

Bijections on r -Shi and r -Catalan Arrangements

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Abstract

Associated with the r -Shi arrangement and r -Catalan arrangement in \mathbb{R}^n , we introduce a cubic matrix for each region to establish two bijections in a uniform way. Firstly, the positions of minimal positive entries in column slices of the cubic matrix will give a bijection from regions of the r -Shi arrangement to O -rooted labeled r -trees. Secondly, the numbers of positive entries in column slices of the cubic matrix will give a bijection from regions of the r -Catalan arrangement to pairings of permutation and r -Dyck path. Moreover, the numbers of positive entries in row slices of the cubic matrix will recover the Pak-Stanley labeling, a celebrated bijection from regions of the r -Shi arrangement to r -parking functions.

Keywords: cubic matrix, Shi arrangement, Catalan arrangement, Pak-Stanley labeling

1 Concepts and Backgrounds

This paper aims to establish two bijections: from regions of the r -Shi arrangement to O -rooted labeled r -trees, and from regions of the r -Catalan arrangement to pairings of permutation and r -Dyck path. To this end, we introduce a cubic matrix for each region to read the combinatorial information from the region.

A **hyperplane arrangement** \mathcal{A} is a finite collection of hyperplanes in a vector space V , see [35, 47]. When V is a real space, the set $V \setminus \cup_{H \in \mathcal{A}} H$ consists of finitely many connected components, called **regions** of \mathcal{A} . Denote by $\mathcal{R}(\mathcal{A})$ the set of regions of \mathcal{A} . For any positive integers r and n , the **r -Shi arrangement** \mathcal{S}_n^r in \mathbb{R}^n consists of the following hyperplanes

$$\mathcal{S}_n^r : x_i - x_j = -r + 1, -r + 2, \dots, 0, 1, \dots, r, \quad 1 \leq i < j \leq n.$$

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The case of $r = 1$ is the classical **Shi arrangement** \mathcal{S}_n introduced by Shi [44] in 1986. Shi further obtained the number of regions of \mathcal{S}_n^r .

Theorem 1.1. [45] *For any positive integers r and n , the number of regions of \mathcal{S}_n^r is*

$$|\mathcal{R}(\mathcal{S}_n^r)| = (rn + 1)^{n-1}.$$

Let $O = \{o_1, \dots, o_r\}$ and $V = \{v_1, \dots, v_n\}$ be two disjoint sets of labeled vertices. First introduced by Harary and Palmer [18] in 1968, an **O -rooted labeled r -tree** T on $O \cup V$ is a graph having the property: there is a valid rearrangement $\nu = (v_{i_1}, \dots, v_{i_n})$ of vertices v_1, \dots, v_n , such that each v_{i_j} with $j \in [n]$ is adjacent to exactly r vertices in $\{o_1, \dots, o_r, v_{i_1}, \dots, v_{i_{j-1}}\}$ and, moreover, these r vertices are themselves mutually adjacent in T . Section 3 will be devoted to some characterizations of the O -rooted labeled r -trees. Denote by \mathcal{T}_n^r the set of all O -rooted labeled r -trees. In the case of $r = 1$, write $\mathcal{T}_n = \mathcal{T}_n^1$, whose members are called **O -rooted labeled trees**. The size of \mathcal{T}_n^r has been counted by Foata [10], Beineke and Pippert [4, 5], Gainer-Dewar and Gessel [16] etc., which extends the Cayley formula $|\mathcal{T}_n| = (n + 1)^{n-1}$ of [7].

Theorem 1.2. [4, 5, 10] *For any positive integers r and n , the cardinality of \mathcal{T}_n^r is*

$$|\mathcal{T}_n^r| = (rn + 1)^{n-1}.$$

Closely related to O -rooted labeled r -trees and r -Shi arrangement, the **r -parking function of length n** is a sequence $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ such that the monotonic rearrangement $a_1 \leq a_2 \leq \dots \leq a_n$ of the numbers $\alpha_1, \dots, \alpha_n$ satisfies $a_i \leq r(i - 1)$. Denote by \mathcal{P}_n^r the set of all r -parking functions of length n . In the case of $r = 1$, write $\mathcal{P}_n = \mathcal{P}_n^1$ whose members are called **parking functions of length n** . Explored by Pitman and Stanley [39], Yan [51] etc., the cardinality of \mathcal{P}_n^r is exactly the same as $\mathcal{R}(\mathcal{S}_n^r)$ and \mathcal{T}_n^r , namely,

$$|\mathcal{P}_n^r| = (rn + 1)^{n-1}.$$

Naturally we may ask if there are some bijections among $\mathcal{R}(\mathcal{S}_n^r)$, \mathcal{T}_n^r , and \mathcal{P}_n^r . A celebrated bijection $\mathcal{R}(\mathcal{S}_n^r) \rightarrow \mathcal{P}_n^r$ (abbreviation for ‘from $\mathcal{R}(\mathcal{S}_n^r)$ to \mathcal{T}_n^r ’) is the Pak-Stanley labeling which was first suggested by I. Pak in the case of $r = 1$, and extended to general r by R. P. Stanley [48, 49]. Later, relevant to the Pak-Stanley labeling, many results on bijections $\mathcal{R}(\mathcal{S}_n^r) \rightarrow \mathcal{P}_n^r$ have been obtained, see [2, 3, 8, 31, 40, 47–49] etc.. For the bijection $\mathcal{P}_n^r \rightarrow \mathcal{T}_n^r$, currently we just know that it can be established by a composition of three other bijections given in [36] by I. Pak and A. Postnikov. In the case of $r = 1$, bijections $\mathcal{P}_n \rightarrow \mathcal{T}_n$ have been well studied since 1968, see [11, 13, 22, 25, 41, 43] etc.. To the best of our knowledge, no explicit bijection $\mathcal{R}(\mathcal{S}_n^r) \rightarrow \mathcal{T}_n^r$ has been established, which is exactly our motivation of this paper. Our first main result is to establish a bijection $\mathcal{R}(\mathcal{S}_n^r) \rightarrow \mathcal{T}_n^r$, see Theorem 2.2. To this end, we will introduce a cubic matrix for r -Shi arrangement, which will also let us define the Pak-Stanley labeling in an easy way, see Theorem 2.4.

Surprisingly, the cubic matrix method can be applied to the **r -Catalan arrangement** \mathcal{C}_n^r in \mathbb{R}^n , the collection of hyperplanes

$$\mathcal{C}_n^r : \quad x_i - x_j = 0, \pm 1, \dots, \pm r, \quad \text{for } 1 \leq i < j \leq n.$$

When $r = 1$, denote $\mathcal{C}_n = \mathcal{C}_n^1$, called the **Catalan arrangement**. The number of regions of \mathcal{C}_n^r was first obtained by Athanasiadis [1] in 2004.

Theorem 1.3. [1] *For any positive integers r and n , the number of regions of \mathcal{C}_n^r is*

$$|\mathcal{R}(\mathcal{C}_n^r)| = n!C(n, r) = \frac{n!}{rn+1} \binom{rn+n}{n}.$$

In Theorem 1.3, the number $C(n, r) = \frac{1}{rn+1} \binom{rn+n}{n}$ is called the **Fuss-Catalan number** or **Raney number**, which counts the number of the r -Dyck paths of length n . As written in [23], the Fuss-Catalan number was first studied by Fuss [15] in 1791, forty-seven years before Catalan investigated the parenthesization problem, see [12, 17, 19, 23, 28, 32, 33, 38] for more results on the Fuss-Catalan number. The paper [20] presented several combinatorial structures which are counted by Fuss-Catalan numbers. In the case of $r = 1$, $C(n, r) = C_n = \frac{1}{n+1} \binom{2n}{n}$ is the **Catalan number**, see [50] for a complete investigation on the Catalan number. Dyck path has many generalizations that have been widely studied in the past, see [6, 9, 14, 21, 26, 27, 29, 30, 42]. As a generalization of Dyck path, a **r -Dyck path of length n** is a lattice path in the x - y plane moving from $(0, 0)$ to (n, rn) with steps $(1, 0)$ and $(0, 1)$ and never going above the line $y = rx$. Denote by \mathcal{D}_n^r the collection of all r -Dyck paths of length n and $\mathcal{D}_n = \mathcal{D}_n^1$ the set of all **Dyck paths** of length n . In 1989, Krattenthaler [24] obtained the number of r -Dyck paths of length n .

Theorem 1.4. [24] *For any positive integers r and n , the cardinality of \mathcal{D}_n^r is*

$$|\mathcal{D}_n^r| = C(n, r).$$

As our second main result, we will establish a bijection $\mathcal{R}(\mathcal{C}_n^r) \rightarrow \mathfrak{S}_n \times \mathcal{D}_n^r$ in Theorem 2.6 via the cubic matrix defined for the r -Catalan arrangement, which will extend the bijection defined in [47, p. 69].

2 Main Results

Our first main result is a bijection $\mathcal{R}(\mathcal{S}_n^r) \rightarrow \mathcal{T}_n^r$, which will be stated in Section 2.1 and proved in Section 4. The second main result is a bijection $\mathcal{R}(\mathcal{C}_n^r) \rightarrow \mathfrak{S}_n \times \mathcal{D}_n^r$ and will be given in Section 2.2.

2.1 Bijection $\mathcal{R}(\mathcal{S}_n^r) \rightarrow \mathcal{T}_n^r$

By introducing a cubic matrix for r -Shi arrangement, in this section we establish a bijection $\mathcal{R}(\mathcal{S}_n^r) \rightarrow \mathcal{T}_n^r$ and present a straightforward way to view the Pak-Stanley labelling. Given a region $\Delta \in \mathcal{R}(\mathcal{S}_n^r)$ and a representative $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \Delta$, define the **cubic matrix** $C_{\mathbf{x}} = (c_{ijk}(\mathbf{x})) \in \mathbb{R}^{n \times n \times r}$ to be

$$c_{ijk}(\mathbf{x}) = \begin{cases} x_i - x_j - k, & \text{if } i < j; \\ 0, & \text{if } i = j; \\ x_i - x_j - k + 1, & \text{if } i > j, \end{cases} \quad (1)$$

which is an r -tuple of square matrices as the index k running from 1 to r . For any $i, j \in [n]$, let

$$\text{row}_i(C_{\mathbf{x}}) = (c_{ijk}(\mathbf{x}))_{j \in [n], k \in [r]} \quad \text{and} \quad \text{col}_j(C_{\mathbf{x}}) = (c_{ijk}(\mathbf{x}))_{i \in [n], k \in [r]},$$

called the *i*-th row slice and *j*-th column slice of $C_{\mathbf{x}}$ respectively. Note that each hyperplane $H \in \mathcal{S}_n^r$ is exactly defined by the equation $H : c_{ijk}(\mathbf{x}) = 0$ for some i, j and k , and all points of Δ lie in the same side of H since $\Delta \cap H = \emptyset$. It follows that $c_{ijk}(\mathbf{x})$ has the same sign for all $\mathbf{x} \in \Delta$, namely, $\text{Sgn}(c_{ijk}(\mathbf{x}))$ is independent of the choice of representatives $\mathbf{x} \in \Delta$ and can be denoted by

$$\text{Sgn}_{ijk}(\Delta) = \text{Sgn}(c_{ijk}(\mathbf{x})). \quad (2)$$

Then $\text{Sgn}(\Delta) = (\text{Sgn}_{ijk}(\Delta))$ automatically defines a bijection

$$\text{Sgn} : \mathcal{R}(\mathcal{S}_n^r) \rightarrow \{\text{Sgn}(\Delta) \mid \Delta \in \mathcal{R}(\mathcal{S}_n^r)\}. \quad (3)$$

The symbol \mathbf{x} is understood as either a point of \mathbb{R}^n or indeterminate depending on its meaning in the context.

Definition 2.1. Let $O = \{o_1, \dots, o_r\}$ and $V = \{v_1, \dots, v_n\}$ be two disjoint sets of labeled vertices. Given a region $\Delta \in \mathcal{R}(\mathcal{S}_n^r)$ and $\mathbf{x} \in \Delta$, for any $j \in [n]$, let $f(v_j) = (f_1(v_j), \dots, f_r(v_j)) \in (O \cup V)^r$ be defined recursively as follows,

(i) if all entries of $\text{col}_j(C_{\mathbf{x}})$ are nonpositive, let $p_j = 0$ and

$$f(v_j) = (o_1, o_2, \dots, o_r),$$

(ii) otherwise, $p_j \neq 0$ and $\text{col}_j(C_{\mathbf{x}})$ has a unique minimal positive entry at (p_j, q_j) , let

$$f(v_j) = (f_1(v_{p_j}), \dots, f_{q_j-1}(v_{p_j}), f_{q_j+1}(v_{p_j}), \dots, f_r(v_{p_j}), v_{p_j}),$$

and let the map $F : V \rightarrow \binom{O \cup V}{r}$ with

$$F(v_j) = \{f_i(v_j) \mid i \in [r]\}.$$

Define the graph $T_{\mathbf{x}}$ on the vertex set $O \cup V$ such that the vertex v_j and vertices in $F(v_j)$ form an $(r+1)$ -clique for all $j \in [n]$.

Below is the first main result of this paper, whose proof is highly nontrivial and will be given in Section 4.

Theorem 2.2. With the same notations as Definition 2.1, the following map is a bijection,

$$\Psi_n^r : \mathcal{R}(\mathcal{S}_n^r) \rightarrow \mathcal{T}_n^r, \quad \Psi_n^r(\Delta) = T_{\mathbf{x}} \quad \text{for any } \mathbf{x} \in \Delta. \quad (4)$$

In the case of $r = 1$, the statements of Definition 2.1 and Theorem 2.2 become quite simple, see Corollary 2.3.

Corollary 2.3. Let $O = \{o_1\}$ and $V = \{v_1, \dots, v_n\}$ be two disjoint sets of labeled vertices. Given a region Δ of \mathcal{S}_n , for any $\mathbf{x} \in \Delta$, define an $n \times n$ matrix $A_{\mathbf{x}} = (a_{ij}(\mathbf{x}))$ with

$$a_{ij}(\mathbf{x}) = \begin{cases} x_i - x_j - 1, & \text{if } i < j; \\ 0, & \text{if } i = j; \\ x_i - x_j, & \text{if } i > j, \end{cases}$$

and a graph $T_{\mathbf{x}}$ on $O \cup V$ such that for each $j \in [n]$, v_j is adjacent to v_{p_j} , where p_j is defined as follows,

if column j of $A_{\mathbf{x}}$ has no positive entry, assume $p_j = 0$ and $v_0 = o_1$;

(ii) otherwise, column j of $A_{\mathbf{x}}$ has a unique minimal positive entry at row p_j .

Then $T_{\mathbf{x}}$ is an O -rooted labeled tree and independent of the choice of representatives $\mathbf{x} \in \Delta$. Moreover, the map $\Psi_n : \mathcal{R}(\mathcal{S}_n) \rightarrow \mathcal{T}_n$ with $\Psi_n(\Delta) = T_{\mathbf{x}}$ is a bijection.

In 1998, a celebrated bijection $\mathcal{R}(\mathcal{S}_n^r) \rightarrow \mathcal{P}_n^r$ was obtained by Stanley [49] and called **the Pak-Stanley labeling**, which is defined recursively as follows. Start with the base region $\Delta_0 \in \mathcal{R}(\mathcal{S}_n^r)$ with

$$\Delta_0 : x_1 > x_2 > \cdots > x_n > x_1 - 1,$$

whose labeling is assumed to be $\lambda(\Delta_0) = (0, \dots, 0) \in \mathbb{Z}_{\geq 0}^n$. Suppose $\Delta \in \mathcal{R}(\mathcal{S}_n^r)$ has been labeled by $\lambda(\Delta) \in \mathbb{Z}_{\geq 0}^n$, and an unlabeled region $\Delta' \in \mathcal{R}(\mathcal{S}_n^r)$ is separated from Δ by a unique hyperplane $H : c_{ijk}(\mathbf{x}) = 0$. Then define the region Δ' to be labeled by $\lambda(\Delta') = \lambda(\Delta) + e_i$. Using the cubic matrix $C_{\mathbf{x}}$, Theorem 2.1 of [49] can be restated as follows.

Theorem 2.4. [49] *Given a region Δ of \mathcal{S}_n^r and $\mathbf{x} \in \Delta$, for any $i \in [n]$, let*

$$\lambda_i(\Delta) = \text{the number of positive signs of } \text{Sgn}(\text{row}_i(C_{\mathbf{x}})).$$

The following map is a bijection

$$\lambda : \mathcal{R}(\mathcal{S}_n^r) \rightarrow \mathcal{P}_n^r, \quad \Delta \mapsto \lambda(\Delta) = (\lambda_1(\Delta), \dots, \lambda_n(\Delta)).$$

Proof. Note that the base region is

$$\Delta_0 = \{\mathbf{y} \in \mathbb{R}^n \mid c_{ijk}(\mathbf{y}) < 0, i, j \in [n], k \in [r]\}.$$

If the region Δ is separated from Δ_0 by the hyperplane $H : c_{ijk}(\mathbf{y}) = 0$, then $\mathbf{x} \in \Delta$ implies $c_{ijk}(\mathbf{x}) > 0$. From the definition of the Pak-Stanley labeling, it is easily seen that $\lambda_i(\Delta)$ is the number of the hyperplanes $H : c_{ijk}(\mathbf{y}) = 0$ separating Δ from Δ_0 . Namely, $\lambda_i(\Delta)$ is the number of positive entries in the i -th row slice of $C_{\mathbf{x}}$. \square

Remark 2.5. *Theoretically, the compositions of our bijection $(\Psi_n^r)^{-1} : \mathcal{T}_n^r \rightarrow \mathcal{R}(\mathcal{S}_n^r)$ in Theorem 2.2 and the Pak-Stanley labeling $\lambda : \mathcal{R}(\mathcal{S}_n^r) \rightarrow \mathcal{P}_n^r$ in Theorem 2.4 will produce a bijection $\mathcal{T}_n^r \rightarrow \mathcal{P}_n^r$, while it seems to be highly complicated and difficult to be stated explicitly.*

2.2 Bijection $\mathcal{R}(\mathcal{C}_n^r) \rightarrow \mathfrak{S}_n \times \mathcal{D}_n^r$

In this section, we will establish a bijection $\mathcal{R}(\mathcal{C}_n^r) \rightarrow \mathfrak{S}_n \times \mathcal{D}_n^r$. Similar as [47, p. 68], the permutation group \mathfrak{S}_n acts on \mathbb{R}^n by permuting coordinates, i.e., if $\pi \in \mathfrak{S}_n$, for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ we have

$$\pi(\mathbf{x}) = (x_{\pi(1)}, \dots, x_{\pi(n)}).$$

Given a region $\Delta \in \mathcal{R}(\mathcal{C}_n^r)$ and $\mathbf{x} \in \Delta$, there is a unique permutation $\pi_{\Delta} \in \mathfrak{S}_n$, independent of the choice of $\mathbf{x} \in \Delta$, such that

$$x_{\pi_{\Delta}(1)} > \cdots > x_{\pi_{\Delta}(n)}.$$

Note that $\mathcal{R}(\mathcal{C}_n^r)$ is \mathfrak{S}_n -invariant, i.e., for any $\pi \in \mathfrak{S}_n$ and $\Delta \in \mathcal{R}(\mathcal{C}_n^r)$, we have

$$\pi(\Delta) = \{\pi(\mathbf{x}) \mid \mathbf{x} \in \Delta\} \in \mathcal{R}(\mathcal{C}_n^r).$$

For $\pi \in \mathfrak{S}_n$, denote by

$$\mathcal{R}_\pi(\mathcal{C}_n^r) = \{\Delta \in \mathcal{R}(\mathcal{C}_n^r) \mid \pi_\Delta = \pi\}.$$

In particular, let

$$\mathcal{R}_1(\mathcal{C}_n^r) = \{\Delta \in \mathcal{R}(\mathcal{C}_n^r) \mid \pi_\Delta = \mathbf{1} \text{ is the identity permutation}\}.$$

It is clear that π is a bijection from $\mathcal{R}_\pi(\mathcal{C}_n^r)$ to $\mathcal{R}_1(\mathcal{C}_n^r)$ and

$$\mathcal{R}(\mathcal{C}_n^r) = \bigsqcup_{\pi \in \mathfrak{S}_n} \mathcal{R}_\pi(\mathcal{C}_n^r).$$

To obtain the bijection $\mathcal{R}(\mathcal{C}_n^r) \rightarrow \mathfrak{S}_n \times \mathcal{D}_n^r$, it is enough to establish a bijection $\mathcal{R}_1(\mathcal{C}_n^r) \rightarrow \mathcal{D}_n^r$. Given a region $\Delta \in \mathcal{R}_1(\mathcal{C}_n^r)$ and a representative $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \Delta$, define the **cubic matrix** $D_{\mathbf{x}} = (d_{ijk}(\mathbf{x})) \in \mathbb{R}^{n \times n \times r}$ to be

$$d_{ijk}(\mathbf{x}) = \begin{cases} x_i - x_j - k, & \text{if } i \neq j; \\ 0, & \text{if } i = j; \end{cases}$$

Similar as before, each hyperplane $H \in \mathcal{C}_n^r$ is exactly defined by the equation $H : d_{ijk}(\mathbf{x}) = 0$ for some $i, j \in [n]$ with $i \neq j$ and $k \in [r]$. So we still have that $\text{Sgn}(d_{ijk}(\mathbf{x}))$ is independent of the choice of representatives $\mathbf{x} \in \Delta$.

For any r -Dyck path $P \in \mathcal{D}_n^r$, if the vertical line $x = i - \frac{1}{2}$ intersects P at the y -coordinate $h_i(P)$, the sequence $\mathbf{h}(P) = (h_1(P), \dots, h_n(P)) \in \mathbb{Z}^n$ is nondecreasing and satisfies $0 \leq h_i(P) \leq r(i-1)$, called the **height sequence** of P . Conversely, it is clear that any nondecreasing sequence $\mathbf{h} = (h_1, \dots, h_n)$ with $0 \leq h_i \leq r(i-1)$ uniquely determines a r -Dyck path P of length n such that $\mathbf{h}(P) = \mathbf{h}$. Indeed, the height sequence of a r -Dyck path is also a r -parking function. Now we are ready to give the bijection $\mathcal{R}_1(\mathcal{C}_n^r) \rightarrow \mathcal{D}_n^r$. Given any region $\Delta \in \mathcal{R}_1(\mathcal{C}_n^r)$ and $\mathbf{x} \in \Delta$, let $\mathbf{h}(\Delta) = (h_1(\Delta), \dots, h_n(\Delta))$ be a sequence defined by

$$h_j(\Delta) = \text{the number of positive signs of } \text{Sgn}(\text{col}_j(D_{\mathbf{x}})), \quad j \in [n]. \quad (5)$$

As we shall see in Theorem 2.6, the sequence $\mathbf{h}(\Delta)$ is exactly the height sequence of a r -Dyck path of length n , say P_Δ , which defines the bijection

$$\mathcal{R}_1(\mathcal{C}_n^r) \rightarrow \mathcal{D}_n^r, \quad \Delta \mapsto P_\Delta. \quad (6)$$

Now for any region $\Delta \in \mathcal{R}(\mathcal{C}_n^r)$, we have $\Delta \in \mathcal{R}_{\pi_\Delta}(\mathcal{C}_n^r)$ and $\Delta' = \pi_\Delta(\Delta) \in \mathcal{R}_1(\mathcal{C}_n^r)$. By abuse of notations, denote by P_Δ the corresponding r -Dyck path $P_{\Delta'}$ obtained from the above bijection (6), namely $P_\Delta = P_{\pi_\Delta(\Delta)}$ for any $\Delta \in \mathcal{R}(\mathcal{C}_n^r)$. Below is our second main result.

Theorem 2.6. *For any positive integers r and n , the following map is a bijection,*

$$\Phi_n^r : \mathcal{R}(\mathcal{C}_n^r) \rightarrow \mathfrak{S}_n \times \mathcal{D}_n^r, \quad \Phi_n^r(\Delta) = (\pi_\Delta, P_\Delta).$$

Proof. Notice from Theorem 1.3 and 1.4 that the both $\mathcal{R}(\mathcal{C}_n^r)$ and $\mathfrak{S}_n \times \mathcal{D}_n^r$ have the same cardinality $n!C(n, r)$. By the above arguments, it is enough to show that the map defined in (6) is injective. For any $\Delta \in \mathcal{R}_1(\mathcal{C}_n^r)$ and $\mathbf{x} = (x_1, \dots, x_n) \in \Delta$, we have $x_1 > x_2 > \dots > x_n$. It is easily seen from the definition of the cubic matrix $D_{\mathbf{x}} = (d_{ijk}(\mathbf{x})) \in \mathbb{R}^{n \times n \times r}$ that

- (a) $d_{ijk}(\mathbf{x}) > 0$ implies $i < j$;

(b) if $i < j < j'$, then $d_{ijk}(\mathbf{x}) > 0$ implies $d_{ij'k}(\mathbf{x}) > 0$ since $d_{ij'k}(\mathbf{x}) > d_{ijk}(\mathbf{x})$.

Note from the definition of (5) that for $j \in [n]$,

$$h_j(\Delta) = \#\{(i, k) \in [n] \times [r] \mid d_{ijk}(\mathbf{x}) > 0\}. \quad (7)$$

The properties (a) and (b) imply $h_j(\Delta) \leq r(j-1)$ for any $j \in [n]$ and $h_1(\Delta) \leq h_2(\Delta) \leq \dots \leq h_n(\Delta)$ respectively. So $\mathbf{h}(\Delta) = (h_1(\Delta), \dots, h_n(\Delta))$ is a height sequence of a r -Dyck path of length n , i.e., the map given in (6) is well-defined. Next we prove the injectivity of the map in (6) by contradiction. Suppose Δ and Ω are two distinct regions in $\mathcal{R}_1(\mathcal{C}_n^r)$ with $\mathbf{h}(\Delta) = \mathbf{h}(\Omega)$ and let $\mathbf{x} \in \Delta$ and $\mathbf{y} \in \Omega$. From $\Delta \neq \Omega$, we have a minimal index $j \in [n]$ such that the hyperplane $H : d_{ijk}(\mathbf{z}) = 0$ separates Δ from Ω . Assume

$$d_{ijk}(\mathbf{x}) = x_i - x_j - k > 0 \quad \text{and} \quad d_{ijk}(\mathbf{y}) = y_i - y_j - k < 0.$$

Since $h_j(\Delta) = h_j(\Omega)$, from (7) there must exist a pairing $(i', k') \neq (i, k)$ such that

$$d_{i'jk'}(\mathbf{x}) = x_{i'} - x_j - k' < 0 \quad \text{and} \quad d_{i'jk'}(\mathbf{y}) = y_{i'} - y_j - k' > 0.$$

By property (a), we have $i, i' < j$. If $i = i'$, we have $k' > x_i - x_j > k$ since $d_{ijk}(\mathbf{x}) > 0$ and $d_{i'jk'}(\mathbf{x}) < 0$ and $k > y_i - y_j > k'$ since $d_{ijk}(\mathbf{y}) < 0$ and $d_{i'jk'}(\mathbf{y}) > 0$, which is a contradiction. If $i < i'$, we have $k > y_i - y_j > y_{i'} - y_j > k'$ since $d_{ijk}(\mathbf{y}) < 0$ and $d_{i'jk'}(\mathbf{y}) > 0$. Consider the hyperplane $H : d_{ii'(k-k')}(\mathbf{z}) = 0$. We have

$$\begin{aligned} d_{ii'(k-k')}(\mathbf{x}) &= d_{ijk}(\mathbf{x}) - d_{i'jk'}(\mathbf{x}) > 0, \\ d_{ii'(k-k')}(\mathbf{y}) &= d_{ijk}(\mathbf{y}) - d_{i'jk'}(\mathbf{y}) < 0, \end{aligned}$$

which means that the hyperplane $H : d_{ii'(k-k')}(\mathbf{z}) = 0$ separates Δ from Ω , a contradiction to the minimality of the index j . By similar arguments as the case $i < i'$, we can obtain a contradiction for the case $i > i'$. So we can conclude that the map in (6) is injective, which completes the proof. \square

It is easily seen that Theorem 2.6 not only extends the bijection $\mathcal{R}(\mathcal{C}_n) \rightarrow \mathfrak{S}_n \times \mathcal{D}_n$ defined in [47, page 69], but make it more straightforward with the help of the cubic matrix. Below is an example to illustrate the construction of the Dyck path from a region in the case of $r = 1$.

Example 2.7. Let $\Delta \in \mathcal{R}(\mathcal{C}_6)$ be the region

$$\Delta = \left\{ \mathbf{x} = (x_1, \dots, x_6) \in \mathbb{R}^6 \left| \begin{array}{l} x_4 > x_3 > x_6 > x_1 > x_2 > x_5, \\ x_3 - x_1 > 1, \quad x_1 - x_2 > 1, \\ x_4 - x_6 < 1, \quad x_6 - x_1 < 1, \quad x_2 - x_5 < 1. \end{array} \right. \right\}.$$

It is obvious that $\pi_\Delta = 436125 \in \mathfrak{S}_6$ and for $1 \leq i < j \leq 6$,

$$\Delta' = \pi_\Delta(\Delta) = \left\{ \mathbf{x} = (x_1, \dots, x_6) \in \mathbb{R}^6 \left| \begin{array}{l} x_1 > x_2 > x_3 > x_4 > x_5 > x_6, \\ x_2 - x_4 > 1, \quad x_4 - x_5 > 1, \\ x_1 - x_3 < 1, \quad x_3 - x_4 < 1, \quad x_5 - x_6 < 1. \end{array} \right. \right\}.$$

It follows that for any $\mathbf{x} \in \Delta'$,

$$\text{Sgn}(c_{ij1}(\mathbf{x})) = \begin{cases} -, & \text{if } (i, j) \in \{(1, 2), (1, 3), (2, 3), (3, 4), (5, 6)\}; \\ +, & \text{if } (i, j) \in \{(1, 4), (2, 4), (1, 5), (2, 5), (3, 5), (4, 5), (1, 6), (2, 6), (3, 6), (4, 6)\}. \end{cases}$$

So we have $\mathbf{h}(\Delta') = (0, 0, 0, 2, 4, 4)$, which is the height sequence of the Dyck path $P_\Delta = P_{\Delta'}$. Namely, $\Phi_n^1(\Delta) = (\pi_\Delta, P_\Delta)$ with $\pi_\Delta = 436125$ and $P_\Delta =$ the red path of Figure-1.

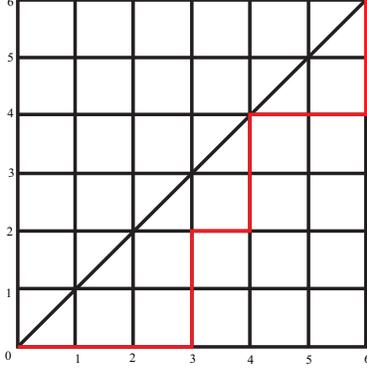


Figure-1: The Dyck path D_Δ

3 O -Rooted Labeled r -Trees

Preparing for Theorem 2.2, we give some characterizations on O -rooted labeled r -trees in this section. For the structural integrity, below we restate the definition of O -rooted labeled r -trees following from Foata [10] in 1971.

Definition 3.1. [10] Let $O = \{o_1, \dots, o_r\}$ and $V = \{v_1, \dots, v_n\}$ be two disjoint sets of labeled vertices. An **O -rooted labeled r -tree** T on $O \cup V$ is a graph having the property: there is a valid rearrangement $\nu = (v_{i_1}, \dots, v_{i_n})$ of vertices v_1, \dots, v_n , such that each v_{i_j} with $j \in [n]$ is adjacent to exactly r vertices in $\{o_1, \dots, o_r, v_{i_1}, \dots, v_{i_{j-1}}\}$ and, moreover, these r vertices are themselves mutually adjacent in T . Let

$$F_T^\nu(v_{i_j}) = \{v \mid v \text{ is adjacent to } v_{i_j} \text{ in } T\} \cap \{o_1, \dots, o_r, v_{i_1}, \dots, v_{i_{j-1}}\},$$

whose members are called **fathers of v_{i_j} in T under ν** .

Remark 3.2. Note from the above definition that the father set of v_{i_1} in T under ν is the root set O , so vertices of O are mutually adjacent in T . In the case of $r = 1$, for any ordinary tree T on the labeled vertices $O \cup V$, suppose that ν is a rearrangement having the property: v_i is ahead of v_j in ν whenever $d_T(v_i, o_1) < d_T(v_j, o_1)$ as distances of two vertices in T . Obviously such ν always exists and is valid for defining T as an O -rooted labeled tree.

To make the above definition more clear, Propositions 3.3-3.5 are characterizations on valid rearrangements and father sets, which might have been obtained by others in the literature but not noticed by us yet. Indeed, the r -trees have been characterized exactly to be the maximal graphs with a given treewidth in [34], and the chordal graphs all of whose maximal cliques are the same size $r + 1$ and all of whose minimal clique separators are also all the same size r in [37].

Proposition 3.3. Let $O = \{o_1, \dots, o_r\}$ and $V = \{v_1, \dots, v_n\}$ be two disjoint sets of labeled vertices, and T an O -rooted labeled r -tree on $O \cup V$. For each $i \in [n]$, the father set $F_T^\nu(v_i)$ is independent of the choice of valid rearrangements ν for T , and denoted by $F_T(v_i)$.

Proof. Without loss of generality, we may assume that $\epsilon = (v_1, \dots, v_n)$ is a valid rearrangement for T . Given a new valid rearrangement ν of vertices v_1, \dots, v_n for T , suppose s is the minimal number $F_T^\epsilon(v_s) \neq F_T^\nu(v_s)$. If $o_j \in F_T^\epsilon(v_s) \setminus F_T^\nu(v_s)$ for some $j \in [r]$, then $o_j \notin F_T^\nu(v_s)$

implies that v_s is not adjacent to o_j in T , a contradiction to $o_j \in F_T^\epsilon(v_s)$. If $v_t \in F_T^\epsilon(v_s) \setminus F_T^\nu(v_s)$, then we have $t < s$ since $v_t \in F_T^\epsilon(v_s)$ and $v_s \notin F_T^\epsilon(v_t) = F_T^\nu(v_t)$ by the minimality of s . Note the fact that v_s is adjacent to v_t in T , a contradiction to $v_s \notin F_T^\nu(v_t)$ and $v_t \notin F_T^\nu(v_s)$. Hence, $F_T^\epsilon(v_i) = F_T^\nu(v_i)$ for all $i \in [n]$. \square

Proposition 3.4. *Let $O = \{o_1, \dots, o_r\}$ and $V = \{v_1, \dots, v_n\}$ be two disjoint sets of labeled vertices, and T an O -rooted labeled r -tree on $O \cup V$. A rearrangement ν of vertices v_1, \dots, v_n is valid for T if and only if v_s is ahead of v_t in ν whenever $v_s \in F_T(v_t)$.*

Proof. The sufficiency is obvious from the definition of O -rooted labeled r -tree. To prove the necessity, we may assume that $\epsilon = (v_1, \dots, v_n)$ is a valid rearrangement for T . Proposition 3.3 implies $F_T^\epsilon(v_i) = F_T(v_i)$. Now suppose that ν is a rearrangement such that v_s is ahead of v_t in ν whenever $v_s \in F_T^\epsilon(v_t)$. Let $G_T^\nu(v_i)$ consist of those vertices o_1, \dots, o_r who are adjacent to v_i in T , and vertices v_1, \dots, v_n who are adjacent to v_i in T and ahead of v_i in ν . Immediately, we have $F_T^\epsilon(v_i) \subseteq G_T^\nu(v_i)$ for all $i \in [n]$ and $G_T^\nu(v_i) \cap O = F_T^\epsilon(v_i) \cap O$. To obtain the necessity, i.e., ν is valid for T , it is enough to show $F_T^\epsilon(v_i) = G_T^\nu(v_i)$. Suppose $G_T^\nu(v_t) \neq F_T^\epsilon(v_t)$ and $v_s \in G_T^\nu(v_t) \setminus F_T^\epsilon(v_t)$ for some $s, t \in [n]$. By the definition of $G_T^\nu(v_t)$, $v_s \in G_T^\nu(v_t)$ implies that v_s is ahead of v_t in ν and adjacent to v_t in T . From the assumption of ν , we have $v_t \notin F_T^\epsilon(v_s)$. Note that v_s and v_t are adjacent, a contradiction to $v_t \notin F_T^\epsilon(v_s)$ and $v_s \in F_T^\epsilon(v_t)$. \square

Proposition 3.5. *Let $O = \{o_1, \dots, o_r\}$ and $V = \{v_1, \dots, v_n\}$ be two disjoint sets of labeled vertices, and $F : V \rightarrow \binom{O \cup V}{r}$. There is an O -rooted labeled r -tree T on $O \cup V$ with $F(v_i) = F_T(v_i)$ for all $i \in [n]$ if and only if F satisfies the following properties:*

- (a) *if $v_{i_1} \in F(v_{i_2}), \dots, v_{i_{j-1}} \in F(v_{i_j})$ for some $i_1, \dots, i_j \in [n]$, then $v_{i_j} \notin F(v_{i_1})$;*
- (b) *if $F(v_j) \neq O$, then there is a vertex $v_i \in F(v_j)$ such that $|F(v_j) \cap F(v_i)| = r - 1$.*

Moreover, both the r -tree T and the vertex v_i in (b) are unique.

Proof. Let's prove the second part first. If T is an O -rooted labeled r -tree on $O \cup V$ with $F(v_i) = F_T(v_i)$, then all vertices of $F(v_i) \cup \{v_i\}$ are mutually adjacent, which exactly form all edges of T . So T is uniquely determined by $F(v_i) = F_T(v_i)$. To prove the uniqueness of v_i in (b), note that (a) implies $v_j \notin F(v_j)$ for all $j \in [n]$. Suppose there is another vertex $v_{i'} \in F(v_j)$ with $i' \neq i$ such that $|F(v_j) \cap F(v_{i'})| = r - 1$. Then we have $v_i \in F(v_{i'})$ and $v_{i'} \in F(v_i)$, a contradiction to (a).

To prove the sufficiency of the first part, we may assume that $\epsilon = (v_1, \dots, v_n)$ is a valid rearrangement for T . From Proposition 3.4, if $v_j \in F(v_i)$, v_j is ahead of v_i in ϵ , i.e., $j < i$. So if $v_{i_1} \in F(v_{i_2}), \dots, v_{i_{j-1}} \in F(v_{i_j})$ for some $i_1, \dots, i_j \in [n]$, then $i_1 < i_j$ which implies $v_{i_j} \notin F(v_{i_1})$ and (a) holds. To prove (b), let i be the largest number with $v_i \in F(v_j)$, $i < j$ obviously. Suppose $v_{i'} \in F(v_j) \setminus F(v_i)$, then $i' < i < j$ and $v_i \notin F(v_{i'})$. Note $v_i, v_{i'} \in F(v_j)$ and all vertices of $F(v_j)$ are mutually adjacent, which is a contradiction to $v_{i'} \notin F(v_i)$ and $v_i \notin F(v_{i'})$. Thus we have $F(v_j) \setminus F(v_i) = \{v_i\}$, i.e., $|F(v_j) \cap F(v_i)| = r - 1$. Moreover, for any $v_{i'} \in F(v_j)$ with $i' < i$, we have at least $v_{i'}, v_i \notin F(v_{i'})$, i.e., $|F(v_j) \cap F(v_{i'})| \leq r - 2$.

To prove the necessity of the first part, from the assumption $v_{i_j} \notin F(v_{i_1})$ whenever $v_{i_1} \in F(v_{i_2}), \dots, v_{i_{j-1}} \in F(v_{i_j})$ for some $i_1, \dots, i_j \in [n]$, we have $v_{i_t} \notin F(v_{i_s})$ for $1 \leq s < t \leq j$, which implies $v_{i_s} \neq v_{i_t}$, i.e., $i_s \neq i_t$ for $1 \leq s < t \leq j$ since $v_{i_s} \in F(v_{i_{s+1}})$ and $v_{i_t} \notin F(v_{i_{s+1}})$, and $j \leq n$ consequently. Suppose $v_{i_1} \in F(v_{i_2}), \dots, v_{i_{k-1}} \in F(v_{i_k})$ for some $i_1, \dots, i_k \in [n]$, where k is maximal possible. The maximality of k implies $v_{i_k} \notin F(v_i)$ for all $i \in [n]$. Next we use induction on the size of V . When $|V| = 1$, note that from (a), $v_1 \in F(v_1)$ produces

$v_1 \notin F(v_1)$, which forces $v_1 \notin F(v_1)$, i.e., $F(v_1) = O$ and the result follows clearly. Let $V' = V \setminus \{v_{i_k}\}$ and $F' : V' \rightarrow \binom{O \cup V'}{r}$ with $F'(v_i) = F(v_i)$ for all $v_i \in V'$. It is clear that (a) and (b) holds for F' . From the induction hypothesis, there is an O -rooted labeled r -tree T' on $O \cup V'$ such that $F'(v_i) = F_{T'}(v_i)$ with $v_i \in V'$. Given a valid rearrangement ν' for T' , let T be a graph on the labeled vertex set $O \cup V$ obtained from T' by adding the r edges between v_{i_k} and each vertex of $F(v_{i_k})$, and let $\nu = (\nu', v_{i_k})$. The case of $F(v_{i_k}) = O$ is clear. Otherwise, from (b) we have $|F(v_{i_k}) \cap F(v_i)| = r - 1$ for some $v_i \in F(v_{i_k})$. Note all vertices of $F(v_i) \cup \{v_i\}$ are mutually adjacent in T' , which implies that all vertices of $F(v_{i_k})$ are also mutually adjacent in T' . Consequently, the graph T is an O -rooted labeled r -tree and ν is a valid rearrangement for T . \square

4 Proof of Theorem 2.2

Roughly speaking, in Definition 2.1 the graph $T_{\mathbf{x}}$ is obtained by the following process,

$$\mathbf{x} \in \Delta \longrightarrow p_j, q_j (\text{if } p_j \neq 0) \longrightarrow f(v_j) \longrightarrow F(v_j) \longrightarrow T_{\mathbf{x}}.$$

which requires that $p_j, q_j (\text{if } p_j \neq 0), f(v_j), F(v_j)$, and $T_{\mathbf{x}}$ are well-defined for all $j \in [n]$, see Proposition 4.1.

Proposition 4.1. *With the same notations as Definition 2.1, the graph $T_{\mathbf{x}}$ is independent of the choice of $\mathbf{x} \in \Delta$. Moreover, $T_{\mathbf{x}}$ is an O -rooted labeled r -tree with $F_{T_{\mathbf{x}}}(v_j) = F(v_j)$ for all $j \in [n]$, namely, the map Ψ_n^r in (4) is well defined.*

Proof. Firstly, we will show that p_j and $q_j (\text{if } p_j \neq 0)$ is independent of the choice of $\mathbf{x} \in \Delta$. Let $\Delta \in \mathcal{R}(\mathcal{S}_n^r)$ and $j \in [n]$. For the case (i) of Definition 2.1, by (2) we have $\text{Sgn}_{i,j,k}(\Delta) \neq +$ for all $i \in [n]$ and $k \in [r]$, which implies $p_j = 0$ for all $\mathbf{x} \in \Delta$. For the case (ii) of Definition 2.1, given any $(i, k) \neq (i', k')$ in $[n] \times [r]$ with $i > i'$, by routine calculations on entries of $C_{\mathbf{x},j}$ we have

$$c_{ijk}(\mathbf{x}) - c_{i'jk'}(\mathbf{x}) = \begin{cases} c_{ii'(k-k')}(\mathbf{x}), & \text{if } k > k', i > j > i'; \\ -c_{ii'(k'-k+1)}(\mathbf{x}), & \text{if } k \leq k', i > j > i'; \\ c_{ii'(k-k'+1)}(\mathbf{x}), & \text{if } k \geq k', i > i' > j \text{ or } j > i > i'; \\ -c_{ii'(k'-k)}(\mathbf{x}), & \text{if } k < k', i > i' > j \text{ or } j > i > i'. \end{cases}$$

For all $\mathbf{x} \in \Delta$, we have $\text{Sgn}(c_{ijk}(\mathbf{x}) - c_{i'jk'}(\mathbf{x})) = \text{Sgn}(c_{ii's}(\mathbf{x}))$ or $-\text{Sgn}(c_{ii's}(\mathbf{x}))$ for some $s \in [r]$, which means that both $\text{Sgn}(c_{ijk}(\mathbf{x}))$ and $\text{Sgn}(c_{ijk}(\mathbf{x}) - c_{i'jk'}(\mathbf{x})) \neq 0$ are independent of the choice of $\mathbf{x} \in \Delta$. It implies that for all $\mathbf{x} \in \Delta$, the j -th column slice $\text{col}_j(C_{\mathbf{x}})$ has a unique minimal positive entry at the same position (p_j, q_j) , i.e., (p_j, q_j) is independent of the choice of $\mathbf{x} \in \Delta$.

Secondly, we will prove that $f(v_j)$ and $F(v_j)$ is well defined. Let

$$\pi : [n] \rightarrow \{0, 1, \dots, n\} \quad \text{with} \quad \pi(j) = p_j, \tag{8}$$

$\pi^{l+1}(j) = \pi(\pi^l(j)) = p_{\pi^l(j)}$ for $l \geq 0$, and $\pi^0(j) = j$. Note that if $p_j \neq 0$, we have $c_{p_j j q_j}(\mathbf{x}) > 0$ and

$$c_{p_j j q_j}(\mathbf{x}) = \begin{cases} x_{p_j} - x_j - q_j, & \text{if } p_j < j; \\ x_{p_j} - x_j - q_j + 1, & \text{if } p_j > j, \end{cases}$$

which implies $x_{p_j} > x_j$ and $p_j \neq j$. Namely, we have $x_{\pi(j)} > x_j$ if $\pi(j) \neq 0$. There exists some $m \in [n]$ such that $\pi(j), \dots, \pi^{m-1}(j) \neq 0$ and $\pi^m(j) = 0$, otherwise $\pi(j), \dots, \pi^n(j) \neq 0$ and $x_j < x_{\pi(j)} < \dots < x_{\pi^n(j)}$ which is obviously impossible for $\mathbf{x} = (x_1, \dots, x_n)$. Moreover $x_j < x_{\pi(j)} < \dots < x_{\pi^{m-1}(j)}$ implies that $j, \pi(j), \dots, \pi^{m-1}(j)$ are mutually distinct. It follows from (i) and (ii) of Definition 2.1 that

$$f(v_{p_{\pi^{m-2}(j)}}) = f(v_{\pi^{m-1}(j)}) = (o_1, \dots, o_r),$$

and for all $l = m - 2, \dots, 1, 0$, we have

$$f(v_{\pi^l(j)}) = (f_1(v_{\pi^{l+1}(j)}), \dots, f_{q_{\pi^l(j)}-1}(v_{\pi^{l+1}(j)}), f_{q_{\pi^l(j)}+1}(v_{\pi^{l+1}(j)}), \dots, f_r(v_{\pi^{l+1}(j)}), v_{\pi^{l+1}(j)}). \quad (9)$$

Proceeding l from $m - 2$ to 0 step by step recursively, finally we can obtain $f(v_j)$, which is well-defined consequently. Moreover, note from the definition that for each $l = m - 2, \dots, 1, 0$,

$$F(v_{\pi^l(j)}) = (F(v_{\pi^{l+1}(j)}) \setminus f_{q_{\pi^l(j)}}(v_{\pi^{l+1}(j)})) \cup \{v_{\pi^{l+1}(j)}\},$$

i.e., $F(v_{\pi^l(j)})$ is obtained from $F(v_{\pi^{l+1}(j)})$ by removing the vertex $f_{q_{\pi^l(j)}}(v_{\pi^{l+1}(j)})$ and adding the vertex $v_{\pi^{l+1}(j)}$. So each $F(v_{\pi^l(j)})$ consists of r members of vertices $o_1, \dots, o_r, v_{\pi^{m-1}(j)}, \dots, v_{\pi^{l+1}(j)}$, which is of size r since $j, \pi(j), \dots, \pi^{m-1}(j)$ are mutually distinct and nonzero. In particular, $|F(v_j)| = |F(v_{\pi^0(j)})| = r$ and $F(v_j)$ is well defined.

Finally, it remains to show that there exists uniquely an O -rooted labeled r -tree $T_{\mathbf{x}}$ on $O \cup V$ with $F_{T_{\mathbf{x}}}(v_j) = F(v_j)$ for all $j \in [n]$. Recall the arguments of the above proof that if $\pi(j) \neq 0$, $F(v_j)$ consists of r members of vertices $o_1, \dots, o_r, v_{\pi^{m-1}(j)}, \dots, v_{\pi(j)}$, where m is the smallest integer with $\pi^m(j) = 0$ and $m \geq 2$. If $v_i \in F(v_j)$, then $\pi(j) = p_j \neq 0$ and

$$v_i \in \{o_1, \dots, o_r, v_{\pi^{m-1}(j)}, \dots, v_{\pi(j)}\},$$

which follows $v_i = v_{\pi^l(j)}$, i.e., $i = \pi^l(j)$ for some positive integer $l \in [m - 1]$. Now suppose $v_{i_1} \in F(v_{i_2}), \dots, v_{i_{j-1}} \in F(v_{i_j})$ for some $i_1, \dots, i_j \in [n]$. There exist some positive integers l_1, \dots, l_{j-1} such that

$$i_1 = \pi^{l_1}(i_2), \dots, i_{j-1} = \pi^{l_{j-1}}(i_j).$$

We have $i_1 = \pi^l(i_j)$ with $l = l_1 + \dots + l_{j-1}$, i.e., $v_{i_1} = v_{\pi^l(i_j)}$ which implies $v_{i_j} \notin F(v_{i_1})$. So the map F satisfies the property (a) of Proposition 3.5. The property (b) is obvious since $v_{p_j} \in F(v_j)$ and $|F(v_j) \cap F(v_{p_j})| = r - 1$. The proof completes by Proposition 3.5. \square

Remark 4.2. (1) From the above proof, notations of Definition 2.1 can be written more precisely as

$$p_j = p_j(\Delta), \quad q_j = q_j(\Delta), \quad f = f_{\Delta}, \quad F = F_{\Delta}, \quad \text{and } T_{\mathbf{x}} = T_{\Delta}, \quad (10)$$

since they are all independent of the choice of $\mathbf{x} \in \Delta$. (2) We also have the following observation

$$V = \{v_1, \dots, v_n\} \not\subseteq \bigcup_{j=1}^n F_{\Delta}(v_j), \quad (11)$$

otherwise, we have $v_{i_1} \in F_{\Delta}(v_{i_2}), v_{i_2} \in F_{\Delta}(v_{i_3}), \dots$ for an infinite sequence i_1, i_2, \dots , which is a contradiction since $i_1 = \pi^{l_1}(i_2), i_2 = \pi^{l_2}(i_3), \dots$ for some positive integers l_1, l_2, \dots

It is easily seen from (1) that for any $i, j, k \in [n]$ and $s, t \in [r]$ with $i > j > k$ and $s+t \leq r$, we have the following facts on linear relations among the entries of the cubic matrix $C_{\mathbf{x}}$,

$$\begin{aligned} \text{(F1)} \quad c_{ijs}(\mathbf{x}) + c_{jkt}(\mathbf{x}) &= c_{ik(s+t-1)}(\mathbf{x}); & \text{(F2)} \quad c_{iks}(\mathbf{x}) + c_{kjt}(\mathbf{x}) &= c_{ij(s+t)}(\mathbf{x}); \\ \text{(F3)} \quad c_{kis}(\mathbf{x}) + c_{ijt}(\mathbf{x}) &= c_{kj(s+t-1)}(\mathbf{x}); & \text{(F4)} \quad c_{kjs}(\mathbf{x}) + c_{jit}(\mathbf{x}) &= c_{ki(s+t)}(\mathbf{x}); \\ \text{(F5)} \quad c_{jks}(\mathbf{x}) + c_{kit}(\mathbf{x}) &= c_{ji(s+t-1)}(\mathbf{x}); & \text{(F6)} \quad c_{jis}(\mathbf{x}) + c_{ikt}(\mathbf{x}) &= c_{jk(s+t)}(\mathbf{x}). \end{aligned}$$

Lemma 4.3. *If $p_j \neq 0$ and q_j are defined as (ii) of Definition 2.1, then entries of $C_{\mathbf{x}}$ have the following sign relations,*

$$\text{Sgn}(c_{ijk}(\mathbf{x})) = \begin{cases} \text{Sgn}(c_{ip_j(k-q_j+1)}(\mathbf{x})), & \text{if } q_j \leq k \text{ and } (i, j, p_j) \text{ is even;} \\ -\text{Sgn}(c_{p_j i(q_j-k)}(\mathbf{x})), & \text{if } q_j > k \text{ and } (i, j, p_j) \text{ is even;} \\ -\text{Sgn}(c_{p_j i(q_j-k+1)}(\mathbf{x})), & \text{if } q_j \geq k \text{ and } (i, j, p_j) \text{ is odd;} \\ \text{Sgn}(c_{ip_j(k-q_j)}(\mathbf{x})), & \text{if } q_j < k \text{ and } (i, j, p_j) \text{ is odd,} \end{cases}$$

where (i, j, p_j) is even if $i < j < p_j$, or $p_j < i < j$, or $j < p_j < i$, and odd otherwise.

Proof. We prove the result in the case of $q_j \leq k$ and $i < j < p_j$ whose arguments can be applied to other cases analogously. When $i < j < p_j$, by the fact (F3) we have

$$c_{p_j j q_j}(\mathbf{x}) = c_{ijk}(\mathbf{x}) - c_{ip_j(k-q_j+1)}(\mathbf{x}),$$

which from the assumption is the unique minimal positive entry in the j -th column slice of $C_{\mathbf{x}}$. If $c_{ijk}(\mathbf{x})$ is positive, by the unique minimality of $c_{p_j j q_j}(\mathbf{x})$ we have $c_{ijk}(\mathbf{x}) > c_{p_j j q_j}(\mathbf{x})$, which implies $c_{ip_j(k-q_j+1)}(\mathbf{x}) > 0$. If $c_{ijk}(\mathbf{x})$ is negative, by the positivity of $c_{p_j j q_j}(\mathbf{x})$ we have $c_{ip_j(k-q_j+1)}(\mathbf{x}) < 0$. Namely, $\text{Sgn}(c_{ijk}(\mathbf{x})) = \text{Sgn}(c_{ip_j(k-q_j+1)}(\mathbf{x}))$. \square

Let $\text{Proj}_j : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ be the projection defined by

$$\text{Proj}_j(x_1, \dots, x_n) = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n).$$

It is clear that $\text{Proj}_j(\Delta) \in \mathcal{R}(\mathcal{S}_{n-1}^r)$ for any $\Delta \in \mathcal{R}(\mathcal{S}_n^r)$. Below is a key lemma to prove the injectivity of the map Ψ_n^r of (4).

Lemma 4.4. *Given $j' \in [n]$, for any $\Delta' \in \mathcal{R}(\mathcal{S}_{n-1}^r)$, $i' \in \{0, 1, \dots, j' - 1, j' + 1, \dots, n\}$, and $k' \in [r]$ (if $i' \neq 0$), there is at most one region $\Delta \in \mathcal{R}(\mathcal{S}_n^r)$ such that $\text{Proj}_{j'}(\Delta) = \Delta'$, $p_{j'}(\Delta) = i'$, $q_{j'}(\Delta) = k'$ (if $i' \neq 0$), and $j' \neq p_j(\Delta)$ for all $j \in [n]$, see (10) for notations $p_j(\Delta)$ and $q_j(\Delta)$.*

Proof. We only consider the case of $j' = n$. For general j' , the arguments are analogous but more tedious. Suppose two regions $\Delta, \Omega \in \mathcal{R}(\mathcal{S}_n^r)$ satisfying that $\text{Proj}_n(\Delta) = \text{Proj}_n(\Omega) = \Delta'$ and

$$p_n(\Delta) = p_n(\Omega) = i', \quad q_n(\Delta) = q_n(\Omega) = k' \text{ (if } i' \neq 0), \quad \text{and } n \neq p_j(\Delta), p_j(\Omega) \text{ for } j \in [n].$$

We will show $\text{Sgn}(\Delta) = \text{Sgn}(\Omega)$, which implies $\Delta = \Omega$ since the map $\text{Sgn} : \mathcal{R}(\mathcal{S}_n^r) \rightarrow \{\text{Sgn}(\Delta) \mid \Delta \in \mathcal{R}(\mathcal{S}_n^r)\}$ is a bijection by (3). Given $\mathbf{z} \in \Delta'$, let $\mathbf{x} \in \Delta$ and $\mathbf{y} \in \Omega$ such that $\text{Proj}_n(\mathbf{x}) = \text{Proj}_n(\mathbf{y}) = \mathbf{z}$. By (2) we have $\text{Sgn}_{ijk}(\Delta) = \text{Sgn}(c_{ijk}(\mathbf{x}))$ and $\text{Sgn}_{ijk}(\Omega) = \text{Sgn}(c_{ijk}(\mathbf{y}))$. It is enough to show that for all $i, j \in [n]$ and $k \in [r]$,

$$\text{Sgn}(c_{ijk}(\mathbf{x})) = \text{Sgn}(c_{ijk}(\mathbf{y})). \tag{12}$$

We first claim that for all $j \in [n]$,

$$p_j(\Delta) = p_j(\Omega) = p_j \quad \text{and} \quad q_j(\Delta) = q_j(\Omega) = q_j \quad (\text{if } p_j \neq 0).$$

Indeed, if $j = j' = n$, it is obvious from the assumptions. If $j \neq j' = n$, note $n \neq p_j(\Delta)$ ($p_j(\Omega)$ resp.) for all $j \in [n]$. By the definitions of $p_j(\Delta)$ and $q_j(\Delta)$ ($p_j(\Omega)$ and $q_j(\Omega)$ resp.), the minimal positive entry of $\text{col}_j(C_{\mathbf{x}})$ ($\text{col}_j(C_{\mathbf{y}})$ resp.) never appears in the n -th row slice of $C_{\mathbf{x}}$ ($C_{\mathbf{y}}$ resp.). It follows that $p_j(\Delta) = p_j(\Delta') = p_j(\Omega)$ and $q_j(\Delta) = q_j(\Delta') = q_j(\Omega)$ for $j \neq j'$, so the claim holds. Notice that (12) holds if $p_j = 0$ since all entries of the j -th column slice of $C_{\mathbf{x}}$ are nonpositive, more precisely, $\text{Sgn}(c_{jjk}(\mathbf{x})) = 0$ and $\text{Sgn}(c_{ijk}(\mathbf{x})) = -$ if $i \neq j$. Now we assume $p_j \neq 0$ and consider the following cases to prove (12).

- i For $i, j \in [n-1]$, since $\text{Proj}_n(\mathbf{x}) = \text{Proj}_n(\mathbf{y}) = \mathbf{z}$, we have $c_{ijk}(\mathbf{x}) = c_{ijk}(\mathbf{y}) = c_{ijk}(\mathbf{z})$. Thus (12) holds in this case.
- (ii) For $j = n$ and $i \in [n]$, note $p_n(\Delta) = p_n(\Omega) = i' \neq n$ and $q_n(\Delta) = q_n(\Omega) = k'$ (if $i' \neq 0$). If $k' \geq k$ and $i < i' < n$, the 3rd identity of Lemma 4.3 implies $\text{Sgn}(c_{ink}(\mathbf{x})) = -\text{Sgn}(c_{i'i(k'-k+1)}(\mathbf{x}))$ and $\text{Sgn}(c_{ink}(\mathbf{y})) = -\text{Sgn}(c_{i'i(k'-k+1)}(\mathbf{y}))$. Note from the case of i above that $\text{Sgn}(c_{i'i(k'-k+1)}(\mathbf{x})) = \text{Sgn}(c_{i'i(k'-k+1)}(\mathbf{y}))$. Thus (12) holds in this case. Other cases can be obtained by similar arguments.

- (iii) For $i = n$ and $j \in [n]$, we have the following four cases.

(C-1). $q_j \geq k$ and $p_j < j < i = n$. From the 3rd identity of Lemma 4.3, we have

$$\text{Sgn}(c_{njk}(\mathbf{x})) = -\text{Sgn}(c_{p_j n(q_j - k + 1)}(\mathbf{x})) \quad \text{and} \quad \text{Sgn}(c_{njk}(\mathbf{y})) = -\text{Sgn}(c_{p_j n(q_j - k + 1)}(\mathbf{y})).$$

(C-2). $q_j > k$ and $j < p_j < i = n$. From the 2nd identity of Lemma 4.3 we have

$$\text{Sgn}(c_{njk}(\mathbf{x})) = -\text{Sgn}(c_{p_j n(q_j - k)}(\mathbf{x})) \quad \text{and} \quad \text{Sgn}(c_{njk}(\mathbf{y})) = -\text{Sgn}(c_{p_j n(q_j - k)}(\mathbf{y})).$$

(C-3). $q_j < k$ and $p_j < j < i = n$. From the 4th identity of Lemma 4.3, we have

$$\text{Sgn}(c_{njk}(\mathbf{x})) = \text{Sgn}(c_{np_j(k - q_j)}(\mathbf{x})) \quad \text{and} \quad \text{Sgn}(c_{njk}(\mathbf{y})) = \text{Sgn}(c_{np_j(k - q_j)}(\mathbf{y})).$$

(C-4). $q_j \leq k$ and $j < p_j < i = n$. From the 1st identity of Lemma 4.3, we have

$$\text{Sgn}(c_{njk}(\mathbf{x})) = \text{Sgn}(c_{np_j(k - q_j + 1)}(\mathbf{x})) \quad \text{and} \quad \text{Sgn}(c_{njk}(\mathbf{y})) = \text{Sgn}(c_{np_j(k - q_j + 1)}(\mathbf{y})).$$

It is obvious from (ii) above that (12) holds in cases (C-1) and (C-2). Next we will show (12) holds in (C-3) and (C-4) simultaneously by induction on k . For $k = 1$, note that (12) holds if $p_j < j$ by (C-1), and also holds if $q_j > k = 1$ and $j < p_j$ by (C-2). In particular, we have $\text{Sgn}(c_{n(n-1)1}(\mathbf{x})) = \text{Sgn}(c_{n(n-1)1}(\mathbf{y}))$ obviously. The remainder case is $q_j = 1$ and $j < p_j$. From the 1st identity of Lemma 4.3, we have $\text{Sgn}(c_{nj1}(\mathbf{x})) = \text{Sgn}(c_{np_j 1}(\mathbf{x}))$. Now consider j from $n-2$ to 1 step by step as follows. For $j = n-2$, we have $p_j = n-1$ since $j < p_j$, and $\text{Sgn}(c_{n(n-2)1}(\mathbf{x})) = \text{Sgn}(c_{n(n-1)1}(\mathbf{x})) = \text{Sgn}(c_{n(n-1)1}(\mathbf{y})) = \text{Sgn}(c_{n(n-2)1}(\mathbf{y}))$. For $j = n-3$, we have $p_j = n-1$ or $n-2$ since $j < p_j$, and $\text{Sgn}(c_{nj1}(\mathbf{x})) = \text{Sgn}(c_{np_j 1}(\mathbf{x})) = \text{Sgn}(c_{np_j 1}(\mathbf{y})) = \text{Sgn}(c_{nj1}(\mathbf{y}))$. Continuing above steps, we finally obtain $\text{Sgn}(c_{nj1}(\mathbf{x})) = \text{Sgn}(c_{nj1}(\mathbf{y}))$ for any $j \in [n]$, which proves (12) for $k = 1$. Now suppose (12) holds in (C-3) and (C-4) for $1, \dots, k-1$. For general

k , since $k - q_j < k$, it is clear from the induction hypothesis that (12) holds in (C-3). In particular, by (C-1) and (C-3) we have $\text{Sgn}(c_{n(n-1)k}(\mathbf{x})) = \text{Sgn}(c_{n(n-1)k}(\mathbf{y}))$ for all $k \in [r]$. For the case (C-4), if $q_j > 1$, then $k - q_j + 1 < k$ and (12) holds in this case by the induction hypothesis. So we only need to consider the case of $q_j = 1$ and $j < p_j$ for (C-4), i.e.,

$$\text{Sgn}(c_{njk}(\mathbf{x})) = \text{Sgn}(c_{npjk}(\mathbf{x})) \quad \text{and} \quad \text{Sgn}(c_{njk}(\mathbf{y})) = \text{Sgn}(c_{npjk}(\mathbf{y})).$$

Similar as the base case $k = 1$, we may consider j from $n - 2$ to 1 step by step, which can prove (12) in this case. E.g. if $j = n - 2 < p_j$, then $p_j = n - 1$ and we have $\text{Sgn}(c_{n(n-2)k}(\mathbf{x})) = \text{Sgn}(c_{n(n-1)k}(\mathbf{x})) = \text{Sgn}(c_{n(n-1)k}(\mathbf{y})) = \text{Sgn}(c_{n(n-2)k}(\mathbf{y}))$. The proof of (12) in case (iii) completes. \square

Proposition 4.5. *The map Ψ_n^r in (4) is injective.*

Proof. We will use induction on the dimension $n \geq 2$. For the induction base $n = 2$, it is easy to see that under the map Ψ_2^r , all $2r + 1$ regions of \mathcal{S}_2^r are 1-1 corresponding to O -rooted labeled r -trees in \mathcal{T}_2^r . Suppose the result holds for $n - 1$, i.e., if $T'_\Delta = \Psi_{n-1}^r(\Delta') = \Psi_{n-1}^r(\Omega') = T'_\Omega$ for any two regions $\Delta', \Omega' \in \mathcal{R}(\mathcal{S}_{n-1}^r)$, then we have $\Delta' = \Omega'$. Now suppose $\Delta, \Omega \in \mathcal{R}(\mathcal{S}_n^r)$ with $T_\Delta = T_\Omega = T \in \mathcal{T}_n^r$. By Proposition 3.3 and Definition 2.1, we have $F_\Delta(v_j) = F_\Omega(v_j) = F_T(v_j)$ for all $j \in [n]$. We claim that $p_j(\Delta) = p_j(\Omega)$ and $q_j(\Delta) = q_j(\Omega)$ (if $p_j(\Delta) = p_j(\Omega) \neq 0$) for all $j \in [n]$, whose proof will be given later. From (11), we can take a vertex $v_{j'} \notin F_\Delta(v_j) = F_\Omega(v_j)$ for all $j \in [n]$. It is clear that

- (a) $j' \neq p_j(\Delta), j' \neq p_j(\Omega)$ for all $j \in [n]$.
- (b) $p_{j'}(\Delta) = p_{j'}(\Omega) = i', q_{j'}(\Delta) = q_{j'}(\Omega) = k'$ (if $i' \neq 0$).

Let $T' \in \mathcal{T}_{n-1}^r$ be the O -rooted labeled r -tree obtained from T by removing the vertex $v_{j'}$ and the edges between $v_{j'}$ and vertices of $F_T(v_{j'})$. Given $\mathbf{x} \in \Delta$ and $\mathbf{y} \in \Omega$, let

$$\text{Proj}_{j'}(\Delta) = \Delta', \quad \text{Proj}_{j'}(\mathbf{x}) = \mathbf{x}', \quad \text{Proj}_{j'}(\Omega) = \Omega', \quad \text{Proj}_{j'}(\mathbf{y}) = \mathbf{y}'.$$

It follows from $\text{Proj}_{j'}(\mathbf{x}) = \mathbf{x}'$ that the cubic matrix $C_{\mathbf{x}'} \in \mathbb{R}^{(n-1) \times (n-1) \times r}$ is obtained from $C_{\mathbf{x}} \in \mathbb{R}^{n \times n \times r}$ by removing the j' -th column and row slices from $C_{\mathbf{x}}$. The above property (a) implies that for each $j \in [n] \setminus \{j'\}$, the minimal positivity entry $c_{p_j q_j}(\mathbf{x})$ of $\text{col}_j(C_{\mathbf{x}})$ never appears in the j' -th row slice of $C_{\mathbf{x}}$. So for $j \in [n] \setminus \{j'\}$, the minimal positivity entry of j -th column slice $\text{col}_j(C_{\mathbf{x}})$ appears in the same position as $\text{col}_j(C_{\mathbf{x}'})$, i.e., $p_j(\Delta) = p_j(\Delta') = p_j$ and $q_j(\Delta) = q_j(\Delta') = q_j$ (if $p_j \neq 0$). By the definition of O -rooted labeled r -tree in Definition 2.1, we have $T' = T_{\mathbf{x}'} = \Psi_{n-1}^r(\Delta')$, and $T' = T_{\mathbf{y}'} = \Psi_{n-1}^r(\Omega')$ similarly. From the induction hypothesis, we obtain $\Delta' = \Omega'$, i.e., $\text{Proj}_{j'}(\Delta) = \text{Proj}_{j'}(\Omega) = \Delta'$. Combining with the above properties (a) and (b), Lemma 4.4 implies $\Delta = \Omega$.

To prove the claim under the assumption $F_\Delta(v_j) = F_\Omega(v_j) = F_T(v_j)$ for all $j \in [n]$, note

$$p_j(\Delta) = 0 \quad \Leftrightarrow \quad F_\Delta(v_j) = O = F_\Omega(v_j) \quad \Leftrightarrow \quad p_j(\Omega) = 0.$$

Now for $p_j(\Delta) \neq 0$, we have $p_j(\Omega) \neq 0$ and from the Definition 2.1,

$$v_{p_j(\Delta)} \in F_\Delta(v_j) = F_\Omega(v_j) \subseteq F_\Omega(v_{p_j(\Omega)}) \cup v_{p_j(\Omega)}.$$

Suppose $p_j(\Delta) \neq p_j(\Omega)$. Then we have $v_{p_j(\Delta)} \in F_\Omega(v_{p_j(\Omega)}) = F_T(v_{p_j(\Omega)})$, and symmetrically $v_{p_j(\Omega)} \in F_\Delta(v_{p_j(\Delta)}) = F_T(v_{p_j(\Delta)})$, which is impossible by Proposition 3.5. Thus $p_j(\Delta) = p_j(\Omega)$ for all $j \in [n]$. Define $\pi : [n] \rightarrow \{0, 1, \dots, n\}$ with $\pi(j) = p_j(\Delta) = p_j(\Omega)$ as (8). If $\pi_j \neq 0$, we have shown that for some $m \geq 2$, we have $\pi(j), \dots, \pi^{m-1}(j) \neq 0$ and $\pi^m(j) = 0$. Next we prove $q_j(\Delta) = q_j(\Omega)$. Let's start with the convenient notations $(\mathbf{u}; i, u)$ and $[\mathbf{u}; i, u]$ for an r -tuple $\mathbf{u} = (u_1, \dots, u_r)$, an element u , and $i \in [r]$, where

$$(\mathbf{u}; i, u) = (u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_r, u) \quad \text{and} \quad [\mathbf{u}; i, u] = \{u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_r, u\}.$$

It is easy to see that if u_1, \dots, u_r and u are mutually distinct, then the following result holds

$$[\mathbf{u}; i, u] = [\mathbf{u}; j, u] \quad \Leftrightarrow \quad i = j \quad \Leftrightarrow \quad (\mathbf{u}; i, u) = (\mathbf{u}; j, u). \quad (13)$$

It follows from (i) and (ii) of Definition 2.1 that

$$f_\Delta(v_{\pi^{m-1}(j)}) = O = f_\Omega(v_{\pi^{m-1}(j)}),$$

For any $l = m - 2, \dots, 1, 0$, it is clear from (9) that

$$\begin{aligned} F_\Delta(v_{\pi^l(j)}) &= [f_\Delta(v_{\pi^{l+1}(j)}); q_{\pi^l(j)}(\Delta), v_{\pi^{l+1}(j)}], \\ F_\Omega(v_{\pi^l(j)}) &= [f_\Omega(v_{\pi^{l+1}(j)}); q_{\pi^l(j)}(\Omega), v_{\pi^{l+1}(j)}], \end{aligned}$$

and $F_\Delta(v_{\pi^l(j)}) = F_\Omega(v_{\pi^l(j)})$. Applying (13) and running l from $m - 2$ to 0, we finally obtain

$$q_j(\Delta) = q_j(\Omega) \quad \text{and} \quad f_\Delta(v_j) = f_\Omega(v_j).$$

□

Proof of Theorem 2.2. We have obtained that the map $\Psi_n^r : \mathcal{R}(\mathcal{S}_n^r) \rightarrow \mathcal{T}_n^r$ of (4) is well defined by Proposition 4.1 and injective by Proposition 4.5, which is enough to conclude that Ψ_n^r is a bijection from the fact that both $\mathcal{R}(\mathcal{S}_n^r)$ and \mathcal{T}_n^r have the same cardinality by Theorem 1.1 and Theorem 1.2.

Corollary 4.6. *Given $j' \in [n]$, for any $\Delta' \in \mathcal{R}(\mathcal{S}_{n-1}^r)$, $i' \in \{0, 1, \dots, j' - 1, j' + 1, \dots, n\}$ and $k' \in [r]$ (if $i' \neq 0$), there is a region $\Delta \in \mathcal{R}(\mathcal{S}_n^r)$ such that $\text{Proj}_{j'}(\Delta) = \Delta'$, $p_{j'}(\Delta) = i'$, $q_{j'}(\Delta) = k'$ (if $i' \neq 0$), and $j' \neq p_j(\Delta)$ for all $j \in [n]$.*

Note that Corollary 4.6 can be easily obtained from the surjectivity of Ψ_n^r . Recall Lemma 4.4 that the uniqueness of the region $\Delta \in \mathcal{R}(\mathcal{S}_n^r)$ (if exists) is crucial to guarantee the injectivity of Ψ_n^r in Proposition 4.5. As a parallel situation, the existence of such region Δ in Corollary 4.6 will produce a proof on the surjectivity of Ψ_n^r . However, similar as the Pak-Stanley labeling at the very beginning appeared in [48], we currently have no direct proof on the surjectivity of Ψ_n^r without using Theorem 1.1 and Theorem 1.2. So it would be of great interest to find a direct proof on Corollary 4.6.

Recall arguments of Proposition 4.5 and Lemma 4.4, which provide an algorithm to construct the region Δ of r -Shi arrangement from an O -rooted labeled r -tree T , i.e., the inverse map

$$(\Psi_n^r)^{-1} : \mathcal{T}_n^r \rightarrow \mathcal{R}(\mathcal{S}_n^r), \quad (\Psi_n^r)^{-1}(T) = \Delta.$$

As a brief look, below is a small example to illustrate the bijection Ψ_n^r by constructing the O -rooted labeled tree from a given region, and its inverse $(\Psi_n^r)^{-1}$ by constructing the region from a given O -rooted labeled tree.

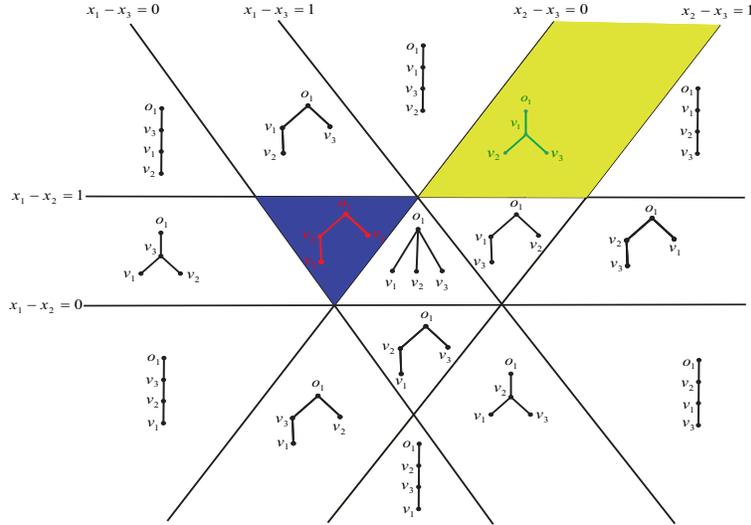


Figure-2: The bijection $\Psi_3 : \mathcal{R}(\mathcal{S}_3) \rightarrow \mathcal{T}_3$

Example 4.7. For $n = 3$ and $r = 1$, Figure-2 describes the complete correspondence between $\mathcal{R}(\mathcal{S}_3)$ and \mathcal{T}_3 under the map Ψ_3 . E.g., let $\Delta \in \mathcal{R}(\mathcal{S}_3)$ be the blue region in Figure-2 defined by

$$\Delta = \{0 < x_1 - x_2 < 1; 0 < x_1 - x_3 < 1; x_2 - x_3 < 0\},$$

and $\mathbf{x} = (0.2, -0.2, 0) \in \Delta$. By Theorem 2.2, we have

$$A_{\mathbf{x}} = \begin{pmatrix} 0 & -0.6 & -0.8 \\ -0.4 & 0 & -1.2 \\ -0.2 & 0.2 & 0 \end{pmatrix},$$

and $p_1(\Delta) = 0, p_2(\Delta) = 3, p_3(\Delta) = 0$. Namely, in the O -rooted labeled tree $T_{\mathbf{x}} = \Psi_3(\Delta)$, the fathers of v_1, v_2 , and v_3 are o_1, v_3 , and o_1 respectively, which exactly determines $T_{\mathbf{x}}$ to be the red tree in Figure 2. Conversely, let $T \in \mathcal{T}_3$ be the green tree in Figure-2 having o_1, v_1 , and v_1 as the fathers of v_1, v_2 , and v_3 respectively. If $\Omega = \Psi_3^{-1}(T)$, it follows from the definition of T in Theorem 2.2 that $p_1(\Omega) = 0, p_2(\Omega) = 1$, and $p_3(\Omega) = 1$. Take the leaf v_3 of T . Let T' be the tree obtained from T by removing v_3 and the edge $v_3 \sim v_1$, and $\Omega' = \text{Proj}_3(\Omega)$. According to the proofs of Lemma 4.4 and Proposition 4.5, we have $\Psi_2^{-1}(T') = \Omega'$ and

$$\text{Sgn}(\Omega') = \begin{pmatrix} 0 & + \\ - & 0 \end{pmatrix} = \begin{pmatrix} \text{Sgn}_{11}(\Omega) & \text{Sgn}_{12}(\Omega) \\ \text{Sgn}_{21}(\Omega) & \text{Sgn}_{22}(\Omega) \end{pmatrix}.$$

Since $p_3(\Omega) = 1$ and $p_1(\Omega) = 0$, we have $\text{Sgn}_{13}(\Omega) = +$ and $\text{Sgn}_{31}(\Omega) = -$. By Lemma 4.3, we have $\text{Sgn}_{23}(\Omega) = \text{Sgn}_{21}(\Omega) = -$ and $\text{Sgn}_{32}(\Omega) = -\text{Sgn}_{13}(\Omega) = -$. The sign matrix $\text{Sgn}(\Omega) = (\text{Sgn}_{ij}(\Omega))_{3 \times 3}$ exactly determines Ω to be the yellow region in Figure-2.

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