The zeta-regularized product of odious numbers

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Abstract

What is the product of all *odious* integers, i.e., of all integers whose binary expansion contains an odd number of 1's? Or more precisely, how to define a product of these integers which is not infinite, but still has a "reasonable" definition? We will answer this question by proving that this product is equal to $\pi^{1/4}\sqrt{2\varphi e^{-\gamma}}$, where γ and φ are respectively the Euler-Mascheroni and the Flajolet-Martin constants.

— *Dedicated to Joseph Kung*

1 Introduction

Extending or generalizing "simple" notions is a basic activity in mathematics. This involves trying to give some sense to an *a priori* meaningless formula, like $\sqrt{-1}$, $1/0$, $\sum_{n\geq 1} n$, etc. Among these attempts is the question of "assigning a reasonable value" to an infinite product of increasing positive real numbers. This question arises for example when trying to define "determinants" for operators on infinite-dimensional vector spaces. One possible approach is to define "zeta-regularization" (see the definition in Section 2 below). The literature on the subject is vast, going from theoretical aspects to explicit computations in mathematics but also in physics (see, e.g., [10, 23]): we will —of course— not give a complete view of the existing references, but rather restrict to a few ones to allude to general contexts where these infinite products take place. Our purpose is modest: to give the value of an infinite arithmetic product (namely the product of all *odious* integers, i.e., of all those integers whose binary expansion contains an odd number of 1's, see, e.g., [19])

$$1 \times 2 \times 4 \times 7 \times 8 \times 11 \times 13 \times 14 \times 16 \times 19 \times 21 \times 22 \times 25 \times 26 \times 28 \times 31 \times 32 \times \dots$$

2 Definitions. First properties. Examples

2.1 Definitions

The remark that

$$\log \prod_{i=1}^{n} \lambda_i = -\left(\frac{\mathrm{d}}{\mathrm{ds}} \sum_{i=1}^{n} \frac{1}{(\lambda_i)^s}\right)_{s=0}$$

suggests a way of defining the infinite product of a sequence $(\lambda_i)_{i\geq 1}$ of positive numbers (see, e.g., [20, 22]) by means of *zeta-regularization*: suppose that the Dirichlet series $\zeta_\Lambda(s) := \sum_i 1/\lambda_i^s$ converges when the real part of s is large enough, that it has a meromorphic continuation to the whole complex plane, and that it has no pole at 0, then the *zeta-regularized* product of the λ_i's is defined by

$$\prod_{i=1}^{\infty} \lambda_i := e^{-\zeta'_\Lambda(0)}$$

(this definition clearly coincides with the usual product when the sequence of λ_i's is finite). If ζ_Λ has as pole at 0, there is a slight generalization of the definition above (see [12, 16], also see [13]):

$$\prod_{i=1}^{\infty} \lambda_i := e^{-\operatorname*{Res}_{s=0} \frac{\zeta_\Lambda(s)}{s^2}}$$

where $\operatorname*{Res}_{s=0} g(s)$ stands for the residue at 0 of the function g. To put some general deep context about this definition, in particular about infinite determinants, the reader can consult [6, 17, 24].

2.2 First properties

The following equalities hold

* For all $N \geq 1$, $\displaystyle\prod_{i=1}^{\infty} \lambda_i = \prod_{i=1}^{N} \lambda_i \left(\prod_{i=N+1}^{\infty} \lambda_i \right)$.

* For $a > 0$, $\displaystyle\prod_{i=1}^{\infty} (a\lambda_i) = \left(\prod_{i=1}^{\infty} \lambda_i \right) a^{\zeta_\lambda(0)}$

* If A and B form a partition of the positive integers, then $\displaystyle\prod_{i=1}^{\infty} \lambda_i = \prod_{i\in A} \lambda_i \prod_{i\in B} \lambda_i$.

2.3 Examples

One can find several examples in the literature, (taken, e.g., from [13, 15, 16, 20, 25] or deduced from properties in Section 2.2):

$$\prod_{n\geq 0}(n+x) = \frac{\sqrt{2\pi}}{\Gamma(x)} \quad \text{(for } x>0\text{; Lerch formula [}*\text{])}$$

$$\prod_{n\geq 1} n = \sqrt{2\pi} \quad \text{(Lerch formula for } x=1\text{), hence, e.g., } \prod_{n\geq 1}(2n) = \sqrt{\pi}, \text{ and } \prod_{n\geq 1}(2n+1) = \sqrt{2}$$

$$\prod_{n\geq 0}(n^2+1) = e^{\pi} - e^{-\pi} \quad \text{(general Lerch formula [}*\text{])}$$

$$\prod_{n\geq 0}(n^2-n+1) = e^{\pi\frac{\sqrt{3}}{2}} + e^{-\pi\frac{\sqrt{3}}{2}} \quad [*]$$

$$\prod_{n\geq 0}(n^4+1) = 2\left(\cosh(\pi\sqrt{2}) - \cos(\pi\sqrt{2})\right) \quad [*]$$

$$\prod_{n\geq 1} n^n = e^{-\zeta'(-1)} = Ae^{-1/12} \quad (A = 1.2824271\ldots \text{ is the Glaisher-Kinkelin constant [}**\text{])}$$

$$\prod_{n\geq 0} a^n = a^{-1/12} \quad \text{(for } a>1\text{, [}***\text{])}$$

$$\prod_{n\in Sq} n = 2\pi \quad \text{(where } Sq \text{ is the set of squarefree positive integers; compare with } \prod_{n\geq 1} n^2 = 2\pi)$$

[$*$] The original formula proved by Lerch (cited in [15, p. 941–942]) reads

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\sum_{n\geq 0}\frac{1}{(n+x)^s}\right)_{s=0} = \log\frac{\Gamma(x)}{\sqrt{2\pi}}\cdot$$

Actually Lerch proved (cited in [15, Equality (2), p. 942]) the general formula

$$\prod_{n\geq 0}((n+x)^2+y^2) = \frac{2\pi}{\Gamma(x+iy)\Gamma(x-iy)}$$

which of course implies the classical Lerch formula. This general Lerch formula was generalized in [15] where $((n+x)^2+y^2)$ is replaced with $((n+x)^m+y^m)$.

[$**$] Recall that the Glaisher-Kinkelin constant $A = 1.2824271\ldots$ can be defined in several ways (see, e.g., [11] and the references therein):

$$A = \lim_{n\to\infty}\frac{1^1\times 2^2\times 3^3\times \ldots \times n^n}{n^{n^2+n/2+1/12}}\, e^{n^2/4} = e^{\frac{1}{12}-\zeta'(-1)} = (2\pi)^{\frac{1}{12}} e^{\frac{\gamma}{12}-\frac{\zeta'(2)}{2\pi^2}} = \left(2\pi e^{\gamma}\prod_{p \text{ prime}} p^{\frac{1}{p^2-1}}\right)^{1/12}.$$

[***] As indicated in [16] one may recall that formally $\sum_{n\geq 1} n = \zeta(-1) = -1/12$.

Another example of zeta-regularized product is given by the Fibonacci numbers $(F_n)_{n\geq 1}$ in [14] where $\prod_{n=1}^{\infty} F_n$ is computed in terms of the Fibonacci factorial constant and the golden ratio or in terms of the derivative of the Jacobi theta function of the first kind and the golden ratio. Of course the result needs the study of the Dirichlet series $\sum 1/F_n^s$: other similar Dirichlet series or zeta-regularized products are studied in [5] and [7].

Up to generalizing the notion of zeta-regularized product ("super-regularization"), one has ([18], also see [21]):

$$\prod_{p \text{ prime}} p = 4\pi^2.$$

3 The zeta-regularized product of odious numbers

In what follows, we let $\mathcal{O}$ denote the set of *odious* positive integers, i.e., of positive integers whose sum of binary digits is odd, and $\mathcal{E}$ the set of *evil* numbers, i.e., of positive integers whose sum of binary digits is even. We let $a(n)$ denote the characteristic function of odious numbers (i.e., $a(n) = 1$ if the sum of binary digits of n is odd, and $a(n) = 0$ if it is even) and $\varepsilon_n = (-1)^{a(n)}$ (note that $\varepsilon_n = 1 - 2a(n)$). Clearly ε_n is equal to $+1$ if the sum of the binary digits of n is even, and to -1 if the sum is odd; in other words $(\varepsilon_n)_{n\geq 1}$ is (up to its first term) the famous Thue-Morse sequence on the alphabet $\{-1, +1\}$ (see, e.g., [4]).

Theorem 3.1 *We have with the notation above, with* $Q = \displaystyle\prod_{n\geq 1}\left(\frac{2n}{2n+1}\right)^{\varepsilon_n}$,

$$\prod_{n\in\mathcal{O}} = (2\pi)^{1/4}Q^{-1/2}$$
$$\prod_{n\in\mathcal{E}} = (2\pi)^{1/4}Q^{1/2}$$

Also $Q = \dfrac{2^{-1/2}e^{\gamma}}{\varphi}$ *where* φ *is the Flajolet-Martin constant [9].*

Proof. For $\Re s > 1$ we have

$$\zeta_{\mathcal{O}}(s) = \sum_{n\in\mathcal{O}}\frac{1}{n^s} = \sum_{n\geq 1}\frac{a(n)}{n^s} = \sum_{n\geq 1}\frac{1-\varepsilon_n}{2n^s} = \frac{1}{2}\zeta(s) - \frac{1}{2}g(s)$$

where ζ is the Riemann zeta function and $g(s) := \displaystyle\sum_{n\geq 1}\frac{\varepsilon_n}{n^s}$.

But, by [2, Theorem 1.2] with $q = 2$ (also see [9, Lemma 1]) g can be analytically continued to the whole complex plane, and it satisfies, for all $s \in \mathbb{C}$,

$$g(s) = -1 + \sum_{k\geq 1}(-1)^{k+1}\binom{s+k-1}{k}\frac{g(s+k)}{2^{s+k}}. \tag{1}$$

This implies that $g(0) = -1$ and

$$\frac{g(s)-g(0)}{s} = \sum_{k\geq 1}(-1)^{k+1}\frac{(s+1)(s+2)\ldots(s+k-1)}{k!}\frac{g(s+k)}{2^{s+k}}$$

hence, by letting s tend to 0:

$$g'(0) = \sum_{k\geq 1}(-1)^{k+1}\frac{g(k)}{k2^k}.$$

On the other hand, mimicking a computation in [2, p. 534], one has

$$\sum_{k\geq 1}(-1)^{k+1}\frac{g(k)}{k2^k} = \sum_{k\geq 1}\frac{(-1)^{k+1}}{k2^k}\sum_{n\geq 1}\frac{\varepsilon_n}{n^k} = \sum_{n\geq 1}\varepsilon_n\sum_{k\geq 1}\frac{(-1)^{k+1}}{k2^kn^k} = \sum_{n\geq 1}\varepsilon_n\log\left(1+\frac{1}{2n}\right) = -\log Q$$

where $Q := \prod_{n\geq 1}\left(\frac{2n}{2n+1}\right)^{\varepsilon_n}$. So that $g'(0) = -\log Q$. We thus obtain

$$\zeta'_{\mathcal{O}}(0) = \frac{1}{2}\zeta'(0) - \frac{1}{2}g'(0) = -\frac{1}{4}\log(2\pi) + \frac{1}{2}\log Q$$

which finally yields

$$\prod_{n\in\mathcal{O}} n = e^{\frac{1}{4}\log(2\pi) - \frac{1}{2}\log Q} = (2\pi)^{1/4}Q^{-1/2}.$$

Now

$$\prod_{n\in\mathcal{O}} n \prod_{n\in\mathcal{E}} n = \prod_{n\geq 1} n = \sqrt{2\pi}$$

which gives

$$\prod_{n\in\mathcal{E}} n = (2\pi)^{1/4}Q^{1/2}.$$

It remains to recall that the Flajolet-Martin constant φ [9] is equal to

$$\varphi := 2^{-1/2}e^{\gamma}\frac{2}{3}\prod_{n\geq 1}\left(\frac{(4n+1)(4n+2)}{4n(4n+3)}\right)^{\varepsilon_n} = 0.77351\ldots$$

and thus ([8, Section 6.8.1], also see [1])

$$\varphi := \frac{2^{-1/2}e^{\gamma}}{Q} \quad \text{hence} \quad Q = \frac{2^{-1/2}e^{\gamma}}{\varphi}\cdot$$

Remark 3.2 Instead of considering the odious and evil numbers, one might have considered –in a rather non-natural way– the shifted odious and evil numbers, namely the sets $\mathcal{O}_S := \{n+1,\ n\in\mathcal{O}\}$ and $\mathcal{E}_S := \{n+1,\ n\in\mathcal{E}\}$. Then $\prod_{n\in\mathcal{O}_S} n = \exp(-\zeta'_{\mathcal{O}_S}(0))$. But, with the notation of the proof of Theorem 3.1, and using that $\varepsilon_0 = 1$,

$$\zeta_{\mathcal{O}_S}(s) = \sum_{n\in\mathcal{O}}\frac{1}{(n+1)^s} = \sum_{n\geq 1}\frac{a(n)}{(n+1)^s} = \frac{1}{2}\sum_{n\geq 1}\frac{1-\varepsilon_n}{(n+1)^s} = \frac{1}{2}\sum_{n\geq 0}\frac{1-\varepsilon_n}{(n+1)^s} = \frac{1}{2}\zeta(s) - \frac{1}{2}f(s)$$

where $f(s) = \sum_{n\geq 0}\frac{\varepsilon_n}{(n+1)^s}$. It was proved in [2] that this function f can be analytically continued to the whole complex plane, and that $f'(0) = \frac{\log 2}{2}$. Hence $\zeta'_{\mathcal{O}_S}(0) = -\frac{1}{4}(\log(2\pi) + \log 2) = -\frac{1}{4}\log 4\pi$. This gives finally

$$\prod_{n\in\mathcal{O}_S} n = 2^{1/2}\pi^{1/4} \quad \text{and hence} \quad \prod_{n\in\mathcal{E}_S} n = \pi^{1/4}.$$

4 Beyond odious and evil

The main tool used in the computation of the zeta-regularized product of odious or of evil numbers above is the "infinite functional equation" (1). A multidimensional analog of this equation exists for "automatic" sequences [3]. We could expect a similar result in the general case of a zeta-regularized product of integers having an automatic characteristic function. What makes things more complicated in the general case is that the involved zeta function occurs as a component of a vector of Dirichlet series satisfying an infinite functional equation, but that the zeta function itself does not necessarily satisfy such an equation. Furthermore this vector is meromorphic but not necessarily holomorphic on $\mathbb{C}$ (in particular 0 might be a pole). As suggested by the referee, looking at the subcase given by the parity of block-counting sequences could well be the right generalization of the Thue-Morse case. Trying to follow this suggestion, we only arrived at a partial result for, e.g., the Golay-Shapiro (also called Rudin-Shapiro) sequence where, instead of considering the parity of the number of 1's in the binary expansions of integers for the Thue-Morse sequence, one considers the number of 11's in the binary expansions of integers. We hope to revisit these questions in the near future.

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