mathcal{M}_8$ is $(t_0+2t_2, t_1+3t_3,\ t_4,\ t_3-t_5, t_2+t_6, 3t_5+t_7, 2t_6+t_8)^T$, and the first column of $\mathcal{M}_9$ is $(5t_0+9t_2, 3t_1+7t_3, t_2+5t_4, t_3-3t_5, 3t_4-t_6, 5t_5+t_7, 7t_6+3t_8, 9t_7+5t_9)^T$.

We conjecture that the same result holds for all r, and we explain what we currently know after the proof. Let us begin with a lemma concerning the dimension of our variety.

Lemma 3.2. *The fradeco variety $\mathcal{T}_{r,2,d}$ is irreducible and has dimension* $\min(2r-3, d)$.

Proof. For $d \geq 5$, the funtf variety $\mathcal{F}_{r,2} \subset (\mathbb{S}^1)^r$ is irreducible, by Theorem 2.1, and hence so is its closure $\mathcal{G}_{r,2}$ in $(\mathbb{P}^1)^r$. While the two special varieties $\mathcal{F}_{3,2}$ and $\mathcal{F}_{4,2}$ are reducible, the analyses in Example 2.2 and Corollary 2.10 show that $\mathcal{G}_{3,2}$ and $\mathcal{G}_{4,2}$ are irreducible.

Regarding $\mathcal{G}_{r,2}$ as an affine variety in $\mathbb{C}^{2\times r}$, we obtain $\mathcal{T}_{r,2,d}$ as its image under the map

$$t_i \;=\; v_{11}^i v_{21}^{d-i} + v_{12}^i v_{22}^{d-i} + v_{13}^i v_{23}^{d-i} + \cdots + v_{1r}^i v_{2r}^{d-i} \qquad \text{for } i = 0, 1, \ldots, d. \tag{23}$$

This proves that $\mathcal{T}_{r,2,d}$ is irreducible. To see that it has the expected dimension, consider the r-th secant variety of the rational normal curve in $\mathbb{P}^d$, which is the image of the map $\mathbb{C}^{2\times r} \dashrightarrow \mathbb{P}^d$ given by (23). It is known that this secant variety has the expected dimension, namely $\min(2r-1, d)$, and the fiber dimension of the map (23) does not jump unless some 2×2-minor of $V = (v_{ij})$ is zero. Since $\operatorname{codim}(\mathcal{G}_{r,2}) = 2$, by Lemma 2.3, the claim follows. $\square$

Proof of Theorem 3.1. We first show that the maximal minors of our matrices $\mathcal{M}_r$ vanish on the fradeco variety $\mathcal{T}_{r,2,d}$ for $r = 3, 4, \ldots, 9$. After substituting the parametrization (23) for $t_0, t_1, \ldots, t_d$, we can decompose these matrices as follows:

$$\mathcal{M}_r \;=\; M_r \cdot \begin{pmatrix} v_{11}^{d-r} & v_{11}^{d-r-1}v_{21} & v_{11}^{d-r-2}v_{21}^2 & \cdots & v_{21}^{d-r} \\ v_{12}^{d-r} & v_{12}^{d-r-1}v_{22} & v_{12}^{d-r-2}v_{22}^2 & \cdots & v_{22}^{d-r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{1r}^{d-r} & v_{1r}^{d-r-1}v_{2r} & v_{1r}^{d-r-2}v_{2r}^2 & \cdots & v_{2r}^{d-r} \end{pmatrix},$$

where

$$M_3 \;=\; \begin{pmatrix} (v_{22}^2 - 3v_{11}^2)v_{21} & (v_{22}^2 - 3v_{12}^2)v_{22} & (v_{23}^2 - 3v_{13}^2)v_{23} \\ (3v_{21}^2 - v_{11}^2)v_{11} & (3v_{22}^2 - v_{12}^2)v_{12} & (3v_{23}^2 - v_{13}^2)v_{13} \end{pmatrix},$$

$$M_4 = \begin{pmatrix} v_{21}^4 + v_{11}^4 & v_{22}^4 + v_{12}^4 & v_{23}^4 + v_{13}^4 & v_{24}^4 + v_{14}^4 \\ v_{11}v_{21}^3 - v_{11}^3 v_{21} & v_{12}v_{22}^3 - v_{12}^3 v_{22} & v_{13}v_{23}^3 - v_{13}^3 v_{23} & v_{14}v_{24}^3 - v_{14}^3 v_{24} \\ v_{11}^2 v_{21}^2 & v_{12}^2 v_{22}^2 & v_{13}^2 v_{23}^2 & v_{14}^2 v_{24}^2 \end{pmatrix},$$

$$M_5 = \begin{pmatrix} v_{21}^5 + 5v_{11}^5 & v_{22}^5 + 5v_{12}^5 & v_{23}^5 + 5v_{13}^5 & v_{24}^5 + 5v_{14}^5 & v_{25}^5 + 5v_{15}^5 \\ v_{11}v_{21}^4 - 3v_{11}^3 v_{21}^2 & v_{12}v_{22}^4 - 3v_{12}^3 v_{22}^2 & v_{13}v_{23}^4 - 3v_{13}^3 v_{23}^2 & v_{14}v_{24}^4 - 3v_{14}^3 v_{24}^2 & v_{15}v_{25}^4 - 3v_{15}^3 v_{25}^2 \\ 3v_{11}^2 v_{21}^3 - v_{11}^4 v_{21} & 3v_{12}^2 v_{22}^3 - v_{12}^4 v_{22} & 3v_{13}^2 v_{23}^3 - v_{13}^4 v_{23} & 3v_{14}^2 v_{24}^3 - v_{14}^4 v_{24} & 3v_{15}^2 v_{25}^3 - v_{15}^4 v_{25} \\ 5v_{11}^3 v_{21}^2 + v_{11}^5 & 5v_{12}^3 v_{22}^2 + v_{12}^5 & 5v_{13}^3 v_{23}^2 + v_{13}^5 & 5v_{14}^3 v_{24}^2 + v_{14}^5 & 5v_{15}^3 v_{25}^2 + v_{15}^5 \end{pmatrix},$$

and similarly for M_6, M_7, M_8 and M_9. We claim that the matrices M_r have rank $< r-1$ whenever $V \in \mathcal{F}_{r,2}$. Equivalently, the $(r-1)\times(r-1)$ minors of M_r lie in the ideal of $\mathcal{G}_{r,2}$. It suffices to consider the leftmost such minor since all minors are equivalent under permuting the columns of V. For each $r \leq 9$, we check that the determinant of that minor factors as

$$(m_{11}m_{22} - m_{12}m_{21}) \cdot \prod_{1\leq i<j\leq r-1} (v_{1i}v_{2j} - v_{2i}v_{1j}), \tag{24}$$

where the left factor is the polynomial of degree $2r-2$ given in (13). That polynomial vanishes on $\mathcal{G}_{r,2}$. This implies $\mathrm{rank}(M_r) \leq r-2$ on $\mathcal{G}_{r,2}$, and hence $\mathrm{rank}(\mathcal{M}_r) \leq r-2$ on $\mathcal{T}_{r,2,d}$.

Fix the lexicographic term order on $\mathbb{C}[t_0, t_1, \ldots, t_d]$. We can check that, for each $r \in \{3, 4, ..., 9\}$, the leading monomial of the leftmost maximal minor of $\mathcal{M}_r$ equals $t_0 t_2 t_4 \cdots t_{r-2}$. Hence all $\binom{d-r+1}{r-1}$ maximal minors of $\mathcal{M}_r$ are squarefree, and they generate the ideal

$$I_{r,d} \;=\; \big\langle\, t_{i_1}t_{i_2}t_{i_3}\cdots t_{i_{r-1}} \;:\; 2 \leq i_1{+}2 \leq i_2,\; i_2{+}2 \leq i_3,\; i_3{+}2 \leq t_4,\; \ldots,\; i_{r-2}{+}2 \leq i_{r-1} \leq d{-}2 \,\big\rangle.$$

This squarefree monomial ideal is pure of codimension $d-2r+3$ and it has degree $\binom{d-r+1}{r-2}$. This follows from [19, Theorem 1.6]. Indeed, in Murai's theory, our ideal $I_{r,d}$ is obtained from the power of the maximal ideal by applying the stable operator given by $a = (2, 4, 6, \ldots)$.

Combinatorial analysis reveals that the ideal $I_{r,d}$ is the intersection of the prime ideals

$$\langle t_{j_0}, t_{j_1}, t_{j_2}, t_{j_3}, \ldots, t_{j_{d-2r+2}} \rangle,$$

where $j_0, j_2, j_4, \ldots$ are even, $j_1, j_3, j_5, \ldots$ are odd, and $0 \leq j_0 < j_1 < j_2 < \cdots < j_{d-2r+2} \leq d$. Note that number of such sequences is $\binom{d-r+1}{d-2r+3} = \binom{d-r+1}{r-2}$. Hence the codimension and degree of $I_{r,d}$ are as expected for the ideal of maximal minors of an $(r{-}1)\times(d{-}r{+}1)$-matrix with linear entries [14, Ex. 19.10]. The monomial ideal $I_{r,d}$ is Cohen-Macaulay because its corresponding simplicial complex is shellable (cf. [24, §III.2]). Indeed, if we list the associated primes in a dictionary order for all sequences $j_0 j_1 j_2 \cdots j_{d-2r+2}$ as above, then this gives a shelling order.

Using Buchberger's S-pair criterion, we check that the maximal minors of $\mathcal{M}_r$ form a Gröbner basis. We only need to consider pairs of minors whose leading terms share variables. Up to symmetry, there are only few such pairs, so this is an easy check for each fixed $r \leq 9$.

Since $I_{r,d}$ is radical of codimension $d-2r+3$, we conclude that the ideal of maximal minors of $\mathcal{M}_r$ is radical and has the same codimension. However, that ideal of minors is contained in the prime ideal of $\mathcal{T}_{r,2,d}$, which has codimension $d-2r+3$ by Lemma 3.2.

Therefore, we now know that $\mathcal{T}_{r,2,d}$ is one of the irreducible components of the variety of maximal minors of $\mathcal{M}_r$. To conclude the proof we need to show that the latter variety is irreducible, so they are equal. To see this, we fix r and we proceed by induction on d. For $d = 2r-2$, when $\mathcal{M}_r$ is a square matrix, this can be checked directly. To pass from d to $d+1$, we factor the matrix as M_r times the rank r Hankel matrix associated with a funtf V. Increasing the value of d to $d+1$ multiplies the i-th row of the Hankel matrix by v_{i1} and it adds one more column. This gives us the value for the new variable t_{d+1}. Now, since that variable occurs linearly in the maximal minors, its value is unique. This implies that the unique rank $r-2$ extension from the old to the new $\mathcal{M}_r$ must come from the funtf V. $\square$

We established Theorem 3.1 assuming that $r \leq 9$, but we believe that it holds for all r:

Conjecture 3.3. *For all $r \geq 3$ there exists a matrix $\mathcal{M}_r$ which satisfies the properties (a),(b) and (c) in Theorem 3.1. When r is odd, the matrix $\mathcal{M}_r$ is given by*

$$\begin{pmatrix} r-4 & & & & & & & \\ & r-6 & & & & & & \\ & & \ddots & & & & & \\ & & & -1 & & & & \\ & & & & 3 & & & \\ & & & & & 5 & & \\ & & & & & & \ddots & \\ & & & & & & & r \end{pmatrix} \begin{pmatrix} t_0 & t_1 & \cdots & t_{d-r} \\ t_1 & t_2 & \cdots & t_{d-r+1} \\ \vdots & \vdots & \ddots & \vdots \\ t_{r-2} & t_{r-1} & \cdots & t_{d-2} \end{pmatrix} + \begin{pmatrix} r & & & & & & & \\ & r-2 & & & & & & \\ & & \ddots & & & & & \\ & & & 3 & & & & \\ & & & & -1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & r-4 \end{pmatrix} \begin{pmatrix} t_2 & t_3 & \cdots & t_{d-r+2} \\ t_3 & t_4 & \cdots & t_{d-r+3} \\ \vdots & \vdots & \ddots & \vdots \\ t_r & t_{r+1} & \cdots & t_d \end{pmatrix}.$$

We do not know yet what the general formula for $\mathcal{M}_r$ should be when r is even. The following systematic construction led to the matrices M_r and $\mathcal{M}_r$ in all cases known to us. Let $\widetilde{P}$ and $\widetilde{Q}$ be the multilinear forms in Lemma 2.3 that define $\mathcal{G}_{r,2}$. Let F_j denote the polynomial of degree $2r-2$ obtained by eliminating v_{1j} and v_{2j} from $\widetilde{P}$ and $\widetilde{Q}$. Let G_j denote the product of all $\binom{r-1}{2}$ minors $v_{1k}v_{2l} - v_{1l}v_{2k}$ of V where $j \notin \{k, l\}$. Each product F_jG_j is a polynomial of degree $r(r-1)$. Note that F_r is $m_{11}m_{22} - m_{12}m_{21}$ in (13), and F_rG_r is (24). Now, the ideal $\langle F_1G_1, F_2G_2, \ldots, F_rG_r\rangle$ is Cohen-Macaulay of codimension 2. By the Hilbert-Burch Theorem, the F_jG_j are the maximal minors of an $(r-1)\times r$-matrix M_r, which can be extracted from the minimal free resolution of $\langle F_1G_1, \ldots, F_rG_r\rangle$. This is precisely our matrix. In order to extend Theorem 3.1, and to find the desired $\mathcal{M}_r$ for even r, we need that all entries of the Hilbert-Burch matrix M_r have the same degree r.

Remark 3.4. The singular locus of $\mathcal{T}_{r,2,d}$ is defined by the $(r-2)\times(r-2)$-minors of $\mathcal{M}_r$. It would be interesting to study this subvariety of $\mathbb{P}^d$ and how it relates to singularities of $\mathcal{F}_{r,2}$. For instance, for $r = 4$, this singular locus is precisely the odeco variety $\mathcal{T}_{2,2,d}$, and, using [21, Proposition 3.6], we can see that its prime ideal is generated by the 2×2-minors of $\mathcal{M}_4$.

In Section 5 we shall see how the matrices $\mathcal{M}_r$ can be used to find a frame decomposition of a given symmetric $2\times 2\times\cdots\times 2$-tensor T. We close with an example that shows how this task differs from the easier problem of constructing a rank r Waring decomposition of T.

Example 3.5. Let $r = 4$, $d = 8$, and consider the sum of two odeco tensors

$$T \;=\; x^8 + y^8 \;+\; (x-y)^8 + (x+y)^8 \;=\; 3x^8 + 56x^6y^2 + 140x^4y^4 + 56x^2y^6 + 3y^8.$$

The coordinates of this tensor are $t_0 = t_8 = 3$, $t_2 = t_4 = t_6 = 2$, and $t_1 = t_3 = t_5 = t_7 = 0$. Here, the 3×5-matrix $\mathcal{M}_4$ has rank 2. This verifies that T lies in $\mathcal{T}_{4,2,8}$, in accordance with Example 2.2. However, the 4×4-matrix $\mathcal{M}_5$ is invertible. This means that T does not lie in $\mathcal{T}_{5,2,8}$. In other words, there is no funtf among the rank 5 Waring decompositions of T. $\diamondsuit$

4 Ternary Forms and Beyond

We now move on to higher dimensions $n \geq 3$. Our object of study is the fradeco variety

$$\mathcal{T}_{r,n,d} \subset \mathbb{P}(\mathrm{Sym}_d(\mathbb{C}^n)).$$

A very first question is: What is the dimension of $\mathcal{T}_{r,n,d}$? In Lemma 3.2, we saw that $\dim(\mathcal{T}_{r,2,d}) = 2r - 3$. The following proposition generalizes that formula to arbitrary n:

Proposition 4.1. *For all $r > n$ and $d \geq 3$, the dimension of $\mathcal{T}_{r,n,d}$ is bounded above by*

$$\min\left\{(n-1)(r-n) + \frac{(n-1)(n-2)}{2} + r - 1\,,\ \binom{n+d-1}{d} - 1\right\}. \tag{25}$$

Proof. The right number is the dimension of the ambient space, so this is an upper bound. The left number is the dimension of $\mathcal{F}_{r,n} \times \mathbb{P}^{r-1}$, by the formula in Theorem 2.1. The formula (3) expresses our variety as the (closure of the) image of a polynomial map

$$\mathcal{F}_{r,n} \times \mathbb{P}^{r-1} \longrightarrow \mathcal{T}_{r,n,d}. \tag{26}$$

The dimension of the image of this map is bounded above by the dimension of the domain. □

We conjecture that the true dimension always agrees with the expected dimension:

Conjecture 4.2. *The dimension of the variety $\mathcal{T}_{r,n,d}$ is equal to (25) for all $r > n$ and $d \geq 3$.*

This conjecture is subtler than it may seem. Let $\sigma_r \nu_d \mathbb{P}^{n-1}$ denote the Zariski closure of the set of tensors of rank $\leq r$ in $\mathbb{P}(\mathrm{Sym}_d(\mathbb{C}^n))$. Geometrically, this is the r-th secant variety of the d-th Veronese embedding of $\mathbb{P}^{n-1}$. It is known that $\sigma_r \nu_d \mathbb{P}^{n-1}$ has the expected dimension in almost all cases. The Alexander-Hirschowitz Theorem (cf. [4, 18]) states that, assuming $d \geq 3$, the dimension of $\sigma_r \nu_d \mathbb{P}^{n-1}$ is lower than expected in precisely four cases:

$$(r, n, d) \in \big\{(5,3,4), (7,5,3), (9,4,4), (14,5,4)\big\}. \tag{27}$$

One might think that in these cases also the fradeco subvariety $\mathcal{T}_{r,n,d}$ has lower than expected dimension. However, the results summarized in Theorem 4.3 suggest that this is not the case.

Theorem 4.3. *Consider the fradeco varieties $\mathcal{T}_{r,n,d}$ in the cases when $n \geq 3$ and $1 \leq \dim(\mathcal{T}_{r,n,d}) \cdot \mathrm{codim}(\mathcal{T}_{r,n,d}) \leq 100$. Table 2 gives their degrees and some defining polynomials. The last column shows the minimal generators of lowest possible degrees in the ideal of $\mathcal{T}_{r,n,d}$.*

Computational Proof. The dimensions are consistent with Conjecture 4.2. They were verified by computing tangent spaces at a generic point using `Bertini` and `Matlab`. The degrees were computed with the monodromy loop method described in Subsection 5.2.1. The numerical Hilbert function method in Subsection 5.2.2 was used to determine how many polynomials of a given degree vanish on $\mathcal{T}_{r,n,d}$. This was followed up with computations in exact arithmetic in `Maple` and `Macaulay2`. These confirmed the earlier numerical results, and they enabled us to find the explicit polynomials in $\mathbb{Q}[T]$ that are listed in Examples 4.4, 4.5 and 4.6. In the cases where we report no equations occurring below a certain degree, this is a combination of Corollary 4.10 and the numerical Hilbert function computation. □

variety	dim	codim	degree	known equations
$\mathcal{T}_{4,3,3}$	6	3	17	3 cubics, 6 quartics
$\mathcal{T}_{4,3,4}$	6	8	74	6 quadrics, 37 cubics
$\mathcal{T}_{4,3,5}$	6	14	191	27 quadrics, 104 cubics
$\mathcal{T}_{5,3,4}$	9	5	210	1 cubic, 6 quartics
$\mathcal{T}_{5,3,5}$	9	11	1479	20 cubics, 213 quartics
$\mathcal{T}_{6,3,4}$	12	2	99	none in degree ≤ 5
$\mathcal{T}_{6,3,5}$	12	8	4269	one quartic
$\mathcal{T}_{7,3,5}$	15	5	≥ 38541	none in degree ≤ 4
$\mathcal{T}_{8,3,5}$	18	2	690	none in degree ≤ 5
$\mathcal{T}_{10,3,6}$	24	3	≥ 16252	none in degree ≤ 7
$\mathcal{T}_{5,4,3}$	10	9	830	none in degree ≤ 4
$\mathcal{T}_{6,4,3}$	14	5	1860	none in degree ≤ 3
$\mathcal{T}_{7,4,3}$	18	1	194	one in degree 194

Table 2: A census of small fradeco varieties

We shall now discuss some of the cases appearing in Theorem 4.3 in more detail.

Example 4.4. The 6-dimensional variety $\mathcal{T}_{4,3,3} \subset \mathbb{P}^9$ has the parametrization

$$\begin{array}{rcc} t_{300} & = & v_{11}^3 + v_{12}^3 + v_{13}^3 + v_{14}^3, \\ t_{030} & = & v_{21}^3 + v_{22}^3 + v_{23}^3 + v_{24}^3, \\ t_{003} & = & v_{31}^3 + v_{32}^3 + v_{33}^3 + v_{34}^3, \\ t_{012} & = & v_{21}v_{31}^2 + v_{22}v_{32}^2 + v_{23}v_{33}^2 + v_{24}v_{34}^2, \\ t_{021} & = & v_{21}^2v_{31} + v_{22}^2v_{32} + v_{23}^2v_{33} + v_{24}^2v_{34}, \\ t_{102} & = & v_{11}v_{31}^2 + v_{12}v_{32}^2 + v_{13}v_{33}^2 + v_{14}v_{34}^2, \\ t_{120} & = & v_{11}v_{21}^2 + v_{12}v_{22}^2 + v_{13}v_{23}^2 + v_{14}v_{24}^2, \\ t_{201} & = & v_{11}^2v_{31} + v_{12}^2v_{32} + v_{13}^2v_{33} + v_{14}^2v_{34}, \\ t_{210} & = & v_{11}^2v_{21} + v_{12}^2v_{22} + v_{13}^2v_{23} + v_{14}^2v_{24}, \\ t_{111} & = & v_{11}v_{21}v_{31} + v_{12}v_{22}v_{32} + v_{13}v_{23}v_{33} + v_{14}v_{24}v_{34}. \end{array} \tag{28}$$

Here the matrix $V = (v_{ij})$ is given by the parametrization of $\mathcal{G}_{4,3}$ seen in (8) of Example 1.1.

Using exact linear algebra in `Maple`, we find that the ideal of $\mathcal{T}_{4,3,3}$ contains no quadrics, but it contains three linearly independent cubics and 36 quartics. One of the cubics is

$$C_{123} + 2C_{145} + 2C_{345} - C_{126} - C_{236} - 4C_{456}, \tag{29}$$

where C_{ijk} denotes the determinant of the 3×3 submatrix with columns i, j, k in

$$C = \begin{pmatrix} t_{300} & t_{210} & t_{120} & t_{201} & t_{111} & t_{102} \\ t_{210} & t_{120} & t_{030} & t_{111} & t_{021} & t_{012} \\ t_{201} & t_{111} & t_{021} & t_{102} & t_{012} & t_{003} \end{pmatrix}.$$

The other two cubics are obtained from this one by permuting the indices. The resulting three cubics define a complete intersection in $\mathbb{P}^9$. However, that complete intersection strictly contains $\mathcal{T}_{4,3,3}$ because the three cubics have only 30 multiples in degree 4, whereas we know that 36 quartics vanish on $\mathcal{T}_{4,3,3}$. Using `Macaulay2`, we identified six minimal ideal generators in degree 4, and we found that the nine known generators generate a Cohen-Macaulay ideal of codimension 3 and degree 17. Using `Bertini`, we independently verified that fradeco variety $\mathcal{T}_{4,3,3}$ has degree 17. This implies that we have found the correct prime ideal. $\diamondsuit$

Example 4.5. The variety $\mathcal{T}_{4,3,4}$ is also 6-dimensional, and it lives in the $\mathbb{P}^{14}$ of ternary quartics. The parametrization is as in (28) but with quartic monomials instead of cubic. Among the ideal generators for $\mathcal{T}_{4,3,4}$ are six quadrics and 37 cubics. One of the quadrics is

$$\begin{gathered}8(t_{013}^2-t_{004}t_{022})+8(t_{031}^2-t_{022}t_{040})+8(t_{211}^2-t_{202}t_{220})+18(t_{112}^2-t_{103}t_{121})+18(t_{121}^2-t_{112}t_{130})\\+(t_{004}t_{040}+19t_{022}^2-20t_{013}t_{031})+(t_{004}t_{220}+t_{022}t_{202}-2t_{013}t_{211})+(t_{040}t_{202}+t_{022}t_{220}-2t_{031}t_{211}).\end{gathered}$$

A `Bertini` computation suggests that the known generators suffice to cut out $\mathcal{T}_{4,3,4}$. We also note that the 27 quadrics for $\mathcal{T}_{4,3,5}$ come from the 6 quadrics for $\mathcal{T}_{4,3,4}$. For instance, replacing each variable t_{ijk} by $t_{i,j,k+1}$ yields the quadric $8t_{014}^2+8t_{032}^2+\cdots+19t_{023}^2$ for $\mathcal{T}_{4,3,5}$. $\diamondsuit$

Example 4.6. The fradeco variety $\mathcal{T}_{5,3,4}$ is especially interesting because $(5,3,4)$ appears on the Alexander-Hirschowitz list (27). The unique cubic that vanishes on $\mathcal{T}_{5,3,4}$ is

$$\begin{gathered}46t_{022}t_{202}t_{220}+73t_{112}t_{121}t_{211}-4t_{004}t_{040}t_{400}+19[t_{013}t_{130}t_{301}]_2-50[t_{004}t_{112}^2]_3-22[t_{004}t_{220}^2]_3\\-18[t_{022}t_{211}^2]_3+50[t_{004}t_{022}t_{202}]_3+26[t_{004}t_{130}t_{310}]_3+100[t_{013}t_{103}t_{112}]_3-53[t_{013}t_{121}t_{310}]_3\\+5[t_{004}t_{022}t_{400}]_6-50[t_{013}^2t_{202}]_6-5[t_{013}^2t_{220}]_6+45[t_{004}t_{031}t_{211}]_6-40[t_{022}t_{202}^2]_6+5[t_{004}t_{022}t_{220}]_6\\+40[t_{022}t_{112}^2]_6-5[t_{004}t_{130}^2]_6-45[t_{004}t_{121}^2]_6-10[t_{004}t_{112}t_{130}]_6-45[t_{013}t_{022}t_{211}]_6+35[t_{013}t_{031}t_{202}]_6\\+10[t_{013}t_{103}t_{130}]_6+10[t_{013}t_{112}t_{121}]_6-80[t_{013}t_{112}t_{301}]_6+80[t_{013}t_{202}t_{211}]_6+8[t_{013}t_{211}t_{220}]_6.\end{gathered}$$

This polynomial has 128 terms: each bracket denotes an orbit of monomials under the S_3-action, and the subscript is the orbit size. In addition, six fairly large quartics vanish on $\mathcal{T}_{5,3,4}$. The seven known generators cut out a reducible variety of dimension 9 in $\mathbb{P}^{14}$. The fradeco variety $\mathcal{T}_{5,3,4}$ is the unique top-dimensional component. But, using `Bertini`, we found two extraneous components of dimension 7. Their degrees are 120 and 352 respectively. $\diamondsuit$

We close this section by examining the geometric interplay between fradeco varieties and secant varieties. We write $\sigma_r\nu_d\mathbb{P}^{n-1}$ for the r-th secant variety of the d-th Veronese embedding of $\mathbb{P}^{n-1}$. This lives in $\mathbb{P}(\mathrm{Sym}_d(\mathbb{C}^n))$ and comprises rank r symmetric tensors (3). The same ambient space contains the fradeco variety $\mathcal{T}_{r,n,d}$ and all its secant varieties $\sigma_s\mathcal{T}_{r,n,d}$.

Theorem 4.7. *For any $r>n\geq d\geq 2$, we have*

$$\sigma_{r-n}\nu_d\mathbb{P}^{n-1}\subset\mathcal{T}_{r,n,d}\subset\sigma_r\nu_d\mathbb{P}^{n-1}, \tag{30}$$

and hence $\mathcal{T}_{r-n,n,d}\subset\mathcal{T}_{r,n,d}$ whenever $r\geq 2n$. Also, if $r=r_1r_2$ with $r_1\geq 2$ and $r_2\geq n$, then

$$\sigma_{r_1}\mathcal{T}_{r_2,n,d}\subseteq\mathcal{T}_{r,n,d}. \tag{31}$$

Proof. We fix d. The right inclusion in (30) is immediate from the definition. For the left inclusion we use the parametrization of $\mathcal{F}_{r,n}$ given in (15) and (16). The point is that the $(r-n)\times n$-matrix W can be chosen freely. Equivalently, the projection of $\mathcal{G}_{r,n} \subset (\mathbb{P}^{n-1})^r$ to any coordinate subspace $(\mathbb{P}^{n-1})^{r-n}$ is dominant. This means that the first $r-n$ summands in (1) are arbitrary powers of linear forms, and this establishes the left inclusion in (30).

To show the inclusion (31), we consider arbitrary frames $V_1, V_2, \ldots, V_{r_1} \in \mathcal{F}_{r_2,n}$. Then the $n\times r$-matrix $V = (V_1, V_2, \ldots, V_{r_1})$ is a frame in $\mathcal{F}_{r,n}$. Each V_i together with a choice of $\lambda_i \in \mathbb{R}^{r_2}$ determines a point on $\mathcal{T}_{r_2,n,d}$. Thus we have r_1 points in $\mathcal{T}_{r_2,n,d}$, and any point on the $\mathbb{P}^{r_1-1}$ spanned by these lies in $\mathcal{T}_{r,n,d}$, where it is represented by V with $\lambda = (\lambda_1, \ldots, \lambda_{r_1}) \in \mathbb{R}^r$. $\square$

Example 4.8. Let $n = 2$ and write $H = (t_{i+j})$ for a Hankel matrix of unknowns with $r+1$ rows and sufficiently many columns. The secant variety $\sigma_r\nu_d\mathbb{P}^1$ is defined by the ideal $I_{r+1}(H)$ of $(r{+}1)\times(r{+}1)$-minors of H. The ideal-theoretic version of (30) states that

$$I_{r-1}(H) \supset I_{r-1}(\mathcal{M}_r) \supset I_{r+1}(H).$$

It is instructive to check this. The left inclusion follows from the Cauchy-Binet Theorem applied to $\mathcal{M}_r = A \cdot H$ where A is the $(r{-}1)\times(r{+}1)$ integer matrix underlying M_r. $\diamondsuit$

Remark 4.9. (a) Since concatenations of frames in $\mathbb{R}^n$ are always frames, (31) generalizes from secant varieties to joins. Namely, if $r = r_1 + r_2$, then $\mathcal{T}_{r_1,n,d} \star \mathcal{T}_{r_2,n,d} \subset \mathcal{T}_{r,n,d}$.

(b) The inclusion in (31) is always strict, with one notable exception: $\sigma_2\mathcal{T}_{2,2,d} = \mathcal{T}_{4,2,d}$.

Theorem 4.7 implies that the Veronese variety $\nu_d\mathbb{P}^{n-1}$ is contained in the fradeco variety $\mathcal{T}_{r,n,d}$ with $r > n$. This is illustrated in Example 4.5 where we wrote the quadric that vanishes on $\mathcal{T}_{4,3,4}$ as a linear combination of the binomials that define $\nu_4\mathbb{P}^2 \subset \mathbb{P}^{14}$. The formula (29) shows that this cubic vanishes on $\sigma_2\nu_3\mathbb{P}^2$. Similarly, we can verify that the cubic in Example 4.6 vanishes on $\sigma_2\nu_4\mathbb{P}^2$ by writing it as a linear combination of the 3×3-minors $C_{ijk,lmm}$ of the 6×6-catalecticant C matrix in (34). One such expression is

$$\begin{gathered}
50C_{012,012} - 30C_{012,123} + 50C_{012,034} - 30C_{012,125} + 50C_{012,045} + 63C_{012,345} - 10C_{013,024} + 10C_{013,234} \\
+5C_{013,015} + 35C_{013,135} + 34C_{013,245} + 5C_{023,023} - 80C_{023,134} + 5C_{023,025} - 26C_{023,235} - 19C_{023,145} \\
-30C_{123,123} + 29C_{123,125} - 10C_{123,345} - 10C_{014,025} + 19C_{014,235} - 53C_{014,145} - 30C_{024,245} + 5C_{034,034} \\
+26C_{034,045} + 5C_{034,345} + 50C_{134,134} + 50C_{134,235} + 30C_{134,145} + 30C_{234,245} + 5C_{015,015} + 26C_{015,135} \\
+50C_{015,245} - 5C_{025,235} - 10C_{025,145} - 10C_{125,345} - 4C_{035,035} + 5C_{135,135} + 50C_{135,245} + 5C_{235,235} \\
+5C_{045,045} + 5C_{045,345} + 50C_{245,245}.
\end{gathered}$$

Theorem 4.7 gives lower bounds on the degrees of the equations defining fradeco varieties:

Corollary 4.10. *All non-zero polynomials in the ideal of $\mathcal{T}_{r,n,d}$ must have degree at least $r-n+1$.*

Proof. The ideal of the Veronese variety $\nu_d\mathbb{P}^{n-1}$ contains no linear forms. It is generated by 2×2 minors of catalecticants. A general result on secant varieties [23, Thm. 1.2] implies that the ideal of $\sigma_{r-n}\nu_d\mathbb{P}^{n-1}$ is zero in degree $\leq r-n$. The inclusion $\sigma_{r-n}\nu_d\mathbb{P}^{n-1} \subset \mathcal{T}_{r,n,d}$ yields the claim. $\square$

In Table 2 we see that $\mathcal{T}_{4,3,4}$, $\mathcal{T}_{4,3,5}$, $\mathcal{T}_{5,3,4}$, $\mathcal{T}_{5,3,5}$ and $\mathcal{T}_{6,3,5}$ have their first minimal generators in the lowest possible degrees. However this is not always the case, as shown dramatically by $\mathcal{T}_{7,4,3}$.

5 Numerical Recipes

Methods from Numerical Algebraic Geometry (NAG) are useful for studying the decomposition of tensors into frames. Many of the results on fradeco varieties $\mathcal{T}_{r,n,d}$ reported in Sections 3 and 4 were discovered using NAG. In this section we discuss the relevant methodologies. Our experiments involve a mixture of using `Bertini` [2], `Macaulay2` [13], `Maple`, and `Matlab`.

All algebraic varieties have an *implicit representation*, as the solution set to a system of polynomial equations. Some special varieties admit a *parametric representation*, as the (closure of the) image of a map whose coordinates are rational functions. Having to pass back and forth between these two representations is a ubiquitous task in computational algebra.

The fradeco variety studied in this paper is given by a mixture of implicit and parametric. Our point of departure is the implicit representation (4) of the funtf variety $\mathcal{F}_{r,n}$, or its homogenization $\mathcal{G}_{r,n}$. Built on top of that is the parametrization (1) of rank r tensors:

$$\begin{array}{ccc} \mathbb{C}^{n\times r}\times\mathbb{C}^r & & \mathrm{Sym}_d(\mathbb{C}^n) \\ \cup & & \cup \\ \mathcal{F}_{r,n}\times\mathbb{C}^r & \xrightarrow{\Sigma_d} & \widehat{\mathcal{T}}_{r,n,d} \\ & & \\ (V,\lambda) & \longmapsto & \lambda_1\mathbf{v}_1^{\otimes d}+\lambda_2\mathbf{v}_2^{\otimes d}+\cdots+\lambda_r\mathbf{v}_r^{\otimes d} \end{array} \tag{32}$$

Here, $\widehat{\mathcal{T}}_{r,n,d}$ denotes the affine cone over the projective variety $\mathcal{T}_{r,n,d}$. The input to our *decomposition problem* is an arbitrary symmetric $n\times n\times\cdots\times n$-tensor T and a positive integer r. The task is to decide whether T lies in $\widehat{\mathcal{T}}_{r,n,d}$, and, if yes, to compute a preimage (V,λ) under the map Σ_d in (32). Any preimage must satisfy the non-trivial constraint $V\in\mathcal{F}_{r,n}$.

5.1 Decomposing fradeco tensors

We discuss three approaches to finding frame decompositions of symmetric tensors.

5.1.1 Tensor power method

Our original motivation for this project came from the case $r=n$ of odeco tensors [21]. If $T\in\widehat{\mathcal{T}}_{n,n,d}$, then the *tensor power method* of [1] reliably reconstructs the decomposition (1) where $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ is an orthonormal basis of $\mathbb{R}^n$. The algorithm is to iterate the rational map $\nabla T:\mathbb{P}^{n-1}\dashrightarrow\mathbb{P}^{n-1}$ given by the gradient vector $\nabla T=(\partial T/\partial x_1,\ldots,\partial T/\partial x_n)$. This map is regular when the hypersurface $\{T=0\}$ is smooth. The fixed points of ∇T are the *eigenvectors* of the tensor T. Their number was given in [7]. The punchline is this: if the multipliers $\lambda_1,\ldots,\lambda_n$ in (1) are positive, then $\mathbf{v}_1,\ldots,\mathbf{v}_n$ are precisely the *robust eigenvectors*, i.e. the attracting fixed points of the gradient map ∇T.

This raises the question whether the tensor power method also works for fradeco tensors. The answer is "no" in general, but it is "yes" in some special cases.

Example 5.1. Let $n=2,r=4,d=5$ and consider the fradeco quintic

$$T\;=\;\alpha x^5+y^5+(x+y)^5+(x-y)^5\quad\in\;\mathcal{T}_{4,2,5},$$

where $\alpha>6$ is a parameter. The eigenvectors of T are the zeros in $\mathbb{P}^1$ of the binary quintic

$$y\frac{\partial T}{\partial x}-x\frac{\partial T}{\partial y}\;=\;5y\cdot\Big((\alpha x-6)x^4\;+\;\big(2xy-\frac{1}{4}y^2\big)^2\;+\;\frac{31}{16}y^4\Big).$$

The point $(1:0)$ is an eigenvector, but there are no other real eigenvectors, as the expression is a sum of squares. Hence the frame decomposition of T cannot be recovered from its eigenvectors. $\diamondsuit$

Example 5.2. For any reals $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$ and any integer $d \geq 5$, we consider the tensor

$$T \;=\; \lambda_1(-5x_1 + x_2 + x_3)^d + \lambda_2(x_1 - 5x_2 + x_3)^d + \lambda_3(x_1 + x_2 - 5x_3)^d + \lambda_4(3x_1 + 3x_2 + 3x_3)^d. \quad (33)$$

This tensor has precisely four robust eigenvectors, namely the columns of the matrix V in (7). Hence the frame decomposition of T can be recovered by the tensor power method. $\diamondsuit$

The following conjecture generalizes this example.

Conjecture 5.3. *Let $r = n+1 < d$ and $T \in \mathcal{T}_{n+1,n,d}$ with $\lambda_1, \ldots, \lambda_{n+1} > 0$ in (1). Then $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$ are the robust eigenvectors of T, so they are found by the tensor power method.*

Example 5.1 shows that Conjecture 5.3 is false for $r \geq n+2$, and it suggests that the Tensor Power Method will not work in general. We next discuss two alternative approaches.

5.1.2 Catalecticant method for frames

The matrices in Theorem 3.1 furnish a practical algorithm for the decomposition problem when $n = 2$. This is a variant of Sylvester's Catalecticant Algorithm, and it works as follows.

Our input is a binary form $T \in \widehat{\mathcal{T}}_{r,2,n}$. We seek to recover the tight frame into which T decomposes. Since we do not know the fradeco rank r in advance, we start with $\mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_5$, etc. and plug in the coordinates t_i of T. The fradeco rank is the first index r with $\mathcal{M}_r$ rank deficient.

If the matrix $\mathcal{M}_r$ is rank deficient, then its rank is at most $r-2$. Let us assume that the rank equals exactly $r-2$. Otherwise T is a singular point (cf. Remark 3.4). Then, up to scaling, we find the unique row vector $\mathbf{w} \in \mathbb{R}^{r-1}$ in the left kernel of $\mathcal{M}_r$. By Theorem 3.1 we know that $\mathcal{M}_r$ is the product of the matrix M_r and an $(r-1) \times (d-r-1)$ matrix with entries $v_{i1}^{d-r-j+1} v_{i2}^{j-1}$, where $V = (v_{ij}) \in \mathcal{G}_{r,2}$ is the desired frame. Moreover, the matrix M_r has rank $r-2$, so the vector $\mathbf{w}$ also lies in the left kernel of M_r, i.e. $\mathbf{w} \cdot M_r = 0$. Thus,

$$0 = \mathbf{w} \cdot M_r \;=\; \big(f(v_{11}, v_{21}), f(v_{12}, v_{22}), \ldots, f(v_{1r}, v_{2r})\big),$$

where $f(x,y)$ is a binary form of degree r. The r roots of $f(x,y)$ in $\mathbb{P}^1$ are the columns of the desired $V = (v_{ij}) \in \mathcal{G}_{r,2}$. Using these v_{ij}, the given binary form has the decomposition

$$T(x,y) \;=\; \sum_{j=1}^{r} \lambda_j (v_{1j}x + v_{2j}y)^d,$$

where the multipliers $\lambda_1, \ldots, \lambda_r$ are recovered by solving a linear system of equations.

Example 5.4. Let $r = 5$ and $d = 8$. We illustrate this method for the binary octic

$$\begin{aligned} T = (-237 - 896\alpha)x^8 + 8(65+241\alpha)x^7y - 28(16+68\alpha)x^6y^2 + 56(5+31\alpha)x^5y^3 + 70(2-56\alpha)x^4y^4 \\ +56(-7+193\alpha)x^3y^5 + 28(32-716\alpha)x^2y^6 + 8(-115+2671\alpha)xy^7 + (435-9968\alpha)y^8, \end{aligned}$$

where $\alpha = \sqrt{3} - 2$. The parenthesized expressions are the coordinates $t_0, \ldots, t_8$. We find

$$\mathcal{M}_5 \;=\; \begin{pmatrix} -13548\alpha + 595 & 3636\alpha - 150 & -996\alpha + 42 & 348\alpha + 18 \\ 2092\alpha - 94 & -548\alpha + 26 & 100\alpha - 22 & 148\alpha + 50 \\ -2092\alpha + 94 & 548\alpha - 26 & -100\alpha + 22 & -148\alpha - 50 \\ 996\alpha - 30 & -348\alpha - 6 & 396\alpha + 90 & -1236\alpha - 317 \end{pmatrix}.$$

This matrix has rank 3 and its left kernel is the span of the vector $\mathbf{w} = (0, 1, 1, 0)$. Therefore,

$$0 \;=\; \mathbf{w}M_5 \;=\; \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} v_{21}^5 + 5v_{11}^5 & v_{22}^5 + 5v_{12}^5 & v_{23}^5 + 5v_{13}^5 & v_{24}^5 + 5v_{14}^5 & v_{25}^5 + 5v_{15}^5 \\ v_{11}v_{21}^4 - 3v_{11}^3v_{21}^2 & v_{12}v_{22}^4 - 3v_{12}^3v_{22}^2 & v_{13}v_{23}^4 - 3v_{13}^3v_{23}^2 & v_{14}v_{24}^4 - 3v_{14}^3v_{24}^2 & v_{15}v_{25}^4 - 3v_{15}^3v_{25}^2 \\ 3v_{11}^2v_{21}^3 - v_{11}^4v_{21} & 3v_{12}^2v_{22}^3 - v_{12}^4v_{22} & 3v_{13}^2v_{23}^3 - v_{13}^4v_{23} & 3v_{14}^2v_{24}^3 - v_{14}^4v_{24} & 3v_{15}^2v_{25}^3 - v_{15}^4v_{25} \\ 5v_{11}^3v_{21}^2 + v_{11}^5 & 5v_{12}^3v_{22}^2 + v_{12}^5 & 5v_{13}^3v_{23}^2 + v_{13}^5 & 5v_{14}^3v_{24}^2 + v_{14}^5 & 5v_{15}^3v_{25}^2 + v_{15}^5 \end{pmatrix}.$$

Hence the five columns of the desired tight frame $V = (v_{ij})$ are the distinct zeros in $\mathbb{P}^1$ of

$$f(v_{1i}, v_{2i}) \;=\; v_{1i}v_{2i}^4 - 3v_{1i}^3v_{2i}^2 + 3v_{1i}^2v_{2i}^3 - v_{1i}^4v_{2i} \qquad \text{for } i = 1, \ldots, 5.$$

We find

$$V \;=\; \begin{pmatrix} 1 & 0 & 1 & \alpha & 1 \\ 0 & 1 & 1 & 1 & \alpha \end{pmatrix} \;\in\; \mathcal{G}_{5,2}.$$

It remains to solve the linear system of nine equations in $\lambda = (\lambda_1, \ldots, \lambda_5)$ given by

$$T \;=\; \lambda_1 x^8 + \lambda_2 y^8 + \lambda_3(x+y)^8 + \lambda_4(\alpha x + y)^8 + \lambda_5(x + \alpha y)^8.$$

The unique solution to this system is $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = 1$ and $\lambda_4 = 1552 + 896\sqrt{3}$. $\diamondsuit$

5.1.3 Waring-enhanced frame decomposition

We now examine the decomposition problem for $n \geq 3$. Since no determinantal representation of $\mathcal{T}_{r,n,d}$ is known, a system of equations must be solved to recover (V, λ) from a given tensor in $\widehat{\mathcal{T}}_{r,n,d}$. In some special situations, we can approach this by taking advantage of known results on Waring decompositions. For instance, in Example 1.1 the Waring decomposition is already the frame decomposition. Example 3.5 shows that this is an exceptional situation.

We demonstrate the "Waring-enhanced" frame decomposition for the ternary quartic

$$\begin{aligned} \textstyle\sum_{i+j+k=4} \frac{24}{i!j!k!} t_{ijk} x^i y^j z^k \;=\; & 467x^4 + 152x^3y + 1448x^3z + 660x^2y^2 - 1488x^2yz + 4020x^2z^2 + 536xy^3 \\ & - 1992xy^2z + 2352xyz^2 + 944xz^3 + 227y^4 - 1000y^3z + 2148y^2z^2 - 1960yz^3 + 1267z^4. \end{aligned}$$

Ternary quartics of rank ≤ 5 form a hypersurface of degree 6 in $\mathbb{P}^{14}$. The equation of this hypersurface is the determinant of the 6×6 catalecticant matrix C. Here the dimension is one less than expected; this is the first entry in the Alexander-Hirschowitz list (27). For the given quartic,

$$C \;=\; \begin{bmatrix} t_{400} & t_{310} & t_{301} & t_{220} & t_{211} & t_{202} \\ t_{310} & t_{220} & t_{211} & t_{130} & t_{121} & t_{112} \\ t_{301} & t_{211} & t_{202} & t_{121} & t_{112} & t_{103} \\ t_{220} & t_{130} & t_{121} & t_{040} & t_{031} & t_{022} \\ t_{211} & t_{121} & t_{112} & t_{031} & t_{022} & t_{013} \\ t_{202} & t_{112} & t_{103} & t_{022} & t_{013} & t_{004} \end{bmatrix} \;=\; \begin{bmatrix} 467 & 38 & 362 & 110 & -124 & 670 \\ 38 & 110 & -124 & 134 & -166 & 196 \\ 362 & -124 & 670 & -166 & 196 & 236 \\ 110 & 134 & -166 & 227 & -250 & 358 \\ -124 & -166 & 196 & -250 & 358 & -490 \\ 670 & 196 & 236 & 358 & -490 & 1267 \end{bmatrix}. \qquad (34)$$

This matrix has rank 5 and its kernel is spanned by the vector corresponding to the quadric $q = 14u^2 - uv - 2uw - 4v^2 - 11vw - 10w^2$. The points $(u : v : w)$ in $\mathbb{P}^2$ that lie on the conic $\{q = 0\}$ represent all the linear forms $ux + vy + wz$ that may appear in a rank 5 decomposition.

Our task is to find five points on the conic $\{q = 0\}$ that form a frame $V \in \mathcal{G}_{5,3}$. This translates into solving a rather challenging system of polynomial equations. One of the solutions is

$$V \;=\; (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5) \;=\; \begin{pmatrix} -1 & 2 & 2 & 1 + 2\sqrt{3} & -1 + 2\sqrt{3} \\ 2 & 2 & -1 & -2 + \sqrt{3} & 2 + \sqrt{3} \\ 0 & 1 & -2 & 5 & -5 \end{pmatrix}.$$

The given ternary quartic has the frame decomposition $\mathbf{v}_1^{\otimes 4} + \mathbf{v}_2^{\otimes 4} + \mathbf{v}_3^{\otimes 4} + \mathbf{v}_4^{\otimes 4} + \mathbf{v}_5^{\otimes 4}$.

5.2 Exploring the fradeco variety

The following tasks make sense for any variety $X \subset \mathbb{P}^N$ arising in an applied context: (i) sample points on X, (ii) compute the dimension and degree of X, (iii) compute an irreducible decomposition of X, (iv) find a parametrization of X, (v) find some polynomials that vanish on X, (vi) determine polynomials that cut out X, (vii) find generators for the ideal of X. Numerical algebraic geometry (NAG) furnishes tools for addressing these points. In our study, X is the fradeco variety $\mathcal{T}_{r,n,d}$. We used NAG to find answers in some cases. In what follows, we explain our computations. Particular emphasis is placed on the results reported in Section 4 for the degree and Hilbert function of $\mathcal{T}_{r,n,d}$. All computations are carried out by working on the affine cone $\widehat{\mathcal{T}}_{r,n,d} \subset \mathrm{Sym}_d(\mathbb{C}^n)$.

5.2.1 Dimension and degree

The dimension and degree of the affine variety $\widehat{\mathcal{T}}_{r,n,d}$ can be computed directly from the mixed parametric-implicit representation in (32). The dimension can be found by selecting a random point on $\mathcal{F}_{r,n} \times \mathbb{R}^r$, determining its tangent space via [26], and then taking the image of this tangent space via the derivative of the map Σ_d. The image is a linear subspace in $\mathrm{Sym}_d(\mathbb{R}^n)$, and its dimension is found via the rank of its defining matrix. These matrices are usually given numerically, in terms of points sampled from $\mathcal{F}_{r,n}$, so we need to use singular value decompositions.

The computation of the degree is carried out using *monodromy*. We obtained the results of Theorem 4.3 by applying essentially the same technique as in [15, 16], adapted to our situation where the mapping is from an implicitly defined source. Here are some highlights of this method for $\widehat{\mathcal{T}}_{r,n,d}$. We performed these computations using `Bertini` and `MatLab`.

Let c denote the codimension of $\widehat{\mathcal{T}}_{r,n,d}$, as given by the formula in Conjecture 4.2. The degree of $\widehat{\mathcal{T}}_{r,n,d}$ is the number of points in the intersection with a random c-dimensional affine subspace of $\mathrm{Sym}_d(\mathbb{C}^n)$. Here we represent the fradeco variety purely numerically, namely as the set of images of points (V, λ) under the parametrization Σ_d shown in (32). This method verifies the dimension of $\widehat{\mathcal{T}}_{r,n,d}$ because the intersection would be empty if the dimension were lower than expected.

As a first step, we compute a numerical irreducible decomposition of the funtf variety $\mathcal{F}_{r,n}$. This also gives its degree and dimension, as shown in Table 1. In particular, we obtain degree-many points of $\mathcal{F}_{r,n}$ that lie in a random linear space of dimension equal to $\mathrm{codim}(\mathcal{F}_{r,n})$.

We take V to be one of these generic points in $\mathcal{F}_{r,n}$, we select a random vector $\lambda \in \mathbb{C}^r$, and we compute the fradeco tensor $\Sigma_d(V, \lambda)$. We also fix a random c-dimensional linear subspace $\mathcal{R}$ of $\mathrm{Sym}_d(\mathbb{C}^n)$ and a random point U in the c-dimensional affine space $\mathcal{R} + U$.

By construction, the affine cone $\widehat{\mathcal{T}}_{r,n,d}$ and the affine space $\mathcal{R} + U$ intersect in $\deg(\widehat{\mathcal{T}}_{r,n,d})$ many points in $\mathrm{Sym}_d(\mathbb{C}^n)$. One of these points is $\Sigma_d(V, \lambda)$. Our goal is to discover all the other intersection points by sequences of parameter homotopies that form monodromy loops. Geometrically, the base space for these monodromies is the vector space quotient $\mathrm{Sym}_d(\mathbb{C}^n)/R$.

We fix two further random points P_1 and P_2 in $\mathrm{Sym}_d(\mathbb{C}^n)$. These represent residue classes modulo the linear subspace $\mathcal{R}$. The data we fixed now define a (triangular) monodromy loop

$$\begin{tikzcd} & (\mathcal{R}+U) \cap \widehat{\mathcal{T}}_{r,n,d} \arrow[dr] & \\ (\mathcal{R}+P_2) \cap \widehat{\mathcal{T}}_{r,n,d} \arrow[ur] & & (\mathcal{R}+P_1) \cap \widehat{\mathcal{T}}_{r,n,d} \arrow[ll] \end{tikzcd}$$

We use `Bertini` to perform each linear parameter homotopy. This constructs a path (V_t, λ_t) in the parameter space. Here t runs from 0 to 3. We start at $(V_0, \lambda_0) = (V, \lambda)$, the point $\Sigma_d(V_i, \lambda_i)$ lies in $(\mathcal{R} + P_i) \cap \widehat{\mathcal{T}}_{r,n,d}$ for $i = 1, 2$, and $\Sigma_d(V_3, \lambda_3)$ is back in $(\mathcal{R} + U) \cap \widehat{\mathcal{T}}_{r,n,d}$. With high probability, $\Sigma_d(V_3, \lambda_3) \neq \Sigma_d(V, \lambda)$ holds, and we have discovered a new point. Then we iterate the process. Let $S_k := \{\Sigma_d(V, \lambda), \ldots, \Sigma_d(V', \lambda')\}$ denote the subset of $(\mathcal{R} + U) \cap \widehat{\mathcal{T}}_{r,n,d}$ that has been found after k steps. In the next monodromy loop we trace the paths of S_k to produce $\tilde{S}_{k+1}$, the endpoints of monodromy loops starting from S_k. Using `MatLab`, we then merge the point sets to form $S_{k+1} = S_k \cup \tilde{S}_{k+1}$. We repeat this process until no new points are found after 20 consecutive monodromy loops. The number of points in S_k is very strong numerical evidence for the degree of $\mathcal{T}_{r,n,d}$. At this point, one can also use the trace test [25] with pseudowitness sets [17] to confirm that degree.

5.2.2 Numerical Hilbert Function

We wish to learn the polynomial equations that vanish on $\mathcal{T}_{r,n,d}$. The set I of all such polynomials is a homogeneous prime ideal in the polynomial ring over $\mathbb{Q}$ whose variables are the entries $t_{i_1 i_2 \cdots i_d}$ of an indeterminate tensor T. We write this polynomial ring as

$$\mathbb{Q}[T] = \bigoplus_{e \geq 0} \mathbb{Q}[T]_e \quad \simeq \quad \mathrm{Sym}_e(\mathrm{Sym}_d(\mathbb{Q}^n)) = \bigoplus_{e \geq 0} \mathrm{Sym}_*(\mathrm{Sym}_d(\mathbb{Q}^n)).$$

The space of all polynomials of degree e in the ideal I is the subspace

$$I_e = I \cap \mathbb{Q}[T]_e \quad \subset \quad \mathbb{Q}[T]_e \simeq \mathrm{Sym}_e(\mathrm{Sym}_d(\mathbb{Q}^n)).$$

A natural approach is to fix some small degree e and to ask for a $\mathbb{Q}$-linear basis of I_e.

The dimensions of these vector spaces are organized into the *Hilbert function*

$$\mathbb{N} \to \mathbb{N}, \; e \mapsto \dim_{\mathbb{Q}}(I_e).$$

We used `Bertini` and `Matlab` to determine specific values of the Hilbert function. In some cases, an independent `Maple` computation was used to construct a basis for the $\mathbb{Q}$-vector space I_e.

Fix values for r, n, d. As discussed above, we can use the parametrization (32) to produce many sample points $T = \Sigma_d(V, \lambda)$ on $\mathcal{T}_{r,n,d}$. The condition $f(T) = 0$ translates into a linear equation in the coefficients of a given polynomial $f \in \mathbb{Q}[T]_e$, and I_e is the solution space to these equations as T ranges over $\mathcal{T}_{r,n,d}$. We write these linear equations as a matrix whose number of columns is $\dim(\mathbb{Q}[T]_e) = \binom{\binom{n+d-1}{d}+e-1}{e}$, and with one row per sample point T. In practice we take enough sample points so that I_e is sure to equal the kernel of that matrix.

This procedure may be carried out in exact arithmetic over $\mathbb{Q}$ when sufficiently many exact points can be found on $\mathcal{F}_{r,n}$. When floating point approximations are used, some care is required in choosing the appropriate number of points and a sufficient degree of precision. This numerical test can become inconclusive in high dimension due to these issues. Using floating point arithmetic and 30,000 points of $\mathcal{F}_{r,n}$ we obtained the values listed in Table 3. The blanks indicate that we did not find conclusive evidence for the exact value of $\dim(I_e)$ in that case. For $\mathcal{T}_{5,4,3}$, $\mathcal{T}_{6,3,4}$, $\mathcal{T}_{6,4,3}$, $\mathcal{T}_{7,3,5}$, and $\mathcal{T}_{8,3,5}$ we also found no conclusive numerical evidence for equations in degrees less than 5.

The calculation of $\dim(I_e)$ is a numerical rank computation via singular value decomposition, so at least in principle it is possible to also extract a basis of I_e. However, in practice, round-off errors yield imprecise values for the coefficients of the basis elements of I_e. This makes it difficult to reliably determine an exact $\mathbb{Q}$-basis of I_e by numerical methods.

ideal \ $\deg e$	2	3	4	5	6
$\dim \mathcal{I}(\mathcal{T}_{5,2,9})_e$	0	0	5	46	235
$\dim \mathcal{I}(\mathcal{T}_{4,3,4})_e$	6	127	1093	5986	
$\dim \mathcal{I}(\mathcal{T}_{4,3,5})_e$	27	651	6370		
$\dim \mathcal{I}(\mathcal{T}_{5,3,4})_e$	0	1	21		
$\dim \mathcal{I}(\mathcal{T}_{5,3,5})_e$	0	20	633		
$\dim \mathcal{I}(\mathcal{T}_{6,3,5})_e$	0	0	1		

Table 3: Numerical computation of the Hilbert functions of fradeco varieties

To discover the explicit ideal generators displayed in Sections 3 and 4, we instead used exact arithmetic in `Maple`. A key step was to produce points in the funtf variety $\mathcal{F}_{r,n}$ that are defined over low-degree extension of $\mathbb{Q}$, and to map them carefully via Σ_d. To accomplish this, we used the representation of $\mathcal{G}_{r,n}$ discussed in Section 2. In our experiments, we found that the `solve` command in `Maple` was able to handle dense linear systems with up to $3,500$ unknowns.

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