

On the automorphism group of the Morse complex

Maxwell Lin, Nicholas A. Scoville

April 2019

Abstract

Let K be a finite, connected, abstract simplicial complex. The Morse complex of K , first introduced by Chari and Joswig, is the simplicial complex constructed from all gradient vector fields on K . We show that if K is neither the boundary of the n -simplex nor a cycle, then $\text{Aut}(\mathcal{M}(K)) \cong \text{Aut}(K)$. In the case where $K = C_n$, a cycle of length n , we show that $\text{Aut}(\mathcal{M}(C_n)) \cong \text{Aut}(C_{2n})$. In the case where $K = \partial\Delta^n$, we prove that $\text{Aut}(\mathcal{M}(\partial\Delta^n)) \cong \text{Aut}(\partial\Delta^n) \times \mathbb{Z}_2$. These results are based on recent work of Capitelli and Minian.

Keywords: Discrete Morse theory, automorphism group, Morse complex, gradient vector field

*2000 MSC:*Primary: 55U05, 08A35 Secondary: 52B05, 57Q05

1 Introduction

In 2005, Chari and Joswig [3] introduced the Morse complex of a simplicial complex. The Morse complex is based on Forman's discrete Morse theory [5, 6] where after fixing a simplicial complex K , one builds a new simplicial complex $\mathcal{M}(K)$ from the collection of all gradient vector fields or arrows on K . Chari and Joswig computed the homotopy type of the Morse complex when K is the n -simplex. Ayala et al. have shown that the pure Morse complex of a tree is collapsible and some other results on the pure Morse complex of an arbitrary graph [1]. Kozlov studied shellability and other properties for trees [9], although the language of the Morse complex was not available to him at the time. Recently, Capitelli and Minian showed that the isomorphism type of the Morse complex completely determines the isomorphism type of the corresponding simplicial complex [2]. Other than these results, very little is known about the Morse complex. Its sheer size alone makes it a notoriously complex object of study.

The goal of this paper is to compute the automorphism group of the Morse complex $\mathcal{M}(K)$. We derive a formula relating $\text{Aut}(\mathcal{M}(K))$ to $\text{Aut}(K)$ for K any finite, connected, abstract simplicial complex. Our main result is the following:

Theorem 1. Let K be a finite, connected, abstract simplicial complex. Then

$$\text{Aut}(\mathcal{M}(K)) \cong \begin{cases} \text{Aut}(K) & \text{if } K \neq \partial\Delta^n, C_n \\ \text{Aut}(C_{2n}) & \text{if } K = C_n \\ \text{Aut}(K) \times \mathbb{Z}_2 & \text{if } K = \partial\Delta^n. \end{cases}$$

Here $\partial\Delta^n$ is the boundary of the n -simplex and C_n is the cycle of length n . Theorem 1 is proved in three parts. The first is Proposition 20 where we prove that $\text{Aut}(\mathcal{M}(K)) \cong \text{Aut}(K)$ for $K \neq C_n, \partial\Delta^n$. In Section 3.1, we show that one can induce an automorphism on $\mathcal{M}(K)$ from an automorphism of K . We then show that there is an injection of $\text{Aut}(K)$ into $\text{Aut}(\mathcal{M}(K))$ in Proposition 19. We are then able to show that $\text{Aut}(\mathcal{M}(K)) \cong \text{Aut}(K)$ for $K \neq \partial\Delta^n, C_n$ by utilizing results of Capitelli and Minian [2]. These results concern when we may pull an automorphism of the Morse complex back to an automorphism of the original complex. The case where $K = C_n$ follows as a corollary of the more general fact that $\text{Aut}(\mathcal{M}(K)) \cong \text{Aut}(\mathcal{H}(K))$ where $\mathcal{H}(K)$ is the Hasse diagram of K . We establish this later isomorphism in Proposition 22 and prove that $\text{Aut}(\mathcal{M}(C_n)) \cong \text{Aut}(C_{2n})$ in Proposition 23.

The case where $K = \partial\Delta^n$ is Proposition 32. In this case, there are automorphisms of the Morse complex which are not induced by a simplicial map on the original complex, called ghost automorphisms. We define what we call the reflection map π which is a cosimplicial map in the sense that if $\alpha \subseteq \beta$, then $\pi(\alpha) \supseteq \pi(\beta)$. This cosimplicial map induces and then generates all the ghost automorphisms of the Morse complex of $\partial\Delta^n$. By studying these ghost automorphisms, we account for all automorphisms of $\mathcal{M}(\partial\Delta^n)$.

The outline of this paper is as follows: Section 2 gives necessary background, terminology, and notation. Section 3 is the heart of the paper where we compute the automorphism group of $\mathcal{M}(K)$. In Section 3.1, we show that any automorphism K induces an automorphisms of $\mathcal{M}(K)$ so that there is an injective homomorphism $\text{Aut}(K) \rightarrow \text{Aut}(\mathcal{M}(K))$. This injection turns out to be an isomorphism in the case where $K \neq C_n$ or $\partial\Delta^n$ by Proposition 20. We then turn to the cases $K = C_n$ and $K = \partial\Delta^n$. We show that $\text{Aut}(\mathcal{M}(C_n)) \cong \text{Aut}(C_{2n})$ in Section 3.2. This follows fairly easily from the fact that $\text{Aut}(\mathcal{M}(K)) \cong \text{Aut}(\mathcal{H}(K))$ (Proposition 22). Finally, Section 3.3 is devoted to computing the automorphism group of $\partial\Delta^n$ via the ghost automorphisms mentioned above.

2 Background

All our simplicial complexes are assumed to be abstract, finite, and connected simplicial complexes. Our reference for the basics of simplicial complexes is [4] or [7].

Let $n \geq 1$ be an integer, and write $[v_n] := \{v_0, v_1, \dots, v_n\}$. We use K to denote a simplicial complex and α, σ etc. to denote a simplex of K . If K is a simplicial complex on $n + 1$ vertices, the set $V(K) := [v_n]$ is the **vertex set** of

K or the set of 0-simplices of K . We use $\sigma^{(i)}$ to denote a simplex of dimension i , and we write $\tau < \sigma^{(i)}$ to denote any face of σ of dimension strictly less than i . The number $\dim(\sigma) - \dim(\tau)$ is called the **codimension of τ with respect to σ** .

Definition 2. A **simplicial map** $f: K \rightarrow L$ is a function induced by a map on the vertex sets $f_V: V(K) \rightarrow V(L)$ with the property that if $\sigma = v_{i_0} v_{i_1} \dots v_{i_m}$ is a simplex in K , then $f(\sigma) := f_V(v_{i_0}) f_V(v_{i_1}) \dots f_V(v_{i_m})$ is a simplex of L .

If v_0, v_1, \dots, v_n are all the vertices of a simplex σ , we will often use the notation $\sigma := \prod_{i=0}^n v_i$.

Lemma 3. If $f: A \rightarrow B$ and $g: B \rightarrow C$ are simplicial maps, then $(g \circ f)_V = g_V \circ f_V$.

Proof. Consider any $\sigma \in V(A)$. Then, $(g \circ f)_V(\sigma) = (g \circ f)(\sigma)$. Since f, g are simplicial maps, we also know that $\dim f(\sigma) = \dim \sigma = 0$, so $f(\sigma) \in V(B)$. Likewise, we know $\dim g(f(\sigma)) = \dim f(\sigma) = 0$, so $g(f(\sigma)) \in V(C)$. Therefore, $(g \circ f)_V(\sigma) = (g \circ f)(\sigma) = (g \circ f_V)(\sigma) = (g_V \circ f_V)(\sigma)$, as desired. \square

Definition 4. A simplicial map which is a bijection is a **simplicial isomorphism**, and if $f: K \rightarrow K$ is a simplicial isomorphism, we say that f is a **simplicial automorphism**. The **automorphism group** of K is defined by

$$\text{Aut}(K) := \{f: K \rightarrow K \mid f \text{ is an automorphism}\}.$$

Because we need to refer to them below, we define a cycle C_n and the boundary of the n -simplex $\partial\Delta^n$.

Definition 5. Let $n \geq 3$ be an integer. Define the **cycle of length n** , denoted C_n , to be the 1-dimensional simplicial complex (graph) with vertex set $V(C_n) := \{v_0, v_1, \dots, v_{n-1}\}$ and edge set given by

$$\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-2}, v_{n-1}\}, \{v_{n-1}, v_0\}.$$

Definition 6. Let $n \geq 1$ be an integer. The **boundary of the n -simplex**, denotes $\partial\Delta^n$ is the simplicial complex given by $\partial\Delta^n := \mathcal{P}([v_n]) - \{\emptyset, \{v_0, \dots, v_n\}\}$.

2.1 The Morse complex

In this section, we recall the basics of the Morse complex. Our references for discrete Morse theory in general are [6, 8, 10] and the Morse complex in particular are [3, 2].

Definition 7. Let K be a simplicial complex. A **discrete vector field** V on K is defined by

$$V := \{(\sigma^{(p)}, \tau^{(p+1)}) : \sigma < \tau, \text{ each simplex of } K \text{ is in at most one pair}\}.$$

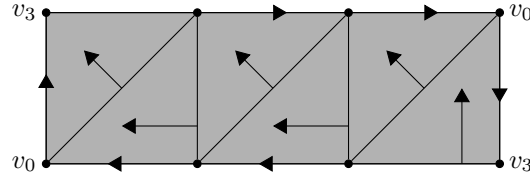
Any pair in $(\sigma, \tau) \in V$ is called a **regular pair**, and σ, τ are called **regular simplices** or just **regular**. If $(\sigma^{(p)}, \tau^{(p+1)}) \in V$, we say that $p+1$ is the **index** of the regular pair. Any simplex in K which is not in V is called **critical**.

Definition 8. Let V be a discrete vector field on a simplicial complex K . A **V -path** is a sequence of simplices

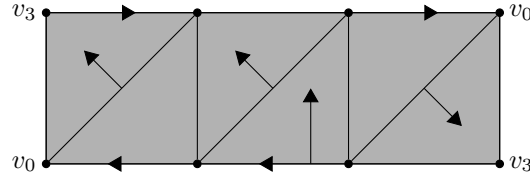
$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \alpha_2^{(p)} \dots, \beta_{k-1}^{(p+1)}, \alpha_k^{(p)}$$

of K such that $(\alpha_i^{(p)}, \beta_i^{(p+1)}) \in V$ and $\beta_i^{(p+1)} > \alpha_{i+1}^{(p)} \neq \alpha_i^{(p)}$ for $0 \leq i \leq k-1$. If $k \neq 0$, then the V -path is called **non-trivial**. A V -path is said to be **closed** if $\alpha_k^{(p)} = \alpha_0^{(p)}$. A discrete vector field V which contains no non-trivial closed V -paths is called a **gradient vector field**.

Example 9. An example of a gradient vector field is given on a triangulation of the Möbius band below:

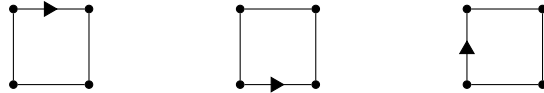


Another gradient vector field on the Möbius band is

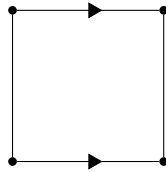


If a gradient vector field V has only one regular pair, we call V **primitive**. Given multiple primitive gradient vector fields, we may sometimes combine them to form a new gradient vector field. This will be accomplished “overlaying” all the arrows of each primitive gradient vector field. Clearly such a construction may or may not yield a gradient vector field.

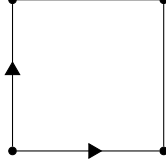
Example 10. Let primitive gradient vector fields V_0, V_1, V_2 be given by



respectively. Then V_0, V_1 combine to form a new gradient vector field V



but clearly combining V_1 and V_2

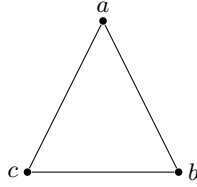


is not a gradient vector field since the bottom left vertex is in two pairs of V .

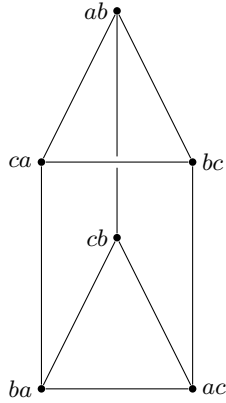
If V, W are two gradient vector fields, write $V \leq W$ whenever the regular pairs of V are also regular pairs of W . In general, we say that a collection of primitive vector fields V_0, V_1, \dots, V_n is **compatible** if there exists a gradient vector field V such that $V_i \leq V$ for all $0 \leq i \leq n$.

Definition 11. The **Morse complex** of K , denote $\mathcal{M}(K)$, is the simplicial complex whose vertices are given by primitive gradient vector fields and whose n -simplices are given by gradient vector fields with $n + 1$ regular pairs. A gradient vector field V is then associated with all primitive gradient vector fields $V := \{V_0, \dots, V_n\}$ with $V_i \leq V$ for all $0 \leq i \leq n$.

Example 12. Let $K = \partial\Delta^1 = C_3$ be the simplicial complex given by



Here we adopt the convention that if (x, xy) is a primitive vector field consisting of a vertex and edge, we denote this by xy . Note that this notation only works for a primitive vector of index 1. Then one checks that the Morse complex $\mathcal{M}(K)$ is given by:



3 The Automorphism group of $\mathcal{M}(K)$

This section is devoted to computing the automorphism group of $\mathcal{M}(K)$. We first show in section 3.1 that certain automorphisms of K induce automorphisms of $\mathcal{M}(K)$. It will then follow by Proposition 19 that $\text{Aut}(K)$ is isomorphic to a subgroup of $\text{Aut}(\mathcal{M}(K))$. We show in Proposition 20 that $\text{Aut}(K) \cong \text{Aut}(\mathcal{M}(K))$ in the case where $K \neq C_n$ or $\partial\Delta^n$. The next two sections compute $\text{Aut}(\mathcal{M}(K))$ in the two excluded cases.

3.1 Induced maps on $\mathcal{M}(K)$

If $f_V: V(K) \rightarrow V(L)$ is a bijection on the vertex sets and induces a simplicial map $f: K \rightarrow L$, then f is an isomorphism. We are interested in isomorphisms from K to K , i.e., automorphisms of K . Given an automorphism of K , we now define an induced automorphism on $\mathcal{M}(K)$.

Definition 13. Let $f: K \rightarrow K$ be an automorphism. Define the **induced automorphism on the Morse complex** $f_{*V}: V(\mathcal{M}(K)) \rightarrow V(\mathcal{M}(K))$ by $f_{*V}(v) = (f(\sigma), f(\tau))$ where $v = (\sigma^{(p)}, \tau^{(p+1)}) \in V(\mathcal{M}(K))$.

We then extend f_{*V} to a simplicial map on all of $\mathcal{M}(K)$. Below we will justify our claims that this yields a well-defined automorphism of $\mathcal{M}(K)$.

Proposition 14. If $v = (\sigma^{(p)}, \tau^{(p+1)}) \in V(\mathcal{M}(K))$, then $f_{*V}(v) \in V(\mathcal{M}(K))$.

Proof. We seek to show that $(f(\sigma), f(\tau))$ is a primitive vector of K . Since f_V is a bijection, we know that $\dim f(\sigma) = \dim \sigma$ and $\dim f(\tau) = \dim \tau$. Hence we have $\dim f(\sigma) = \dim \sigma = \dim \tau - 1 = \dim f(\tau) - 1$. Since $\sigma \subseteq \tau$, $f(\sigma) \subseteq f(\tau)$ so that $(f(\sigma), f(\tau))$ is primitive and $(f(\sigma), f(\tau)) \in V(\mathcal{M}(K))$. \square

Proposition 15. Let $f: K \rightarrow K$ be a simplicial automorphism. Then the induced map f_* is simplicial.

Proof. Suppose for the sake of contradiction that $f_*: \mathcal{M}(K) \rightarrow \mathcal{M}(K)$ is not a simplicial map. Then there is some $\alpha = v_{i_0} v_{i_1} \cdots v_{i_m} \in \mathcal{M}(K)$ such that $f_*(\alpha) = f_{*V}(v_{i_0}) f_{*V}(v_{i_1}) \cdots f_{*V}(v_{i_m}) \notin \mathcal{M}(K)$. This implies that the induced function $f_*(\alpha)$ either does not induce a gradient vector field (i.e. two vertices in $f_*(\alpha)$ are not compatible), or it induces a discrete vector field that contains a nontrivial closed V -path.

Case 1: Suppose for the sake of contradiction that there are at least two distinct simplices, say $v_{i_j} = (\sigma_j, \tau_j)$ and $v_{i_k} = (\sigma_k, \tau_k)$, which are incompatible. This implies that exactly one of the following holds:

- (1) $f(\sigma_i) = f(\sigma_j)$
- (2) $f(\tau_i) = f(\tau_j)$
- (3) $f(\tau_j) = f(\sigma_k)$

$$(4) \ f(\sigma_j) = f(\tau_k).$$

Suppose (1) is true. Since f is an automorphism, this implies $\sigma_i = \sigma_j$. However, since $\tau_i \neq \tau_j$, we cannot simultaneously have both (σ_i, τ_i) and (σ_i, τ_j) , so this would imply $\alpha \notin \mathcal{M}(K)$, a contradiction. The cases in which (2), (3), or (4) are true lead to similar contradictions, again due to the fact that f is a simplicial automorphism.

Case 2: Suppose for the sake of contradiction that $f_*(\alpha)$ contains a nontrivial closed V-path. Then, $f_*(\alpha)$ contains some vertices

$$\begin{aligned} & f_*(\sigma_0, \tau_0), f_*(\sigma_1, \tau_1), f_*(\sigma_2, \tau_2), \dots, f_*(\sigma_k, \tau_k) \\ &= (f(\sigma_0), f(\tau_0)), (f(\sigma_1), f(\tau_1)), (f(\sigma_2), f(\tau_2)), \dots, (f(\sigma_{k-1}), f(\tau_{k-1})) \end{aligned}$$

such that $f(\sigma_0), f(\tau_0), f(\sigma_1), f(\tau_1), \dots, f(\sigma_{k-1}), f(\tau_{k-1}), f(\sigma_0)$ is a closed V-path, in which $f(\sigma_i) < f(\tau_{i-1})$ for $0 \leq i \leq k-1$ (where the indices are taken mod k). Since f is an automorphism, it follows that $\sigma_0, \tau_0, \sigma_1, \tau_1, \dots, \sigma_{k-1}, \tau_{k-1}, \sigma_0$ must also be a nontrivial closed V-path. Since $(\sigma_0, \tau_0), (\sigma_1, \tau_1), \dots, (\sigma_{k-1}, \tau_{k-1})$ are vertices in α , it follows that $\alpha \notin \mathcal{M}(K)$, a contradiction. Therefore, $f_*(\alpha) \in \mathcal{M}(K)$, so f_* must be a simplicial map. \square

Example 16. We give an example to show that if the simplicial map $f: K \rightarrow K$ is not an automorphism, then f_* is not necessarily a simplicial map. Indeed, consider the simplicial complex $K = \Delta^2$ with vertex set $\{a, b, c\}$. Define $f_V: V(K) \rightarrow V(K)$ by $f_V(v) = a$ for all vertices v . It is easy to verify that f is a simplicial map. Let $f_*: \mathcal{M}(K) \rightarrow \mathcal{M}(K)$ be induced by f . Notice that $(a, ab) \in \mathcal{M}(K)$, but $f_*((a, ab)) = (f(a), f(ab)) = (a, a) \notin \mathcal{M}(K)$. In order to avoid these "degenerate" primitive vectors, we must impose the constraint that f be an isomorphism.

We now show that a simplicial automorphism on a simplicial complex K gives rise to a simplicial automorphism on $\mathcal{M}(K)$.

Proposition 17. Let $f: K \rightarrow K$ be a simplicial automorphism. Then the induced map $f_*: \mathcal{M}(K) \rightarrow \mathcal{M}(K)$ is a simplicial automorphism.

Proof. We know by Proposition 15 that f_* is simplicial. It thus suffices to show that $f_{*V}: V(\mathcal{M}(K)) \rightarrow V(\mathcal{M}(K))$ is a bijection. Consider any $(\sigma, \tau) \in V(\mathcal{M}(K))$. We know that f is a simplicial isomorphism, so it has an inverse $g: K \rightarrow K$. Thus, $(g(\sigma), g(\tau)) \in V(\mathcal{M}(K))$, as an argument similar to that in Proposition 14 verifies that $(g(\sigma), g(\tau))$ is primitive. Hence, $f_{*V}((g(\sigma), g(\tau))) = (f \circ g(\sigma), f \circ g(\tau)) = (\sigma, \tau)$. Therefore, f_{*V} is surjective. Since $V(\mathcal{M}(K))$ is finite, this implies f_{*V} is bijective. \square

We can then show that the induced map respects composition.

Lemma 18. Let $f, g: K \rightarrow K$ be simplicial automorphisms with induced automorphisms $f_*, g_*: \mathcal{M}(K) \rightarrow \mathcal{M}(K)$, respectively. Then $(f \circ g)_* = f_* \circ g_*$.

Proof. It suffices to show that $(f \circ g)_{*V} = (f_* \circ g_*)_V$. Consider any $(\sigma, \tau) \in \mathcal{M}(K)$. We have

$$\begin{aligned} (f \circ g)_{*V}((\sigma, \tau)) &= (f \circ g(\sigma), f \circ g(\tau)) \\ &= f_{*V}(g(\sigma), g(\tau)) \\ &= (f_{*V} \circ g_{*V})((\sigma, \tau)). \end{aligned}$$

Hence, $(f \circ g)_{*V} = f_{*V} \circ g_{*V}$. By Lemma 3, we know that $f_{*V} \circ g_{*V} = (f_* \circ g_*)_V$, so $(f \circ g)_{*V} = (f_* \circ g_*)_V$. \square

Because automorphisms of K induce automorphisms of $\mathcal{M}(K)$, we next show that we are able to obtain not only elements of $\text{Aut}(\mathcal{M}(K))$, but that $\text{Aut}(K)$ actually corresponds to a subgroup of $\text{Aut}(\mathcal{M}(K))$.

Proposition 19. The group $\text{Aut}(K)$ is isomorphic to a subgroup of $\text{Aut}(\mathcal{M}(K))$. In particular, there is an injective homomorphism $\phi: \text{Aut}(K) \rightarrow \text{Aut}(\mathcal{M}(K))$.

Proof. Consider the function $\phi: \text{Aut}(K) \rightarrow \text{Aut}(\mathcal{M}(K))$ that sends each simplicial automorphism $f \in \text{Aut}(K)$ to its induced simplicial automorphism $f_* \in \text{Aut}(\mathcal{M}(K))$. We first show that ϕ is a homomorphism. Consider any $f, g \in \text{Aut}(K)$, and an arbitrary $\sigma = \prod_{i=0}^m (\sigma_i, \tau_i) \in \mathcal{M}(K)$ where each (σ_i, τ_i) is a vertex of σ . Since $f, g \in \text{Aut}(K)$, $f \circ g \in \text{Aut}(K)$. This induces $(f \circ g)_*: \mathcal{M}(K) \rightarrow \mathcal{M}(K)$. Notice that

$$\begin{aligned} \phi(f \circ g)(\sigma) &= (f \circ g)_*(\sigma) \\ &= \prod_{i=0}^m (f \circ g)_{*V}((\sigma_i, \tau_i)) \\ &= \prod_{i=0}^m ((f \circ g)(\sigma_i), (f \circ g)(\tau_i)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (\phi(f) \circ \phi(g))(\sigma) &= (f_* \circ g_*)(\sigma) \\ &= f_* \left(\prod_{i=0}^m g_{*V}((\sigma_i, \tau_i)) \right) \\ &= f_* \left(\prod_{i=0}^m (g(\sigma_i), g(\tau_i)) \right) \\ &= \prod_{i=0}^m ((f \circ g)(\sigma_i), (f \circ g)(\tau_i)) \end{aligned}$$

Thus $\phi(f \circ g) = \phi(f) \circ \phi(g)$.

We now need to show that ϕ is injective. Consider any $f \in \text{Ker}(\phi)$. We claim that $f = \text{id}_K$. Again, consider an arbitrary $\sigma = \prod_{i=0}^m (\sigma_i, \tau_i) \in \mathcal{M}(K)$. Notice f induces a simplicial map $f_*: \mathcal{M}(K) \rightarrow \mathcal{M}(K)$. Since $f \in \text{Ker}(\phi)$, we have

$$\begin{aligned}\phi(f)(\sigma) &= \sigma \\ f_*(\sigma) &= \sigma \\ \prod_{i=0}^m f_{*v}((\sigma_i, \tau_i)) &= \prod_{i=0}^m (\sigma_i, \tau_i) \\ \prod_{i=0}^m (f(\sigma_i), f(\tau_i)) &= \prod_{i=0}^m (\sigma_i, \tau_i).\end{aligned}$$

As this holds for any choice of σ , we conclude that $f = \text{id}_K$. Since $\text{Ker}(\phi)$ is trivial, it follows that ϕ is injective. \square

Proposition 19 guarantees that $\text{Aut}(K) \leq \text{Aut}(\mathcal{M}(K))$ by inducing an automorphism of $\mathcal{M}(K)$ from an automorphism of K . Thus if every automorphism of $\mathcal{M}(K)$ is induced by an automorphism of K , then $\text{Aut}(\mathcal{M}(K)) = \text{Aut}(K)$. We will first show that this is indeed the case for $K \neq C_n, \partial\Delta^n$.

Proposition 20. If K is a simplicial complex other than C_n or $\partial\Delta^n$, then $\text{Aut}(\mathcal{M}(K)) \cong \text{Aut}(K)$.

Proof. As in Proposition 19, define a function $\phi: \text{Aut}(K) \rightarrow \text{Aut}(\mathcal{M}(K))$ that sends each simplicial automorphism $f \in \text{Aut}(K)$ to its induced simplicial automorphism $f_* \in \text{Aut}(\mathcal{M}(K))$. We know that ϕ is an injective homomorphism by Proposition 19.

We now show that there is a surjection onto $\text{Aut}(\mathcal{M}(K))$. If so, since $\mathcal{M}(K)$ and K are finite, this implies that ϕ is an isomorphism. Let $F \in \text{Aut}(\mathcal{M}(K))$. If K is a 1-dimensional simplicial complex (i.e., a graph) other than C_n , then F is induced by a simplicial isomorphism $f: K \rightarrow K$ by [2, Theorem 3.5]. Thus the result for K a graph other than C_n .

Now suppose that $K \neq \partial\Delta^n$ is a simplicial complex of dimension greater than or equal to 2, and let $F \in \text{Aut}(\mathcal{M}(K))$. In the proof of Theorem A [2], Capitelli and Minian construct a simplicial isomorphism $f: K \rightarrow K$ that induces the given $F: \mathcal{M}(K) \rightarrow \mathcal{M}(K)$. Their construction of f relies on a condition that is not satisfied for $K = \partial\Delta^n$ according to the contrapositive of [2, Theorem 4.2]. Thus the result for $K \neq \partial\Delta^n$ a simplicial complex of dimension greater than 2. \square

3.2 The Morse complex of the Hasse diagram

In this section, we will show that computing the automorphism group of $\mathcal{M}(K)$ is equivalent to computing the automorphism group of $\mathcal{H}(K)$. We will then

immediately be able to compute the Morse complex of C_n , one of the two cases excluded in Proposition 20. Section 3.3 is then devoted to computing the Morse complex of our final special case, $K = \partial\Delta^n$. We briefly recall here the definition of the Hasse diagram.

Definition 21. The **Hasse diagram** of K , denoted \mathcal{H}_K or \mathcal{H} , is defined as the partially ordered set of simplices of K ordered by the face relations. We view \mathcal{H} as a graph.

We adopt the convention that if $\sigma^{(p)} < \tau^{(p+1)}$ are two nodes of the Hasse diagram, the edge joining them is denoted $\sigma\tau$.

Proposition 22. For any simplicial complex K , $\text{Aut}(\mathcal{M}(K)) \cong \text{Aut}(\mathcal{H}(K))$.

Proof. We will construct an isomorphism $\phi: \text{Aut}(\mathcal{H}(K)) \rightarrow \text{Aut}(\mathcal{M}(K))$. Consider an arbitrary automorphism $f \in \text{Aut}(\mathcal{H}(K))$. Define a function $m: E(\mathcal{H}(K)) \rightarrow V(\mathcal{M}(K))$ by $m(\sigma\tau) = (\sigma, \tau)$, where σ is a codimension 1 face of τ . Notice that m has an inverse $m^{-1}: V(\mathcal{M}(K)) \rightarrow E(\mathcal{H}(K))$ given by $m^{-1}(\sigma, \tau) = \sigma\tau$. Define a function $g_V: V(\mathcal{M}(K)) \rightarrow V(\mathcal{M}(K))$ by $g_V := m \circ f \circ m^{-1}$. Clearly $g_V((\sigma, \tau))$ is another vertex of $V(\mathcal{M}(K))$. Since f is a simplicial isomorphism, it has an inverse $f^{-1}: \mathcal{H}(K) \rightarrow \mathcal{H}(K)$. It is then clear that g_V has an inverse $g_V^{-1} := m^{-1} \circ f^{-1} \circ m$. Hence g_V is a bijection. We now show that g is simplicial. Consider any $\sigma := (\sigma_1, \tau_1)(\sigma_2, \tau_2) \cdots (\sigma_k, \tau_k) \in \mathcal{M}(K)$. Observe that

$$\begin{aligned} g(\sigma) &= (m \circ f \circ m^{-1})(\sigma) \\ &= (m \circ f) \left(\prod_{i=1}^k \sigma_i \tau_i \right) \\ &= m \left(\prod_{i=1}^k f_V(\sigma_i) f_V(\tau_i) \right) \\ &= \prod_{i=1}^k m(f_V(\sigma_i) f_V(\tau_i)). \end{aligned}$$

Suppose that $\prod_{i=1}^k m(f_V(\sigma_i) f_V(\tau_i)) = \prod_{i=1}^k (f_V(\sigma_i), f_V(\tau_i))$ as the case $\prod_{i=1}^k m(f_V(\sigma_i) f_V(\tau_i)) = \prod_{i=1}^k (f_V(\tau_i), f_V(\sigma_i))$ is identical. We claim that each $(f_V(\sigma_i), f_V(\tau_i))$ is a primitive vector in $\mathcal{M}(K)$. Since $\sigma_i \tau_i \in \mathcal{H}(K)$, by the construction of $\mathcal{H}(K)$, we know that either σ_i is a codimension 1 face of τ_i , or vice versa, and since f is an automorphism, $f_V(\sigma_i) f_V(\tau_i) \in \mathcal{H}(K)$. Again, by construction of $\mathcal{H}(K)$, we know that either $f_V(\sigma_i)$ is a codimension 1 face of $f_V(\tau_i)$, or vice versa. Thus, applying m to $f_V(\sigma_i) f_V(\tau_i)$ will form a primitive vector $(f_V(\sigma_i) f_V(\tau_i)) \in \mathcal{M}(K)$.

We now must show that all primitive vectors in $g(\sigma) = \prod_{i=1}^k (f_V(\sigma_i), f_V(\tau_i))$ are compatible. Suppose for the sake of contradiction they are not all compatible. The first possibility is that there are two compatible primitive vectors $(\sigma_k, \tau_k), (\sigma_\ell, \tau_\ell) \in V(\mathcal{M}(K))$ such that

$$(f_V(\sigma_k), f_V(\tau_k))(f_V(\sigma_\ell), f_V(\tau_\ell)) \notin \mathcal{M}(K).$$

However, this implies that $\{f_V(\sigma_k), f_V(\tau_k)\} \cap \{f_V(\sigma_\ell), f_V(\tau_\ell)\} \neq \emptyset$. Since f_V is bijective, this implies that $\{\sigma_k, \tau_k\} \cap \{\sigma_\ell, \tau_\ell\} \neq \emptyset$, so that $(\sigma_k, \tau_k), (\sigma_\ell, \tau_\ell)$ are not compatible, which is a contradiction. The second possibility is that $g(\sigma)$ contains a nontrivial closed V -path. In a similar manner to the first case, this nontrivial closed V -path in the image of g can be pulled back to obtain a nontrivial closed V -path in σ , a contradiction. We conclude that $g(\sigma) \in \mathcal{M}(K)$ so that g is a simplicial map.

Since g_V is a bijection and g is simplicial, it follows that g is a simplicial automorphism on $\mathcal{M}(K)$. Define $\phi(f) := g$. We first show that ϕ is a homomorphism. Suppose we have $a, b \in \text{Aut}(\mathcal{H}(K))$. We seek to show that $\phi(a \circ b) = \phi(a) \circ \phi(b)$. By the definition of a simplicial map, it suffices to show that $\phi(a \circ b)_V = (\phi(a) \circ \phi(b))_V$. We have

$$\begin{aligned} \phi(a \circ b)_V &= m \circ a \circ b \circ m^{-1} \\ &= (m \circ a \circ m^{-1}) \circ (m \circ b \circ m^{-1}) \\ &= \phi(a)_V \circ \phi(b)_V. \end{aligned}$$

By Lemma 3, we know that $\phi(a \circ b)_V = \phi(a)_V \circ \phi(b)_V = (\phi(a) \circ \phi(b))_V$, as desired.

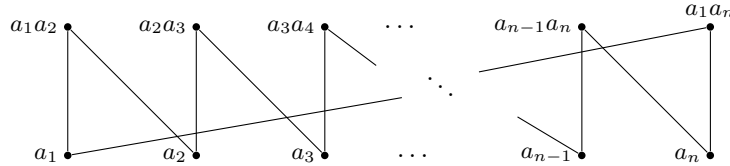
To see that ϕ is injective, suppose that $\phi(a) = \phi(b)$. Then we have $m \circ a \circ m^{-1} = m \circ b \circ m^{-1}$ which implies that $a = b$.

Finally, we show that ϕ is surjective. For any $g \in \text{Aut}(\mathcal{M}(K))$, g must be induced by some $g_V : V(\mathcal{M}(K)) \rightarrow V(\mathcal{M}(K))$. Then construct an $f : \mathcal{H}(K) \rightarrow \mathcal{H}(K)$ defined by $f = m^{-1} \circ g \circ m$. We see that $\phi(f) = m \circ (m^{-1} \circ g \circ m) \circ m^{-1} = g$, as desired. Therefore, we conclude that $\text{Aut}(\mathcal{H}(K)) \cong \text{Aut}(\mathcal{M}(K))$. \square

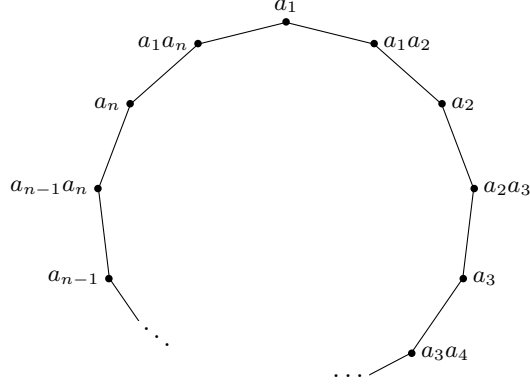
We now are able to easily compute the automorphism group of C_n .

Proposition 23. If $K = C_n$, then $\text{Aut}(\mathcal{M}(C_n)) \cong \text{Aut}(C_{2n})$.

Proof. It suffices to show $\text{Aut}(\mathcal{H}(C_n)) \cong \text{Aut}(C_{2n})$. We construct the Hasse diagram of C_n :



It is clear that this Hasse Diagram can be redrawn as:



which is C_{2n} . Therefore, $\text{Aut}(\mathcal{H}(C_n)) \cong \text{Aut}(C_{2n})$.

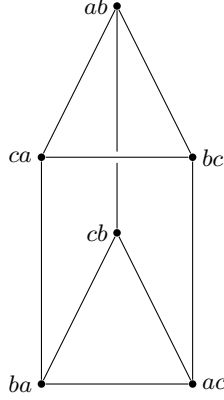
□

Remark 24. Let D_n be the dihedral group of order n . It is well known that $D_{2n} \cong D_n \times \mathbb{Z}_2$ for n odd. Since $\text{Aut}(C_n) \cong D_{2n}$, we have $\text{Aut}(\mathcal{M}(C_n)) \cong \text{Aut}(C_{2n}) \cong \text{Aut}(C_n) \times \mathbb{Z}_2$ whenever n is odd, yielding the same formula as the automorphism group of $\partial\Delta^n$ that we show in Section 3.3.

3.3 The Morse complex of $\partial\Delta^n$

We now investigate the case where $K = \partial\Delta^n$. In this case, as in the $K = C_n$ case, there are automorphisms of the Morse complex which are not induced by an automorphism of the original complex. While these automorphisms of the Morse complex are not induced by simplicial maps, we will show below that they are induced by what we are calling the reflection map. This is not a simplicial map, but rather a “cosimplicial map,” a term we define in Definition 27. The automorphisms induced by the simplicial maps and those induced by this cosimplicial map will then be shown to generate all possible automorphisms of the Morse complex, allowing us to compute $\text{Aut}(\mathcal{M}(\partial\Delta^n))$ in Theorem 32. To illustrate, we first look at an example.

Example 25. Let $K = \partial\Delta^1 = C_3$, and recall that we computed the Morse complex of K in Example 10. For reference, we give the Morse complex here, noting again the convention that ab is shorthand for the primitive vector (a, ab) .



We see that $\text{Aut}(\partial\Delta^1)$ is the symmetries of a triangle, and has six automorphisms. Meanwhile, $\text{Aut}(\mathcal{M}(\partial\Delta^1))$ has twelve automorphisms. Six of the automorphisms of $\mathcal{M}(\partial\Delta^1)$ arise from automorphisms of $\partial\Delta^1$, but the other six do not. For example, define $F: \mathcal{M}(\partial\Delta^1) \rightarrow \mathcal{M}(\partial\Delta^1)$ by

$$\begin{aligned} F(ab) &= cb \\ F(ba) &= ca \\ F(ca) &= ba \\ F(cb) &= ab \\ F(ac) &= bc \\ F(bc) &= ac. \end{aligned}$$

Then it is easy to see that F is a simplicial automorphism. However, it is not induced by any automorphism of $\partial\Delta^1$. Furthermore, composition of F with any automorphism induced by an automorphism of $\partial\Delta^1$ yields a new automorphism of $\mathcal{M}(\partial\Delta^1)$ that is not induced by an automorphism of $\partial\Delta^1$. For example, if $f: \partial\Delta^1 \rightarrow \partial\Delta^1$ is given by $f(a) = a, f(b) = c$, and $f(c) = b$, then $F \circ f_*$ is an automorphism of $\mathcal{M}(\partial\Delta^1)$ which is not equal to any automorphism induced from $\partial\Delta^1$.

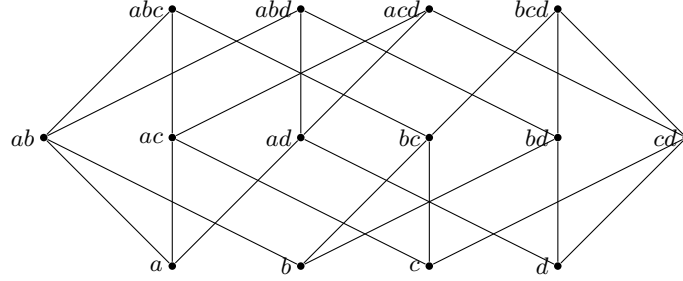
The map F in Example 25 is what in general we call π_* , the induced map of the reflection map $\pi_n: \partial\Delta^n \rightarrow \partial\Delta^n$ given below in Definition 26. We will show in Lemma 30 that π_* is a simplicial automorphism of $\mathcal{M}(\partial\Delta^n)$ and that it generates all the “missing” automorphisms of $\mathcal{M}(\partial\Delta^n)$ in Theorem 32.

Definition 26. Let $\partial\Delta^n$ be the boundary of the n -simplex on the vertices $\{v_0, v_1, \dots, v_n\}$ and write $\delta := v_0 v_1 \cdots v_n$. Define the **reflection map** $\pi_n = \pi: \partial\Delta^n \rightarrow \partial\Delta^n$ by $\pi(\sigma) := \delta - \sigma$.

The reflection map is a cosimplicial map in the following sense:

Definition 27. Let K be a simplicial complex, $f: K \rightarrow K$ a function such that if σ is a simplex in K , $f(\sigma)$ is a simplex of K . Then f is called a **cosimplicial map** if whenever $\tau \subseteq \sigma$, then $f(\tau) \supseteq f(\sigma)$. If in addition f is a bijection on the simplices of K , we say that f is a **cosimplicial automorphism**.

Example 28. We now illustrate how the reflection map $K = \partial\Delta^2$ gives rise to an automorphism on the Hasse diagram of $\partial\Delta^2$.



Observe that $\partial\Delta^2$ demonstrates rotational symmetry about the center of the diagram. (Notice that $K = C_n$ also demonstrates rotational symmetry, as can be seen in the Hasse diagram in Proposition 23.) An automorphism of $\partial\Delta^1$ induces an automorphism of the Hasse diagram which permutes the 0-simplices, but the rotational symmetry of the Hasse diagram arises from the reflection map. For instance, the map induced by the reflection map sends $a \leftrightarrow bcd$, $b \leftrightarrow acd$, and so on, which visually rotates the Hasse Diagram upside down. For simplicial complexes other than $K = \partial\Delta^n$ and C_n , the Hasse diagram does not exhibit this rotational symmetry. Hence all automorphisms of the Hasse diagrams correspond to permutations of the 0-simplices, which in turn correspond to automorphisms on the original K . (This follows from Proposition 20.)

We now give several basic properties of the reflection map.

Lemma 29. The reflection map is a cosimplicial automorphism that commutes with all members of $\text{Aut}(K)$.

Proof. Clearly if σ is a simplex in $\partial\Delta^n$, then $\delta - \sigma$ is a simplex in $\partial\Delta^n$ since $\partial\Delta^n$ by definition is made up of all proper, nonempty subsets of δ . If $\sigma \subseteq \tau$, then $\pi(\sigma) = \delta - \sigma \supseteq \delta - \tau = \pi(\tau)$. Hence π is a cosimplicial map. Next notice that we have $\pi(\pi(\sigma)) = \sigma$, so π is its own right and left inverse. It follows that π is a cosimplicial automorphism.

To see that π commutes with all simplicial automorphisms of $\partial\Delta^n$, let $f \in \text{Aut}(\partial\Delta^n)$. Then f is induced by a bijection $f_V: V(K) \rightarrow V(K)$. Consider any simplex $\sigma \in K$. We seek to show that $\pi(f(\sigma)) = f(\pi(\sigma))$. It thus suffice to show that $\delta - f(\sigma) = f(\delta - \sigma)$. We proceed by subset inclusion. Consider any vertex $v \in \delta - f(\sigma)$. Since f is an automorphism, we can express $v = f(w)$ for some vertex w . Then we have $f(w) \notin f(\sigma)$, so $w \notin \sigma$. Thus, $w \in \delta - \sigma$, so it follows that $v = f(w) \in f(\delta - \sigma)$. Hence $\delta - f(\sigma) \subseteq f(\delta - \sigma)$. For the other direction,

consider any $v \in f(\delta - \sigma)$. Again, we can express $v = f(w)$ for some vertex w , hence $w \in \delta - \sigma$. Then $w \notin \sigma$, so $f(w) \notin f(\sigma)$. So $v = f(w) \in \delta - f(\sigma)$. Therefore, $f(\delta - \sigma) \subseteq \delta - f(\sigma)$. We conclude that $\delta - f(\sigma) = f(\delta - \sigma)$ so that the reflection map commutes with all of $\text{Aut}(\partial\Delta^n)$. \square

As in Proposition 15, the reflection map induces a function on the Morse complex $\pi_{*V} : V(\mathcal{M}(\partial\Delta^n)) \rightarrow V(\mathcal{M}(\partial\Delta^n))$ defined by $\pi_{*V}((\sigma, \tau)) = (\pi(\tau), \pi(\sigma))$. Even though π is not a simplicial map, the induced map is a map on the vertex set of $\mathcal{M}(\partial\Delta^n)$ which induces a simplicial map on $\mathcal{M}(\partial\Delta^n)$. The following lemma shows that this induced map on the Morse complex behaves in a similar way to the cosimplicial automorphism on $\partial\Delta^n$.

Lemma 30. Let $\pi_n = \pi : \partial\Delta^n \rightarrow \partial\Delta^n$ be the reflection map, and $\pi_{*V} : V(\mathcal{M}(\partial\Delta^n)) \rightarrow V(\mathcal{M}(\partial\Delta^n))$ the induced function on the Morse complex. Then π_{*V} is a bijection that commutes with all bijections $g_{*V} : V(\mathcal{M}(\partial\Delta^n)) \rightarrow V(\mathcal{M}(\partial\Delta^n))$ that are induced by some $g \in \text{Aut}(\partial\Delta^n)$. Moreover, the induced function π_* is a simplicial map that commutes with all members of $\text{Aut}(\mathcal{M}(\partial\Delta^n))$.

Proof. That π_{*V} is a bijection follows from the fact that π_{*V} is its own inverse; that is,

$$\pi_{*V} \circ \pi_{*V}((\sigma, \tau)) = \pi_{*V}((\pi(\tau), \pi(\sigma))) = (\pi(\pi(\sigma)), \pi(\pi(\tau))) = (\sigma, \tau).$$

To see that π_{*V} commutes with any induced bijection g_{*V} observe that

$$\begin{aligned} \pi_{*V} \circ g_{*V}((\sigma, \tau)) &= \pi_{*V}((g(\sigma), g(\tau))) \\ &= (\pi \circ g(\tau), \pi \circ g(\sigma)) \\ &= (g \circ \pi(\tau), g \circ \pi(\sigma)) \\ &= g_{*V}(\pi(\tau), \pi(\sigma)) \\ &= g_{*V} \circ \pi((\sigma, \tau)) \end{aligned}$$

where Lemma 29 justifies the fact that π and g commute.

Since $\pi_{*V} \circ g_{*V} = g_{*V} \circ \pi_{*V}$, they induce the same function on $\mathcal{M}(K)$. Then $\pi_{*V} \circ g_{*V}$ induces $\pi_* \circ g_*$, and $g_{*V} \circ \pi_{*V}$ induces $g_* \circ \pi_*$, thus $\pi_* \circ g_* = g_* \circ \pi_*$.

It remains to verify that this is a simplicial map. Consider any $\sigma = \prod_{i=0}^k (\alpha_i, \beta_i) \in \mathcal{M}(K)$. We seek to show that $\pi_*(\sigma) \in \mathcal{M}(K)$. As we already know that π_{*V} is a bijection, it remains to show that if $\pi_*(\sigma)$ contains incompatible primitive vectors, so does σ .

Case 1: There exists two primitive vectors $\pi_*((\alpha_i, \beta_i)) = (\pi(\beta_i), \pi(\alpha_i))$ and $\pi_*((\alpha_j, \beta_j)) = (\pi(\beta_j), \pi(\alpha_j))$ of $\pi_*(\sigma)$ that are not compatible. Then, $\{\pi(\beta_i), \pi(\alpha_i)\} \cap \{\pi(\beta_j), \pi(\alpha_j)\} \neq \emptyset$. As π is bijective, it follows that $\{\beta_i, \alpha_i\} \cap \{\beta_j, \alpha_j\} \neq \emptyset$, so (α_i, β_i) and (α_j, β_j) are not compatible.

Case 2: There exists a nontrivial V -path

$$(\pi(\beta_{i_0}), \pi(\alpha_{i_0})), (\pi(\beta_{i_1}), \pi(\alpha_{i_1})), (\pi(\beta_{i_2}), \pi(\alpha_{i_2})), \dots, (\pi(\beta_{i_m}), \pi(\alpha_{i_m}))$$

where each $\pi(\beta_{i_j})$ is a codimension 1 face of $\pi(\alpha_{i_{j-1}})$ for each $1 \leq j \leq m$, and $\pi(\beta_{i_0})$ is a codimension 1 face of $\pi(\alpha_{i_m})$. However, since π is a cosimplicial automorphism, we must have that each $\alpha_{i_{j-1}}$ is a codimension 1 face of β_{i_j} for $1 \leq j \leq m$, and α_{i_m} is a codimension 1 face of β_{i_0} . However, this would imply that $(\alpha_{i_m}, \beta_{i_m}), (\alpha_{i_{m-1}}, \beta_{i_{m-1}}), (\alpha_{i_{m-2}}, \beta_{i_{m-2}}), \dots, (\alpha_{i_0}, \beta_{i_0}) \in \sigma$ is a nontrivial closed V-path, and are therefore not compatible.

Thus π_* is a simplicial map, and we therefore conclude that $\pi_* \in \text{Aut}(\mathcal{M}(K))$. \square

We will refer to the simplicial automorphism $\pi_* \in \text{Aut}(\mathcal{M}(K))$ induced by π_{*V} (and in general, any automorphism of $\mathcal{M}(\partial\Delta^n)$ that is not induced by a simplicial automorphism of $\partial\Delta^n$) as a **ghost automorphism** on $\mathcal{M}(\partial\Delta^n)$. In the proof of Theorem 32, we will see that this ghost automorphism generates all the other ghost automorphisms of $\mathcal{M}(\partial\Delta^n)$.

Next we will compute the cardinality of $\text{Aut}(\mathcal{M}(\partial\Delta^n))$. We first fix some notation and terminology. Let \mathcal{H}_i denote the set of nodes of the Hasse diagram that correspond to simplices in $\partial\Delta^n$ of dimension i , and let $H = \{\mathcal{H}_0, \mathcal{H}_1, \dots, \mathcal{H}_{n-1}\}$. We also define $\mathcal{H}_{-1} = \mathcal{H}_n = \emptyset$ for convenience. Call \mathcal{H}_i the *i th layer* of the Hasse diagram of $\partial\Delta^n$. Observe that $|\mathcal{H}_i| = \binom{n+1}{i+1}$ for indices $0 \leq i \leq n-1$. Abusing language, we will use simplex to mean both a simplex of $\partial\Delta^n$ and the corresponding vertex of $\mathcal{H}(\partial\Delta^n)$. We also define the **degree** of layer \mathcal{H}_i , denoted as $\text{Deg } \mathcal{H}_i$, as $\text{Deg } \sigma$ where $\sigma \in \mathcal{H}_i$. Note that this is well-defined since for $\partial\Delta^n$, it is clear that if $\sigma, \tau \in \mathcal{H}_i$, then $\text{Deg } \sigma = \text{Deg } \tau$. We say that two layers $\mathcal{H}_i, \mathcal{H}_j$ are **connected** if there exist $\sigma \in \mathcal{H}_i, \tau \in \mathcal{H}_j$ that are connected in $\mathcal{H}(\partial\Delta^n)$. It is clear by the construction of the Hasse diagram that two layers are connected if and only if i and j are consecutive. It is also clear that connectivity of layers is preserved under automorphisms of $\mathcal{H}(K)$.

Lemma 31. $|\text{Aut}(\mathcal{M}(\partial\Delta^n))| = 2|\text{Aut}(\partial\Delta^n)|$.

Proof. By Proposition 22, it suffices to show that $|\text{Aut}(\mathcal{H}(\partial\Delta^n))| = 2|\text{Aut}(\partial\Delta^n)|$. Consider an arbitrary automorphism $f \in \text{Aut}(\mathcal{H}(\partial\Delta^n))$. We claim that the image of any \mathcal{H}_i under f will either be \mathcal{H}_i or \mathcal{H}_{n-i-1} . We first establish that f takes \mathcal{H}_0 to \mathcal{H}_0 or \mathcal{H}_{n-1} , and then proceed by induction on n . To that end, observe that if $\sigma \in \mathcal{H}_0$ or \mathcal{H}_{n-1} , then $\text{Deg } \sigma = n$. If $\sigma \in \mathcal{H}_j$ for $1 \leq j \leq n-2$, then σ has $j+1$ faces of dimension $(j-1)$ and $(n+1) - (j+1)$ cofaces of dimension $j+1$. Hence $\text{Deg } \sigma = (j+1) + (n+1) - (j+1) = n+1$. Therefore, the only layer with the same degree as \mathcal{H}_0 is \mathcal{H}_{n-1} , so that any automorphism must send a node of \mathcal{H}_0 into either \mathcal{H}_0 or \mathcal{H}_{n-1} .

Next we establish that $f(\mathcal{H}_0) = \mathcal{H}_0$ or \mathcal{H}_{n-1} . Let ab be a 1-simplex of $\partial\Delta^n$ and suppose for the sake of contradiction that f sends $a, b \in \mathcal{H}_0$ to two different layers, say $f(a) \in \mathcal{H}_0$ and $f(b) \in \mathcal{H}_{n-1}$. Now ab is connected to both a and b , and since f is an automorphism, $f(ab)$ must be connected to both $f(a)$ and $f(b)$. Since $f(a) \in \mathcal{H}_0$ and $f(ab)$ is connected to $f(a)$, it follows that $f(ab) \in \mathcal{H}_1$. Similarly, since $f(b)$ and $f(ab)$ are connected, $f(ab) \in \mathcal{H}_{n-2}$. Thus, $\mathcal{H}_{n-2} = \mathcal{H}_1$,

so we must have $n = 3$. But it is easily seen by inspection of the $n = 3$ case that such an automorphism is impossible. Thus $f(\mathcal{H}_0) = \mathcal{H}_0$ or \mathcal{H}_{n-1} .

Having established that $f(\mathcal{H}_0) = \mathcal{H}_0$ or \mathcal{H}_{n-1} for any automorphism f , we now show by induction that $f(\mathcal{H}_i) = \mathcal{H}_i$ or \mathcal{H}_{n-i-1} for all $1 \leq i \leq n-1$. For the first case, suppose that $f(\mathcal{H}_0) = \mathcal{H}_0$, and suppose the inductive hypothesis that for some integer $0 < k < n-1$, we have $f(\mathcal{H}_j) = \mathcal{H}_j$ for all integers $0 \leq j \leq k$. We seek to show that $f(\mathcal{H}_{k+1}) = \mathcal{H}_{k+1}$. Notice that \mathcal{H}_{k+1} is connected to \mathcal{H}_k . Thus, $f(\mathcal{H}_{k+1})$ is connected to $f(\mathcal{H}_k)$. Additionally, the only layers connected to \mathcal{H}_k are \mathcal{H}_{k-1} and \mathcal{H}_{k+1} . Since $f(\mathcal{H}_k) = \mathcal{H}_k$ by the inductive hypothesis, we know $f(\mathcal{H}_{k+1}) \subseteq \mathcal{H}_{k-1} \cup \mathcal{H}_{k+1}$. However, by the inductive hypothesis, we know that $f(\mathcal{H}_{k-1}) = \mathcal{H}_{k-1}$. Since f is an isomorphism, this means we cannot send any simplices of \mathcal{H}_{k+1} to \mathcal{H}_{k-1} under f . Therefore, $f(\mathcal{H}_{k+1}) \subseteq \mathcal{H}_{k+1} \implies f(\mathcal{H}_{k+1}) = \mathcal{H}_{k+1}$.

The case where $f(\mathcal{H}_0) = \mathcal{H}_{n-1}$ is similar. Suppose that for some integer $0 < k < n-1$, we have $f(\mathcal{H}_j) = \mathcal{H}_{n-j-1}$ for all integers $0 \leq j \leq k$. We seek to show that $f(\mathcal{H}_{k+1}) = \mathcal{H}_{n-k-2}$. Since \mathcal{H}_{k+1} is connected to \mathcal{H}_k , $f(\mathcal{H}_{k+1})$ is connected to $f(\mathcal{H}_k) = \mathcal{H}_{n-k-1}$. The only layers connected to \mathcal{H}_{n-k-1} are \mathcal{H}_{n-k-2} , \mathcal{H}_{n-k} , so $f(\mathcal{H}_{k+1}) \subseteq \mathcal{H}_{n-k-2} \cup \mathcal{H}_{n-k}$. By the inductive hypothesis, $f(\mathcal{H}_{k-1}) = \mathcal{H}_{n-k}$. Since f is injective, we cannot send any simplices of \mathcal{H}_{k+1} to \mathcal{H}_{n-k} . Thus, $f(\mathcal{H}_{k+1}) \subseteq \mathcal{H}_{n-k-2}$. We know that

$$\begin{aligned} |f(\mathcal{H}_{k+1})| &= |\mathcal{H}_{k+1}| \\ &= \binom{n+1}{k+2} \\ &= \binom{n+1}{n-k-1} \\ &= |\mathcal{H}_{n-k-2}|. \end{aligned}$$

Hence $f(\mathcal{H}_{k+1}) = \mathcal{H}_{n-k-2}$, as desired.

Having established the behaviour of each layer under automorphism, we now establish a group action to count $|\text{Aut}(\mathcal{H}(\partial\Delta^n))|$. Define an action of $\text{Aut}(\mathcal{H}(\partial\Delta^n))$ on $H = \{\mathcal{H}_0, \mathcal{H}_1, \dots, \mathcal{H}_{n-1}\}$ by

$$(g, \mathcal{H}_i) \mapsto g\mathcal{H}_i := \{g(\sigma) : \sigma \in \mathcal{H}_i\}.$$

We verify this is indeed a group action by noting that, $\text{id}_{\mathcal{H}(\partial\Delta^n)}(\mathcal{H}_i) = \mathcal{H}_i$ for all \mathcal{H}_i , and that if $g, h \in \text{Aut}(\mathcal{H}(\partial\Delta^n))$, we have

$$\begin{aligned} (gh)(\mathcal{H}_i) &= \{(g \circ h)(\sigma) : \sigma \in \mathcal{H}_i\} \\ &= g(\{h(\sigma) : \sigma \in \mathcal{H}_i\}) \\ &= g(h(\mathcal{H}_i)). \end{aligned}$$

By the Orbit Stabilizer theorem, we have $|\text{Aut}(\mathcal{H}(\partial\Delta^n))| = |\text{Orb}(\mathcal{H}_0)| |\text{Stab}(\mathcal{H}_0)|$. Suppose $f \in |\text{Aut}(\mathcal{H}(\partial\Delta^n))|$ fixes \mathcal{H}_0 . Then, f is bijective on the set of vertices of $\partial\Delta^n$ so that it corresponds with an automorphism of $\partial\Delta^n$. Likewise, any automorphism of $\partial\Delta^n$ is induced by a bijective map $V(\partial\Delta^n) \rightarrow V(\partial\Delta^n)$, so

must correspond with an automorphism on $\mathcal{H}(\partial\Delta^n)$ that fixes \mathcal{H}_0 . Therefore, $|\text{Stab}(\mathcal{H}_0)| = |\text{Aut}(\partial\Delta^n)|$. We also know that $\text{Orb}(\mathcal{H}_0) = 2$ since automorphisms of the Hasse Diagram send \mathcal{H}_0 to either \mathcal{H}_0 or \mathcal{H}_{n-1} , as we showed earlier in this proof. We conclude that $|\text{Aut}(\mathcal{H}(\partial\Delta^n))| = 2|\text{Aut}(\partial\Delta^n)|$, as desired. \square

We are now able to compute the automorphism group of the Morse complex in the case where $K = \partial\Delta^n$.

Theorem 32. If $K = \partial\Delta^n$, with $n \geq 2$, then $\text{Aut}(\mathcal{M}(\partial\Delta^n)) \cong \text{Aut}(\partial\Delta^n) \times \mathbb{Z}_2$.

Proof. For any function f , write $f^n := \underbrace{f \circ f \circ \dots \circ f}_n$. We construct an isomor-

phism $\phi: \text{Aut}(\partial\Delta^n) \times \mathbb{Z}_2 \rightarrow \text{Aut}(\mathcal{M}(\partial\Delta^n))$. For each $(f, i) \in \text{Aut}(K) \times \mathbb{Z}_2$, $i = 0, 1$, define $\phi((f, i)) := f_* \circ \pi_*^i$, where f_* is the automorphism of $\mathcal{M}(\partial\Delta^n)$ induced by f , and π_* is the ghost automorphism induced by the reflection map.

We first show that ϕ is a homomorphism. Suppose we have $(f, i), (g, j) \in \text{Aut}(\partial\Delta^n) \times \mathbb{Z}_2$. By Lemma 18 and the fact that π_* commutes with all f_* , we have

$$\begin{aligned} \phi((f, i)(g, j)) &= \phi((f \circ g, i + j)) \\ &= (f \circ g)_* \circ \pi_*^{i+j} \\ &= f_* \circ g_* \circ \pi_*^i \circ \pi_*^j \\ &= (f_* \circ \pi_*^i) \circ (g_* \circ \pi_*^j) \\ &= \phi((f, i))\phi((g, j)), \end{aligned}$$

as desired.

Next, we will show that ϕ is a bijection. We first show that ϕ is injective. Suppose we have $\phi((f, i)) = \phi((g, j))$ for some $(f, i), (g, j) \in \text{Aut}(\partial\Delta^n) \times \mathbb{Z}_2$. Then, we have $f_* \circ \pi_*^i = g_* \circ \pi_*^j$. If $i = j$, then, we have $f_* = g_*$, so $f = g$ by Proposition 19. We claim that $i \neq j$ is impossible. Suppose by contradiction that $i = 0, j = 1$, so that $f_* = g_* \circ \pi_*$. However, this implies that $g_* \circ \pi_*$ is induced by some simplicial automorphism on $\partial\Delta^n$. Consider any $(\sigma, \tau) \in V(\mathcal{M}(\partial\Delta^n))$ with $\dim \sigma = 0$. We have $f_*((\sigma, \tau)) = (f(\sigma), f(\tau))$. We know that $\dim \sigma = \dim f(\sigma) = 0$. However, $g_* \circ \pi_*((\sigma, \tau)) = g_*(\pi(\tau), \pi(\sigma)) = (g(\pi(\tau)), g(\pi(\sigma)))$. We know that $\dim \pi(\tau) = n - \dim \tau = n - 1$. Then, $\dim g(\pi(\tau)) = n - 1 > 0$, so $\dim f(\sigma) \neq \dim(g(\pi(\tau)))$, a contradiction. Thus, $i \neq j$ is not possible. Hence ϕ is injective.

Finally, by Proposition 22 and Lemma 31, we see that $|\text{Aut}(\mathcal{M}(\partial\Delta^n))| = |\text{Aut}(\mathcal{H}(\partial\Delta^n))| = 2|\text{Aut}(\partial\Delta^n)| = |\text{Aut}(\partial\Delta^n) \times \mathbb{Z}_2|$. Since these groups are finite, ϕ is a bijection. We conclude that ϕ is an isomorphism. \square

Combining Propositions 20, 23, and Theorem 32 thus yields Theorem 1 as promised.

References

- [1] R. Ayala, L. M. Fernández, A. Quintero, and J. A. Vilches, *A note on the pure Morse complex of a graph*, Topology Appl. **155** (2008), no. 17-18, 2084–2089.
- [2] N. A. Capitelli and E. G. Minian, *A simplicial complex is uniquely determined by its set of discrete Morse functions*, Discrete Comput. Geom. **58** (2017), no. 1, 144–157.
- [3] M. K. Chari and M. Joswig, *Complexes of discrete Morse functions*, Discrete Math. **302** (2005), no. 1-3, 39–51.
- [4] D. L. Ferrario and R. A. Piccinini, *Simplicial structures in topology*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011, Translated from the 2009 Italian original by Maria Nair Piccinini.
- [5] R. Forman, *Morse theory for cell complexes*, Adv. Math. **134** (1998), no. 1, 90–145.
- [6] ———, *A user’s guide to discrete Morse theory*, Sémin. Lothar. Combin. **48** (2002), Art. B48c, 35.
- [7] J. Jonsson, *Introduction to simplicial homology*, 2011, Available at the website <https://people.kth.se/~jakobj/homology.html>.
- [8] K. Knudson, *Morse theory: Smooth and discrete*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.
- [9] D. N. Kozlov, *Complexes of directed trees*, J. Combin. Theory Ser. A **88** (1999), no. 1, 112–122.
- [10] ———, *Combinatorial algebraic topology*, Algorithms and Computation in Mathematics, vol. 21, Springer, Berlin, 2008.