# On the Borel Submonoid of a Symplectic Monoid 

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#### Abstract

In this article, we study the Bruhat-Chevalley-Renner order on the complex symplectic monoid $M S p_{n}$. After showing that this order is completely determined by the Bruhat-Chevalley-Renner order on the linear algebraic monoid of $n \times n$ matrices $M_{n}$, we focus on the Borel submonoid of $M S p_{n}$. By using this submonoid, we introduce a new set of type B set partitions. We determine their count by using the "folding" and "unfolding" operators that we introduce. We show that the Borel submonoid of a rationally smooth reductive monoid with zero is rationally smooth. Finally, we analyze the nilpotent subsemigroups of the Borel semigroups of $M_{n}$ and $M S p_{n}$. We show that, contrary to the case of $M S p_{n}$, the nilpotent subsemigroup of the Borel submonoid of $M_{n}$ is irreducible.


Keywords: Symplectic monoid, Renner monoid, Borel submonoid, rationally smooth, set partitions, (un)folding
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## 1 Introduction

Let $M$ be a complex reductive monoid with unit group $G$, and let $B$ be a Borel subgroup in $G$. Then we have a square of inclusions as in the following diagram

where $\bar{B}$ is the Zariski closure of $B$ in $M$; we will call $\bar{B}$ the Borel submonoid determined by $B$. Although its combinatorics and geometry are relatively less explored compared to that of the ambient reductive monoid, the Borel submonoid is a very important object for
the study of the representation theory of $M$ [11, Theorem 3.4]. In the special case of the linear algebraic monoid of $n \times n$ matrices, the $B \times B$-orbits in $\bar{B}$ are parametrized by the set partitions of $\{1, \ldots, n\}$, providing a gateway to an unchartered domain for combinatorialists, see [7]. In this regard, our goal in this paper is to present first combinatorial results, whose analogous versions are obtained in [7], for the Borel submonoid of a "symplectic monoid" that we will define next.

Let $l$ be a positive integer, and set $n:=2 l$. The set of all $l \times l$ matrices with entries from $\mathbb{C}$ will be denoted by $M_{l}$. We let $J$ denote the $n \times n$ matrix, $J=\left[\begin{array}{cc}0 & J_{l} \\ -J_{l} & 0\end{array}\right]$, where $J_{l}$ is the unique antidiagonal $l \times l$ permutation matrix, that is,

$$
J_{l}=\left[\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & . & \vdots & \vdots \\
0 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{array}\right]
$$

The symplectic group is defined by $S p_{n}:=\left\{A \in G L_{n}: A^{\top} J A=J\right\}$. This is the group of linear automorphisms of $\mathbb{C}^{n}$ that preserve the skew-bilinear form that is defined by $J$. (Once we fix the even integer $n=2 l$, in the sequel, it will be convenient for us to denote $S p_{n}$ by $G$.) Let us denote the central extension of $S p_{n}$ in $G L_{n}$ by $G S p_{n}$. This is the smallest reductive subgroup of $G L_{n}$ that contains both of the subgroups $G$ and the group of invertible scalar matrices $\left\{c I_{n}: c \in \mathbb{C}^{*}\right\}$, where $I_{n}$ denotes the $n \times n$ identity matrix. The Zariski closure of $G S p_{n}$ in $M_{n}$ is called the $n$-th symplectic monoid. Such monoids were first considered by Grigor'ev [13]. The following concrete description of the $n$-th symplectic monoid, which we will denote by $M S p_{n}$, is due to Doty [10, Proposition 4.3]: $M S p_{n}:=$ $\left\{A \in M_{n}: A^{\top} J A=A J A^{\top}=c J, c \in \mathbb{C}\right\}$. Basic geometric ingredients (the Renner monoid, the cross section lattice, and a cell decomposition) of $M S p_{n}$ are described explicitly by Li and Renner in [16]. An in-depth analysis of the rational points of $M S p_{n}$ over finite fields, including some fascinating combinatorial formulas about its Renner monoids, are described by Cao, Lei, and Li in [9]. To describe the main results of our paper, next, we will briefly review the Renner monoid of $M S p_{n}$ in relation with the rook monoid.

To keep our notation simple, let us denote by $B$ the Borel subgroup consisting of the upper triangular matrices in $G S p_{n}$. The natural action of $B \times B$ on $M S p_{n}$ is defined by $\left(b_{1}, b_{2}\right) \cdot x=$ $b_{1} x b_{2}^{-1}$, where $b_{1}, b_{2} \in B, x \in M S p_{n}$. This action has finitely many orbits [21, 16] and moreover the orbits are parametrized by a finite inverse semigroup, $M S p_{n}=\bigsqcup_{\sigma \in \mathcal{R}_{G}} B \sigma B$. The finite inverse semigroup $\mathcal{R}_{G}$ is called the symplectic Renner monoid; it is the symplectic version of the rook monoid $\mathcal{R}_{n}$, which consists of $0 / 1$ square matrices of size $n$ with at most one 1 in each row and each column. In fact, $\mathcal{R}_{G}$ is a submonoid of $\mathcal{R}_{n}$. The elements of $\mathcal{R}_{n}$ are called rooks, and we will call the elements of $\mathcal{R}_{G}$ the symplectic rooks. The Bruhat-Chevalley-Renner order on $\mathcal{R}_{n}$ is defined by

$$
\begin{equation*}
\sigma \leq \tau \Longleftrightarrow B_{n} \sigma B_{n} \subseteq \overline{B_{n} \tau B_{n}} \tag{1.1}
\end{equation*}
$$

for $\sigma, \tau \in \mathcal{R}_{n}$. (We will introduce the most general form of the Bruhat-Chevalley-Renner order in the preliminaries section.) An explicit combinatorial description of $\leq$, in the spirit of Deodhar's criteria, is obtained in [8]. By using this explicit characterization of $\leq$, it is shown in [6] that $\left(\mathcal{R}_{n}, \leq\right)$ is a graded, bounded, EL-shellable poset.

The first main observation in our paper, Theorem 3.9, states that, for $\sigma, \tau \in \mathcal{R}_{G}$, we have

$$
\sigma \leq \tau \text { in } \mathcal{R}_{G} \Longleftrightarrow \sigma \leq \tau \text { in } \mathcal{R}_{n} .
$$

An important family of subposets of $\mathcal{R}_{n}$ are defined as follows. Let $k$ be an integer in $\{0,1, \ldots, n\}$, and let

$$
\mathcal{B}_{n}(k):=\left\{\sigma \in \mathcal{R}_{n}: \sigma \text { is upper triangular and } \operatorname{rank}(\sigma)=k\right\}
$$

and

$$
\mathcal{B}_{n}:=\bigsqcup_{k=0}^{n} \mathcal{B}_{n}(k)
$$

Then $\mathcal{B}_{n}$ parametrizes the $B_{n} \times B_{n}$-orbits in the Borel monoid $\overline{B_{n}}$. Actually, $\mathcal{B}_{n}$ is the lower interval $[0,1]=\left\{x \in \mathcal{R}_{n}: x \leq 1\right\}$ in $\mathcal{R}_{n}$. Therefore, $\mathcal{B}_{n}$ is also EL-shellable. Generalizing this observation, in [7], joint with Cherniavsky, the first author showed that each poset $\left(\mathcal{B}_{n}(k), \leq\right)(k \in\{0,1, \ldots, n\})$ is a graded, bounded, EL-shellable poset. In fact, it turns out that $\left(\mathcal{B}_{n}(k), \leq\right)$ is a union of $\binom{n}{k}$ maximal subintervals all of which have the same minimum element. An important combinatorial aspect of this development is that, as a set, $\mathcal{B}_{n}(k)$ is in bijection with the set partitions of $\{1, \ldots, n+1\}$ with $k$ blocks. In particular, the cardinality $\left|\mathcal{B}_{n}(k)\right|$ is given by the Stirling numbers of the second kind, $S(n+1, k)$. In our second main result, we obtain similar results for the rank $k$ elements of the Borel submonoid $\bar{B}$ in $M S p_{n}$. We should mention that the type BC analogs of the set partitions with respect to "refinement order" is well known [19]. For a more recent study of their combinatorial properties, see [1].

We will denote by $\mathcal{B}_{G}$ the submonoid of all upper triangular elements in the symplectic Renner monoid $\mathcal{R}_{G}$. In other words, $\mathcal{B}_{G}=\left\{x \in \mathcal{R}_{G}: x \leq 1\right\}=[0,1]$ in $\mathcal{R}_{G}$. The $k$-th symplectic Stirling poset, denoted by $\mathcal{B}_{G}(k)$, is the subposet defined by

$$
\begin{equation*}
\mathcal{B}_{G}(k):=\left\{x \in \mathcal{B}_{G}: \operatorname{rank}(x)=k\right\} . \tag{1.2}
\end{equation*}
$$

In Theorem 4.6, we prove that the $k$-th symplectic Stirling poset is a graded bounded poset with unique minimum element, and with $\binom{l}{k} 2^{k}$ maximal elements, all of which are rank $k$ diagonal idempotents. It is now a natural question to find the cardinality of each of the posets $\mathcal{B}_{G}(k)$. We answer this question in Theorem 5.13. It turns out that

$$
\left|\mathcal{B}_{G}(k)\right|=\sum_{a+b+c=k} 2^{a+c} 3^{b}\binom{l}{b} S(l+1, l+1-a) S(l+1, l+1-c),
$$

where $(a, b, c) \in \mathbb{Z}_{\geq 0}^{3}$. Here, for integers $s, t \in \mathbb{Z}$ such that $s \leq t, S(s, t)$ stands for the $(s, t)$-th Stirling numbers of the second kind.

Reductive monoids are regular in the semigroup theory sense. Geometrically, the only smooth reductive monoids with one-dimensional center are the monoids of $n \times n$ matrices [20]. A complex algebraic variety of dimension $n$ is called rationally smooth if for every $x \in X$, the local cohomology groups $H^{i}(X, X \backslash\{x\})$ are zero for $i \neq 2 n$, and $H^{2 n}(X, X \backslash\{x\})=$ $\mathbb{Q}$. It turns out that the rationally smooth reductive monoids have rich combinatorial and geometric structures [22, 23]. Their classification has been completed by Renner [22, 24]. In particular, $M S p_{n}$ is a rationally smooth monoid. Gonzales showed that the rationally smooth reductive monoids are GKM manifolds, see [12]. This means that the relevant (equivariant) cohomological data for such a monoid can be recovered from the knowledge of torus invariant points and curves alone.

In Theorem 6.3, we show that, if $M$ is a rationally smooth reductive monoid with zero, and $\bar{B}$ is a Borel submonoid in $M$, then $\bar{B}$ is rationally smooth as well. Although we do not exploit this information here, we can now use Renner's $H$-polymomials for computing the intersection cohomology Poincaré polynomials of many Borel submonoids. In particular, this idea is applicable to the case of $M S p_{n}$. We plan to revisit this topic in a future paper.

We now describe the organization of our paper. In Section 2 we briefly summarize some basic properties of the symplectic groups and monoids. Section 3 is devoted to the study of the Bruhat-Chevalley-Renner order on $M S p_{n}$. In this section we prove our first result, Theorem 3.9. Empowered by the concrete description of the partial order, we begin our study of the Borel submonoid of $M S p_{n}$ in Section 4. In particular, in this section, we give a count of the number of rank $k$ elements of the symplectic upper triangular rooks, Theorem 5.13. The purpose of Section 6 is to show that the Borel submonoids of rationally smooth reductive monoids with zeros are rationally smooth, Theorem 6.3. In the final part of our paper, we return to our study of the symplectic monoids. We observe that, unlike the case of the monoid of $n \times n$ matrices, the subsemigroup of nilpotent elements of the Borel submonoid of $M S p_{n}$ is not irreducible for $n \geq 2$.

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## 2 Preliminaries

In this section we will review the basic ingredients of our objects.

### 2.1 Symplectic groups.

Let $n$ be a positive integer of the form $n=2 l$ for some $l \in \mathbb{Z}$. Let us denote by $H$ (resp. $G$ ) the special linear group $S L_{n}$ (resp. the symplectic group $S p_{n}$ ). Then $G \subseteq H$ with equality if $n=2$. We will denote by $T_{H}$ and $B_{H}$ the maximal diagonal torus and the Borel subgroup consisting of upper triangular matrices in $H$, respectively. Then the intersections $T_{G}:=G \cap T_{H}$ and $B_{G}:=G \cap B_{H}$ are, respectively, the maximal diagonal torus and a Borel subgroup containing $T_{G}$ in $G$.

Let $\theta: H \rightarrow H$ denote the following involutory automorphism:

$$
\theta(A)=J\left(A^{\top}\right)^{-1} J^{-1} \quad A \in H
$$

Then the fixed subgroup of $\theta$ in $H$ is $G$. In other words, we have $H^{\theta}=G$. Also, it is easy to verify that $B_{G}=B_{H}^{\theta}$ and that $T_{G}=T_{H}^{\theta}$. With this choice of $T_{H}$ and $B_{H}$, we know that the normalizer of $T_{H}$ in $H$, that is, $N_{H}\left(T_{H}\right)$ is equal to the $n \times n$ monomial matrices in $H$, and the elements of the Weyl group, $W_{H}:=N_{H}\left(T_{H}\right) / T_{H}$, are represented by the permutation matrices of size $n$. We will denote $W_{H}$ by $\mathcal{S}_{n}$. The one-line notation of an element $w$ of $S_{n}$ is the sequence $\left(w_{1}, \ldots, w_{n}\right)$, where $w_{i}=w(i)$ for $i \in\{1, \ldots, n\}$. In this notation, the Weyl group of $\left(G, T_{G}\right)$ has a convenient description as the fixed point subgroup of the induced involution, $\theta: S_{n} \rightarrow S_{n}$ which is defined by

$$
\theta(w):=\left(n+1-w_{n}, n+1-w_{n-1}, \ldots, n+1-w_{1}\right) \quad w \in S_{n}
$$

In other words, we have

$$
W_{G}=\left\{w \in S_{n}: \theta(w)=w\right\} .
$$

By working with the root system corresponding to $\left(G, B_{G}, T_{G}\right)$, one knows that ( $W_{G}, S_{G}$ ), as a Coxeter group, is generated by

$$
S_{G}=\left\{r_{i} r_{n-i}: 1 \leq i \leq l-1\right\} \cup\left\{r_{l}\right\},
$$

where $r_{j}(j \in\{1, \ldots, n-1\})$ denotes the simple transposition $r_{j}=(j, j+1)$ in $S_{n}$. Let us define $s_{1}, \ldots, s_{l}$ by setting

$$
s_{j}:= \begin{cases}r_{j} r_{n-j} & \text { if } j \in\{1, \ldots, l-1\}  \tag{2.1}\\ r_{l} & \text { if } j=l\end{cases}
$$

In this notation, the Coxeter-Dynkin diagram of $\left(G, B_{G}, T_{G}\right)$ can be depicted as in Figure 2.1. This labeling is consistent with the labeling that is given in [4].


Figure 2.1: The Coxeter-Dynkin diagram of type $\mathrm{C}_{l}$.

### 2.2 Symplectic monoids.

Let $M$ be a reductive monoid with unit group $G$. Then, by definition, $G$ is a connected reductive group. Let $B$ be a Borel subgroup in $G$, and let $T$ be a maximal torus of $G$ that is contained in $B$. Then the Weyl group of $G$ is given by $W:=N_{G}(T) / T$. The BruhatChevalley decomposition of $G$ is the finite decomposition $G=\bigsqcup_{w \in W} B \dot{w} B$. Likewise, the $B \times B$-orbits in $M$ are parametrized by a finite inverse semigroup, which is called the Renner
monoid of $M$; it is defined as the quotient $R:=\overline{N_{G}(T)} / T$, where $\overline{N_{G}(T)}$ denotes the Zariski closure of $N_{G}(T)$ in $M$. Then the Bruhat-Chevalley-Renner decomposition of $M$ is given by

$$
M=\bigsqcup_{r \in R} B \dot{r} B
$$

The dot on $r$ indicates that we are choosing a representative of $r$ from $\overline{N_{G}(T)}$. In general, it is not true that $R$ is a submonoid of $M$. An excellent survey of the Renner monoids of classical monoids can be found in [15].

Notation 2.2. Let $G$ denote, as before, the symplectic group $S p_{n}$. Then the Renner monoid of the symplectic monoid $M S p_{n}$ will be denoted by $\mathcal{R}_{G}$. The Weyl group of $G$ will be denoted by $W_{G}$.

1) Let $\theta$ denote the involution that we introduced before, that is, $\theta(i)=n-i+1$ for $i \in\{1, \ldots, n\}$. A subset $S \subseteq\{1, \ldots, n\}$ is called an admissible subset if $\theta(S) \cap S=\emptyset$. For $i, j \in\{1, \ldots, n\}$, let $E_{i j}$ denote the $(i, j)$-th elementary matrix. Then the $(k, l)$-th entry of $E_{i j}$ is 1 if $(k, l)=(i, j)$, and it is 0 if $(k, l) \neq(i, j)$. In [16, Theorem 3.1.8], it is shown that

$$
\begin{equation*}
\mathcal{R}_{G}=\left\{\sum_{i \in I, w \in W_{G}} E_{i, w i}: I \text { is admissible }\right\} . \tag{2.3}
\end{equation*}
$$

2) An injective partial transformation on $\{1, \ldots, n\}$ is an injective map $f: D \rightarrow R$, where $D=D(f)$ and $R=R(f)$ are two subsets from $\{1, \ldots, n\}$ with equal cardinalities.

Definition 2.4. The set of all injective partial transformations on $\{1, \ldots, n\}$ is called the rook monoid; we will denote it by $\mathcal{R}_{n}$. As we mentioned before, $\mathcal{R}_{n}$ is the Renner monoid of $M_{n}$.

In [16, Theorem 3.1.10], Li and Renner show that

$$
\begin{equation*}
\mathcal{R}_{G}=\left\{x \in \mathcal{R}_{n}: D(x) \text { and } R(x) \text { are admissible, and } x \text { is singular }\right\} \cup W_{G} . \tag{2.5}
\end{equation*}
$$

3) For $k \in\{0,1, \ldots, n\}$, let $e_{k}$ denote the diagonal idempotent, $e_{k}:=E_{11}+E_{22}+\cdots+E_{k k}$. Also, let $e_{0}$ denote the $n \times n 0$-matrix. The cross-section lattice of $\mathcal{R}_{G}$ is then given by

$$
\begin{equation*}
\Lambda_{G}:=\left\{e_{0}, e_{1}, e_{2}, \ldots, e_{l}, e_{n}\right\} \tag{2.6}
\end{equation*}
$$

(Notice the jump in the indices of the last two idempotents $e_{l}$ to $e_{n}$. This is not a typo!) In this notation, the Renner monoid of $M S p_{n}$ is given by

$$
\begin{equation*}
\mathcal{R}_{G}=\bigsqcup_{e_{i} \in \Lambda} W_{G} e_{i} W_{G} \tag{2.7}
\end{equation*}
$$

Since $e_{n}$ is the identity element, the subset $W_{G} e_{n} W_{G}$ is equal to $W_{G}$. Therefore, the rank of a singular element in $\mathcal{R}_{G}$ is at most $l$.

## 3 The Bruhat-Chevalley-Renner order

Let $M$ be a reductive monoid with unit group $G$. Let $B$ be a Borel subgroup in $G$, and let $T$ be a maximal torus in $G$ such that $T \subseteq B$. Let us denote the Renner monoid of $M$ by $R$. The Bruhat-Chevalley-Renner order on $R$ is the following partial order: for $x, y \in R$,

$$
x \leq_{B C R} y \Longleftrightarrow B x B \subseteq \overline{B y B},
$$

where the bar over $B y B$ stands for the Zariski closure in $M$. Whenever it is clear from the context, we will omit writing the subscript $B C R$ in $\leq_{B C R}$. Note that the restriction of $\leq$ to $W$ is known as the Bruhat-Chevalley order on $W$, which is defined by the same formulation,

$$
x \leq y \Longleftrightarrow B x B \subseteq \overline{B y B} \text { for } x, y \in W
$$

where the bar over $B y B$ stands for the Zariski closure in $G$.
The Weyl group $W$ is a graded poset with the rank function $\ell_{W}: W \rightarrow \mathbb{Z}_{\geq 0}$ defined by

$$
\ell_{W}(w)=\operatorname{dim} B w B-\operatorname{dim} B \text { for } w \in W .
$$

Note that $W$ is a Coxeter group and it has a system of Coxeter generators, denoted by $S$. For $w \in W, \ell_{W}(w)$ can also be defined as the minimal number of simple reflections $s_{i_{1}}, \ldots, s_{i_{r}}$ from $S$ with $w=s_{i_{1}} \cdots s_{i_{r}}$. A subgroup that is generated by a subset $I \subset S$ will be denoted by $W_{I}$ and it will be called a parabolic subgroup of $W$. For $I \subseteq S$, we will denote by $D_{I}$ the following set:

$$
\begin{equation*}
D_{I}:=\left\{x \in W: \ell_{W}(x w)=\ell_{W}(x)+\ell_{W}(w) \text { for all } w \in W_{I}\right\} \tag{3.1}
\end{equation*}
$$

The type-map, $\lambda: \Lambda \rightarrow 2^{S}$, is defined by $\lambda(e):=\{s \in S: s e=e s\}$ for $e \in \Lambda$. The containment ordering between $G \times G$-orbit closures in $M$ is transferred via $\lambda$ to a sublattice of the Boolean lattice on $S$. Associated with $\lambda(e)$ are the following sets: $\lambda_{*}(e):=\cap_{f \leq e} \lambda(f)$ and $\lambda^{*}(e):=\cap_{f \geq e} \lambda(f)$. We define the subgroups

$$
W(e):=W_{\lambda(e)}, \quad W_{*}(e):=W_{\lambda_{*}(e)}, \quad W^{*}(e):=W_{\lambda^{*}(e)}
$$

Then we have

1. $W(e)=\{a \in W: a e=e a\}$,
2. $W^{*}(e)=\cap_{f \geq e} W(f)$,
3. $W_{*}(e)=\cap_{f \leq e} W(f)=\{a \in W: a e=e a=e\}$.

We know from [18, Chapter 10] that $W(e), W^{*}(e)$, and $W_{*}(e)$ are parabolic subgroups of $W$, and furthermore, we know that $W(e) \cong W^{*}(e) \times W_{*}(e)$. If $W(e)=W_{I}$ and $W_{*}(e)=W_{K}$ for some subsets $I, K \subset S$, then we define $D(e):=D_{I}$ and $D_{*}(e):=D_{K}$.

Theorem/Definition (Pennell-Putcha-Renner): For every $x \in W e W$ there exist elements $a \in D_{*}(e), b \in D(e)$, which are uniquely determined by $x$, such that

$$
\begin{equation*}
x=a e b^{-1} . \tag{3.2}
\end{equation*}
$$

The decomposition of $x$ in (3.2) will be called the standard form of $x$. Let $e, f$ be two elements from $\Lambda$. It is proven in [17] that if $x=a e b^{-1}$ and $y=c f d^{-1}$ are two elements in standard form in $R$, then

$$
\begin{equation*}
x \leq y \Longleftrightarrow e \leq f, a \leq c w, w^{-1} d^{-1} \leq b^{-1} \quad \text { for some } w \in W(f) W(e) \tag{3.3}
\end{equation*}
$$

We will occasionally write $D(e)^{-1}$ to denote the set $\left\{b^{-1}: b \in D(e)\right\}$.

### 3.1 Deodhar's criteria.

Our goal in this section is to present a practical description of the Bruhat-Chevalley-Renner order on the rook monoid.

Let us denote by $B_{n}$ the Borel subgroup of invertible upper triangular matrices in $M_{n}$. The Renner monoid $\mathcal{R}_{n}$ is a graded poset with the following rank function [21]:

$$
\ell(x)=\operatorname{dim}\left(B_{n} x B_{n}\right), x \in \mathcal{R}_{n} .
$$

There is a combinatorial formula for computing the values of $\ell$ [8].
We represent elements of $\mathcal{R}_{n}$ by $n$-tuples. For $x=\left(x_{i j}\right) \in \mathcal{R}_{n}$ we define the sequence $\left(x_{1}, \ldots, x_{n}\right)$ by

$$
x_{j}= \begin{cases}0 & \text { if the } j \text {-th column consists of zeros },  \tag{3.4}\\ i & \text { if } x_{i j}=1\end{cases}
$$

By abuse of notation, we denote both the matrix and the sequence $\left(x_{1}, \ldots, x_{n}\right)$ by $x$. For example, the associated sequence of the partial permutation matrix

$$
x=\left[\begin{array}{llll}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right]
$$

is $x=(3,0,4,0)$.
Next, we define a useful partial order on finite sets of integers. Let $\left\{i_{1}, \ldots, i_{k}\right\}$ and $\left\{j_{1}, \ldots, j_{k}\right\}$ be two equinumerous sets of integers such that $i_{1}<\cdots<i_{k}$ and $j_{1}<\cdots<j_{k}$. We will write

$$
\begin{equation*}
\left\{i_{1}, \ldots, i_{k}\right\} \leqslant\left\{j_{1}, \ldots, j_{k}\right\} \Longleftrightarrow i_{1} \leq j_{1}, i_{2} \leq j_{2}, \ldots, i_{k} \leq j_{k} \tag{3.5}
\end{equation*}
$$

Let $x=\left(x_{1}, \ldots, x_{n}\right)$ be an element from $\mathcal{R}_{n}$. For $i \in\{1, \ldots, n\}$, we define

$$
\tilde{x}(i):=\left\{x_{1}, \ldots, x_{i}\right\} .
$$

In this notation, the main result of [8] is as follows.

Theorem 3.6. Let $x=\left(x_{1}, \ldots, x_{n}\right)$ and $y=\left(y_{1}, \ldots, y_{n}\right)$ be two elements from $\mathcal{R}_{n}$. Then $x \leq_{B C R} y$ if and only if for every $i \in\{1, \ldots, n-1\}$ we have $\tilde{x}(i) \leqslant \tilde{y}(i)$.

Example 3.7. Let $x=(3,1,5,2,4)$ and $y=(5,2,4,3,1)$ be two elements from $\mathcal{R}_{5}$. Since

$$
\begin{gathered}
\tilde{x}(1)=\{3\} \leqslant\{5\}=\tilde{y}(1), \\
\tilde{x}(2)=\{1,3\} \leqslant\{2,5\}=\tilde{y}(2), \\
\tilde{x}(3)=\{1,3,5\} \leqslant\{2,4,5\}=\tilde{y}(3), \\
\tilde{x}(4)=\{1,2,3,5\} \leqslant\{2,3,4,5\}=\tilde{y}(4),
\end{gathered}
$$

we see that $x \leq_{B C R} y$.
Theorem 3.6 is quite useful for explicit computations.

### 3.2 The Bruhat-Chevalley-Renner order on $\mathcal{R}_{G}$.

As before, let $n$ be an even number, $n=2 l, l \in \mathbb{Z}_{+}$. Recall that $W_{G}$ denotes the Weyl group of $S p_{n}$. Then $W_{G}$ is the centralizer in $\mathcal{S}_{n}$ of the involution $\theta=(1, n)(2, n-1) \cdots(l, l+1)$. As a Coxeter group, $W_{G}$ has type $\mathrm{BC}_{l}$. In [2, Corollary 8.1.9], it is shown that for two elements $u$ and $v$ from $W_{G}$,

$$
\begin{equation*}
u \leq v \text { in } W_{G} \Longleftrightarrow u \leq v \text { in } W_{H}=\mathcal{S}_{n} \tag{3.8}
\end{equation*}
$$

We will extend (3.8) to the Renner monoid of $M S p_{n}$.
Theorem 3.9. Let $x$ any $y$ be two elements from $\mathcal{R}_{G}$. Then

$$
x \leq y \text { in } \mathcal{R}_{G} \Longleftrightarrow x \leq y \text { in } \mathcal{R}_{n}
$$

Proof. We write $x$ and $y$ in their standard form, $x=a e b^{-1}$ and $y=c f d^{-1}$, where $a \in$ $D_{*}(e), b \in D(e), c \in D_{*}(f), d \in D(f)$. Of course, $a, b, c$, and $d$ are elements of $W_{G}$. By (3.3) we know that

$$
x \leq y \text { in } \mathcal{R}_{G} \Longleftrightarrow e \leq f, a \leq c w, w^{-1} d^{-1} \leq b^{-1}
$$

for some $w \in W_{G}(f) W_{G}(e)$. The idempotents of $\mathcal{R}_{G}$ are in $\mathcal{R}_{n}$, hence, the Bruhat-ChevalleyRenner order on them is the one that is induced from $\mathcal{R}_{n}$. It follows from (3.8) that the relations $a \leq c w$ and $w^{-1} d^{-1} \leq b^{-1}$ hold in $W_{G}$ if and only if they hold in $\mathcal{S}_{n} \subseteq \mathcal{R}_{n}$. Therefore, the relation $x \leq y$ holds in $\mathcal{R}_{G}$ if and only if it holds in $\mathcal{R}_{n}$.

Corollary 3.10. Let $x=\left(x_{1}, \ldots, x_{n}\right)$ and $y=\left(y_{1}, \ldots, y_{n}\right)$ be two elements from $\mathcal{R}_{G}$. Then $x \leq_{B C R} y$ if and only if for every $i \in\{1, \ldots, n-1\}$ we have $\tilde{x}(i) \leqslant \tilde{y}(i)$.

Proof. It follows from Theorem 3.9 that $x \leq_{B C R} y$ in $\mathcal{R}_{G}$ if and only if $x \leq_{B C R} y$ in $\mathcal{R}_{n}$. The rest of the proof follows from Theorem 3.6.

## 4 The Borel Submonoid of $M S p_{n}$

We will follow our convention that if the even integer $n=2 l$ is fixed, then $G$ stands for $S p_{n}$. We will denote by $B_{G}$, as before, the Borel subgroup in $S p_{n}$ that is defined by

$$
B_{G}:=B_{n} \cap S p_{n} .
$$

The Borel subgroup of the unit group $G S p_{n}$ of $M S p_{n}$ is given by $B:=\mathbb{C}^{*} B_{G}$, and the corresponding Borel submonoid of $M S p_{n}$ is the Zariski closure of $B$ in $M S p_{n}$. Evidently, $B$ is a connected, hence irreducible, algebraic group. Then $\bar{B}$ is an irreducible $B \times B$-variety. The orbits of $B \times B$ are parametrized by $x \in \mathcal{R}_{G}$ such that $x \leq 1$. Indeed, we have

$$
\mathcal{B}_{G}=\mathcal{B}_{n} \cap \mathcal{R}_{G}:=\left\{x \in \mathcal{R}_{G}: x \leq 1\right\} .
$$

We depict the Bruhat-Chevalley-Renner order on $\mathcal{B}_{S p_{4}}$ in Figure 4.1.
We are going to reformulate the description of $\mathcal{B}_{G}$ in two different ways.

1. It is observed in [17, Lemma 2.3] that an element $r$ from $\mathcal{R}_{G}$ satisfies $r \leq 1$ if and only if $a \leq b$, where $a e b^{-1}$ is the standard form of $r$. Thus, we have

$$
\begin{equation*}
\mathcal{B}_{G}=\left\{a e b^{-1}: a e b^{-1} \text { is in standard form, } e \in \Lambda_{G}, a \in D_{*}(e), b \in D(e), \text { and } a \leq b\right\} . \tag{4.1}
\end{equation*}
$$

2. Let $D(x)$ (resp. $R(x)$ ) denote the domain (resp. range) of an element $x \in \mathcal{R}_{n}$. The data of $D(x)$ and $R(x)$ are not enough to recover $x$. One needs to know the (bijective) assignment between them to uniquely determine $x$. Let $D(x)$ and $R(x)$ be as follows:

$$
D(x)=\left\{i_{1}, \ldots, i_{k}\right\} \text { and } R(x)=\left\{j_{1}, \ldots, j_{k}\right\}
$$

where

$$
x\left(i_{t}\right)=j_{t} \text { for } t \in\{1, \ldots, k\} .
$$

We will assume that the entries of $D(x)$ are listed in the increasing order as in $1 \leq$ $i_{1}<\cdots<i_{k} \leq n$, however, we may not have the same ordering on the corresponding elements of $R(x)$.
Notice that in order for $x$ be $\leq 1$ in the Bruhat-Chevalley-Renner order its matrix representation has to have all of its nonzero entries on or above the main diagonal. Since $D(x)$ gives the column indices of the nonzero entries in $x$, and since $R(x)$ gives the row indices of the nonzero entries in $x$, we see that

$$
x \leq 1 \Longleftrightarrow i_{t} \geq j_{t} \text { for every } t \in\{1, \ldots, k\}
$$

By (2.5), for the elements $x$ in $\mathcal{R}_{G} \backslash\{1\}$, both of the subsets $D(x), R(x) \subseteq\{1, \ldots, n\}$ are admissible.


Figure 4.1: Bruhat-Chevalley-Renner order on $\mathcal{B}_{S p_{4}}$.

Question 4.2. What is the cardinality of $\mathcal{B}_{G}$ ? By the second item, our problem is equivalent to the question of finding, for every $k \in\{1, \ldots, l\}$, the number of pairs of admissible subsets

$$
I=\left\{i_{1}, \ldots, i_{k}\right\} \text { and } J=\left\{j_{1}, \ldots, j_{k}\right\}
$$

such that

1. $1 \leq i_{1}<\cdots<i_{k} \leq n$ and $J \subset\{1, \ldots, n\}$; there are no order constraints on the elements of $J$.
2. $i_{t} \geq j_{t} \geq 1$ for every $t \in\{1, \ldots, k\}$.

It turns out that the number of admissible subsets of $\{1, \ldots, n\}$ has a pleasant formula. We begin with a simple lemma.

Lemma 4.3. Let $n$ and $k$ be two positive integers such that $1 \leq k \leq n$. Assume that $n$ is an even number, $n=2 l$. Let $A_{n, k}$ denote the set of admissible subsets of $\{1, \ldots, n\}$ with $k$ elements. If $a_{n, k}$ denotes the cardinality of $A_{n, k}$, then

$$
a_{n, k}=\sum_{r=0}^{k}\binom{l}{r}\binom{l-r}{k-r}=\binom{l}{k} 2^{k} .
$$

Proof. Clearly, by the pigeon-hole principle, if $k>l$, then $A_{n, k}=\emptyset$. Also, in this case, $\binom{l-r}{k-r}$ is 0 for every $r \in\{0, \ldots, k\}$, hence, $a_{n, k}=0$. Therefore, we will assume that $k \leq l$.

Let $A=\left\{i_{1}, \ldots, i_{k}\right\}\left(i_{1}<\cdots<i_{k}\right)$ be an element from $A_{n, k}$. The entries of $A$ satisfy the inequalities

$$
1 \leq i_{1}<\cdots<i_{r} \leq l<i_{r+1}<\cdots<i_{k} \leq n .
$$

We will determine the number of such $A$. Clearly, the first $r$ entries, $i_{1}, \ldots, i_{r}$, can be chosen in $\binom{l}{r}$ ways. Then the remaining entries, $i_{r+1}, \ldots, i_{k}$, cannot be contained in the set $\left\{\theta\left(i_{s}\right)=n-i_{s}+1: s \in\{1, \ldots, r\}\right\}$. In other words, $\left\{i_{r+1}, \ldots, i_{k}\right\} \subseteq\{l+1, \ldots, 2 l\} \backslash\left\{\theta\left(i_{s}\right)=\right.$ $\left.n-i_{s}+1: s \in\{1, \ldots, r\}\right\}$. Then, the number of possibilities for $\left\{i_{r+1}, \ldots, i_{k}\right\}$ is given by $\binom{l-r}{k-r}$. Therefore, in total, we have $\sum_{r=0}^{k}\binom{l}{r}\binom{l-r}{k-r}$ possibilities for $A$. This finishes the proof of the first equality. To prove the second equality, we manipulate the summation as follows:

$$
\begin{equation*}
\sum_{r=0}^{k}\binom{l}{r}\binom{l-r}{k-r}=\sum_{r=0}^{k} \frac{l!}{(l-r)!r!} \frac{(l-r)!}{(l-k)!(k-r)!}=\sum_{r=0}^{k} \frac{l!}{r!(l-k)!(k-r)!} \tag{4.4}
\end{equation*}
$$

Let us multiply and divide each summand in the last sum in (4.4) by $k$ !. Then by reorganizing the terms we get

$$
\sum_{r=0}^{k} \frac{l!}{(l-k)!k!} \frac{k!}{r!(k-r)!}=\sum_{r=0}^{k}\binom{l}{k}\binom{k}{r}=\binom{l}{k} \sum_{r=0}^{k}\binom{k}{r}=\binom{l}{k} 2^{k} .
$$

This finishes the proof.

Since the empty set is admissible, we set $a_{n, 0}=1$.
Corollary 4.5. The total number of admissible subsets of $\{1, \ldots, n\}$, that is, $\left|\cup_{k=0}^{l} A_{n, k}\right|$, is equal to $3^{l}$.

Proof. We will determine the number $\sum_{k=0}^{l} a_{n, k}=\sum_{k=0}^{l}\binom{l}{k} 2^{k}$. But, by the binomial theorem, $f(2)=\sum_{k=0}^{l}\binom{l}{k} 2^{k}$, where $f(x)=(1+x)^{l}$.

Theorem 4.6. The $k$-th symplectic Stirling poset $\mathcal{B}_{G}(k)$ is a graded bounded poset with a unique minimum element. There are $\binom{l}{k} 2^{k}$ maximal elements in $\mathcal{B}_{G}(k)$.

Proof. If $k=n$ (resp. $k=0$ ), then $\mathcal{B}_{G}(k)=\{i d\}$ (resp. $\mathcal{B}_{G}(k)=\{0\}$ ), hence, in these cases there is nothing to prove. We proceed with the assumption that $1 \leq k \leq l$. Since $\mathcal{B}_{G}$ is equal to the intersection $\mathcal{R}_{G} \cap \mathcal{B}_{n}$, we have

$$
\mathcal{B}_{G}(k)=\mathcal{B}_{n}(k) \cap \mathcal{R}_{G} \text { for } k \in\{1, \ldots, l\} .
$$

Notice that the rook $i d(k):=(0, \ldots, 0,1,2, \ldots, k)$ is a symplectic rook. In fact, $i d(k)$ is the unique minimum of $\mathcal{B}_{n}(k)$. It follows from Theorem 3.9 that $i d(k)$ is the unique minimum element in $\mathcal{B}_{G}(k)$ as well. Next, we will show that $\mathcal{B}_{G}(k)$ is a graded poset. To this end, it will suffice to show that every maximal element of $\mathcal{B}_{G}(k)$ has the same rank. In [7, Lemma 5.1], it is shown that the maximal elements of $\mathcal{B}_{G}(k)$ are given by the diagonal idempotents of rank $k$ in $\mathcal{R}_{n}$. Clearly, any diagonal idempotent of rank $k$ whose domain and range are admissible subsets in $\{1, \ldots, n\}$ are contained in $\mathcal{B}_{G}(k)$. But for a diagonal matrix, the domain and the range agree, therefore, the number of diagonal idempotents in $\mathcal{B}_{G}(k)$ is equal to the number of admissible subsets in $\{1, \ldots, n\}$. This number is equal to $\binom{l}{k} 2^{k}$ by Lemma 4.3. Next, we will show that there are no other maximal elements in $\mathcal{B}_{G}(k)$. Towards a contradiction, let $a=\left(a_{1}, \ldots, a_{n}\right)$ be a maximal element in $\mathcal{B}_{G}(k)$ which is not a diagonal idempotent. Since $a$ is an upper triangular rook, we know that $a_{i} \leq i$ for every $i \in\{1, \ldots, n\}$. Let $i \in\{1, \ldots, n\}$ be the smallest index such that $0<a_{i} \lesseqgtr i$. Let $j$ denote $a_{i}$. Then we know that $a_{j}=0$. Let $b$ denote the rook matrix that is obtained by interchanging $a_{i}$ and $a_{j}$. It is easy to verify that $b$ is an element of $\mathcal{B}_{G}(k)$ such that $a<b$. This contradicts with our assumption that $a$ is a maximal element. Hence, the proof of our theorem is finished.

Remark 4.7. For each $d \in\{1, \ldots, l\}$, by (3.5), we have a very special poset structure on $A_{n, d}$. It is easily seen from [14, Section 6.1.1] that $\left(A_{n, d}, \leqslant\right)$ is isomorphic to the BruhatChevalley order on the Grassmann variety $G / P_{d}$, where $G=S p_{n}$ and $P_{d}$ is the maximal parabolic subgroup corresponding to the set of simple generators $S_{G} \backslash\left\{s_{d}\right\}$ in $W_{G}$.

Proposition 4.8. Let $\Lambda_{G}$ be the cross section lattice of $G$ as in (2.6). If $e$ is an element from $\Lambda_{G} \backslash\{1\}$, then $W_{G}(e)$ is a maximal parabolic subgroup in $W_{G}$. Conversely, any maximal parabolic subgroup of $W_{G}$ is obtained this way.

Proof. The cross section lattice $\Lambda_{G}$ is part of the cross section lattice of the monoid $M_{n}$. It is easy to verify that, if the matrix rank of $e$ is $d$, then the centralizer of $e$ in $W_{H}=S_{n}$ is the maximal parabolic subgroup generated by the set $\left\{r_{1}, \ldots, r_{n-1}\right\} \backslash\left\{r_{d}\right\}$. Let $s_{1}, \ldots, s_{l}$ denote, as defined in (2.1), the simple Coxeter generators for $W_{G}$. Now it is easy to check that, for every $j \in\{1, \ldots, l\} \backslash\{d\}$, we have

$$
s_{j} e=e s_{j}, \text { hence, } W_{G}(e)=\left\langle s_{j}: j \in\{1, \ldots, l\} \backslash\{d\}\right\rangle .
$$

Our second assertion is now easy to verify. This finishes the proof.
Next, we will compute the stabilizer of an element $e$ from $\Lambda_{G} \backslash\{1\}$.
Proposition 4.9. Let $e$ be an element from $\Lambda_{G} \backslash\{1\}$. If the matrix rank of $e$ is $d$, where $1 \leq d<l$, then $\left(W_{G}\right)_{*}(e)$ is generated by the simple Coxeter generators $s_{d+1}, \ldots, s_{l}$. If $d=l$, then $\left(W_{G}\right)_{*}(e)=\{1\}$.

Proof. Once again, the proof will follow from the corresponding computation that is performed in the rook monoid (the Renner monoid of $M_{n}$ ). In that case, by explicitly computing the matrix products $r_{j} e(j \in\{1, \ldots, n-1\})$, one sees that $r_{j} e=e r_{j}=e$ if and only if $j \in\{d+1, \ldots, n-1\}$. It follows immediately from this observation that $s_{j}=r_{j} r_{n-j}$ stabilizes $e$ if and only if $j \in\{d+1, \ldots, l\}$. It also follows that if the rank of $e$ is $l$, then $s_{l}$ does not stabilize $e$, hence, $\left(W_{G}\right)_{*}(e)=\{1\}$. This finishes the proof.

## 5 Folding, unfolding

Let $n$ be a positive integer. A collection $S_{1}, \ldots, S_{r}$ of subsets of the set $S:=\{1, \ldots, n\}$ is said to be a set partition of $S$ if $S_{i}$ 's $(i=1, \ldots, r)$ are mutually disjoint and $\cup_{i=1}^{r} S_{i}=S$. In this case, the $S_{i}$ 's are called the blocks of the partition. The collection of all set partitions of $S$ is denoted by $\Pi_{n}$. We will often drop set parentheses and commas and just put vertical bars between blocks. If $S_{1}, \ldots, S_{k}$ are the blocks of a set partition $\pi$ from $\Pi_{n}$, then the standard form of $\pi$ is defined as $S_{1}\left|S_{2}\right| \cdots \mid S_{k}$, where we assume that min $S_{1}<\cdots<\min S_{k}$ and the elements of each block are listed in increasing order. For example, $\pi=136|2459| 78$ is a set partition from $\Pi_{9}$. Set partitions can be visualized by using "arc-diagrams" which we will define next.

A linearly ordered poset is called a chain. We will identify chains by their Hasse diagrams; we draw a Hasse diagram by placing the smallest entry on the left and connecting the vertices by arcs. For example, in Figure 5.1, we have the chain on 9 vertices, where each arc represents a covering relation.


Figure 5.1: A chain on 9 vertices.

Definition 5.1. A labeled chain is a chain whose vertices are labeled by distinct numbers. An arc-diagram on $n$ vertices is a disjoint union of labeled chains where the labels are from $\{1, \ldots, n\}$ and each label $i \in\{1, \ldots, n\}$ is used exactly once. We depict an example in Figure 5.2.


Figure 5.2: An arc-diagram on 9 vertices

Clearly, in an arc-diagram subchains represents the blocks of the corresponding set partition. We know from [3, Lemma 1.17] that the number of set partitions of $S=\{1, \ldots, n\}$ into $k$ blocks, denoted by $S(n, k)$, and called the $(n, k)$-th Stirling number of the second kind, is given by the formula $S(n, k)=\frac{1}{k!} \sum_{i=1}^{k}(-1)^{i}\binom{k}{i}(k-i)^{n}$. The recurrence formula for the Stirling numbers of the second kind is well-known:

$$
S(l+1, k)=S(l, k-1)+k S(l, k),
$$

where

$$
S(l, k)= \begin{cases}1 & \text { if } l=k=0 \\ 0 & \text { if } l>0 \text { and } k=0 \\ 0 & \text { if } l<0 \text { or } k<0 \text { or } l<k .\end{cases}
$$

Let $\mathcal{B}_{n}$ denote the submonoid of $\mathcal{R}_{n}$ such that if $x \in \mathcal{B}_{n}$, then $x$ is an upper triangular matrix. The subsemigroup of nilpotent elements in $\mathcal{B}_{n}$ will be denoted by $\mathcal{B}_{n}^{\text {nil }}$. For each $A$ in $\mathcal{B}_{n}$, there exists a unique $(n+1) \times(n+1)$ nilpotent matrix, $\tilde{A}$, which is obtained from $A$ by appending to it a column and a row of zeros as follows:

$$
A \longmapsto \tilde{A}:=\left[\begin{array}{ccc}
0 & &  \tag{5.2}\\
\vdots & A & \\
0 & \ldots & 0
\end{array}\right] \in \mathcal{B}_{n+1} \quad\left(A \in \mathcal{B}_{n}\right)
$$

In this notation, it is easily verified that (5.2) defines a set bijection $\mathcal{B}_{n} \longrightarrow \mathcal{B}_{n+1}^{\text {nil }}$. There is a simple bijection between $\mathcal{B}_{n+1}^{\text {nil }}$ and $\Pi_{n+1}$ which is defined as follows: the matrix corresponding to the set partition $A$ has an entry equal to 1 in row $i$ and and column $j$ if and only if $(i, j)$ is an $\operatorname{arc}$ of $A$. Therefore, for $k \in\{1, \ldots, n+1\}$, we have

$$
\begin{equation*}
S(n+1, k)=\left|\left\{A \in \mathcal{B}_{n}: \operatorname{rank} A=n+1-k\right\}\right| \tag{5.3}
\end{equation*}
$$

It follows from the bijections above that the number of elements of $\mathcal{B}_{n}$ is given by the summation $b_{n+1}:=\sum_{k=0}^{n+1} S(n+1, k)$, which is called the $(n+1)$ th Bell number. As a convention, we set $b_{0}=1$ and $b_{1}=1$.

It is easy to check that the number of elements of $\mathcal{R}_{n}$ of rank $k$ is given by

$$
\begin{equation*}
\left|\left\{A \in \mathcal{R}_{n}: \operatorname{rank}(A)=k\right\}\right|=\binom{n}{k} \frac{n!}{(n-k)!} . \tag{5.4}
\end{equation*}
$$

We will express this cardinality by using Stirling numbers of the second kind.
Every element $A$ of $\mathcal{R}_{n}$ has a triangular decomposition in $\mathcal{R}_{n}$,

$$
\begin{equation*}
A=A_{l}+A_{d}+A_{u} \tag{5.5}
\end{equation*}
$$

where $A_{l}$ is a strictly lower triangular matrix, $A_{d}$ is a diagonal matrix, and $A_{u}$ is a strictly upper triangular matrix.

Proposition 5.6. Let $S_{a, b, c}(n)$ denote the number of elements $A \in \mathcal{R}_{n}$ such that $\operatorname{rank}\left(A_{l}\right)=$ $a, \operatorname{rank}\left(A_{d}\right)=b$, and $\operatorname{rank}\left(A_{u}\right)=c$, where $A_{l}, A_{d}$, and $A_{u}$ are as in (5.5). Then we have

$$
\begin{aligned}
\binom{n}{k} \frac{n!}{(n-k)!} & =\sum_{a+b+c=k} S_{a, b, c}(n) \\
& =\sum_{a+b+c=k}\binom{n}{b} S(n+1, n+1-a) S(n+1, n+1-c) .
\end{aligned}
$$

Proof. The number of strictly upper triangular elements of rank $k$ in $\mathcal{R}_{n}$ is equal to the number of strictly lower triangular rank $k$ elements in $\mathcal{R}_{n}$. Now the proof of the first equality follows from the equality in (5.4) and the uniqueness of the triangular decomposition in (5.5). The proof of the second equality follows from (5.3) together with the fact that there are exactly $\binom{n}{b}$ diagonal $0 / 1$ matrices of rank $b$.

We proceed with the assumption that $n$ is an even number of the form $n=2 l, l \in \mathbb{Z}_{+}$. In the sequel, we will count the number of elements of $\mathcal{B}_{G}$, where $G=S p_{n}$, by a technique that we call unfolding. But before that we want to demonstrate that the elements of $\mathcal{R}_{G}$ behave well under "folding". We already mentioned the result of Li and Renner [16, Theorem 3.1.10], which states that, if an element $A$ from $\mathcal{R}_{G}$ is singular, then both of the domain and the range of $A$ are admissible subsets in $\{1, \ldots, n\}$. Furthermore, the elements of $D(A)$ correspond to the indices of the nonzero columns of $A$, and the elements of $R(A)$ correspond to the indices of the nonzero rows of $A$. This shows that $A$ can be folded vertically as well as horizontally. We will demonstrate what we mean here by an example.

Example 5.7. In this example, we fold an element of $\mathcal{R}_{S p_{8}}$ horizontally from top to bottom:

$$
\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \longrightarrow\left[\begin{array}{llllllll}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

Example 5.8. In this example, we fold the matrix of the previous example vertically from left to right:

$$
\left[\begin{array}{llll:lll}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

Definition 5.9. We will denote the horizontal folding operation from top to bottom by $F_{T B}$. Likewise, we will denote the vertical folding operation from left to right by $F_{L R}$.

Clearly, the operations $F_{T B}$ and $F_{L R}$ can be composed. In fact, they commute,

$$
\begin{equation*}
F_{T B} F_{L R}=F_{L R} F_{T B} \tag{5.10}
\end{equation*}
$$

Let $F$ denote the composition of the folding operators as in (5.10). We will refer to $F$ by the folding map.

Proposition 5.11. The folding map is a surjective map from $\mathcal{R}_{S p_{n}}$ onto the rook monoid $\mathcal{R}_{l}$. Furthermore, the restricted folding map, $\left.F\right|_{\mathcal{B}_{G}}$, which we will denote by $F^{\prime}$, is surjective as well.

Proof. If we show that $F^{\prime}$ is surjective, then the surjectivity of $F$ will follow. To this end, let $A$ be an element from $\mathcal{R}_{l}$, and let $A=A_{l}+A_{d}+A_{u}$ be its triangular decomposition. Recall from the introduction that $J_{l}$ denotes the $l \times l$ permutation matrix with 1 's on its anti-diagonal. Now we define an $n \times n$ matrix $B$ as follows: $B:=\left[\begin{array}{cc}\mathbf{0} & \widetilde{A}_{l} \\ \mathbf{0} & A_{d}+A_{u}\end{array}\right]$, where $\mathbf{0}$ is the $l \times l 0$ matrix, and $\widetilde{A}_{l}:=J_{l} A_{l}$. In other words, $\widetilde{A}_{l}$ is the matrix whose $i$ th row is the $(l-i+1)$ th row of $A_{l}$. Since the indices of the nonzero columns of $B$ are contained in the set
$\{l+1, \ldots, n\}$, the domain of $B$ is an admissible subset in $\{1, \ldots, n\}$. It is also easy to check that the set of row indices of $B$ is an admissible subset of $\{1, \ldots, n\}$. Clearly, $B$ is upper triangular matrix, therefore, $B \in \mathcal{B}_{G}$. Finally, by its construction, the image of $B$ under $F^{\prime}$ is equal to $A, F^{\prime}(B)=F(B)=A$. This finishes the proofs of our assertions.

We are now ready to count the number of elements of $\mathcal{B}_{G}$ by "unfolding" the elements of $\mathcal{R}_{l}$ first horizontally from bottom to top, and then vertically from right to left. We will demonstrate our count by an example.
Example 5.12. We will compute the preimage of $J_{2}=\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]$ under the restricted folding $\operatorname{map} F^{\prime}: \mathcal{B}_{S p_{4}} \rightarrow \mathcal{R}_{2}$. Equivalently, we will determine the set $F_{L R}^{-1}\left(F_{T B}^{-1}\left(J_{2}\right)\right) \cap \mathcal{B}_{S p_{4}}$. Since we are looking for the upper triangular elements in the preimage, the lower halves of the $4 \times 2$ matrices in $F_{T B}^{-1}\left(J_{2}\right)$ must be upper triangular. The following matrices are the possibilities:

$$
\left[\begin{array}{cc}
1 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0
\end{array}\right]
$$

Let $A_{1}$ denote the $4 \times 2$ matrix that is on the top-left position, and let $A_{2}$ denote the $4 \times 2$ matrix that is on the top-right position. Then, the following two matrices are mapped onto $A_{1}$ by $F_{L R}$ :

$$
\left[\begin{array}{ll:ll}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{array}\right],\left[\begin{array}{ll:l}
0 & 1 & 0 \\
0 & 0 \\
0 & 0 & 0
\end{array} 0\right.
$$

Likewise, the following two matrices are folded onto $A_{2}$ by $F_{L R}$ :

$$
\left[\begin{array}{ll:ll}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right],\left[\begin{array}{ll:ll}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right] \quad \xrightarrow{F_{L R}}\left[\begin{array}{ll}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{array}\right]
$$

We find that, in total, there are four matrices that fold onto $J_{2}$.
We are now ready to present a formula for the number of elements of $\mathcal{B}_{G}$ that lie in the preimage of the folding map $F^{\prime}$.
Theorem 5.13. The number of elements of rank $k$ in $\mathcal{B}_{G}$ is given by the formula

$$
\begin{equation*}
\sum_{a+b+c=k} 2^{a+c} 3^{b}\binom{l}{b} S(l+1, l+1-a) S(l+1, l+1-c) \tag{5.14}
\end{equation*}
$$

where $(a, b, c) \in \mathbb{Z}_{\geq 0}^{3}$.
Proof. Let $A$ be an element from $\mathcal{R}_{l}$ with the triangular decomposition $A=A_{l}+A_{d}+A_{u}$. Then it is easy to verify that

$$
\begin{equation*}
\left|F^{\prime-1}(A) \cap \mathcal{B}_{S p_{n}}\right|=\left|F^{\prime-1}\left(A_{l}\right) \cap \mathcal{B}_{S p_{n}}\right|\left|F^{\prime-1}\left(A_{d}\right) \cap \mathcal{B}_{S p_{n}}\right|\left|F^{\prime-1}\left(A_{u}\right) \cap \mathcal{B}_{S p_{n}}\right| \tag{5.15}
\end{equation*}
$$

Let us denote the matrix ranks of $A_{l}, A_{d}$, and $A_{u}$ by $a, b$, and $c$, respectively. Then we denote the three factors on the right hand side of (5.15) by the notation $f_{a}(A), f_{b}(A)$, and $f_{u}(A)$, respectively. Our choice of the subscripts for $f_{a}, f_{b}, f_{c}$ will be clarified in the next two paragraphs.

As it was shown for the special case of $J_{2}$ in Example 5.12, if $B$ is an element from $F^{\prime-1}(A)$, then the lower $l \times l$ half of the $2 l \times l$ matrix $F_{L R}(B)$ must be an upper triangular matrix. In other words, when we unfold $A$ to a $2 l \times l$ matrix, all of the nonzero entries of $A_{l}$ are moved into the upper $l \times l$ portion of the resulting matrix, hence, there is a unique $2 l \times l$ matrix $A_{l}^{\prime}$ such that $F_{T B}\left(A_{l}^{\prime}\right)=A_{l}$. Moreover, for every subset of the nonzero entries of $A_{l}^{\prime}$, there exists a unique $A_{l}^{\prime \prime}$ in $\mathcal{B}_{G}$ such that $F_{L R}\left(A_{l}^{\prime \prime}\right)=A_{l}$. It follows from these arguments that

$$
\begin{equation*}
f_{a}(A)=\left|F_{L R}^{-1}\left(A_{l}^{\prime}\right)\right|=\binom{a}{0}+\binom{a}{1}+\cdots+\binom{a}{a}=2^{a} \tag{5.16}
\end{equation*}
$$

A similar argument shows that

$$
\begin{equation*}
f_{c}(A)=\binom{c}{0}+\binom{c}{1}+\cdots+\binom{c}{c}=2^{c} \tag{5.17}
\end{equation*}
$$

We now consider the possible unfolding of the diagonal matrix $A_{d}$. Recall that the rank of $A_{d}$ is $b$. For every $s$ element subset of the set of nonzero entries of $A_{d}$, there exists a unique $2 l \times l$ matrix $A_{d}^{\prime}$ such that $F_{T B}\left(A_{d}^{\prime}\right)=A_{d}$ and $\operatorname{rank}\left(A_{d}^{\prime}\right)=s$. Likewise, for every $r$ element subset of the set of nonzero entries of $A_{d}^{\prime}$, there exists a unique $2 l \times 2 l$ matrix $A_{d}^{\prime \prime}$ such that $F_{L R}\left(A_{d}^{\prime \prime}\right)=A_{d}^{\prime}$ and $\operatorname{rank}\left(A_{d}^{\prime \prime}\right)=r$. In total, there exist $\sum_{s=0}^{b} \sum_{r=0}^{s}\binom{s}{r}\binom{b}{s}$ elements in the preimage $F^{\prime-1}\left(A_{d}\right)$. But this double sum has a closed form:

$$
\begin{equation*}
f_{b}(A)=\sum_{s=0}^{b} \sum_{r=0}^{s}\binom{s}{r}\binom{b}{s}=3^{b} . \tag{5.18}
\end{equation*}
$$

By combining (5.16), (5.17), and (5.18), we find that $\left|F^{\prime-1}(A) \cap \mathcal{B}_{S p_{n}}\right|=2^{a} 2^{c} 3^{b}=2^{a+c} 3^{b}$, which actually depends only on the ranks of the matrices $A_{l}, A_{d}$, and $A_{c}$. Our formula now follows from Proposition 5.6.

Remark 5.19. It is easy to check that $\mathcal{B}_{S p_{n}}(1)$ is equal to $\mathcal{B}_{n}(1)$. The Hasse diagram of $\left(\mathcal{B}_{n}(1), \leq\right)$ is a fishnet, see [7, Figure 1.9]. By Theorem 3.9, we know that, for every $k \in\{1, \ldots, l\},\left(\mathcal{B}_{S p_{n}}(k), \leq\right)$ is a subposet of $\left(\mathcal{B}_{n}(k), \leq\right)$. However, unless $k=1$, the inclusion $\operatorname{map} \mathcal{B}_{S p_{n}}(k) \hookrightarrow \mathcal{B}_{n}(k)$ does not preserve the interval structure. Indeed, already for $k=2$, the cardinalities of $\mathcal{B}_{S p_{4}}(2)$ and $\mathcal{B}_{4}(2)$ are different.

## 6 Rationally Smooth Borel Submonoids

Let $M$ be a reductive monoid with zero. Let $G$ denote its unit group, which is a connected reductive group. Then $M$ is called a semisimple monoid if $G$ has a one-dimensional center. The classification of semisimple (smooth) monoids is due to Renner [20]. It turns out that a semisimple monoid $M$ is smooth if and only if $M$ is isomorphic to the monoid of $n \times n$ matrices, $M_{n}$, for some $n \in \mathbb{Z}_{+}$. Note that the situation for general reductive monoids is not very different; a reductive monoid with zero is smooth if and only if $M$ is of the form

$$
M=\left(G_{0} \times \prod_{i=1}^{r} M_{n_{i}}\right) / Z
$$

where $Z$ is a finite central torus that does not intersect the unit group of $\prod_{i=1}^{r} M_{n_{i}}$, and $G_{0}$ is a semisimple subgroup [25, Section 11]. The semisimple monoids whose cohomological properties are as good as one hopes for are identified by Renner also [22]. They are called "rationally smooth" monoids.

Let $X$ be a complex algebraic variety with $\operatorname{dim} X=n$, and let $x$ be a point in $X$. The variety $X$ is called rationally smooth at $x$ if there exists an open neighborhood $U$ of $x$ such that for all $y \in U$, the following holds:

$$
H^{m}(X, X \backslash\{y\})= \begin{cases}\{0\} & \text { if } m \neq 2 n \\ \mathbb{Q} & \text { if } m=2 n\end{cases}
$$

$X$ is called rationally smooth if it is rationally smooth at every point $x$ in $X$. The classification as well as various characterizations of rationally smooth reductive monoids is given in [22, 24].

We will now adapt another result of Renner [24, Theorem 2.2] in our setting.
Lemma 6.1. Let $X$ and $Y$ be two (normal) Borel submonoids of the (normal) reductive monoids $M$ and $N$, respectively. Assume that both of $M$ and $N$ have zero elements, and assume that there exists a finite dominant morphism of algebraic monoids $g: M \rightarrow N$. Under these assumptions, $X$ is rationally smooth if and only if $Y$ is rationally smooth.

Proof. By abuse of notation, we will denote the restriction $\left.g\right|_{X}$ by $g$ also. By our assumptions, the algebraic monoids $X$ and $Y$ have zero elements, denoted by $0_{X}$ and $0_{Y}$, respectively. Let $B$ denote the Borel subgroup contained in $X$, and let $T$ be a maximal torus contained in $B$. Then $0_{X}$ is the unique closed $B \times B$ orbit in $X$, hence, $X \backslash\{0\}$ is (rationally) smooth. Let $\mu: \mathbb{C}^{*} \times M \rightarrow M$ be a central (in $B$ ) one-parameter subgroup action on $M$ such
that $\lim _{t \rightarrow 0} \mu(t, x)=0_{X}$ for every $x \in M$. Then the quotients $\mathbb{P}_{X}:=\left(X \backslash\left\{0_{X}\right\}\right) / \mathbb{C}^{*}$ and $\mathbb{P}_{M}:=\left(M \backslash\left\{0_{X}\right\}\right) / \mathbb{C}^{*}$ are projective $T \times T$ varieties such that $\mathbb{P}_{X} \subsetneq \mathbb{P}_{M}$. Furthermore, $\mathbb{P}_{X}$ and $\mathbb{P}_{M}$ are rationally smooth. Similarly, we have the rationally smooth, projective $T^{\prime} \times T^{\prime}$ varieties $\mathbb{P}_{Y} \subsetneq \mathbb{P}_{N}$, where $T^{\prime}$ is the maximal torus in $g(T) \subset Y$.

By result of Brion [5, Lemma 1.3] we know that $X$ (resp. $Y$ ) is rationally smooth if and only if the Euler characteristic of $\mathbb{P}_{X}$ (resp. the Euler characteristic of $\mathbb{P}_{Y}$ ) is equal to the number of $T \times T$ fixed points in $\mathbb{P}_{X}$ (resp. the number of $T^{\prime} \times T^{\prime}$ fixed points in $\mathbb{P}_{Y}$ ). Let us denote by $C(M)$ (resp. by $C(N))$ the closed $G \times G$-orbit in $\mathbb{P}_{M}$ (resp. the closed $G^{\prime} \times G^{\prime}$-orbit in $\mathbb{P}_{N}$ ), where $G$ (resp. $G^{\prime}$ ) is the unit group of $M$ (resp. of $N$ ). Since the $T \times T$ fixed points in $\mathbb{P}_{X}$ lie in the closed intersection $\mathbb{P}_{X} \cap C(M)$, and since $\left.g\right|_{C(M)}: C(M) \rightarrow C(N)$ is a bijection, we see that the Euler characteristics of $X$ and $Y$ are equal. In particular, $X$ is rationally smooth if and only if $Y$ is rationally smooth.

Two reductive monoids $M$ and $N$ are said to be equivalent if there exists a reductive monoid $L$ with two finite dominant morphisms $L \rightarrow M$ and $L \rightarrow N$. If $M$ and $N$ are equivalent monoids, then we will write $M \sim_{0} N$. It is easy to verify that $\sim_{0}$ is an equivalence relation. Let $M$ be a reductive monoid with zero. According to [24, Theorem 2.4],

$$
\begin{equation*}
M \text { is rationally smooth } \Longleftrightarrow M \sim_{0} \prod_{i=1}^{r} M_{n_{i}} . \tag{6.2}
\end{equation*}
$$

We are now ready to prove the main result of this section.
Theorem 6.3. Let $M$ be a rationally smooth reductive monoid with zero. Let $B$ be a Borel subgroup in $M$ and let $\bar{B}$ denote the corresponding Borel submonoid. Then $\bar{B}$ is a rationally algebraic semigroup.

Proof. Since $M$ is rationally smooth, we know from (6.2) that there exists a reductive monoid $L$ admitting two finite dominant morphisms:


Without loss of generality, we may assume that $L$ has a zero. Let $B_{L}$ be a Borel subgroup of $L$. As $f$ and $g$ are finite and dominant morphisms, they are surjective. In particular, the subgroups $f\left(B_{L}\right)$ and $g \underline{\left(B_{L}\right)}$ are Borel subgroups in $M$ and $\prod_{i=1}^{r} M_{n_{i}}$, respectively. We set $X:=\overline{f\left(B_{L}\right)}$ and $Y:=\overline{g\left(B_{L}\right)}$. Then $X$ and $Y$ are Borel submonoids in $M$ and $\prod_{i=1}^{r} M_{n_{i}}$, respectively. Since $Y$ is (rationally) smooth, by Lemma 6.1, so is $\overline{B_{L}}$. Once again by using Lemma 6.1, we see that $X$ is rationally smooth. This finishes the proof of our theorem.

As an application of Theorem 6.3, we consider the symplectic monoid $M S p_{n}$. By the Renner's classification of rationally smooth simple group embeddings [23, Corollary 3.5], $M S p_{n}$ is a rationally smooth semisimple monoid. Therefore, by Theorem 6.3, its Borel submonoid is rationally smooth.

## 7 Final Remarks: Nilpotent Subsemigroups

In this section we will contrast some properties of the Borel submonoids of $M_{n}$ and $M S p_{n}$. We begin with a general observation.

Lemma 7.1. Let $M$ be a reductive monoid with the Bruhat-Chevalley-Renner decomposition $M=\sqcup_{r \in R} B \dot{r} B$, where $B$ is a Borel subgroup in $M$, and $R$ is the Renner monoid of $M$. If an element $r$ from $R$ satisfies the following two properties, then every element of the orbit $B \dot{r} B$, where $\dot{r}$ is a representative of $r$ in $\overline{N_{G}(T)}$, is nilpotent:
(1) $r$ is nilpotent in $R$, that is, $r^{m}=0$ for some $m \in \mathbb{Z}_{+}$;
(2) $r \leq 1$.

Proof. Since $M$ is a linear algebraic monoid, it admits an embedding into $M_{n}$ for a sufficiently large positive integer $n$. By conjugating with an element of $G L_{n}$, we assume that $B$ is contained the upper triangular Borel submonoid of $M_{n}$. Clearly, if we can prove our assertion for $M=M_{n}$ and $B=B_{n}$, then the general case will follow. In this case, the Renner monoid is given by the rook monoid $\mathcal{R}_{n}$, and we can identify $\mathcal{R}_{n}$ as a submonoid of $M_{n}$. An element $r$ from $\mathcal{R}_{n}$ satisfies the two properties in our hypotheses if and only if it is a strictly upper triangular rook. But the product of an upper triangular matrix with a strictly upper triangular matrix is strictly upper triangular. Therefore, any element of $B r B$ is strictly upper triangular, hence, nilpotent. This finishes the proof of our assertion.

We should note that we cannot replace any of the two requirements in Lemma 7.1. Indeed, if $r$ is not nilpotent, then any of its representatives $\dot{r}$, which is contained in $B \dot{r} B$, is not nilpotent. For the second item, we consider the matrices $r=\left[\begin{array}{ll}0 & 0 \\ 1 & 0\end{array}\right]$ and $b_{2}=\left[\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right]$. Then, we have $r b_{2}=\left[\begin{array}{ll}0 & 0 \\ 1 & 1\end{array}\right]$, which is not nilpotent. Evidently, the set of nilpotent elements in a Borel submonoid is a closed subset.

Definition 7.2. The subvariety $\bar{B} \bar{B}^{\text {nil }}:=\left\{x \in \bar{B}: x^{m}=0\right.$ for some $\left.m \in \mathbb{Z}_{+}\right\}$will be called the nilpotent semigroup of $\bar{B}$.

Corollary 7.3. Let $B$ be a Borel subgroup in a reductive monoid $M$ with zero. Then the nilpotent semigroup of $\bar{B}$ is a $B \times B$-stable algebraic subsemigroup of $M$.

Proof. By definition, $\bar{B}^{\text {nil }}$ is defined by the polynomial relations $x^{m}=0\left(m \in \mathbb{Z}_{+}\right)$, therefore, it is a closed subset of $\bar{B}$. By Lemma 7.1, we know that $\bar{B}^{\text {nil }}$ is $B \times B$-stable. In fact, the proof of this lemma shows that $\bar{B}^{n i l}$ is a semigroup.

Next, we will show that ${\overline{B_{n}}}^{\text {nil }}$ is an irreducible variety.
Proposition 7.4. The nilpotent semigroup of $M_{n}$ is an irreducible algebraic semigroup of dimension $\binom{n}{2}$.

Proof. By Lemma 7.1, we know that ${\overline{B_{n}}}^{\text {nil }}=\bigsqcup B_{n} r B_{n}$, where the union is over all strictly upper triangular rooks in $\mathcal{R}_{n}$. It is easy to check that

1. $r_{0}:=(0,1,2, \ldots, n-1)$ is a strictly upper triangular rook;
2. if $r$ is a strictly upper triangular rook, then $r \leq r_{0}$.

These two conditions imply that $\overline{B_{n} r_{0} B_{n}}={\overline{B_{n}}}^{\text {nil }}$. It is easy to check that $\ell\left(r_{0}\right)=1+2+$ $\cdots+(n-1)=\binom{n}{2}$. Since the orbit $B_{n} r_{0} B_{n}$ is an irreducible variety, so is ${\overline{B_{n}}}^{\text {nil }}$. Thus, in light of Corollary 7.3, the proofs of our assertions are finished.

Unfortunately the nice situation as in Proposition 7.4 does not hold for the nilpotent semigroup of $\overline{B_{S p_{n}}}$. It turns out that $\overline{B_{S p_{n}}}$ nil has many irreducible components in varying dimensions. See Figure 7.1 for an example.


Figure 7.1: The Hasse diagram of $\overline{\mathcal{B}}_{S p_{4}}{ }^{\text {nil }}$.

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