

UNIFORM TIGHT FRAMES AS OPTIMAL SIGNALS

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ABSTRACT. Non-orthogonal communication is a promising technique for future wireless networks (e.g., 6G and Wi-Fi 7). In the vector channel model, designing efficient non-orthogonal communication schemes amounts to the following extremum problem:

$$\max \min_k \frac{|v_k|^2}{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2}$$

where the maximum is taken among vector systems $(v_k)_1^N \subset \mathbb{R}^d$ satisfying $c_1 \leq |v_k|^2 \leq c_2$ for every k , and the parameter $\sigma > 0$ corresponds to the noise of the channel. We show that in the case $\sigma = 0$, uniform tight frames are the only optimal configurations. We also give quantitative bounds on the optimal capacity of vector channels with relatively small noise.

1. INTRODUCTION

As essential elements in wireless communications, orthogonal frequency division multiplexing (OFDM) and multiple-input multiple-output (MIMO) have been widely deployed in cellular communications (e.g., 4G and 5G) and Wi-Fi networks. In an OFDM system, the transmitter and receiver (for example, base station, access point, smart phones and other user devices) uses multiple orthogonal subcarriers to transmit information. In a typical MIMO system, on the other hand, the transmitter and receiver are usually equipped with multiple antennas to enhance information transmission efficiency and reliability [7, 18].

To further improve transmission efficiency, non-orthogonal communication schemes attract much attention both from academia and industry. In this work, we study the non-orthogonal communication problem in the following simple yet essential vector channel model [13]:

$$(1) \quad y = \sum_{k=1}^N v_k + w,$$

where $v_k \in \mathbb{R}^d$ represents the vector sent by the k -th transmitter, and $y \in \mathbb{R}^d$ is the vector received at the receiver, with $w \in \mathbb{R}^d$ being the Gaussian noise vector with $w \sim N(0, \sigma^2)$. As we are approaching the massive machine type communications beyond 5G, it is highly required to support huge number of low rate users with limited channel dimensions d . Specifically, the aim is to

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find the proper communication scheme (i.e., construct v_k with $k = 1, \dots, N$ for $N \geq d$) so that the optimal channel capacity (i.e., the maximum rate of reliable communication) may be achieved.

For the purpose of error correction, we may choose the vectors v_i so that the distance of any two of them is as large as possible. This is closely related to the *spherical coding* (or *packing*) problem, in which the goal is to find a set of N points (codewords) on the unit sphere S^{d-1} of \mathbb{R}^d so that the minimum distance between the N points is as large as possible [4, 9]. The spherical code method has gained popularity in connection with the construction of spreading sequences for Code-Division Multiple-Access (CDMA) systems [22].

In the present article, we set off to maximize channel capacity. This is defined to be the theoretical smallest upper bound on the information rate of data that can be communicated at an arbitrarily low error rate using an average received signal power S through an analog communication channel subject to additive white Gaussian noise (AWGN) power Φ , where the unit is bits/symbol. The classical Shannon-Hartley Theorem [15, 20] states that the channel capacity C is given by

$$C = \frac{1}{2} \log_2 \left(1 + \frac{S}{\Phi} \right).$$

Returning to the vector channel model (1), if the k -th transmitter is assigned to a codeword $v_k \in \mathbb{R}^d$, then its received signal power S is $|v_k|^2$, the squared Euclidean norm of v_k (see [13]). The noise of k -th transmitter consists of two parts: one originates of its own channel noise $w \sim N(0, \sigma^2)$, while the other part is yielded by the interference with the other transmitters, which is expressed by the quantity $\sum_{l \neq k} \langle v_k, v_l \rangle^2$. Therefore, the power of noise may be expressed as $\Phi = \sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2$ (see [17, 21]), and the channel capacity from the k -th transmitter to the receiver can be formulated as

$$\log \left(1 + \frac{|v_k|^2}{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2} \right),$$

where $\frac{1}{2}$ is omitted in the equation (throughout the article we use the convention $1/0 = \infty$.) The aim is to find $v_1, \dots, v_N \in \mathbb{R}^d$ such that the minimal channel capacity is as large as possible. Accordingly, we address the following question:

Problem 1. *Assume that $d \geq 2$, $N \geq d$, $0 < c_1 < c_2$ are positive bounds, and $\sigma \geq 0$ is a constant. Determine the quantity*

$$(2) \quad \max_{\substack{v_1, \dots, v_N \in \mathbb{R}^d \\ c_1 \leq |v_i|^2 \leq c_2 \quad \forall i}} \min_{1 \leq k \leq N} \log \left(1 + \frac{|v_k|^2}{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2} \right).$$

In the present article, we solve Problem 1 in the special case $\sigma = 0$, and obtain a stability version for small values of $\sigma > 0$. The latter is essential for practical applications in signal processing.

We start by a trivial simplification. Note that for any strictly monotone increasing function f , the maxima of

$$\min_{1 \leq k \leq N} f \left(\frac{|v_k|^2}{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2} \right)$$

and

$$\min_{1 \leq k \leq N} \frac{|v_k|^2}{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2}$$

are attained at the same vector configurations (subject to arbitrary boundary conditions). Since $\log(1+x)$ is strictly monotone increasing on $[0, \infty)$, we may consider the latter target function when seeking the solution of Problem 1.

In order to formulate our results, we introduce a couple of notions. We are going to call a vector system $(v_i)_1^N \subset \mathbb{R}^d$ *uniform* if $|v_i| = c$ holds for every i with some constant $c > 0$. Equivalently, $(v_i)_1^N \subset cS^{d-1}$, where S^{d-1} denotes the unit sphere in \mathbb{R}^d . The vector system v_1, \dots, v_N is a *uniform tight frame* of norm c , if $|v_k| = c$ holds for every $k \in [N]$ (where $[N] = \{1, \dots, N\}$), and

$$\sum_{i=1}^N v_i \otimes v_i = \frac{Nc^2}{d} I_d.$$

Some basic properties of tight frames are collected in the subsequent section.

First, we study the $\sigma = 0$ case, that is, when the channel is assumed to be noise-free. According to the above remarks, our task is to find the vector systems maximizing

$$(3) \quad M(v_1, \dots, v_N) = \min_{1 \leq k \leq N} \frac{|v_k|^2}{\sum_{l \neq k} \langle v_k, v_l \rangle^2}$$

subject to $c_1 \leq |v_i|^2 \leq c_2$ for every $1 \leq i \leq N$. We are going to call vector systems for which the maximum is attained to be *extremal*.

When $N \leq d$, (3) is maximized when $(v_i)_1^N$ is an orthogonal system. In this case, the denominator is 0 for every k , thus, $M(v_1, \dots, v_N) = \infty$. Clearly, only orthogonal systems correspond to this value. Thus, from now on we may assume that the number of the vectors exceeds d , hence, $M(v_1, \dots, v_N) < \infty$.

Theorem 1. *Assume that $2 \leq d < N$, and $0 < c_1 < c_2$. The vector system $v_1, \dots, v_N \subset \mathbb{R}^d$ is a maximizer of $M(v_1, \dots, v_N)$ defined in (3) subject to the condition $c_1 \leq |v_i|^2 \leq c_2$ for every $i \in [N]$ if and only if $(v_i)_1^N$ is a uniform tight frame of norm $\sqrt{c_1}$.*

By a simple calculation (see (15)) we obtain the optimal estimate for the capacity of a noise-free channel.

Corollary 1. *The answer to Problem 1 when $\sigma = 0$ is*

$$\max_{\substack{v_1, \dots, v_N \in \mathbb{R}^d \\ c_1 \leq |v_i|^2 \leq c_2 \quad \forall i}} \min_{1 \leq k \leq N} N \log \left(1 + \frac{|v_k|^2}{\sum_{l \neq k} \langle v_k, v_l \rangle^2} \right) = N \log \left(1 + \frac{d}{c_1(N-d)} \right).$$

We note that the answer to Problem 1 clearly depends on the value of σ : not only the optimal capacity does so, but the structure of the extremal vector systems as well. To illustrate this, assume that σ is very large compared to c_2N . In this case, the dominant term of $\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2$ is the first one. Therefore, the extremum of (2) is attained when $|v_i|^2 = c_2$ for every i – that is, the vector norms are *maximal*, as opposed to the case $\sigma = 0$.

However, in practical applications, we may assume that the noise is relatively small. This is the situation that we are going to study. First, we restrict the search to uniform vector systems.

Theorem 2. *Assume that $\sigma \leq c_1 \sqrt{(N-d)/d}$. Then there is a uniform tight frame of norm $\sqrt{c_1}$ which maximizes*

$$(4) \quad \min_{1 \leq k \leq N} \frac{|v_k|^2}{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2}$$

among uniform vector systems $(v_i)_1^N \subset \sqrt{c}S^{d-1}$ with $c_1 \leq c \leq c_2$.

Calculating the corresponding channel capacity (see (18)) yields:

Corollary 2. *Assuming that $\sigma \leq c_1 \sqrt{(N-d)/d}$ and that $(v_i)_1^N$ is a uniform vector system,*

$$(5) \quad \max_{\substack{v_1, \dots, v_N \in \sqrt{c}S^{d-1} \\ c_1 \leq c \leq c_2}} \min_{1 \leq k \leq N} N \log \left(1 + \frac{|v_k|^2}{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2} \right) \\ = N \log \left(1 + \frac{c_1}{\sigma^2 + c_1^2(N-d)/d} \right).$$

Next, we consider the general case. Although extremal vector systems are not necessarily uniform, we show that for small σ , there exists an extremal vector system containing relatively few vectors of non-minimal norm.

Theorem 3. *Assuming that $\sigma < c_1/\sqrt{d}$, there exists a vector system which is extremal with respect to Problem 1 containing at most*

$$d \frac{c_1^2 - \sigma^2}{c_1^2 - d\sigma^2}$$

vectors of norm strictly larger than $\sqrt{c_1}$.

For channels with a larger amount of noise, we provide the following bound on the number vectors of non-minimal norm (note that this is indeed weaker for large values of c_2/c_1).

Theorem 4. *For all $\sigma \geq 0$, there exists a vector system which is extremal with respect to Problem 1 containing at most*

$$(6) \quad d \frac{(2\sigma^2 + 2c_2^2 - c_1^2)}{c_1^2}$$

vectors of norm strictly larger than $\sqrt{c_1}$.

We conclude the article by proving the following stability estimate for the channel capacity in the general case under the assumptions that the noise of the channel is not too large, and the number of vectors is sufficiently large.

Theorem 5. Assume that $\sigma < c_1/\sqrt{2d}$ and that $N > 2dc_2^2/c_1^2$. Then

$$\begin{aligned} & \max_{\substack{v_1, \dots, v_N \in \mathbb{R}^d \\ c_1 \leq |v_i|^2 \leq c_2 \quad \forall i}} \min_{1 \leq k \leq N} \log \left(1 + \frac{|v_k|^2}{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2} \right) \\ & \leq \log \left(1 + \frac{d}{c_1(N-d) + \sigma^2 \cdot \frac{d}{c_1} - \frac{d^2}{c_1 N} (c_2^2 - c_1^2) \frac{c_1^2 - \sigma^2}{c_1^2 - d\sigma^2}} \right). \end{aligned}$$

This provides a quantitative estimate on the difference between (2) and (5), showing that for practical applications, using a uniform tight frame of norm $\sqrt{c_1}$ as the set of possible codewords is a well-justified choice.

2. TIGHT FRAMES

From the theoretical viewpoint, Problem 1 is closely related to the notion of *frames*, introduced originally by Duffin and Schaeffer [10]. A vector system $(v_i)_1^N \subset \mathbb{R}^d$ is called a *frame* if there exist $0 < A \leq B < \infty$ such that

$$A|w|^2 \leq \sum_{i=1}^N \langle w, v_i \rangle^2 \leq B|w|^2$$

holds for every vector $w \in \mathbb{R}^d$. If $A = B$ holds above, the vector system is a tight frame. Frame theory has become a well-studied topic in recent years, with plenty of real-world applications. Of the excessive literature on frame theory and its application in information theory, we only pick the volumes [6] and [19], in which the interested reader may find ample literature on the subject.

An alternative definition of tight frames involves the notion of the *tensor product* of the vectors $u, v \in \mathbb{R}^d$, which is the $\mathbb{R}^d \rightarrow \mathbb{R}^d$ linear map $u \otimes v$ satisfying

$$(u \otimes v)z = u \langle z, v \rangle$$

for every $z \in \mathbb{R}^d$. Given a vector system $(v_i)_1^N \subset \mathbb{R}^d$, we define its *frame operator* [3] A by

$$(7) \quad A(v_1, \dots, v_N) = \sum_{i=1}^N v_i \otimes v_i.$$

A set of vectors v_1, \dots, v_N in \mathbb{R}^d is called a *tight frame* if its frame operator is a constant multiple of the identity operator, that is,

$$(8) \quad \sum_{i=1}^N v_i \otimes v_i = \lambda I_d$$

with a real constant $\lambda \in \mathbb{R}$. This is equivalent to requiring that

$$\sum_{i=1}^N \langle w, v_i \rangle^2 = \lambda |w|^2$$

holds for every vector $w \in \mathbb{R}^d$.

A uniform vector system $(v_i)_1^N \subset \mathbb{R}^d$ which satisfies (8) is called a *uniform tight frame*. In the special case when the common norm is 1, we talk about a

unit norm tight frame (UNTF). By comparing traces in (8), it immediately follows that in this latter case, $\lambda = N/d$. The complete characterization of unit norm tight frames was given by Benedetto and Fickus [3] – it also follows that UNTF's exist for every $N \geq d$ (see [14] as well, and [16] for the non-uniform case).

We associate to a vector system $(v_i)_1^N \subset \mathbb{R}^d$ its *frame potential* (or 2-frame potential [11]) defined by

$$FP(v_1, \dots, v_N) = \sum_{i,j=1}^N \langle v_i, v_j \rangle^2.$$

The frame potential was introduced by Duffin and Schaeffer [10] (see [3], [12] and [8] for further applications and generalizations).

Let $G(v_1, \dots, v_N)$ denote the Gram matrix corresponding to the vector system $(v_i)_1^N$, that is, the $N \times N$ matrix G satisfying

$$G(v_1, \dots, v_N)_{ij} = \langle v_i, v_j \rangle.$$

If L denotes the $N \times d$ matrix with rows $v_1^\top, \dots, v_N^\top$, then

$$(9) \quad G(v_1, \dots, v_N) = LL^\top,$$

and on the other hand,

$$(10) \quad A(v_1, \dots, v_N) = L^\top L.$$

The frame potential of the vector system may be expressed as

$$FP(v_1, \dots, v_N) = \operatorname{tr} G^2 = \sum_{i,j=1}^N G_{ij}^2 = \|G\|_{HS}^2,$$

the square of the Hilbert-Schmidt norm of G . Thus, using (9), (10), and the property that for arbitrary $N \times N$ matrices R, S , $\operatorname{tr}(RS^\top) = \operatorname{tr}(R^\top S)$,

$$(11) \quad FP((v_i)_1^N) = \|G\|_{HS}^2 = \operatorname{tr}(LL^\top LL^\top) = \operatorname{tr}(L^\top LL^\top L) = \|A\|_{HS}^2.$$

The above formula is called the *frame potential duality*, which lies at the core of the proof of the characterization of UNTF's [3].

3. THE NOISE-FREE CASE

Proof of Theorem 1. Let (v_1, \dots, v_N) be an extremal vector system, and introduce

$$(12) \quad m_k = \frac{|v_k|^2}{\sum_{l \neq k} \langle v_k, v_l \rangle^2}$$

for every $k = 1, \dots, N$. Then, by (3), $M(v_1, \dots, v_N) = \min_k m_k$, and since $(v_i)_1^N$ is extremal, $M(v_1, \dots, v_N)$ is maximal among the suitable vector systems. Call a direction vector $u \in S^{d-1}$ *minimal*, if $u = v_k/|v_k|$ for some $k \in [N]$ with $m_k = M(v_1, \dots, v_N)$. Denote by $\mathcal{M}(v_1, \dots, v_N)$ the set of minimal directions corresponding to the vector system $(v_i)_1^N$.

We will show that extremal vector systems are uniform. To that end, assume $|v_i| > \sqrt{c_1}$ for some $i \in [N]$. We alter the vector system by defining

$$\tilde{v}_k = \begin{cases} v_k, & \text{for } k \neq i \\ \frac{\sqrt{c_1}}{|v_k|} v_k, & \text{for } k = i \end{cases}$$

for every $k \in [N]$. Accordingly, introduce

$$(13) \quad \tilde{m}_k = \frac{|\tilde{v}_k|^2}{\sum_{l \neq k} \langle \tilde{v}_k, \tilde{v}_l \rangle^2}$$

for every $k \in [N]$.

Claim 1. *The vector system $(\tilde{v}_1, \dots, \tilde{v}_N)$ described above is also extremal. Moreover, $\mathcal{M}(\tilde{v}_1, \dots, \tilde{v}_N) \subseteq \mathcal{M}(v_1, \dots, v_N)$ holds, with equality if and only if v_i is orthogonal to every direction in $\mathcal{M}(v_1, \dots, v_N)$ different from $v_i/|v_i|$.*

Proof. Clearly, $m_i = \tilde{m}_i$. Taking any $k \in [N] \setminus \{i\}$, we have that $\langle \tilde{v}_k, \tilde{v}_i \rangle^2 \leq \langle v_k, v_i \rangle^2$, where equality holds if and only if $\langle v_k, v_i \rangle = 0$. Since all the terms $\langle v_k, v_l \rangle$ not involving v_i remain unchanged, we see that $\tilde{m}_k \geq m_k$ for every $k \in [N]$. In particular, $\min_{k \in [N]} \tilde{m}_k \geq \min_{k \in [N]} m_k$, and since this latter is globally maximal, we derive that $(\tilde{v}_1, \dots, \tilde{v}_N)$ must be extremal too.

For the second statement, the inclusion is trivial by the above argument. Notice that $m_k = \tilde{m}_k$ holds if and only if $k = i$ or $\langle v_k, v_i \rangle = 0$. Thus, if $\mathcal{M}(\tilde{v}_1, \dots, \tilde{v}_N) = \mathcal{M}(v_1, \dots, v_N)$, then every minimal direction is either the direction of v_i , or orthogonal to it. \square

Applying Claim 1 repeatedly to each vector of norm greater than $\sqrt{c_1}$ leads to a uniform vector system of norm $\sqrt{c_1}$ which is extremal. By scaling, we may assume that $c_1 = 1$. Next, we characterize uniform extremal vector systems using an argument along the lines of Theorem 6.2. in [3].

Clearly,

$$\min_{1 \leq k \leq N} \frac{1}{\sum_{l \neq k} \langle v_k, v_l \rangle^2}$$

is maximized if and only if its reciprocal is minimized. Thus, we may study the extremum problem

$$\min_{v_1, \dots, v_N \in S^{d-1}} \max_{1 \leq k \leq N} \sum_{l \neq k} \langle v_k, v_l \rangle^2.$$

Since $|v_k| = 1$ for every $k \in [N]$, this is attained at the same configurations as the minmax of

$$E_k := \sum_{l=1}^N \langle v_k, v_l \rangle^2.$$

By frame potential duality (11),

$$\begin{aligned} N \max_{1 \leq k \leq N} E_k &\geq \sum_{k,l=1}^N \langle v_k, v_l \rangle^2 \\ &= \|G(v_1, \dots, v_N)\|_{HS}^2 \\ &= \|A(v_1, \dots, v_N)\|_{HS}^2. \end{aligned}$$

Since (7) shows that $\text{tr}A = N$, the Cauchy-Schwarz inequality applied to the diagonal entries of A implies that

$$(14) \quad \|A(v_1, \dots, v_N)\|_{HS}^2 \geq \frac{N^2}{d},$$

therefore,

$$\min_{v_1, \dots, v_N \in S^{d-1}} \max_k \sum_{l=1}^N \langle v_k, v_l \rangle^2 \geq \frac{N}{d}.$$

Note that by (7), diagonal entries of $A(v_1, \dots, v_N)$ are non-negative. Thus, equality may hold in (14) only if all diagonal entries of $A(v_1, \dots, v_N)$ are equal, and all off-diagonal entries are 0. Therefore, $A = \frac{N}{d}I_d$, that is, the vectors v_i form a UNTF. In this case, the above bounds are indeed achieved.

This completes the characterization of uniform extremal systems: these are uniform tight frames of norm $\sqrt{c_1}$. Then,

$$(15) \quad \frac{|v_k|^2}{\sum_{l \neq k} \langle v_k, v_l \rangle^2} = \frac{d}{c_1(N-d)}$$

holds for every $1 \leq k \leq N$. Thus, $v_k/|v_k|$ is a minimal direction for every $k \in [N]$.

Let us return to the general case. Let $(v_i)_1^N$ be an extremal vector system. Claim 1 implies that $\mathcal{M}(v_1, \dots, v_N)$ contains the direction of every vector v_k , which is only possible if each vector of norm exceeding $\sqrt{c_1}$ is orthogonal to all the other vectors. Thus, the system $(v_i)_1^N$ must be the union of an orthogonal base of an r -dimensional subspace H consisting of vectors of norm in $(\sqrt{c_1}, \sqrt{c_2}]$, and a $\sqrt{c_1}$ -norm tight frame of S^\perp consisting of $N - r$ vectors. However, in this case, the value of (3) is

$$M(v_1, \dots, v_N) = \frac{d-r}{c_1(N-d)}$$

by (15). This shows that the vector system may only be extremal when $r = 0$, that is, the vector system is a uniform tight frame. \square

4. RESULTS FOR $\sigma^2 > 0$

Proof of Theorem 2. Let $|v_i|^2 = c$ for every i with $c \in [c_1, c_2]$. Clearly, maximizing (4) on $\sqrt{c}S^{d-1}$ is equivalent to solving

$$(16) \quad \min_{v_1, \dots, v_N \in \sqrt{c}S^{d-1}} \max_{1 \leq k \leq N} \frac{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2}{c}.$$

For a fixed value of c , the contribution of the term σ^2/c is constant, therefore it may be omitted from the target function, and the results of the previous section apply. Therefore, the extremum value is attained when the vector system is a uniform tight frame of norm \sqrt{c} , and the extremal value of (16) is

$$(17) \quad \frac{\sigma^2}{c} + c \frac{N-d}{d}.$$

Thus, we need to minimize the above quantity as a function of c over the interval $[c_1, c_2]$. Since $N > d$, (17) is decreasing on the interval $[0, \sigma\sqrt{d/(N-d)}]$

and is increasing for $c > \sigma\sqrt{d/(N-d)}$. Thus, when $c_1 \geq \sigma\sqrt{d/(N-d)}$, the minimum over the interval $[c_1, c_2]$ is attained at $c = c_1$. \square

In the extremal case, by (16) and (17),

$$(18) \quad \frac{|v_k|^2}{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2} = \frac{c_1}{\sigma^2 + c_1^2(N-d)/d},$$

which proves Corollary 2.

Proof of Theorem 3. Let $(v_i)_1^N$ be a vector system satisfying the boundary conditions $c_1 \leq |v_i|^2 \leq c_2$ for every i .

Let $I \subset [N]$ be a subset of indices with $|I| \geq 2$ so that $|v_i|^2 > c_1$ for every $i \in I$ (we will specify I later). Introduce the *simultaneous scaling* corresponding to I by a factor $\lambda < 1$ of $(v_i)_1^N$ by setting

$$\tilde{v}_i = v_i$$

for $i \notin I$, and

$$\tilde{v}_i = \lambda v_i$$

for $i \in I$. If $\lambda < 1$ is close enough to 1, all vectors of the simultaneously scaled configuration have norm between $\sqrt{c_1}$ and $\sqrt{c_2}$.

As in (12) and (13), let

$$\mu_k = \frac{|v_k|^2}{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2}$$

and

$$\tilde{\mu}_k = \frac{|\tilde{v}_k|^2}{\sigma^2 + \sum_{l \neq k} \langle \tilde{v}_k, \tilde{v}_l \rangle^2}.$$

We study the effect of simultaneous scaling on the values μ_k :

Claim 2. *Assume that for the index set $I \subset [N]$ consisting of at least 2 indices,*

$$(19) \quad \sigma^2 < \sum_{l \in I \setminus \{k\}} \langle v_k, v_l \rangle^2$$

holds for every $k \in I$. Then for sufficiently small values of $\varepsilon > 0$, the simultaneous scaling corresponding to I with factor $\lambda = 1 - \varepsilon$ does not decrease any of the terms μ_k . That is, $\tilde{\mu}_k \geq \mu_k$ holds for every $k \in [N]$. In particular, if $(v_i)_1^N$ is extremal, then $(\tilde{v}_i)_1^N$ needs to be extremal as well.

Proof. If $k \notin I$, then $|v_k|$ is unchanged, while the denominator does not increase (it decreases if and only if there is $i \in I$ with $\langle v_i, v_k \rangle \neq 0$). Thus,

$$\tilde{\mu}_k \geq \mu_k$$

for every $k \notin I$.

Assume now that $k \in I$. Then,

$$\tilde{\mu}_k = \frac{\lambda^2 |v_k|^2}{\sigma^2 + \lambda^2 \sum_{l \notin I} \langle v_k, v_l \rangle^2 + \lambda^4 \sum_{l \in I \setminus \{k\}} \langle v_k, v_l \rangle^2}.$$

Calculating the derivative of $\tilde{\mu}_k$ with respect to λ at $\lambda = 1$, one obtains that its sign agrees to that of

$$(20) \quad \sigma^2 - \sum_{l \in I \setminus \{k\}} \langle v_k, v_l \rangle^2.$$

Therefore, (19) implies that the derivative is strictly negative for every $k \in I$, which suffices for the proof. \square

Let now $(v_i)_1^N$ be an extremal vector system which, among the extremal configurations, minimizes $\sum_{i=1}^N |v_i|^2$. Denote by M the number of vectors of norm strictly larger than $\sqrt{c_1}$ – we may and do assume that $M \geq d$ and these vectors are v_1, \dots, v_M . The following classical bound guarantees the existence of two of these vectors whose inner product is large in absolute value.

Lemma 1 (Welch [23]). *Assume that M vectors $w_1, \dots, w_M \subset \mathbb{R}^d$ are given so that $|w_i|^2 \geq c_1$ for every i . Then*

$$(21) \quad \max_{i \neq j} \langle w_i, w_j \rangle^2 \geq \frac{c_1^2(M-d)}{d(M-1)}.$$

We note that an alternative bound has recently been proven by Bukh and Cox [5], which is stronger for $M \approx d + \sqrt{d}$. Yet, for our needs, the above estimate is sufficient.

Let now $i, j \in [M]$ be the indices provided by Lemma 1, and set $I = \{i, j\}$. Perform the simultaneous scaling corresponding to the index set I with some factor $\lambda < 1$. Due to the minimality of $\sum_{i=1}^N |v_i|^2$, the scaled vector system may not be extremal. Therefore, the condition of Claim 2 must be violated:

$$\sigma^2 \geq \langle v_i, v_j \rangle^2.$$

Thus, by (21),

$$\sigma^2 \geq \frac{c_1^2(M-d)}{d(M-1)}.$$

Rearranging for M we derive

$$(22) \quad M < d \frac{c_1^2 - \sigma^2}{c_1^2 - d\sigma^2}$$

provided that $c_1^2 - d\sigma^2 > 0$ holds. \square

Proof of Theorem 4. Instead of Lemma 1, we now apply

Lemma 2. *Assume that Q is an $M \times M$ symmetric matrix with nonnegative entries. Then there exists an index set $J \subset [M]$ such that for every $k \in J$,*

$$(23) \quad \sum_{l \in J} Q_{kl} \geq \frac{\sum_{i,j=1}^M Q_{ij}}{2M} + \frac{Q_{kk}}{2}.$$

Proof. Suppose on the contrary that the above inequality is not true. Starting with $[M]$, remove the indices one-by-one, selecting in each step the index k of a row with minimal sum of the principal minor corresponding to the current index set. Removing this index results in deleting the corresponding

row and column from the minor. By the above assumption, the sum of the entries removed is strictly less than

$$2 \left(\frac{\sum_{i,j=1}^M Q_{ij}}{2M} + \frac{Q_{kk}}{2} \right) - Q_{kk} = \frac{\sum_{i,j=1}^M Q_{ij}}{M}.$$

Since this holds for every step, the sum of all the entries removed during the M steps of the process is strictly less than $\sum_{i,j=1}^M Q_{ij}$, which contradicts to the fact that we remove all entries of Q . \square

As before, let $(v_i)_1^N$ be an extremal vector system with minimal $\sum_{i=1}^N |v_i|^2$, and assume that the vectors which have norm $> \sqrt{c_1}$ are exactly v_1, \dots, v_M . Our goal is to show that (6) holds. Assume on the contrary that

$$(24) \quad M > d \frac{(2\sigma^2 + 2c_2^2 - c_1^2)}{c_1^2}.$$

Let Q be the $M \times M$ matrix defined by $Q_{i,j} = \langle v_i, v_j \rangle^2$. By (11) and the Cauchy-Schwarz inequality,

$$(25) \quad \begin{aligned} \sum_{i,j=1}^M Q_{i,j} &= \left\| \sum_{i=1}^M v_i \otimes v_i \right\|_{HS}^2 \\ &\geq d \left(\frac{\sum_{i=1}^M |v_i|^2}{d} \right)^2 > d \left(\frac{Mc_1}{d} \right)^2 = \frac{M^2 c_1^2}{d}. \end{aligned}$$

Thus, Lemma 2 implies that we may select a set of indices $J \subset [M]$ for which

$$(26) \quad \sum_{l \in J} Q_{kl} \geq \frac{\sum_{i,j=1}^M Q_{ij}}{2M} + \frac{Q_{kk}}{2} > \frac{Mc_1^2}{2d} + \frac{c_1^2}{2}$$

holds for every $k \in J$.

Next, we show that J may not be a singleton. Indeed, suppose that $J = \{k\}$. Then, by (23) and (25),

$$Q_{kk} \geq \frac{\sum_{i,j=1}^M Q_{ij}}{M} > \frac{Mc_1^2}{d}.$$

On the other hand, $Q_{kk} = |v_k|^4 \leq c_2^2$. This implies that $M < dc_2^2/c_1^2$, which contradicts (24).

Thus, we may assume that $|J| \geq 2$. By (26), for all $k \in J$,

$$(27) \quad \sum_{l \in J \setminus \{k\}} \langle v_k, v_l \rangle^2 > \frac{Mc_1^2}{2d} + \frac{c_1^2}{2} - c_2^2.$$

Note that (24) implies that

$$\sigma^2 < \frac{Mc_1^2}{2d} + \frac{c_1^2}{2} - c_2^2.$$

Thus, by (27), the conditions of Claim 2 are satisfied. Hence, the simultaneous scaling corresponding the index set J and factor $1 - \varepsilon$ for sufficiently small ε yields another extremal vector system. This contradicts to the minimality of $\sum_{i=1}^N |v_i|^2$ among extremal vector systems. \square

Finally, we prove a stability version of the estimate for the channel capacity.

Proof of Theorem 5. Assume that $(v_i)_1^N$ is an extremal vector system provided by Theorem 3. Let $A = \sum_{i=1}^N v_i \otimes v_i$ be the associated frame operator. As before,

$$(28) \quad \|A\|_{HS}^2 \geq \frac{\left(\sum_{i=1}^N |v_i|^2\right)^2}{d}.$$

Let

$$\mu = \min_k \frac{|v_k|^2}{\sigma^2 + \sum_{l \neq k} \langle v_k, v_l \rangle^2}$$

be the quantity for which we have to provide an upper bound. Then

$$|v_k|^2 \geq \mu \sigma^2 + \mu \sum_{l \neq k} \langle v_k, v_l \rangle^2 = \mu \sigma^2 + \mu \sum_{l=1}^N \langle v_k, v_l \rangle^2 - \mu |v_k|^4$$

holds for every k . By summing over k ,

$$(29) \quad \sum_{k=1}^N |v_k|^2 \geq N \mu \sigma^2 + \mu \|A\|_{HS}^2 - \mu \sum_{k=1}^N |v_k|^4.$$

Introduce $R = \sum_{k=1}^N |v_k|^2$. By Theorem 3,

$$(30) \quad N c_1 \leq R \leq N c_1 + d \frac{c_1^2 - \sigma^2}{c_1^2 - d \sigma^2} (c_2 - c_1)$$

and

$$\sum_{k=1}^N |v_k|^4 \leq N c_1^2 + d (c_2^2 - c_1^2) \frac{c_1^2 - \sigma^2}{c_1^2 - d \sigma^2}.$$

Therefore, (28) and (29) lead to

$$R \geq \mu \left(N \sigma^2 + \frac{R^2}{d} - N c_1^2 - d (c_2^2 - c_1^2) \frac{c_1^2 - \sigma^2}{c_1^2 - d \sigma^2} \right).$$

Since $R \geq N c_1$, the conditions $\sigma < c_1 / \sqrt{2d}$ and $N > 2dc_2^2/c_1^2$ ensure that the second term of the right-hand side is strictly positive. Then

$$(31) \quad \mu \leq \frac{R}{N \sigma^2 + \frac{R^2}{d} - N c_1^2 - d (c_2^2 - c_1^2) \frac{c_1^2 - \sigma^2}{c_1^2 - d \sigma^2}}.$$

In order to obtain an upper bound for μ , we maximize this quantity as a function of R over the interval given by (30). By a simple calculation one obtains that the conditions on σ and N imply that

$$N^2 c_1^2 > N d (c_1^2 - \sigma^2) + d^2 (c_2^2 - c_1^2) \frac{c_1^2 - \sigma^2}{c_1^2 - d \sigma^2}.$$

Therefore, (31) is decreasing over the whole interval defined by (30). Thus, its maximum value is attained at $R = N c_1$, which by (31) leads to the bound

$$\mu \leq \frac{c_1}{\sigma^2 + \left(\frac{N}{d} - 1\right) c_1^2 - \frac{d}{N} (c_2^2 - c_1^2) \frac{c_1^2 - \sigma^2}{c_1^2 - d \sigma^2}}. \quad \square$$

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