# SYMMETRY FROM SECTIONAL INTEGRALS FOR CONVEX DOMAINS 

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#### Abstract

Let $\Omega$ be a bounded convex domain in $\mathbb{R}^{n}(n \geq 2)$. In this work, we prove that if there exists an integrable function $f$ such that it's Radon transform over ( $n-1$ )-dimensional hyperplanes intersecting the domain $\Omega$ is a strictly positive function of distance to the nearest parallel supporting hyperplane to $\Omega$, then $\Omega$ is a ball and the function $f$ is a unique radial function about the center of $\Omega$.


## 1. Introduction

Let $\Omega$ be a domain in $\mathbb{R}^{n}$ and $f$ be an integrable function in $\Omega$. We extend $f$ to be identically 0 outside of $\Omega$. Let $\langle$,$\rangle denotes the usual inner product in \mathbb{R}^{n}$. Throughout the article $\mathcal{H}^{k}\llcorner G$ denotes the $k$-dimensional Hausdorff measure restricted to $G$, a Borel measurable subset of $\mathbb{R}^{n}$ for $1 \leq k \leq n$.

The ray transform integrates scalar functions over straight lines. The family of oriented lines can be parametrized by the points on the manifold

$$
T \mathbb{S}^{n-1}=\left\{(x, \xi) \in \mathbb{R}^{n} \times \mathbb{R}^{n}:\langle x, \xi\rangle=0,|\xi|=1\right\} \subset \mathbb{R}^{n} \times \mathbb{R}^{n}
$$

which is the tangent bundle of the unit sphere. The ray transform $I$ of an integrable function $f$ is a function defined on $T \mathbb{S}^{n-1}$ as

$$
\begin{equation*}
I f(x, \xi)=\int_{-\infty}^{+\infty} f(x+t \xi) d t \tag{1.1}
\end{equation*}
$$

The Radon transform integrates the functions over the hyperplanes. The Radon transform $R$ of a function $f \in L^{1}(\Omega)$ is the function defined on $\mathbb{S}^{n-1} \times \mathbb{R}$ by

$$
\begin{equation*}
R f(w, p)=\int_{\Sigma_{\omega, p}} f(x) d \mathcal{H}^{n-1}\left\llcorner\Sigma_{\omega, p}\right. \tag{1.2}
\end{equation*}
$$

where $\Sigma_{w, p}=\left\{x \in \mathbb{R}^{n} \mid\langle x, \omega\rangle=p\right\}$ denotes the hyperplane with $p$ as the perpendicular distance from the origin and $\omega$ is normal to the plane. $\left.I f\right|_{\Omega}$ denotes the ray transform of $f$ along all the lines intersecting the domain $\Omega$. Similar definition stands for the notation $\left.R f\right|_{\Omega}$. The operators (1.1) and (1.2) have been well studied and has many applications in computer tomography. For more detailed study of the operators $I$ and $R$ we refer [2, 4].

In dimension $n=2$ both operators $I$ and $R$ coincide. The characterization of range of these operators have been well studied in case of Schwartz class functions $\mathcal{S}\left(\mathbb{R}^{n}\right)$. It is known that $I$ and $R$ are linear isomorphisms between the Schwartz spaces $\mathcal{S}\left(\mathbb{R}^{n}\right)$ to $\mathcal{S}\left(T \mathbb{S}^{n-1}\right)$ and $\mathcal{S}\left(\mathbb{S}^{n-1} \times \mathbb{R}\right)$ respectively. Both the operators $I$ and $R$ are injective, i.e., if $I f=0$ imply $f=0$ and $R f=0$ imply $f=0$.

The question one is interested in is,
Question 1. Are non-zero constant functions in the range of ray transforms, i.e., does $\exists f \in L^{1}(\Omega)$ such that $\left.I f\right|_{\Omega}=c(\neq 0)$ ?

We came to know about this problem when second named author in this paper attended a conference talk by Joonas Ilmavirta titled "Functions of constant $X$-ray transform" based on his joint work with Gabriel Paternain in University of Jyväskylä, Finland on "Inverse problems: PDE and Geometry", where they gave a positive answer to the above question (see [3]). They showed that in Euclidean spaces of

[^0]dimension $n \geq 2$, strictly convex domains $\Omega$ admitting functions of constant $X$-ray transform are balls and the function is unique and radial. They also considered the analogue of the problem on Riemannian manifolds. The function $f(x)=\frac{\chi\{|x|<R\}}{\pi \sqrt{R^{2}-|x|^{2}}}$ in the ball of radius $R$ centered at origin of $\mathbb{R}^{n}$ has constant X-ray transform in its support in $\mathbb{R}^{n}$.

Similarly, one can ask the same question for the Radon transform in dimensions $n \geq 2$.
Question 2. Are non-zero constant functions in the range of Radon transforms in dimension $n \geq 2$, i.e., does $\exists f \in L^{1}(\Omega)$ such that $\left.R f\right|_{\Omega}=c(\neq 0)$ ?

We address this question in section 2 and show that one cannot expect constants to be in the range of Radon transform in dimensions $n \geq 3$. Instead we prove the following analogue. If $\Omega$ is a bounded convex domain in $\mathbb{R}^{n}(n \geq 2)$ and if $\exists f \in L^{1}(\Omega)$ such that $\left.R f\right|_{\Omega}$ is a strictly positive function of distance to nearest parallel supporting hyperplane to $\Omega$ alone [see Theorem 2.1 for precise definition], then $\Omega$ is a ball and $f$ is a unique radial function about the center of $\Omega$.

The proofs utilize the zeroth and the first order moments of the function which was originally used in context of ray transform in the plane by Joonas Ilmavirta \& Gabriel Paternain.

Subsequently, as a corollary to the main result in our paper, we obtain the result for the ray transform $\left.I f\right|_{\Omega}=c(\neq 0)$ in Euclidean space under milder assumptions requiring $\Omega$ to be a bounded domain in $\mathbb{R}^{2}$ or a bounded convex domain in $\mathbb{R}^{n}$ for $n \geq 3$. To our knowledge, the main result in Section 2 is completely new.

Remark 1.1. If $\Omega$ is bounded convex domain in $\mathbb{R}^{n}$, any line $\ell$ intersecting $\Omega$ intersects $\partial \Omega$ at exactly two points. Also if $p \in \partial \Omega$ then there exists at least one supporting $(n-1)$-dimensional hyperplane of $\Omega$ passing through $p$.

## 2. RADON TRANSFORM AS A DISTANCE FUNCTION

We begin by stating the main result.
Theorem 2.1. Let, $\Omega \subset \mathbb{R}^{n}$ be a bounded convex domain $(n \geq 2)$ and $G:[0, \infty) \rightarrow \mathbb{R}$ be a strictly positive locally integrable function. If, $\exists f: \Omega \rightarrow \mathbb{R}$ an integrable function such that,

$$
\begin{equation*}
\int_{\Omega \cap \Sigma} f(x) d \mathcal{H}^{n-1}\left\llcorner\Sigma=G\left(\min _{j=1,2} \operatorname{dist}\left(\Sigma, \Pi_{j}\right)\right)\right. \tag{2.1}
\end{equation*}
$$

for all $(n-1)$-dimensional hyperplanes $\Sigma$ satisfying $\mathcal{H}^{n-1}(\Omega \cap \Sigma)>0$ and $\Pi_{j}$ (for, $j=1,2$ ) are the pair of supporting $(n-1)$-dim hyperplanes to $\Omega$ that are parallel to $\Sigma$, then $\Omega$ is a ball and $f$ is a unique radial function.

Remark 2.2. Before going into the proof of Theorem 2.1 let us address Question 2. Suppose, $\exists f \in L^{1}(\Omega)$ such that $\left.R f\right|_{\Omega}=1$. In view of Theorem 2.1, letting $G \equiv 1$ we see that $\Omega$ has to be a ball, without loss of generality assume $\Omega=B(0, R) \subset \mathbb{R}^{n}$ for some $R>0$. Let us consider the implication of this result on the Fourier slices of the function $f$. Given $\xi \in \mathbb{S}^{n-1}$, let us choose an orthogonal coordinate frame for the space $\mathbb{R}^{n}$ s.t., $x=\left(t, x^{\prime}\right) \in \mathbb{R}_{\xi} \times \mathbb{R}_{\xi^{\perp}}^{n-1}$ and use the notation $\Sigma_{\xi}^{t}$ to denote the $(n-1)$-dimensional hyperplane having $\xi \in \mathbb{S}^{n-1}$ as the normal and passing through the point $\left(t, 0^{\prime}\right) \in \mathbb{R}_{\xi} \times \mathbb{R}_{\xi^{\perp}}^{n-1}$. Now, by Fubini's theorem we have,
$\mathcal{F}(f)(\xi)=\int_{\mathbb{R}^{n}} e^{-i\langle x, \xi\rangle} f(x) d x=\int_{\mathbb{R}_{\xi}} e^{-i|\xi| t}\left(\int_{\mathbb{R}_{\xi \perp}^{n-1}} f\left(t, x^{\prime}\right) d \mathcal{H}^{n-1}\left\llcorner\Sigma_{\xi}^{t}\left(x^{\prime}\right)\right) d t=\int_{-R}^{R} e^{-i|\xi| t} d t=\frac{2 \sin (R|\xi|)}{|\xi|}\right.$.
where, in the third equality we used hypothesis (2.1) with $G \equiv 1$.

It is known that for $n \geq 2$,

$$
\mathcal{F}^{-1}\left(\frac{e^{i t|\xi|}}{|\xi|}\right)(x)= \begin{cases}\frac{1}{2 \pi} \operatorname{sgn}(t) \chi_{\{|x|<|t|\}}\left(t^{2}-|x|^{2}\right)^{-\frac{1}{2}}, & \text { when, } n=2 \\ C_{n} \lim _{\epsilon \downarrow 0}\left(|x|^{2}-(t+i \epsilon)^{2}\right)^{-\frac{n-1}{2}}, & \text { when, } n \geq 3\end{cases}
$$

where, $C_{n}=\frac{\Gamma\left(\frac{n+1}{2}\right)}{(n-1) \pi^{\frac{n+1}{2}}}$ and $t \in \mathbb{R}$, which is integrable when $n=2$ but certainly not in $L^{1}\left(\mathbb{R}^{n}\right)$ for $n \geq 3$. Therefore, there does not exist integrable functions such that $\left.R f\right|_{\Omega}=c(\neq 0)$ in dimensions $n \geq 3$.

Proof of Theorem [2.1. Let $\Omega$ be a bounded convex domain in $\mathbb{R}^{n}(n \geq 2)$. Let us fix an orthogonal coordinate system such that the origin is equidistant from each pair of parallel supporting $(n-1)$ dimensional hyperplanes to $\Omega$ sharing $x_{j}$-axis as their common normal, for $j=1(1) n$. Let us extend the function $f$ in $\Omega^{c}$ by 0 and consider the restriction of first moment of $f$ on the unit sphere in $\mathbb{R}^{n}$,

$$
\begin{align*}
g(\xi) & :=\int_{\mathbb{R}^{n}}\langle x, \xi\rangle f(x) d x, \text { for } \xi \in \mathbb{S}^{n-1}  \tag{2.2}\\
& =\int_{\mathbb{R}_{\xi}^{n-1}} \int_{\mathbb{R}_{\xi}} t f\left(t, x^{\prime}\right) d t d \mathcal{H}^{n-1}\left\llcorner\Sigma_{\xi}^{t}\left(x^{\prime}\right)\right. \tag{2.3}
\end{align*}
$$

where, we decomposed $x=\left(t, x^{\prime}\right) \in \mathbb{R}_{\xi} \times \mathbb{R}_{\xi^{\perp}}^{n-1}$ and in our notation $\Sigma_{\xi}^{t}$ denotes the ( $n-1$ )-dimensional hyperplane having $\xi \in \mathbb{S}^{n-1}$ as the normal and passing through the point $\left(t, 0^{\prime}\right) \in \mathbb{R}_{\xi} \times R_{\xi^{\perp}}^{n-1}$. Let $\Pi_{j}^{\xi}$ for $j=1,2$ be the pair of supporting $(n-1)$-dim hyperplanes of $\Omega$ orthogonal to $\xi$, so that if $r_{1}(\xi):=\inf \left\{t \in \mathbb{R}_{\xi}: \mathcal{H}^{n-1}\left(\Sigma_{\xi}^{t} \cap \Omega\right)>0\right\}$ and $r_{2}(\xi):=\sup \left\{t \in \mathbb{R}_{\xi}: \mathcal{H}^{n-1}\left(\Sigma_{\xi}^{t} \cap \Omega\right)>0\right\}$ then $\Pi_{j}^{\xi}=\Sigma_{\xi}^{r_{j}(\xi)}$ for $j=1,2$. Now, by given condition (2.1) we have,

$$
\begin{equation*}
\int_{\mathbb{R}_{\xi \perp}^{n-1}} f\left(t, x^{\prime}\right) d \mathcal{H}^{n-1}\left\llcorner\Sigma_{\xi}^{t}\left(x^{\prime}\right)=G\left(\min _{j=1,2} \operatorname{dist}\left(\Sigma_{\xi}^{t}, \Pi_{j}^{\xi}\right)\right)\right. \tag{2.4}
\end{equation*}
$$

for $t \in \mathbb{R}_{\xi}$ such that the plane $\Sigma_{\xi}^{t}$ satisfies $\mathcal{H}^{n-1}\left(\Sigma_{\xi}^{t} \cap \Omega\right)>0$. Using equation (2.4) in equation (2.3) and writing $r_{j}=r_{j}(\xi)(j=1,2)$ for brevity we have,

$$
\begin{align*}
g(\xi) & =\int_{\mathbb{R}_{\xi}} t\left(\int_{\mathbb{R}_{\xi \perp}^{n-1}} f\left(t, x^{\prime}\right) d \mathcal{H}^{n-1}\left\llcorner\Sigma_{\xi}^{t}\left(x^{\prime}\right)\right) d t\right. \\
& =\int_{r_{1}}^{r_{2}} t\left(\int_{\mathbb{R}_{\xi \perp}^{n-1}} f\left(t, x^{\prime}\right) d \mathcal{H}^{n-1}\left\llcorner\Sigma_{\xi}^{t}\left(x^{\prime}\right)\right) d t\right. \\
& =\int_{\frac{r_{2}+r_{1}}{2}}^{r_{2}} t G\left(r_{2}-t\right) d t+\int_{r_{1}}^{\frac{r_{2}+r_{1}}{2}} t G\left(t-r_{1}\right) d t \\
& =\int_{0}^{\frac{2_{2}-r_{1}}{2}}\left(r_{2}-t\right) G(t) d t+\int_{0}^{\frac{r_{2}-r_{1}}{2}}\left(t+r_{1}\right) G(t) d t \\
& =\left(r_{2}(\xi)+r_{1}(\xi)\right) \int_{0}^{\frac{r_{2}-r_{1}}{2}(\xi)} G(t) d t . \tag{2.5}
\end{align*}
$$

Similarly we have,

$$
\begin{align*}
K=\int_{\Omega} f(x) d x & =\int_{\mathbb{R}_{\xi \perp}^{n-1}} \int_{\mathbb{R}_{\xi}} f\left(t, x^{\prime}\right) d t d \mathcal{H}^{n-1}\left\llcorner\Sigma_{\xi}^{t}\left(x^{\prime}\right)\right. \\
& =\int_{r_{1}(\xi)}^{r_{2}(\xi)}\left(\int_{\mathbb{R}_{\xi \perp}^{n-1}} f\left(t, x^{\prime}\right) d \mathcal{H}^{n-1}\left\llcorner\Sigma_{\xi}^{t}\left(x^{\prime}\right)\right) d t\right. \\
& =2 \int_{0}^{\frac{r_{2}-r_{1}}{2}(\xi)} G(t) d t, \forall \xi \in \mathbb{S}^{n-1} . \tag{2.6}
\end{align*}
$$

Therefore, $g(\xi)=\frac{K}{2}\left(r_{2}(\xi)+r_{1}(\xi)\right)$. We note that by our choice of origin we have $r_{2}\left( \pm e_{j}\right)+r_{1}\left( \pm e_{j}\right)=0$ and hence $g\left( \pm e_{j}\right)=0$ for each $j=1(1) n$ where, $e_{j}$ is the unit vector along $x_{j}$-axis. By definition $g$ is restriction of a linear function on $\mathbb{S}^{n-1}$ and $g\left( \pm e_{j}\right)=0$ for $j=1(1) n$. Therefore, $g \equiv 0$ in $\mathbb{S}^{n-1}$ and hence, $r_{2}(\xi)=-r_{1}(\xi), \forall \xi \in \mathbb{S}^{n-1}$.

Now, the function $u(r)=\int_{0}^{r} G(t) d t$ is injective (since, $G>0$ by hypothesis). From equation (2.6) we have, $u\left(\frac{r_{2}(\xi)-r_{1}(\xi)}{2}\right)=\frac{K}{2}$, for all $\xi \in \mathbb{S}^{n-1}$, therefore we conclude $r_{2}(\xi)-r_{1}(\xi)=2 R$ (for some positive constant $R$ ). That is the domain $\Omega$ has constant width in the sense that the distance between any pair of parallel ( $n-1$ )-dim supporting hyperplanes of $\Omega$, with common normal $\xi \in \mathbb{S}^{n-1}$, is a constant and equals $2 R$.

Combining these two facts we see that $\left|r_{2}(\xi)\right|=\left|r_{1}(\xi)\right|=R$ for all $\xi \in \mathbb{S}^{n-1}$, i.e., the distance of any ( $n-1$ )-dim supporting hyperplane of $\Omega$ from origin is constant and equals $R$.

Now, by convexity of $\Omega$ it must lie in the intersection of all closed half-spaces $\bar{H}_{\xi}$ containing origin whose boundaries $\partial H_{\xi}$ are $(n-1)$-dimensional supporting hyperplanes of $\Omega$ (with $\xi \in \mathbb{S}^{n-1}$ as outward normal to $\partial H_{\xi}$ ). Since, $\operatorname{dist}\left(0, \partial H_{\xi}\right)=R$, we conclude $\Omega \subseteq \bigcap_{\xi \in \mathbb{S}^{n-1}} \bar{H}_{\xi}=\overline{B(0, R)}$. On the other hand suppose $p \in \partial \Omega$ and $\Pi_{p}$ is a supporting ( $n-1$ )-dimensional hyperplane of $\Omega$ passing through $p$, then $\operatorname{dist}(0, p) \geq \operatorname{dist}\left(0, \Pi_{p}\right)=R$. Hence, $\operatorname{dist}(0, p)=R$ for all $p \in \partial \Omega$. Consequently, $\Omega$ is the ball $B(0, R)$ in $\mathbb{R}^{n}$.

Now, $f$ is radial follows from uniqueness of Radon transform and the fact that $\Omega$ is a ball. We observe that the function $f \circ A$, where $A$ is any orthogonal matrix, also satisfies hypothesis (2.1) and hence from uniqueness of Radon transform we conclude $f=f \circ A$, for all orthogonal matrices $A$. Therefore $f$ is a radial function. This completes the proof.

Any integrable positive radial function in a ball in $\mathbb{R}^{n}$ satisfies the hypothesis (2.1). However, the following example is particularly interesting.
Example 2.1. The functions $f(x)=\left(R^{2}-|x|^{2}\right)^{\gamma}$ for $\gamma>-1$ satisfies the hypothesis in (2.1) with,

$$
\int_{\Omega \cap \Sigma^{d}} f d \mathcal{H}^{n-1}\left\llcorner\Sigma^{d}=c_{n}(\gamma)\left(R^{2}-d^{2}\right)^{\frac{n-1}{2}+\gamma}\right.
$$

where, $\Sigma^{d}$ is a ( $n-1$ )-dimensional hyperplane at a distance $d$ from the origin, therefore at a distance $(R-d)$ from the nearest parallel supporting hyperplane of $\Omega=B(0, R)$ and $G(R-d)=c_{n}(\gamma)\left(R^{2}-d^{2}\right)^{\frac{n-1}{2}+\gamma}$ for, $0 \leq d<R$. The constant $c_{n}(\gamma)=\frac{1}{2}(n-1) \alpha_{n-1} \int_{0}^{1} r^{\frac{n-3}{2}}(1-r)^{\gamma} d r=\frac{1}{2}(n-1) \alpha_{n-1} \beta\left(\frac{n-1}{2}, \gamma+1\right)$, (where, $\alpha_{n-1}$ is the volume of $(n-1)$-dim unit ball and $\beta(x, y)$ denotes the Beta function) depends only on the dimension $n$ and $\gamma$.

Now, we obtain the results for ray transform as corollaries to Theorem 2.1. integrable function $f: \Omega \rightarrow \mathbb{R}$ such that,

$$
\begin{equation*}
\int_{\Omega \cap \ell} f(x) d \mathcal{H}^{1}\llcorner\ell=1 \tag{2.7}
\end{equation*}
$$

for all (1-dim) lines $\ell$ with $\mathcal{H}^{1}(\Omega \cap \ell)>0$, then $\Omega$ is a ball and $f$ is a unique radial function about the center of $\Omega$.

Proof. Case-I: We start by analyzing the simplest case when $\Omega$ is a bounded convex domain in the $\mathbb{R}^{2}$. Then, as a consequence of Theorem 2.1 with $G \equiv 1$ we conclude that $\Omega$ is a disk in $\mathbb{R}^{2}$.

The function $f(x)=\frac{\chi_{\{|x|<R\}}}{\pi \sqrt{R^{2}-|x|^{2}}}$ satisfies the hypothesis (2.7) in the domain $\Omega=B(0, R) \subset \mathbb{R}^{2}$ and is the required radial function. Uniqueness of $f$ follows from injectivity of ray transform.

Now, for a general bounded domain $\Omega \subset \mathbb{R}^{2}$, we note that requiring a line $\ell$ to intersect $\Omega$ i.e., $\mathcal{H}^{1}(\Omega \cap \ell)>0$ is equivalent to $\mathcal{H}^{1}(\operatorname{Conv}(\Omega) \cap \ell)>0$, where $\operatorname{Conv}(\Omega)$ denotes the convex hull of $\Omega$. Suppose, $\exists f \in L^{1}(\Omega)$ satisfying hypothesis (2.7) and $\operatorname{Conv}(\Omega) \backslash \Omega \neq \phi$. Then, from the previous consideration, $\operatorname{Conv}(\Omega)$ is a ball and $f$ is as before, which would contradict that fact that $f \equiv 0$ on $\operatorname{Conv}(\Omega) \backslash \Omega$. Therefore, $\operatorname{Conv}(\Omega)=\Omega$.

Case-II: Now, we address the case of bounded convex domains $\Omega$ in $\mathbb{R}^{n}$ when, $n \geq 3$.
If, $f$ is a function satisfying (2.7) in a convex set $\Omega \subset \mathbb{R}^{n}$, then every 2 -dimensional section of $\Omega$ i.e., $\Omega \cap \Pi$ is convex (where, $\Pi$ is a 2 -dimensional plane intersecting $\Omega$ s.t., $\mathcal{H}^{2}(\Omega \cap \Pi)>0$ ) and admits the function $f$. From the previous case $(n=2)$ we conclude that the 2 -dimensional section $\Omega \cap \Pi$ must be equivalent to a disk in $\mathbb{R}^{2}$. Therefore by Lemma 2.4 the domain $\Omega$ admitting $f$ is a ball $B\left(p_{c}, R\right)$ (say) and $f(x)=\frac{\chi_{\left\{\left|x-p_{c}\right|<R\right\}}}{\pi \sqrt{R^{2}-\left|x-p_{c}\right|^{2}}}$ satisfying the hypothesis (2.7) is the unique radial function.
Lemma 2.4. Let, $n \geq 3$ and $\Omega$ be a bounded convex domain in $\mathbb{R}^{n}$ such that $\Omega \cap \Pi$ is equivalent to a disk in $\mathbb{R}^{2}$ for all 2 -dimensional planes $\Pi$ with $\mathcal{H}^{2}(\Omega \cap \Pi)>0$, then $\Omega$ is a ball in $\mathbb{R}^{n}$.

Proof. Let, $\ell_{m}$ be a maximal diametric line in $\Omega$ i.e., $\mathcal{H}^{1}\left(\Omega \cap \ell_{m}\right)=\sup \mathcal{H}^{1}(\Omega \cap \ell)=2 R$ (say) and let us denote the midpoint of $\ell_{m} \cap \partial \Omega$ with $p_{c}$. Let $p$ be any point on $\partial \Omega \backslash\left\{\ell_{m}^{\ell} \cap \partial \Omega\right\}$ and $\Pi_{p}$ be a 2-dimensional plane passing through $p$ and containing the line $\ell_{m}$. Then, $\Pi_{p}$ must intersect the domain $\Omega$ in a set equivalent to a 2 -dimensional disk of radius $R$ by the maximality of $\mathcal{H}^{1}\left(\Omega \cap \ell_{m}\right)$. Therefore, $\operatorname{dist}\left(p_{c}, p\right)=R$ for all $p \in \partial \Omega$, i.e., $\Omega$ is the ball $B\left(p_{c}, R\right)$ in $\mathbb{R}^{n}$. This completes the proof of the lemma.

In case $n \geq 3$ and $\Omega$ is a strictly convex domain, we note that information on lines close enough to the boundary of $\Omega$ is sufficient to conclude that $\Omega$ is a ball.

Corollary 2.5. Let, $n \geq 3$ and $\Omega$ be a strictly convex domain. Suppose, for every $p \in \partial \Omega$ there is a neighbourhood $\mathcal{U}_{p}\left(\subset \mathbb{R}^{n}\right)$ of $p$ s.t., hypothesis (2.7) holds for all lines $\ell$ with $\Omega \cap \ell \subset \mathcal{U}_{p}$ and $\mathcal{H}^{1}(\Omega \cap \ell)>0$. Then, $\Omega$ is a ball.

Proof. Arguing as before ( $n=2$ case) we have $\Omega \cap \Sigma$ is equivalent to an open disk in $\mathbb{R}^{2}$ for every 2 dimensional plane $\Sigma$ s.t., $\Omega \cap \Sigma \subset \mathcal{U}_{p}$ and $\mathcal{H}^{2}(\Omega \cap \Sigma)>0$. Therefore, the Dupin's Indicatrix (see [1, pp. 363-365]) of $p$ along any 2 -dimensional plane spanned by a pair of principal directions is a circle, meaning $p$ must be umbilical. Therefore, boundary of $\Omega$ being all umbilical must be a sphere.

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