# DP color functions versus chromatic polynomials 

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#### Abstract

For any graph $G$, the chromatic polynomial of $G$ is the function $P(G, m)$ which counts the number of proper $m$-colorings of $G$ for each positive integer $m$. The DP color function $P_{D P}(G, m)$ of $G$, introduced by Kaul and Mudrock in 2019, is a generalization of $P(G, m)$ with $P_{D P}(G, m) \leq P(G, m)$ for each positive integer $m$. Let $P_{D P}(G) \approx P(G)$ (resp. $\left.P_{D P}(G)<P(G)\right)$ denote the property that $P_{D P}(G, m)=P(G, m)$ (resp. $P_{D P}(G, m)<$ $P(G, m))$ holds for sufficiently large integers $m$. It is an interesting problem of finding graphs $G$ for which $P_{D P}(G) \approx P(G)$ (resp. $\left.P_{D P}(G, m)<P(G, m)\right)$ holds. Kaul and Mudrock showed that if $G$ has an even girth, then $P_{D P}(G)<P(G)$ and Mudrock and Thomason recently proved that $P_{D P}(G) \approx P(G)$ holds for each graph $G$ which has a dominating vertex. We shall generalize their results in this article. For each edge $e$ in $G$, let $\ell(e)=\infty$ if $e$ is a bridge of $G$, and let $\ell(e)$ be the length of a shortest cycle in $G$ containing $e$ otherwise. We first show that if $\ell(e)$ is even for some edge $e$ in $G$, then $P_{D P}(G)<P(G)$ holds. However, the converse statement of this conclusion fails with infinitely many counterexamples. We then prove that $P_{D P}(G) \approx P(G)$ holds for every graph $G$ that contains a spanning tree $T$ such that for each $e \in E(G) \backslash E(T), \ell(e)$ is odd and $e$ is contained in a cycle $C$ of length $\ell(e)$ with the property that $\ell\left(e^{\prime}\right)<\ell(e)$ for each $e^{\prime} \in E(C) \backslash(E(T) \cup\{e\})$. Some open problems are proposed in this article.


Keywords: proper coloring; listing coloring; DP-coloring; chromatic polynomial; DP color function; spanning tree; cycle

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## 1 Introduction

In this article, we consider simple graphs only, unless otherwise stated. For any graph $G$, let $V(G)$ and $E(G)$ be its vertex set and edge set respectively. For any nonempty subset $S$ of $V(G)$, let $G[S]$ be the subgraph of $G$ induced by $S$, i.e., the subgraph with vertex set $S$ and edge set $\{u v \in E(G): u, v \in S\}$, where $u v$ denotes the edge joining $u$ and $v$, and let $G-S$ be the subgraph $G[V(G) \backslash S]$ when $S \neq V(G)$. In particular, if $S=\{v\}$ for $v \in V(G)$, write $G-v$ for $G-S$. For $A \subseteq E(G)$, let $G\langle A\rangle$ denote the spanning subgraph of $G$ with edge set $A$, and let $G-A$ be $G\langle E(G) \backslash A\rangle$. In particular, for $e \in E(G), G-\{e\}$ is written as $G-e$. For two disjoint subsets $S_{1}$ and $S_{2}$ of $V(G)$, let $E_{G}\left(S_{1}, S_{2}\right)$ (or simply $E\left(S_{1}, S_{2}\right)$ ) denote the set $\left\{u v \in E(G): u \in S_{1}, v \in S_{2}\right\}$. For any $u \in V(G)$, let $N_{G}(u)$ (or simply $N(u)$ ) be the set of neighbors of $u$ in $G$ and $d_{G}(u)$ (or simply $d(u)$ ) be the degree of $u$ in $G$. The reader may refer to [5] for other terminology and notation.

### 1.1 Proper coloring, list coloring and DP coloring

Let $\mathbb{N}$ denote the set of positive integers. For any $n \in \mathbb{N}$, let $\llbracket n \rrbracket=\{1,2, \cdots, n\}$. For any graph $G$ and $k \in \mathbb{N}$, a proper $k$-coloring of $G$ is a mapping of $f: V(G) \rightarrow \llbracket k \rrbracket$ such that $f(u) \neq f(v)$ for each edge $u v \in E(G)$. The chromatic polynomial $P(G, k)$ of $G$, introduced by Birkhoff [3] in 1912, is the function which counts the number of proper $k$-colorings of $G$ for each $k \in \mathbb{N}$. Note that $P(G, k)$ is indeed a polynomial in $k$ for each $k \in \mathbb{N}$ (see [4, 8, [17, 23]). The chromatic number of $G$, denoted by $\chi(G)$, is the minimum number $k \in \mathbb{N}$ such that $G$ admits a proper $k$-coloring. Obviously, $\chi(G)$ is the minimum number $k \in \mathbb{N}$ such that $P(G, k)>0$. For more details on chromatic polynomials, we refer the readers to [3, 4, 8, 9, 13, 17, 18, 19, 23].

List coloring was introduced by Vizing [21] and Erdős, Rubin and Taylor [11] independently. A list coloring of $G$ is associated with a list assignment $L$, where $L(v)$ is a subset of $\mathbb{N}$ for each $v \in V(G)$. Given a list assignment $L$ of $G$, a proper $L$-coloring of $G$ is a mapping $f: V(G) \rightarrow \mathbb{N}$ such that $f(v) \in L(v)$ for each $v \in V(G)$ and $f(u) \neq f(v)$ for each edge $u v \in E(G)$. If $L(v)=\llbracket k \rrbracket$ for each $v \in V(G)$, then a proper $L$-coloring of $G$ is a proper $k$-coloring of $G$. If $|L(v)|=m$ for each $v \in V(G)$, then $L$ is called an m-assignment. The list chromatic number of $G$, denoted by $\chi_{l}(G)$, is the smallest $m$ such that $G$ has a proper $L$-coloring for every m-assignment $L$ of $G$. By definition, $\chi(G) \leq \chi_{l}(G)$. Due to Noel, Reed and Wu [16], $\chi(G)=\chi_{l}(G)$ holds whenever $\chi(G) \geq(|V(G)|-1) / 2$.

For each list-assignment $L$ of $G$, let $P(G, L)$ be the number of proper $L$-colorings. For each $m \in \mathbb{N}$, let $P_{l}(G, m)$ be the minimum value of $P(G, L)$ among all $m$-assignments $L$. We call
$P_{l}(G, m)$ the list color function of $G$. By definition, $P_{l}(G, m) \leq P(G, m)$ for each $m \in \mathbb{N}$. Wang, Qian and Yan [22] showed that $P_{l}(G, m)=P(G, m)$ holds when $G$ is connected and $m>(|E(G)|-1) / \ln (1+\sqrt{2})$. The survey by Thomassen [20] provided some known results and open questions on the list color function.

DP-coloring was introduced by Dvorák and Postle [10] for the purpose of proving that every planar graph without cycles of lengths 4 to 8 is 3 -choosable. DP-coloring is a generalization of list coloring, and a formal definition is given below. For a graph $G$, a cover of $G$ is an ordered pair $\mathcal{H}=(L, H)$, where $H$ is a graph and $L$ is a mapping from $V(G)$ to the power set of $V(H)$ satisfying the four conditions below:
(i). the sets $\{L(u): u \in V(G)\}$ is a partition of $V(H)$ of size $|V(G)|$;
(ii). for every $u \in V(G), H[L(u)]$ is a complete graph;
(iii). if $u$ and $v$ are non-adjacent vertices in $G$, then $E_{H}(L(u), L(v))=\emptyset$; and
(iv). for each edge $u v \in E(G), E_{H}(L(u), L(v))$ is a matching.

An $\mathcal{H}$-coloring of $G$ is an independent set $I$ of $H$ with $|I|=|V(G)|$. Clearly, for each $\mathcal{H}$ coloring $I$ of $G,|I \cap L(u)|=1$ holds for each $u \in V(G)$. A cover $\mathcal{H}=(L, H)$ of $G$ is called an $m$-fold cover if $|L(u)|=m$ for each $u \in V(G)$. The $D P$-chromatic number of $G$, denoted by $\chi_{D P}(G)$, is the minimum integer $m$ such that $G$ has a $\mathcal{H}$-coloring for every $m$-fold cover $\mathcal{H}=(L, H)$. By definition, $\chi(G) \leq \chi_{l}(G) \leq \chi_{D P}(G)$. Bernshteyn, Kostochka and Zhu [2] showed that for any $n \in \mathbb{N}$, if $r(n)$ is the minimum number $r \in \mathbb{N}$ such that $\chi(G)=\chi_{D P}(G)$ holds for every $n$-vertex graph $G$ with $\chi(G) \geq r$, then $n-r(n)=\Theta(\sqrt{n})$.

For any cover $\mathcal{H}$ of $G$, let $P_{D P}(G, \mathcal{H})$ be the number of $\mathcal{H}$-colorings of $G$. For each $m \in \mathbb{N}$, let $P_{D P}(G, m)$ be the minimum value of $P_{D P}(G, \mathcal{H})$ among all $m$-fold covers $\mathcal{H}$ of $G$. We call $P_{D P}(G, m)$ the $D P$ color function of $G$, which was introduced by Kaul and Mudrock [14]. For any m-assignment $L$ of $G, P(G, L)=P_{D P}(G, \mathcal{H})$ holds for the $m$-fold cover $\mathcal{H}=\left(L^{\prime}, H\right)$, where $L^{\prime}(v)=\{(v, j): j \in L(v)\}$ for each $v \in V(G)$ and for each edge $u v \in E(G), E_{H}\left(L^{\prime}(u), L^{\prime}(v)\right)=$ $\left\{(u, j)(v, j): j \in \mathbb{N},(u, j) \in L^{\prime}(u),(v, j) \in L^{\prime}(v)\right\}$. Thus, $P_{D P}(G, m) \leq P_{l}(G, m) \leq P(G, m)$ holds for each $m \in \mathbb{N}$.

### 1.2 Main results

For any graph $G$, by definition, $P_{D P}(G, m) \leq P(G, m)$ holds for all integers $m \in N$. Thus, exactly one of the following three properties holds:
(i). there exists $N \in \mathbb{N}$ such that $P_{D P}(G, m)=P(G, m)$ for all integers $m \geq N$;
(ii). there exists $N \in \mathbb{N}$ such that $P_{D P}(G, m)<P(G, m)$ for all integers $m \geq N$; and
(iii). there exist two infinite sets $\left\{m_{i} \in \mathbb{N}: i \in \mathbb{N}\right\}$ and $\left\{n_{i} \in \mathbb{N}: i \in \mathbb{N}\right\}$ such that for all $i \in \mathbb{N}$, both $P_{D P}\left(G, m_{i}\right)=P\left(G, m_{i}\right)$ and $P_{D P}\left(G, n_{i}\right)<P\left(G, n_{i}\right)$ hold.

Two questions proposed by Kaul and Mudrock [14] are closed related to property (iii), and there would be no graphs having property (iii) if the answer to any one of them had been yes. Question 7 in [14] asks if, for any graph $G$, there always exist an $N \in \mathbb{N}$ and a polynomial $p(m)$ such that $P_{D P}(G, m)=p(m)$ whenever $m \geq N$. Halberg, Kaul, Liu, Mudrock, Shin and Thomason [12] showed that this question has a positive answer for each graph $G$ with a vertex $v$ such that $G-v$ is acyclic. Question 15 in [14] asks if $P_{D P}\left(G, m_{0}\right)=P\left(G, m_{0}\right)$ for some $m_{0} \geq \chi(G)$ implies that $P_{D P}(G, m)=P(G, m)$ for all $m \geq m_{0}$. Unfortunately, Bui, Kaul, Maxfield, Mudrock, Shin and Thomason [6] found graphs with negative answer to the second question.

For any one of the above properties, it is an interesting problem of knowing which graphs have this property. For convenience purposes, let $P_{D P}(G) \approx P(G)$ (resp. $\left.P_{D P}(G)<P(G)\right)$ denote property (i) (resp. property (ii)) above for a graph $G$.

Problem 1. Is it true that for each graph $G$, either $P_{D P}(G) \approx P(G)$ or $P_{D P}(G)<P(G)$ ?

So far the comparison of DP color functions with chromatic polynomials focuses on following problem.

Problem 2. Determine the set of graphs $G$ such that $P_{D P}(G) \approx P(G)$ holds and the set of graphs $G$ such that $P_{D P}(G)<P(G)$ holds.

Kaul and Mudrock [14] obtained some important results on the study of Problem 2, For example, they showed that if there exists an edge $e$ in $G$ such that $P(G-e, m)<m P(G, m) /(m-1)$, then $P_{D P}(G, m)<P(G, m)$ holds (see Theorem (6).

For each edge $e$ in $G$, if $e$ is a bridge of $G$, let $\ell_{G}(e)=\infty$; otherwise, let $\ell_{G}(e)$ be the length of a shortest cycle containing $e$ in $G$. Write $\ell_{G}(e)$ as $\ell(e)$ when $G$ is clear from the context. Thus, the girth $g$ of $G$ is the minimum value of $\ell(e)$ among all edges $e$ in $G$. Kaul and Mudrock [14] showed that if $G$ has an even girth, then $P_{D P}(G)<P(G)$. We apply Theorem 6 to generalize this result below.

Theorem 1. For any graph $G$, if $\ell(e)$ is even for some edge $e$ in $G$, then $P_{D P}(G)<P(G)$.

The converse statement of Theorem 1 fails, and counterexamples will be given in Section 3 ,
Theorem 2. There exist infinitely many graphs $G$ such that $P_{D P}(G)<P(G)$ and $\ell(e)=3$ for each edge e in $G$.

For a disconnected graph $G$, if $P_{D P}\left(G_{i}\right)<P\left(G_{i}\right)$ for some component $G_{i}$ of $G$, then $P_{D P}(G)<$ $P(G)$ obviously holds. This conclusion also holds for connected graphs.

Theorem 3. For a connected graph $G$, if $P_{D P}\left(G_{i}\right)<P\left(G_{i}\right)$ for some block $G_{i}$ of $G$, then $P_{D P}(G)<P(G)$ holds.

Some results on the study of graphs with the property $P_{D P}(G) \approx P(G)$ have been obtained. Kaul and Mudrock [14] showed that $P_{D P}(G) \approx P(G)$ holds for the graph $G$ obtained from any two odd cycle graphs $C_{2 k+1}$ and $C_{2 r+1}$ by identifying one edge in $C_{2 k+1}$ with one edge in $C_{2 r+1}$. For two vertex-disjoint graphs $G$ and $G^{\prime}$, let $G \vee G^{\prime}$ denote the join of $G$ and $G^{\prime}$, i.e., the graph obtained from $G$ and $G^{\prime}$ by adding all edges in $\left\{u v: u \in V(G), v \in V\left(G^{\prime}\right)\right\}$. Kaul and Mudrock [14] asked that for every graph $G$, does there exist $p \in \mathbb{N}$ such that $P_{D P}\left(K_{p} \vee G\right) \approx P\left(K_{p} \vee G\right)$, where $K_{p}$ is the complete graph with $p$ vertices? Recently, Mudrock and Thomason [15] showed that the problem has a positive answer for $p=1$. Obviously, a graph is isomorphic to $K_{1} \vee G$ for some graph $G$ if and only if it has a dominating vertex (i.e., a vertex which is adjacent to all other vertices in the graph).

For any graph $G$ and any integer $m>0$, there is a special $m$-fold cover of $G$ which corresponds to proper $m$-colorings. Let $\mathcal{H}_{0}(G, m)$ denote the $m$-fold cover $(L, H)$ of $G$, where $L(u)=$ $\{(u, i): i \in \llbracket m \rrbracket\}$ for each $u \in V(G)$ and $E_{H}(L(u), L(v))=\{(u, i)(v, i): i \in \llbracket m \rrbracket\}$ for each edge $u v$ in $G$. The graph $H$ in $\mathcal{H}_{0}(G, m)=(L, H)$ is denoted by $H_{0}(G, m)$ (or simply $H_{0}(m)$ ). Obviously, $P_{D P}\left(G, \mathcal{H}_{0}(G, m)\right)=P(G, m)$ holds for each $m \in \mathbb{N}$.

Let $\mathcal{D} \mathcal{P}^{*}$ denote the set of graphs $G$ for which there exists $M \in \mathbb{N}$ such that for every $m$-fold cover $\mathcal{H}=(L, H)$ of $G$, if $H \not \neq H_{0}(G, m)$, then $P_{D P}(G, \mathcal{H})>P(G, m)$ holds for all integers $m \geq M$. By definition, $P_{D P}(G) \approx P(G)$ holds for each graph $G$ in $\mathcal{D} \mathcal{P}^{*}$. But it is unknown if the converse statement is also true.

Problem 3. Is it true that if $P_{D P}(G) \approx P(G)$, then $G \in \mathcal{D P}^{*}$ ?

Our next result provides a sufficient condition for a graph $G$ to be in $\mathcal{D} \mathcal{P}^{*}$ and therefore $P_{D P}(G) \approx P(G)$ holds.

Theorem 4. If a graph $G$ contains a spanning tree $T$ such that for each edge e in $E(G) \backslash E(T)$, $\ell(e)$ is odd and $e$ is contained in a cycle $C$ of length $\ell(e)$ with the property that $\ell\left(e^{\prime}\right)<\ell(e)$ holds for each $e^{\prime} \in E(C) \backslash(E(T) \cup\{e\})$, then $G \in \mathcal{D P}^{*}$ and hence $P_{D P}(G) \approx P(G)$.

A vertex $u$ in a graph $G$ is called simplicial if either $d_{G}(u)=0$ or $G[N(u)]$ is a complete graph. A graph $G$ is called chordal if for each cycle $C$ in $G, G[V(C)]$ contains 3-cycles. Due to Dirac [7], a graph $G$ is chordal if and only if there exists an ordering $v_{1}, v_{2}, \cdots, v_{n}$ of its vertices, called a perfect elimination ordering, such that each $v_{i}$ is simplicial in the subgraph of $G$ induced by $\left\{v_{j}: j \in \llbracket i \rrbracket\right\}$. Due to Kaul and Mudrock [14], for any chordal graph $G$, $P_{D P}(G, m)=P(G, m)$ holds for all $m \in \mathbb{N}$, and hence $P_{D P}(G) \approx P(G)$ holds. We notice that this conclusion does not follow from Theorem 4. But the next result is its generalization.

Theorem 5. For any graph $G$ with a simplicial vertex $u$, if $P_{D P}(G-u) \approx P(G-u)$, then $P_{D P}(G) \approx P(G)$; also, if $G-u \in \mathcal{D} \mathcal{P}^{*}$, then $G \in \mathcal{D} \mathcal{P}^{*}$.

Theorems 2 and 3 are proved in Section 3, while Theorems 1, 4 and 5are proved in Sections 2, 4 and 5 respectively.

## 2 Proof of Theorem 1

The following result due to Kaul and Mudrock [14] will be applied to study graphs $G$ with the property $P_{D P}(G)<P(G)$.

Theorem 6 ([14]). Let $G$ be a graph with an edge e. If $m \geq 2$ and $P(G-e, m)<\frac{m}{m-1} P(G, m)$, then $P_{D P}(G, m)<P(G, m)$.

In this section, we shall apply two fundamental properties of the chromatic polynomial $P(G, x)$ of $G$. The variable $x$ in $P(G, x)$ can be considered a real number. By the inclusion-exclusion principle, it can be proved that

$$
\begin{equation*}
P(G, x)=\sum_{A \subseteq E(G)}(-1)^{|A|} x^{c(A)} \tag{1}
\end{equation*}
$$

where $c_{G}(A)$ (or simply $c(A)$ ) is the number of components in the spanning subgraph $G\langle A\rangle$ of $G$ (see [23]). Note that (11) holds even if $G$ has parallel edges or loops.

The deletion-contraction theorem of chromatic polynomials (see [8, [17, 18]) states that for each edge $e$ in a graph $G$,

$$
\begin{equation*}
P(G, x)=P(G-e, x)-P(G / e, x), \tag{2}
\end{equation*}
$$

where $G / e$ is the graph obtained by contracting edge $e$ (i.e., the graph obtained from $G-e$ by identifying the two ends of $e$ ). Clearly, $G / e$ may have parallel edges. By (2), for any $e \in E(G)$,
when $x \neq 1$,

$$
\begin{align*}
P(G-e, x)-\frac{x}{x-1} P(G, x) & =P(G-e, x)-\frac{x}{x-1}(P(G-e, x)-P(G / e, x)) \\
& =\frac{1}{x-1}(x P(G / e, x)-P(G-e, x)) \tag{3}
\end{align*}
$$

For any edge $e$ in $G$, let $\mathcal{C}(e)$ denote the set of cycles in $G$ that contain $e$ and are of length $\ell(e)$. Obviously, $\mathcal{C}(e) \neq \emptyset$ if $e$ is not a bridge of $G$.

Proposition 7. Let $G$ be a simple graph and $e$ be an edge in $G$ with $\ell(e)<\infty$. Then, the leading term in the polynomial $x P(G / e, x)-P(G-e, x)$ is $(-1)^{\ell(e)-1}|\mathcal{C}(e)| x^{n-\ell(e)+2}$.

Proof. Note that $G-e$ and $G / e$ have the same edge set, i.e., $E(G) \backslash\{e\}$, and when $\ell(e)=3$, $G / e$ has parallel edges. Applying (1) to both $G-e$ and $G / e$, we have

$$
\begin{equation*}
P(G-e, x)=\sum_{A \subseteq E(G) \backslash\{e\}}(-1)^{|A|} x^{c_{G}(A)} \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
P(G / e, x)=\sum_{A \subseteq E(G) \backslash\{e\}}(-1)^{|A|} x^{c_{G / e}(A)} . \tag{5}
\end{equation*}
$$

Let $u, v$ be the two ends of $e$, and let $\mathcal{E}_{e}$ be the set of subsets $A$ of $E(G) \backslash\{e\}$, such that $u$ and $v$ are in the same component of the spanning subgraph $G\langle A\rangle$ of $G$. Let $\mathcal{E}_{e}^{\prime}$ be the set of subsets $A$ of $E(G) \backslash\{e\}$ with $A \notin \mathcal{E}_{e}$. If $A \in \mathcal{E}_{e}$, then $c_{G}(A)=c_{G / e}(A)$; and if $A \in \mathcal{E}_{e}^{\prime}$, then $c_{G}(A)=c_{G / e}(A)+1$. Thus, (4) and (5) imply that

$$
\begin{equation*}
x P(G / e, x)-P(G-e, x)=\sum_{A \in \mathcal{E}_{e}}(-1)^{|A|} x^{c_{G}(A)}(x-1) . \tag{6}
\end{equation*}
$$

For each $A \in \mathcal{E}_{e}$, let $G_{A}$ denote the component of $G\langle A\rangle$ that contains both vertices $u$ and $v$. Then $G_{A}$ has a $(u, v)$-path $P$, implying that $\left|V\left(G_{A}\right)\right| \geq|V(P)| \geq \ell(e)$. If $\left|V\left(G_{A}\right)\right|=\ell(e)$, then $V\left(G_{A}\right)=V(P)$ and $P$ must be a path $C-e$ for some cycle $C \in \mathcal{C}(e)$. As each cycle in $G$ containing $e$ must be of length at least $\ell(e),\left|V\left(G_{A}\right)\right|=\ell(e)$ implies that $G_{A}$ is a path $C-e$ for some cycle $C \in \mathcal{C}(e)$.

Consequently, for each $A \in \mathcal{E}_{e}, c_{G}(A) \leq n-\ell(e)+1$ holds, and $c_{G}(A)=n-\ell(e)+1$ if and only if $A \cup\{e\}$ is the edge set of some cycle $C$ in $\mathcal{C}(e)$. Thus, $c_{G}(A)=n-\ell(e)+1$ holds for exactly $|\mathcal{C}(e)|$ subsets $A \in \mathcal{E}_{e}$, and for each of them, $|A|=\ell(e)-1$.

By (6) and the above conclusions, $x P(G / e, x)-P(G-e, x)$ is a polynomial of degree $n-\ell(e)+2$ and the coefficient of its leading term is $(-1)^{\ell(e)-1}|\mathcal{C}(e)|$.

Hence the result holds.
We are now going to prove Theorem [1.
Proof of Theorem [1: Let $e$ be an edge in $G$ such that $\ell(e)$ is even. By the equality of (3) and Proposition 7, there exists $M \in \mathbb{N}$ such that $P(G-e, m)<\frac{m}{m-1} P(G, m)$ for all integer $m \geq M$. The result then follows from Theorem 6,

## 3 Proof of Theorems 2 and 3

Let $\omega(G)$ denote the clique number of a graph $G$. For any vertex-disjoint graphs $G_{1}$ and $G_{2}$ and $k \in \mathbb{N}$ with $k \leq \min \left\{\omega\left(G_{i}\right): i=1,2\right\}$, let $\mathscr{G}\left(G_{1} \cup_{k} G_{2}\right)$ denote the set of graphs obtained from $G_{1}$ and $G_{2}$ by identifying a $k$-clique in $G_{1}$ with a $k$-clique in $G_{2}$. Due to Zykov [24], the following identity on chromatic polynomials holds for any $G \in \mathscr{G}\left(G_{1} \cup_{k} G_{2}\right)$ and all $m \geq k$ :

$$
\begin{equation*}
P(G, m)=\frac{P\left(G_{1}, m\right) P\left(G_{2}, m\right)}{m(m-1) \cdots(m-k+1)} . \tag{7}
\end{equation*}
$$

If $u$ is a simplicial vertex of a graph $G$, the following identity on chromatic polynomials follows from (7) (also see [8, [18]):

$$
\begin{equation*}
P(G, m)=\left(m-d_{G}(u)\right) P(G-u, m), \quad \forall m \in \mathbb{N} \tag{8}
\end{equation*}
$$

It is natural to ask if (8) holds for the DP color function.
Problem 4. If $u$ is a simplicial vertex of $G$, is it true that for all integers $m \geq d(u)$,

$$
\begin{equation*}
P_{D P}(G, m)=(m-d(u)) P_{D P}(G-u, m) ? \tag{9}
\end{equation*}
$$

As $P_{D P}(G, m) \geq(m-d(u)) P_{D P}(G-u, m)$ by definition, to prove the equality of (9), it suffices to show that $P_{D P}(G, m) \leq(m-d(u)) P_{D P}(G-u, m)$ for all integers $m \geq d(u)$. It is trivial that Problem 4 has a positive answer when $d(u)=0$, and due to Theorem 10, it also has a positive answer when $d(u)=1$. In this section, we show that it has a positive answer for $d(u)=2$. Applying this conclusion, we are able to prove that the converse statement of Theorem fails.

Proposition 8. If $u$ is a simplicial vertex of $G$ with $d(u)=2$, then for each integer $m \geq 2$,

$$
\begin{equation*}
P_{D P}(G, m)=(m-2) P_{D P}(G-u, m) \tag{10}
\end{equation*}
$$

Proof. Let $m \geq 2$. If $m<\chi_{D P}(G-u)$, then $m<\chi_{D P}(G-u) \leq \chi_{D P}(G)$, implying that $P_{D P}(G-u, m)=P_{D P}(G, m)=0$. It follows that (10) holds in this case.

As $u$ is a simplicial vertex of $G$ with degree $2, \chi_{D P}(G) \geq \chi(G) \geq 3$, implying that $P_{D P}(G, 2)=$ 0 . Thus (10) also holds when $m=2$.

Now let $m \geq \max \left\{3, \chi_{D P}(G-u)\right\}$ and let $\mathcal{H}^{\prime}=\left(L^{\prime}, H^{\prime}\right)$ be an $m$-fold cover of $G-u$ such that $P_{D P}\left(G-u, \mathcal{H}^{\prime}\right)=P_{D P}(G-u, m)$ and $\left|E\left(H^{\prime}\right)\right|$ has the maximum value. It is clear that $E_{H^{\prime}}\left(L^{\prime}\left(v_{1}\right), L^{\prime}\left(v_{2}\right)\right)$ is a matching in $H^{\prime}$ of size $m$ for each pair of adjacent vertices $v_{1}$ and $v_{2}$ in $G-u$.

Let $N_{G}(u)=\left\{u_{1}, u_{2}\right\}$. Assume that $\left(u_{1}, j\right)$ and $\left(u_{2}, \pi(j)\right)$ are adjacent in $H^{\prime}$ for each $j \in \llbracket m \rrbracket$, where $\pi$ is a bijection from $\llbracket m \rrbracket$ to $\llbracket m \rrbracket$.

Let $H$ be the graph obtained from $H^{\prime}$ and a complete graph with vertex set $\{(u, j): j \in \llbracket m \rrbracket\}$ by adding edges joining $(u, j)$ to both $\left(u_{1}, j\right)$ and $\left(u_{2}, \pi(j)\right)$ for each $j \in \llbracket m \rrbracket$. Let $\mathcal{H}=(L, H)$ be the $m$-fold cover of $G$, where $L(u)=\{(u, j): j \in \llbracket m \rrbracket\}$ and $L(v)=L^{\prime}(v)$ for all $v \in V(G)-\{u\}$.

Let $I$ be any member in $\mathcal{I}\left(H^{\prime}\right)$. Assume that $\left(u_{1}, j_{1}\right) \in I \cap L\left(u_{1}\right)$ and $\left(u_{2}, \pi\left(j_{2}\right)\right) \in I \cap L\left(u_{2}\right)$. As $\left(u_{1}, j_{1}\right)$ and $\left(u_{2}, \pi\left(j_{1}\right)\right)$ are adjacent in $H, j_{1} \neq j_{2}$. Then, $I$ can be extended to exactly $(m-2)$ independent sets of $H$ of the form $I \cup\{(u, j)\}$, where $j \in \llbracket m \rrbracket \backslash\left\{j_{1}, j_{2}\right\}$. Thus,

$$
\begin{equation*}
P_{D P}(G, \mathcal{H})=(m-2) P_{D P}\left(G-u, \mathcal{H}^{\prime}\right)=(m-2) P_{D P}(G-u, m), \tag{11}
\end{equation*}
$$

by which $P_{D P}(G, m) \leq(m-2) P_{D P}(G-u, m)$. On the other hand, it is obvious that $P_{D P}(G, m) \geq(m-2) P_{D P}(G-u, m)$. Thus, the result follows.

For any graph $Q$ with at least one edge, let $\Phi(Q)$ be the family of graphs defined below:
(i). $Q \in \Phi(Q)$; and
(ii). if $Q^{\prime} \in \Phi(Q)$, then $\mathscr{G}\left(Q^{\prime} \cup_{2} K_{3}\right) \subseteq \Phi(Q)$.

For example, $G_{1} \in \Phi\left(C_{4}\right)$ and $G_{2} \in \Phi\left(C_{6}\right)$, where $C_{k}$ is the cycle graph of length $k$, and $G_{1}$ and $G_{2}$ are graphs in Figure 1 .

By (8) and Proposition 8, for any graph $G \in \Phi(Q)$ and any integer $m \geq 2$,

$$
\begin{equation*}
P(G, m)=(m-2)^{|V(G)|-|V(Q)|} P(Q, m), \quad P_{D P}(G, m)=(m-2)^{|V(G)|-|V(Q)|} P_{D P}(Q, m) . \tag{12}
\end{equation*}
$$

By (12), we have the following observation.

(a) $G_{1}$

(b) $G_{2}$

Figure 1: $G_{1} \in \Phi\left(C_{4}\right)$ and $G_{2} \in \Phi\left(C_{6}\right)$

Proposition 9. For any graph $Q$ with at least one edge and any $G \in \Phi(Q)$, if $P_{D P}(Q) \approx P(Q)$, then $P_{D P}(G) \approx P(G)$; also, if $P_{D P}(Q)<P(Q)$, then $P_{D P}(G)<P(G)$.

We can now easily prove Theorem 2,
Proof of Theorem圆 Let $Q$ be any graph with $P_{D P}(Q)<P(Q)$. Clearly, $Q$ contains edges. By Proposition 9, $P_{D P}(G)<P(G)$ holds for every $G \in \Phi(Q)$. By the definition of $\Phi(Q)$, there are infinitely many graphs $G \in \Phi(Q)$ such that $\ell_{G}(e)=3$ for each edge $e$ in $G$. For example, for graph $G_{1}$ in Figure 1 (a), if $G$ is a graph in $\Phi\left(G_{1}\right)$, then $P_{D P}(G)<P(G)$ and $\ell_{G}(e)=3$ holds for each edge $e$ in $G$.

Theorem 2 holds.
Theorem 3 will be proved directly by applying the following result due to Becker, Hewitt, Kaul, Maxfield, Mudrock, Spivey, Thomason and Wagstrom [1].

Theorem $10([1])$. For any connected graph $G$ with blocks $G_{1}, G_{2}, \cdots, G_{r}$, where $r \geq 2$,

$$
\begin{equation*}
P_{D P}(G, m) \leq \frac{1}{m^{r-1}} \prod_{i=1}^{r} P_{D P}\left(G_{i}, m\right) \tag{13}
\end{equation*}
$$

Proof of Theorem 3 Let $G_{1}, G_{2}, \cdots, G_{r}$ be the blocks of $G$. By identity (7) and Theorem 10, we have

$$
\begin{equation*}
P_{D P}(G, m) \leq \frac{1}{m^{r-1}} \prod_{i=1}^{r} P_{D P}\left(G_{i}, m\right) \leq \frac{1}{m^{r-1}} \prod_{i=1}^{r} P\left(G_{i}, m\right)=P(G, m) \tag{14}
\end{equation*}
$$

By (14), if $P_{D P}\left(G_{i}\right)<P\left(G_{i}\right)$ for some $i$, then $P_{D P}(G)<P(G)$ holds.
It is natural to ask the following problem.
Problem 5. For a connected graph $G$, if $P_{D P}\left(G_{i}\right) \approx P\left(G_{i}\right)$ holds for each block $G_{i}$ of $G$, is it true that $P_{D P}(G) \approx P(G)$ ?

## 4 Proof of Theorem 4

### 4.1 A set of ordered pairs $(G, T)$, where $T$ is a spanning tree of $G$

Let $\mathcal{G} \mathcal{T}$ be the set of ordered pairs $(G, T)$, where $G$ is a connected graph and $T$ is a spanning tree of $G$ such that for each edge $e$ in $E(G) \backslash E(T), \ell_{G}(e)$ is odd and $e$ is contained in a cycle $C$ of length $\ell_{G}(e)$ with the property that $\ell_{G}\left(e^{\prime}\right)<\ell_{G}(e)$ holds for each $e^{\prime} \in E(C) \backslash(E(T) \cup\{e\})$.

Note that $\mathcal{G \mathcal { T }}$ contains a subfamily $\mathcal{G} \mathcal{T}_{0}$ of ordered pairs $(G, T)$, where $T$ is a spanning tree of $G$ such that for each $e \in E(G) \backslash E(T), \ell(e)$ is odd and the fundamental cycle $C_{T}(e)$ of $e$ with respect to $T$ is of length $\ell(e)$.

Let $\mathcal{G}$ (resp. $\mathcal{G}_{0}$ ) be the set of graphs $G$ such that $(G, T) \in \mathcal{G \mathcal { T }}$ (resp. $\left.(G, T) \in \mathcal{G} \mathcal{T}_{0}\right)$ for some spanning tree $T$ of $G$. For example, for $i=1,2, G_{i} \in \mathcal{G}_{0}$, where $G_{1}$ and $G_{2}$ (i.e., the Petersen graph) are the graphs in Figure 2 (a) and (b) respectively. It is also obvious that $\mathcal{G}_{0}$ contains every graph that has a dominating vertex. But, it can be verified that $G_{3}$ in Figure 2 (c) belongs to $\mathcal{G} \backslash \mathcal{G}_{0}$.

(a) $G_{1}$

(b) $G_{2}$

(c) $G_{3}$

Figure 2: $G_{i} \in \mathcal{G}_{0} \subset \mathcal{G}$ for $i=1,2$ and $G_{3} \in \mathcal{G} \backslash \mathcal{G}_{0}$
Let $(G, T) \in \mathcal{G} \mathcal{T}$. By definition, $\ell_{G}(e)$ is odd for each $e \in E(G) \backslash E(T)$. But, it does not guarantee directly that $\ell_{G}(e)$ is not even for any $e \in E(T)$. By Theorem [1, if $\ell_{G}(e)$ is even for some $e \in E(T)$, then Theorem 4 fails. Thus, before proving Theorem 4, it is necessary to show that $\ell_{G}(e)$ is not even for every $e \in E(T)$.

For $(G, T) \in \mathcal{G} \mathcal{T}$, if $E(G)=E(T)$, let $\ell(G, T)=\infty$; otherwise, let $\ell(G, T)=\max _{e \in E(G) \backslash E(T)} \ell_{G}(e)$.
Proposition 11. Let $(G, T) \in \mathcal{G} \mathcal{T}$. For each edge $e \in E(T)$, if $e$ is not a bridge of $G$, then $\ell_{G}(e)$ is odd and $\ell_{G}(e) \leq \ell(G, T)$.

Proof. We prove the result by induction on $|E(G)|$. Note that $|E(G)| \geq|E(T)|$. The result is obvious when $|E(G)| \leq|E(T)|+1$. Now assume that $|E(G)| \geq|E(T)|+2$ and the result holds for every ordered pair $\left(G^{\prime}, T^{\prime}\right) \in \mathcal{G \mathcal { T }}$ with $\left|E\left(G^{\prime}\right)\right| \leq|E(G)|-1$.

Choose an edge $e_{1}$ in $E(G) \backslash E(T)$ such that $\ell_{G}\left(e_{1}\right)=\ell(G, T)<\infty$. Clearly, $T$ is a spanning tree of $G-e_{1}$. We first show that $\left(G-e_{1}, T\right) \in \mathcal{G} \mathcal{T}$.

Let $G^{\prime}$ denote $G-e_{1}$ and let $e$ be any edge in $E\left(G^{\prime}\right) \backslash E(T)$. As $e \in E(G) \backslash E(T)$, by definition, $\ell_{G}(e)$ is odd and $e$ is contained in a cycle $C$ in $G$ of length $\ell_{G}(e)$ such that $\ell_{G}\left(e^{\prime}\right)<\ell_{G}(e)$ for each $e^{\prime} \in E(C) \backslash(E(T) \cup\{e\})$. By the choice of $e_{1}, \ell_{G}\left(e_{1}\right) \geq \ell_{G}(e)$, implying that $e_{1} \notin E(C)$. Thus, $C$ is in $G^{\prime}$ and $\ell_{G}(e)=\ell_{G^{\prime}}(e)$.

Hence, by definition, $\left(G^{\prime}, T\right) \in \mathcal{G} \mathcal{T}$ and $\ell_{G^{\prime}}(e)=\ell_{G}(e)$ for each $e \in E\left(G^{\prime}\right) \backslash E(T)$, implying that $\ell\left(G^{\prime}, T\right) \leq \ell(G, T)$.

By inductive assumption, the conclusion holds for $\left(G^{\prime}, T\right) \in \mathcal{G} \mathcal{T}$. Now suppose $e_{0} \in E(T)$ and $e_{0}$ is not a bridge in $G$. Then, either $e_{0}$ is a bridge of $G^{\prime}$ or $\ell_{G^{\prime}}\left(e_{0}\right)$ is odd. Furthermore, if $e_{0}$ is not a bridge of $G^{\prime}$, then $\ell_{G^{\prime}}\left(e_{0}\right) \leq \ell\left(G^{\prime}, T\right) \leq \ell(G, T)=\ell_{G}\left(e_{1}\right)$. We shall show that $\ell_{G}\left(e_{0}\right)$ is odd and $\ell_{G}\left(e_{0}\right) \leq \ell(G, T)$.

Case 1: $e_{0}$ is a bridge of $G^{\prime}$ (i.e., $G-e_{1}$ ).
In this case, for each cycle $C$ in $G$, either $E(C) \cap\left\{e_{0}, e_{1}\right\}=\emptyset$ or $\left\{e_{0}, e_{1}\right\} \subseteq E(C)$, implying that $\ell_{G}\left(e_{0}\right)=\ell_{G}\left(e_{1}\right)=\ell(G, T)$ is odd.

Case 2: $\ell_{G^{\prime}}\left(e_{0}\right)$ is odd.
In this case, $\ell_{G^{\prime}}\left(e_{0}\right) \leq \ell\left(G^{\prime}, T\right) \leq \ell(G, T)=\ell_{G}\left(e_{1}\right)$. If $\ell_{G}\left(e_{0}\right)<\ell_{G^{\prime}}\left(e_{0}\right)$, then $e_{0}$ is contained in a cycle $C$ in $G$ with $|E(C)|=\ell_{G}\left(e_{0}\right)$. Since $|E(C)|=\ell_{G}\left(e_{0}\right)<\ell_{G^{\prime}}\left(e_{0}\right), C$ is not in $G^{\prime}$ and thus $e_{1} \in E(C)$, implying that $\ell_{G}\left(e_{1}\right) \leq|E(C)|$. Hence

$$
\ell_{G}\left(e_{1}\right) \leq|E(C)|=\ell_{G}\left(e_{0}\right)<\ell_{G^{\prime}}\left(e_{0}\right) \leq \ell\left(G^{\prime}, T\right) \leq \ell_{G}\left(e_{1}\right)
$$

a contradiction. Hence $\ell_{G}\left(e_{0}\right)=\ell_{G^{\prime}}\left(e_{0}\right)$ is odd. Obviously, $\ell_{G}\left(e_{0}\right)=\ell_{G^{\prime}}\left(e_{0}\right) \leq \ell\left(G^{\prime}, T\right) \leq$ $\ell(G, T)$.

Hence the result holds.
Remark: From the proof of Proposition 11, for any $(G, T) \in \mathcal{G} \mathcal{T}, G$ can be obtained from $T$ by adding a sequence of edges. Actually, $G$ is the last graph $G_{k}$ in a sequence of graphs $G_{0}, G_{1}, G_{2}, \cdots, G_{k}$, where $k=|E(G)|-|V(G)|+1, G_{0}=T$ and each graph $G_{i+1}$, where $0 \leq i \leq k-1$, can be obtained from $G_{i}$ by adding a new edge joining two nonadjacent vertices $u$ and $v$ in $G_{i}$ in which there is a shortest $(u, v)$-path $P$ such that $|E(P)| \geq \ell_{G_{i}}(e)-1$ for each $e \in E\left(G_{i}\right) \backslash E(T)$ and $|E(P)|>\ell_{G_{i}}(e)-1$ for each $e \in E(P) \backslash E(T)$.

### 4.2 Proof of Theorem 4

We are now going to prove Theorem 4.
Proof of Theorem \& Let $G \in \mathcal{G}$ and $n=|V(G)|$. The result is trivial for $n=1$. Now assume that $n \geq 2$. By definition, $(G, T) \in \mathcal{G \mathcal { T }}$ for some spanning tree $T$ of $G$. Thus, for each $e \in E(G) \backslash E(T), \ell_{G}(e)$ is odd and $e$ is contained in a cycle $C$ of length $\ell_{G}(e)$ with the property that $\ell_{G}\left(e^{\prime}\right)<\ell_{G}(e)$ holds for each $e^{\prime} \in E(C) \backslash(E(T) \cup\{e\})$.

Let $\mathcal{H}=(L, H)$ be any $m$-fold cover of $G$ such that $H \not \not H_{0}(G, m)$. As $T$ is a spanning tree of $G$, by Proposition 21 in [14], we may assume that $L(v)=\{(v, j): j \in \llbracket m \rrbracket\}$ for each $v \in V(G)$ and $E_{H}(L(u), L(v)) \subseteq\{(u, j)(v, j): j \in \llbracket m \rrbracket\}$ for each $u v \in E(T)$. Note that relabeling vertices in $L(u)$ for any $u \in V(G)$ does not affect the condition that $H \not \not H_{0}(G, m)$.

If $E_{H}(L(u), L(v)) \subseteq\{(u, j)(v, j): j \in \llbracket m \rrbracket\}$ holds for each $u v \in E(G) \backslash E(T)$, then $H \not \approx$ $H_{0}(G, m)$ implies that $H$ is a proper spanning subgraph of $H_{0}(G, m)$. Without loss of generality, assume that $(u, 1)(v, 1) \notin E_{H}(L(u), L(v))$ for some edge $u v \in E(G)$. Then, for $m \geq n-2$,

$$
P_{D P}(G, \mathcal{H})-P(G, m) \geq P_{D P}\left(G-\{u, v\}, \mathcal{H}^{\prime}\right)>0
$$

where $\mathcal{H}^{\prime}=\left(L^{\prime}, H^{\prime}\right)$ is the $(m-1)$-fold cover of $G-\{u, v\}, L^{\prime}(w)=L(w) \backslash\{(w, 1)\}$ for all $w \in V(G) \backslash\{u, v\}$ and $H^{\prime}=H\left[\cup_{w \in V(G) \backslash\{u, v\}} L^{\prime}(w)\right]$. Thus, the result holds in this case.

Now assume that $E_{H}(L(u), L(v)) \nsubseteq\{(u, j)(v, j): j \in \llbracket m \rrbracket\}$ for some $u v \in E(G) \backslash E(T)$. By definition, adding any possible edge to $H$ does not increase the value of $P_{D P}(G, \mathcal{H})$. Thus, we can assume that $\left|E_{H}(L(u), L(v))\right|=m$ for each edge $u v \in E(G)$, and in particular, $E_{H}(L(u), L(v))=\{(u, j)(v, j): j \in \llbracket m \rrbracket\}$ for each $u v \in E(T)$.

For each $e=u v \in E(G)$, let $X_{e}=E_{H}(L(u), L(v)) \backslash\{(u, j)(v, j): j \in \llbracket m \rrbracket\}$. As $\left|E_{H}(L(u), L(v))\right|=$ $m, X_{e}=\emptyset$ if and only if $E_{H}(L(u), L(v))=\{(u, j)(v, j): j \in \llbracket m \rrbracket\}$. By the assumption above, $X_{e}=\emptyset$ for each $e \in E(T)$, but $X_{e} \neq \emptyset$ for some edge $e \in E(G) \backslash E(T)$. For $s \geq 3$, let

$$
\begin{equation*}
\mathscr{X}_{s}=\bigcup_{\substack{e \in E(G) E(T) \\ \ell_{G}(e)=s}} X_{e} . \tag{15}
\end{equation*}
$$

By the given condition, $\ell_{G}(e) \geq 3$ is odd for each $e \in E(G) \backslash E(T)$, implying that $\mathscr{X}_{s}=\emptyset$ for each even $s \geq 4$. Now assume that $r$ is the minimum integer such that $\mathscr{X}_{r} \neq \emptyset$. So $r \geq 3$ and $r$ is odd. We will prove Theorem 4 by an approach similar to the proof of Theorem 7 in [15].

We first find an expression for $P_{D P}(G, \mathcal{H})$ which is similar to (11) for $P(G, m)$. Let $\mathscr{S}$ be the set of subsets $S$ of $V(H)$ with $|S \cap L(v)|=1$ for each $v \in V(G)$. For each edge $e=u v \in E(G)$,
let $\mathscr{S}_{e}$ be the set of $S \in \mathscr{S}$ such that the two vertices in $S \cap(L(u) \cup L(v))$ are adjacent in $H$. For each $A \subseteq E(G)$, let

$$
\begin{equation*}
\mathscr{S}_{A}=\bigcap_{e \in A} \mathscr{S}_{e} \tag{16}
\end{equation*}
$$

As $P_{D P}(G, \mathcal{H})=|\mathscr{S}|-\left|\cup_{e \in E(G)} \mathscr{S}_{e}\right|$, by the inclusion-exclusion principle, we have

$$
\begin{equation*}
P_{D P}(G, \mathcal{H})=\sum_{A \subseteq E(G)}(-1)^{|A|}\left|\mathscr{S}_{A}\right| \tag{17}
\end{equation*}
$$

For each $U \subseteq V(G)$, let $\left.\mathscr{S}\right|_{U}$ be the set of subsets $S$ of $V(H)$ such that $|S \cap L(v)|=1$ for each $v \in U$ and $S \cap L(v)=\emptyset$ for each $v \in V(G) \backslash U$. For any subgraph $G_{0}$ of $G$ and $\left.S \in \mathscr{S}\right|_{V\left(G_{0}\right)}$, let $\left.H[S]\right|_{G_{0}}$ denote the spanning subgraph of $H[S]$ with edge set $\left\{\left(u, j_{1}\right)\left(v, j_{2}\right) \in E(H): u v \in\right.$ $\left.E\left(G_{0}\right), u, v \in V\left(G_{0}\right),\left(u, j_{1}\right),\left(v, j_{2}\right) \in S\right\}$. Equivalently, $\left.H[S]\right|_{G_{0}}$ can be obtained from $H[S]$ by deleting all those edges $\left(u, j_{1}\right)\left(v, j_{2}\right)$ in $H[S]$ with $u v \notin E\left(G_{0}\right)$. Clearly, $\left.H[S]\right|_{G_{0}}$ is $H[S]$ when $G_{0}$ is a subgraph of $G$ induced by $V\left(G_{0}\right)$. For any $\left.S \in \mathscr{S}\right|_{V\left(G_{0}\right)},\left|E\left(\left.H[S]\right|_{G_{0}}\right)\right| \leq\left|E\left(G_{0}\right)\right|$ holds, and the following statements are equivalent:
(a) $\left.H[S]\right|_{G_{0}} \cong G_{0}$;
(b) $\left|E\left(\left.H[S]\right|_{G_{0}}\right)\right|=\left|E\left(G_{0}\right)\right|$; and
(c) for each $u v \in E\left(G_{0}\right)$, the two vertices in $S \cap(L(u) \cup L(v))$ are adjacent in $H$.

Let $\mathscr{H}\left(G_{0}\right)$ be the set of subgraphs $\left.H[S]\right|_{G_{0}}$ of $H$, where $\left.S \in \mathscr{S}\right|_{V\left(G_{0}\right)}$, such that $\left.H[S]\right|_{G_{0}} \cong G_{0}$. Recall that for $A \subseteq E(G), G\langle A\rangle$ is the spanning subgraph of $G$ with edge set $A$, and $c(A)$ is the number of components of $G\langle A\rangle$. By the definition of $\mathscr{S}_{A}$, the following claim holds.

Claim 1. For any $A \subseteq E(G)$, if $G_{1}, G_{2}, \cdots, G_{c(A)}$ are the components of $G\langle A\rangle$, then

$$
\left|\mathscr{S}_{A}\right|=\prod_{i=1}^{c(A)}\left|\mathscr{H}\left(G_{i}\right)\right| .
$$

Claim 2. Let $G_{0}$ be a connected subgraph of $G$. If $\left.H\left[S_{1}\right]\right|_{G_{0}},\left.H\left[S_{2}\right]\right|_{G_{0}} \in \mathscr{H}\left(G_{0}\right)$, where $S_{1},\left.S_{2} \in \mathscr{S}\right|_{V\left(G_{0}\right)}$, then either $S_{1}=S_{2}$ or $S_{1} \cap S_{2}=\emptyset$. Hence $\left|\mathscr{H}\left(G_{0}\right)\right| \leq m$, where the equality holds if $X_{e}=\emptyset$ holds for each edge $e \in E\left(G_{0}\right)$.

Proof. Suppose that $\left.H\left[S_{1}\right]\right|_{G_{0}},\left.H\left[S_{2}\right]\right|_{G_{0}} \in \mathscr{H}\left(G_{0}\right)$. Then, $\left.\left.H\left[S_{1}\right]\right|_{G_{0}} \cong H\left[S_{2}\right]\right|_{G_{0}} \cong G_{0}$, implying that whenever $u v \in E\left(G_{0}\right)$, the two vertices in $S_{i} \cap(L(u) \cup L(v))$ are adjacent in $H$ for $i=1,2$. Let $u v$ be any edge in $G_{0}$. As $E_{H}(L(u), L(v))$ is a matching of $H$ of size $m$, each vertex in $L(u)$ is only adjacent to one vertex in $L(v)$. If $\left.H\left[S_{1}\right]\right|_{G_{0}}$ and $\left.H\left[S_{2}\right]\right|_{G_{0}}$ have a common vertex
in $L(u)$, then $\left.H\left[S_{1}\right]\right|_{G_{0}}$ and $\left.H\left[S_{2}\right]\right|_{G_{0}}$ must have a common vertex in $L(v)$. As $G_{0}$ is connected, we conclude that either $S_{1} \cap S_{2}=\emptyset$ or $S_{1}=S_{2}$. Thus, $\left|\mathscr{H}\left(G_{0}\right)\right| \leq m$ holds.

If $X_{e}=\emptyset$ holds for each edge $e \in E\left(G_{0}\right)$, then $\left.H\left[S_{j}\right]\right|_{G_{0}} \in \mathscr{H}\left(G_{0}\right)$ for each $j \in \llbracket m \rrbracket$, where $S_{j}=\left\{(u, j): u \in V\left(G_{0}\right)\right\}$. Thus, $\left|\mathscr{H}\left(G_{0}\right)\right|=m$ and Claim 2 holds.
$\square$
Claim 3. Let $G_{0}$ be a connected subgraph of $G$. If $X_{e}=\emptyset$ holds for each $e \in E\left(G_{0}\right)$ that is not a bridge of $G_{0}$, then $\left|\mathscr{H}\left(G_{0}\right)\right|=m$.

Proof. Assume that $X_{e}=\emptyset$ holds for each $e \in E\left(G_{0}\right)$ that is not a bridge of $G_{0}$. Let $B$ be any block of $G_{0}$. If $B$ is trivial (i.e., it consists of a bridge $e=u v$ of $G_{0}$ only), then, it is clear that $\mathscr{H}(B)$ has exactly $m$ members which correspond to the $m$ edges in $E_{H}(L(u), L(v))$. If $B$ is an non-trivial block of $G_{0}$, we have $X_{e}=\emptyset$ for each $e \in E(B)$, and $\mathscr{H}(B)$ has exactly $m$ members $\left.H\left[S_{j}\right]\right|_{B}$ for $j \in \llbracket m \rrbracket$, where $S_{j}=\{(v, j): v \in V(B)\}$. Thus, $\left|\mathscr{H}\left(G_{0}\right)\right|=m$ if $G_{0}$ has only one block.

Suppose that $G_{0}$ has at least two blocks and $B_{0}$ is a block of $G_{0}$ which has only one vertex $u$ shared by other blocks of $G_{0}$. Let $G^{\prime}$ denote $G_{0}-\left(V\left(B_{0}\right) \backslash\{u\}\right)$. Assume that both $\mathscr{H}\left(G^{\prime}\right)$ and $\mathscr{H}\left(B_{0}\right)$ have exactly $m$ members. Each member $\left.H\left[S^{\prime}\right]\right|_{G^{\prime}}$ of $\mathscr{H}\left(G^{\prime}\right)$ can be extended to exactly one member of $\mathscr{H}\left(G_{0}\right)$ by combining $\left.H\left[S^{\prime}\right]\right|_{G^{\prime}}$ with the member in $\mathscr{H}\left(B_{0}\right)$ which shares a vertex in $L(u)$ with $\left.H\left[S^{\prime}\right]\right|_{G^{\prime}}$. Hence $\left|\mathscr{H}\left(G_{0}\right)\right|=m$.

Claim 3 holds.
The next claim follows from Claims 1, 2 and 3 directly.
Claim 4. For each $A \subseteq E(G),\left|\mathscr{S}_{A}\right| \leq m^{c(A)}$ holds. If $X_{e}=\emptyset$ holds for each $e \in A$ that is not a bridge of $G\langle A\rangle$, then $\left|\mathscr{S}_{A}\right|=m^{c(A)}$.

By Claim 4, the next claim follows.
Claim 5. For any $A \subseteq E(G)$, if $|A|$ is odd, $(-1)^{|A|}\left(\left|\mathscr{S}_{A}\right|-m^{c(A)}\right)=m^{c(A)}-\left|\mathscr{S}_{A}\right| \geq 0$.
Let $\mathscr{E}$ be the set of subsets $A$ of $E(G)$ such that $X_{e} \neq \emptyset$ holds for some $e \in A$ that is not a bridge of $G\langle A\rangle$. Note that such an edge $e$ may be not unique. By (11), (17) and Claim 4, we have

$$
\begin{equation*}
P_{D P}(G, \mathcal{H})-P(G, m)=\sum_{A \in \mathscr{E}}(-1)^{|A|}\left(\left|\mathscr{S}_{A}\right|-m^{c(A)}\right) . \tag{18}
\end{equation*}
$$

The following claim presents some properties of members in $\mathscr{E}$.
Claim 6. For each $A \in \mathscr{E}, G\langle A\rangle$ has a component $G_{1}$ and an edge $e$ in some cycle of $G_{1}$ with $X_{e} \neq \emptyset$. Furthermore, $\left|V\left(G_{1}\right)\right| \geq r$ and $c(A) \leq n-r+1$, and $|A|=r$ whenever $c(A)=n-r+1$.

Proof. As $A \in \mathscr{E}, G\langle A\rangle$ has an edge $e$ that is not a bridge of $G\langle A\rangle$ such that $X_{e} \neq \emptyset$. Let $G_{1}$ be the component of $G\langle A\rangle$ containing $e$. As $X_{e} \neq \emptyset$, we have $\ell_{G}(e) \geq r$. Thus, each cycle in $G_{1}$ containing $e$ has at least $r$ edges, implying that $\left|V\left(G_{1}\right)\right| \geq r$, and hence $c(A) \leq n-r+1$.

Assume that $c(A)=n-r+1$. Then $\left|V\left(G_{1}\right)\right| \geq r$ implies that $\left|V\left(G_{1}\right)\right|=r$ and all other components of $G\langle A\rangle$ are isolated vertices. As $e$ is in a cycle $C$ of length $r$ in $G_{1}$ and each cycle containing $e$ is of length at least $r,\left|V\left(G_{1}\right)\right|=r$ implies that $G_{1} \cong C$ and $|A|=r$.

Claim 6 holds.
Assume that $\left\{e_{1}, e_{2}, \cdots, e_{s}\right\}$ is the set of edges in $E(G) \backslash E(T)$ with $\ell_{G}\left(e_{i}\right)=r$. By the given condition, for each $i \in \llbracket s \rrbracket, e_{i}$ is contained in a cycle, denoted by $C_{i}$, such that $\left|V\left(C_{i}\right)\right|=r$ and $\ell_{G}\left(e^{\prime}\right)<r$ for each $e^{\prime} \in E\left(C_{i}\right) \backslash\left(E(T) \cup\left\{e_{i}\right\}\right)$. Thus, $E\left(C_{i}\right) \cap\left\{e_{j}: j \in \llbracket s \rrbracket\right\}=\left\{e_{i}\right\}$ for each $i \in \llbracket s \rrbracket$, implying that $C_{1}, C_{2}, \cdots, C_{s}$ are pairwise distinct.

Claim 7. For each $i \in \llbracket s \rrbracket,\left|\mathscr{H}\left(C_{i}\right)\right|=m-\left|X_{e_{i}}\right|$.

Proof. Without loss of generality, let $V\left(C_{i}\right)=\left\{v_{1}, v_{2}, \cdots, v_{r}\right\}$ and let $v_{1} v_{2} \cdots v_{r}$ be the path $C_{i}-e_{i}$ in $G$. Obviously, $e_{i}$ is the edge $v_{1} v_{r}$. By the definition of $r, X_{e^{\prime}}=\emptyset$ holds for each $e^{\prime} \in E(G) \backslash E(T)$ with $\ell_{G}\left(e^{\prime}\right)<r$. By the given condition on $C_{i}, X_{e^{\prime}}=\emptyset$ holds for each edge $e^{\prime}$ in the path $v_{1} v_{2} \cdots v_{r}$, implying that the subgraph obtained from $H[S]$, where $S=\left\{\left(v_{q}, j\right): q \in[r], j \in \llbracket m \rrbracket\right\}$, by removing all edges in $E_{H}\left(L\left(v_{1}\right), L\left(v_{r}\right)\right)$, consists of $m$ disjoint paths $\left(v_{1}, j\right)\left(v_{2}, j\right) \cdots\left(v_{r}, j\right)$ for $j=1,2, \cdots, m$. Assume that

$$
\begin{equation*}
E_{H}\left(L\left(v_{1}\right), L\left(v_{r}\right)\right) \backslash X_{e_{i}}=\left\{\left(v_{1}, j\right)\left(v_{r}, j\right): 1 \leq j \leq m-\left|X_{e_{i}}\right|\right\} . \tag{19}
\end{equation*}
$$

Then $\left(v_{1}, j\right)\left(v_{r}, j\right) \notin E(H)$ for each $j$ with $m-\left|X_{e_{i}}\right|<j \leq m$. Let $S_{j}=\left\{\left(v_{q}, j\right): q \in[r]\right\}$ for each $j \in \llbracket m \rrbracket$. Clearly, $\left.H\left[S_{j}\right]\right|_{C_{i}} \cong C_{i}$ if and only if $1 \leq j \leq m-\left|X_{e_{i}}\right|$. On the other hand, for any $\left.S^{\prime} \in \mathscr{S}\right|_{V\left(C_{i}\right)}$, if $\left.H\left[S^{\prime}\right]\right|_{C_{i}} \cong C_{i}$, then $\left.H\left[S^{\prime}\right]\right|_{C_{i}}$ must contain a path $\left(v_{1}, j\right)\left(v_{2}, j\right) \cdots\left(v_{r}, j\right)$ for some $j \in \llbracket m \rrbracket$, implying that $S^{\prime}=S_{j}$ for some $j \in \llbracket m \rrbracket$. Thus, $\left|\mathscr{H}\left(C_{i}\right)\right|=m-\left|X_{e_{i}}\right|$. Now we are going to apply Claims 5 and 6 to prove the next claim.

Claim 8. The following result holds:

$$
\begin{equation*}
\sum_{\substack{A \in \mathscr{E} \\ c(A)=n-r+1}}(-1)^{|A|}\left(\left|\mathscr{S}_{A}\right|-m^{c(A)}\right) \geq\left|\mathscr{X}_{r}\right| m^{n-r} \tag{20}
\end{equation*}
$$

Proof. By Claim 6, $|A|=r$ for each $A \in \mathscr{E}$ with $c(A)=n-r+1$. As $r$ is odd, by Claim 5,
for any $\mathscr{E}_{0} \subseteq\{A \in \mathscr{E}: c(A)=n-r+1\}$, we have

$$
\begin{equation*}
\sum_{\substack{A \in \mathscr{E} \\ c(A)=n-r+1}}(-1)^{|A|}\left(\left|\mathscr{S}_{A}\right|-m^{c(A)}\right) \geq \sum_{A \in \mathscr{E}_{0}}\left(m^{n-r+1}-\left|\mathscr{S}_{A}\right|\right) \tag{21}
\end{equation*}
$$

For each $i \in \llbracket s \rrbracket, G\left\langle E\left(C_{i}\right)\right\rangle$ consists of exactly $n-r+1$ components, i.e., $C_{i}$ and $n-r$ isolated vertices in $V(G) \backslash V\left(C_{i}\right)$. By Claim 7, $\left|\mathscr{H}\left(C_{i}\right)\right|=m-\left|X_{e_{i}}\right|$. Thus, by Claim 1,

$$
\begin{equation*}
\left|\mathscr{S}_{E\left(C_{i}\right)}\right|=\left|\mathscr{H}\left(C_{i}\right)\right| m^{n-r}=\left(m-\left|X_{e_{i}}\right|\right) m^{n-r}=m^{n-r+1}-\left|X_{e_{i}}\right| m^{n-r} . \tag{22}
\end{equation*}
$$

Let $\mathscr{E}_{0}=\left\{E\left(C_{i}\right): i \in \llbracket s \rrbracket\right\}$. By (21) and (22),

$$
\begin{equation*}
\sum_{\substack{A \in \mathscr{E} \\ c(A)=n-r+1}}(-1)^{|A|}\left(\left|\mathscr{S}_{A}\right|-m^{c(A)}\right) \geq \sum_{i=1}^{s}\left(m^{n-r+1}-\left|\mathscr{S}_{E\left(C_{i}\right)}\right|\right)=\sum_{i=1}^{s}\left|X_{e_{i}}\right| m^{n-r}=\left|\mathscr{X}_{r}\right| m^{n-r} \tag{23}
\end{equation*}
$$

Claim 8 holds.
Claim 9. For any subgraph $G_{1}$ of $G$, if $\ell_{G}(e) \leq r$ for each edge $e \in E\left(G_{1}\right)$, then $\left|\mathscr{H}\left(G_{1}\right)\right| \geq$ $m-2\left|\mathscr{X}_{r}\right|$.

Proof. For each $j \in \llbracket m \rrbracket$, let $S_{j}=\left\{(u, j): u \in V\left(G_{1}\right)\right\}$ and $Q_{j}=\left.H\left[S_{j}\right]\right|_{G_{1}}$. By the definition of $\left.H\left[S_{j}\right]\right|_{G_{1}}, Q_{j} \in \mathscr{H}\left(G_{1}\right)$ if and only if $(u, j)(v, j) \in E(H)$ for each $u v \in E\left(G_{1}\right)$.

Let $S=\cup_{j \in \llbracket m \rrbracket} S_{j}$, and let $\psi: S \rightarrow\{0,1\}$ be the mapping defined below:

$$
\psi((u, j))= \begin{cases}1, & \text { if }(u, j)\left(v, j^{\prime}\right) \in E(H) \text { for some } v \in N_{G_{1}}(u) \text { and } j^{\prime} \neq j  \tag{24}\\ 0, & \text { otherwise }\end{cases}
$$

If $\psi((u, j))=1$, by definition, $(u, j)$ is one end of some edge $(u, j)\left(v, j^{\prime}\right)$ of $X_{e}$, where $e=u v \in$ $E\left(G_{1}\right)$. Thus,

$$
\begin{align*}
\sum_{(u, j) \in S} \psi((u, j)) & \leq \sum_{e \in E\left(G_{1}\right)} \sum_{(u, j)\left(v, j^{\prime}\right) \in X_{e}}\left(\psi((u, j))+\psi\left(\left(v, j^{\prime}\right)\right)\right) \\
& =2 \sum_{e \in E\left(G_{1}\right)}\left|X_{e}\right| \\
& \leq 2\left|\mathscr{X}_{r}\right| \tag{25}
\end{align*}
$$

where the last inequality follows from the facts that for each $e \in E\left(G_{1}\right), \ell_{G}(e) \leq r$ holds, and $\ell_{G}(e)<r$ implies that $X_{e}=\emptyset$.

By the definition of $\psi, Q_{j} \not \approx G_{1}$ if and only if $\psi((u, j))=1$ for some $u \in V\left(G_{1}\right)$. Then, by (25), there are at most $2\left|\mathscr{X}_{r}\right|$ numbers $j \in \llbracket m \rrbracket$ such that $Q_{j} \not \approx G_{1}$, implying that

$$
\begin{equation*}
\left|\mathscr{H}\left(G_{1}\right)\right| \geq m-2\left|\mathscr{X}_{r}\right| . \tag{26}
\end{equation*}
$$

Thus, Claim 9 holds.
Claim 10. For any $A \in \mathscr{E}$ with $c(A)=n-r$, we have $\left|\mathscr{S}_{A}\right| \geq\left(m-2\left|\mathscr{X}_{r}\right|\right) m^{m-r-1}$.

Proof. Let $A \in \mathscr{E}$ with $c(A)=n-r$. By Claim 6, $G\langle A\rangle$ has a component $G_{1}$ with $\left|V\left(G_{1}\right)\right| \geq$ $r$. Let $G_{2}, \cdots, G_{n-r}$ be the components of $G\langle A\rangle$ different from $G_{1}$ with $\left|V\left(G_{2}\right)\right| \geq \cdots \geq$ $\left|V\left(G_{n-r}\right)\right|$. As $c(A)=n-r$, one of the two cases below happens:
(i). $\left|V\left(G_{1}\right)\right|=r,\left|V\left(G_{2}\right)\right|=2$ and $\left|V\left(G_{i}\right)\right|=1$ for all $3 \leq i \leq n-r$, or
(ii). $\left|V\left(G_{1}\right)\right|=r+1$ and $\left|V\left(G_{i}\right)\right|=1$ for all $2 \leq i \leq n-r$.

In both Cases (i) and (ii) above, by Claim 3, $\left|\mathscr{H}\left(G_{i}\right)\right|=m$ holds for all $i=2,3, \cdots, n-r$. By Claim 1, it remains to show that $\left|\mathscr{H}\left(G_{1}\right)\right| \geq m-2\left|\mathscr{X}_{r}\right|$.

In both cases above, by Claim 6, there is an edge $e$ with $X_{e} \neq \emptyset$ which is in some cycle of $G_{1}$. Such an edge may be not unique. As $X_{e} \neq \emptyset$, we have $\ell_{G}(e) \geq r$. Thus, each cycle in $G_{1}$ containing $e$ must be of length at least $r$. In Case (i), $G_{1}$ can only be a cycle of length $r$. In Case (ii), it can be verified that $G_{1}$ is one of the graphs in Figure 3,


Figure 3: Possible structures of $G_{1}$ when $\left|V\left(G_{1}\right)\right|=r+1$
As each cycle in $G_{1}$ is of length at most $r+1$, for each edge $e^{\prime}$ in cycles of $G_{1}$, we have $\ell_{G}\left(e^{\prime}\right) \leq r+1$. As $\ell_{G}\left(e^{\prime}\right)$ is odd, we have $\ell_{G}\left(e^{\prime}\right) \leq r$ for such edges $e^{\prime}$. Thus, if $G_{1}$ contains an edge $e^{\prime}$ with $\ell_{G}\left(e^{\prime}\right) \geq r+2$, then $e^{\prime}$ must be a bridge of $G_{1}$.

If $G_{1}$ has no bridge, then $\ell_{G}(e) \leq r$ for each $e \in E\left(G_{1}\right)$. By Claim 9, $\mathscr{H}\left(G_{1}\right) \geq m-2\left|\mathscr{X}_{r}\right|$.

If $G_{1}$ has bridges, then $G_{1}$ is the graph in Figure 3 (a), where $G_{1}-v_{r+1}$ is a cycle. By Claim 9 again, $\mathscr{H}\left(G_{1}-v_{r+1}\right) \geq m-2\left|\mathscr{X}_{r}\right|$. Clearly, each member of $\mathscr{H}\left(G_{1}-v_{r+1}\right)$ can be extended to a member of $\mathscr{H}\left(G_{1}\right)$, even when $X_{v_{r} v_{r+1}} \neq \emptyset$. Thus, Claim 10 also holds in this case.

Claim 10 is proved.
$\square$
For any $k \in \llbracket n-r \rrbracket$, let $\phi_{k}$ be the number of elements $A$ of $\mathscr{E}$ such that $c(A)=k$ and $|A|$ is even.

Claim 11. The following inequality holds:

$$
\begin{equation*}
\sum_{\substack{A \in \mathscr{E} \\ c(A)=n-r}}(-1)^{|A|}\left(\left|\mathscr{S}_{A}\right|-m^{c(A)}\right) \geq-2 \phi_{n-r}\left|\mathscr{X}_{r}\right| m^{n-r-1} \tag{27}
\end{equation*}
$$

Proof. By Claim 5,

$$
\begin{equation*}
\sum_{\substack{A \in \mathscr{E} \\ c(A)=n-r}}(-1)^{|A|}\left(\left|\mathscr{S}_{A}\right|-m^{c(A)}\right) \geq \sum_{\substack{A \in \mathcal{E}, c(A)=n-r \\|A| \text { is even }}}\left(\left|\mathscr{S}_{A}\right|-m^{c(A)}\right) \tag{28}
\end{equation*}
$$

For each $A \in \mathscr{E}$ with $c(A)=n-r$, by Claim 10,

$$
\begin{equation*}
\left|\mathscr{S}_{A}\right|-m^{c(A)} \geq\left(m-2\left|\mathscr{X}_{r}\right|\right) m^{n-r-1}-m^{n-r}=-2\left|\mathscr{X}_{r}\right| m^{n-r-1} \tag{29}
\end{equation*}
$$

Then Claim 11 follows from the definition of $\phi_{n-r}$.
Claim 12. For each $k \in \llbracket n-r-1 \rrbracket$, we have

$$
\begin{equation*}
\sum_{\substack{A \in \mathscr{E} \\ c(A)=k}}(-1)^{|A|}\left(\left|\mathscr{S}_{A}\right|-m^{c(A)}\right) \geq-\phi_{k} m^{k} \tag{30}
\end{equation*}
$$

Proof. For each $A \in \mathscr{E}$ with $c(A)=k$, if $|A|$ is even,

$$
\begin{equation*}
(-1)^{|A|}\left(\left|\mathscr{S}_{A}\right|-m^{c(A)}\right)=\left|\mathscr{S}_{A}\right|-m^{c(A)} \geq-m^{k} \tag{31}
\end{equation*}
$$

Thus Claim 12 follows from Claim 5 and the definition of $\phi_{k}$.
Let $\phi_{k}^{\prime}$ be the number of subsets $A \subseteq E(G)$ such that $c(A)=k, G\langle A\rangle$ is not a forest and $|A|$ is even. Obviously, $\phi_{k}^{\prime} \geq \phi_{k}$. By the expression of (18) and Claims 6, 8, 11 and 12,

$$
P_{D P}(G, \mathcal{H})-P(G, m) \geq\left|\mathscr{X}_{r}\right| m^{n-r}-2 \phi_{n-r}\left|\mathscr{X}_{r}\right| m^{n-r-1}-\sum_{k=1}^{n-r-1} \phi_{k} m^{k}
$$

$$
\begin{align*}
& \geq m^{n-r}-2 \phi_{n-r} m^{n-r-1}-\sum_{k=1}^{n-r-1} \phi_{k} m^{k} \\
& \geq m^{n-r}-2 \phi_{n-r}^{\prime} m^{n-r-1}-\sum_{k=1}^{n-r-1} \phi_{k}^{\prime} m^{k} \tag{32}
\end{align*}
$$

where the second inequality holds when $m \geq 2 \phi_{n-r}$. As $\phi_{k}^{\prime}$ is independent of the value of $m$, by (32), there must be a number $M_{r} \in \mathbb{N}$ such that $P_{D P}(G, \mathcal{H})-P(G, m)>0$ for all $m \geq M_{r}$.

Let $M=\max \left\{M_{r}: 3 \leq r \leq n, r\right.$ is odd $\}$. Then, we conclude that for any $m \geq M$ and any $m$-fold cover $\mathcal{H}=(L, H)$ of $G$, if $H \not \not H_{0}(G, m)$, then $P_{D P}(G, \mathcal{H})-P(G, m)>0$ holds.

Hence Theorem 4 holds.
We end this section with an application of Theorem 4 to the generalized $\theta$-graphs. For any $k$ numbers $a_{1}, a_{2}, \cdots, a_{k} \in \mathbb{N}$, where $k \geq 2$, let $G=\Theta_{a_{1}, a_{2}, \cdots, a_{k}}$ denote the generalized $\theta$ graph obtained by connecting two distinct vertices with $k$ internally disjoint paths of lengths $a_{1}, a_{2}, \cdots, a_{k}$ respectively.

Assume that $a_{1} \leq a_{2} \leq \cdots \leq a_{k}$ and $a_{1}+a_{2} \geq 3$. Halberg, Kaul, Liu, Mudrock, Shin and Thomason 12 showed that $P_{D P}\left(\Theta_{a_{1}, a_{2}, \cdots, a_{k}}\right) \approx P\left(\Theta_{a_{1}, a_{2}, \cdots, a_{k}}\right)$ if $a_{1}+a_{i}$ is odd for each $i \in \llbracket k \rrbracket \backslash\{1\}$, and $P_{D P}\left(\Theta_{a_{1}, a_{2}, \cdots, a_{k}}\right)<P\left(\Theta_{a_{1}, a_{2}, \cdots, a_{k}}\right)$ otherwise.

In the case that $a_{1}+a_{i}$ is odd for each $i \in \llbracket k \rrbracket \backslash\{1\}, \Theta_{a_{1}, a_{2}, \cdots, a_{k}}$ belongs to the set $\mathcal{G}_{0}$, and thus $\Theta_{a_{1}, a_{2}, \cdots, a_{k}} \in \mathcal{D} \mathcal{P}^{*}$ by Theorem 4, implying that $P_{D P}\left(\Theta_{a_{1}, a_{2}, \cdots, a_{k}}\right) \approx P\left(\Theta_{a_{1}, a_{2}, \cdots, a_{k}}\right)$.

## 5 Proof of Theorem 5

For a chordal graph $G, P_{D P}(G, m)=P(G, m)$ for all $m \in \mathbb{N}$ (see [14]), and thus, $P_{D P}(G) \approx$ $P(G)$ holds. In the following, we first generalize this conclusion to some non-chordal graphs containing simplicial vertices.

Proposition 12. Let $u$ be a simplicial vertex of $G$. For each $m \in \mathbb{N}$ with $m \geq d(u)+1$, if $P_{D P}(G-u, m)=P(G-u, m)$, then $P_{D P}(G, m)=P(G, m)$.

Proof. Assume that $P_{D P}(G-u, m)=P(G-u, m)$. For any $m$-fold cover $\mathcal{H}=(L, H)$ of $G$,
$P_{D P}(G, \mathcal{H}) \geq(m-d(u)) P_{D P}\left(G-u, \mathcal{H}_{u}^{\prime}\right) \geq(m-d(u)) P_{D P}(G-u, m)=(m-d(u)) P(G-u, m)$,
where $\mathcal{H}_{u}^{\prime}=\left(L^{\prime}, H^{\prime}\right)$ is the $m$-fold cover of $G-u$ with $L^{\prime}(w)=L(w)$ for each $w \in V(G) \backslash\{u\}$
and $H^{\prime}=H-L(u)$. Thus, $P_{D P}(G, m) \geq(m-d(u)) P(G-u, m)=P(G, m)$ by (8). On the other hand, $P_{D P}(G, m) \leq P(G, m)$. Thus, the result follows.

The first part of Theorem 5 follows from Proposition 12 directly. In order to prove the second part of Theorem [5, we need to introduce some preliminary results.

For any cover $\mathcal{H}=(L, H)$ of $G$, let $\mathcal{I}(H)$ denote the set of independent sets $I$ in $H$ with $|I|=|V(G)|$. Thus, $P_{D P}(G, \mathcal{H})=|\mathcal{I}(H)|$. The coloring number of $G$, denoted by $\operatorname{col}(G)$, is the smallest integer $d$ for which there exists an ordering, $v_{1}, v_{2}, \cdots, v_{n}$ of the elements in $V(G)$, where $n=|V(G)|$, such that $\left|N_{G}\left(v_{i}\right) \cap\left\{v_{1}, v_{2}, \cdots, v_{i-1}\right\}\right|<d$ for each $i \in \llbracket n \rrbracket$. Obviously, $\chi_{D P}(G) \leq \operatorname{col}(G) \leq n$. If $|L(v)| \geq \operatorname{col}(G)$ for all $v \in V(G)$, then $\mathcal{I}(H) \neq \emptyset$.

The following fundamental property is important for the study of DP coloring.
Proposition 13. Let $\mathcal{H}=(L, H)$ be a cover of $G$ with $|L(v)| \geq|V(G)|$ for each $v \in V(G)$. Then, each independent set $A$ of $H$ is a subset of some set $I$ in $\mathcal{I}(H)$.

Proof. If $A=\emptyset$, then the conclusion follows from the the fact that $|V(G)| \geq \operatorname{col}(G)$.
Now assume that $A=\left\{\left(v_{i}, \pi_{i}\right): i \in \llbracket k \rrbracket\right\}$, where $k \geq 1$. Clearly, $v_{1}, v_{2}, \cdots, v_{k}$ are pairwise distinct. Let $\mathcal{H}^{\prime}=\left(L^{\prime}, H^{\prime}\right)$ be the cover of the subgraph $G^{\prime}=G-\left\{v_{i}: i \in \llbracket k \rrbracket\right\}$, where $L^{\prime}(v)=$ $L(v) \backslash N_{H}(A)$ for each $v \in V\left(G^{\prime}\right)$ and $H^{\prime}$ is the subgraph of $H$ induced by $\bigcup_{v \in V\left(G^{\prime}\right)} L^{\prime}(v)$.

Observe that $\left|L^{\prime}(v)\right| \geq|L(v)|-k \geq\left|V\left(G^{\prime}\right)\right|$ for each $v \in V\left(G^{\prime}\right)$. By the conclusion for $A=\emptyset$, there exists $I^{\prime} \in \mathcal{I}\left(H^{\prime}\right)$, implying that $I=A \cup I^{\prime} \in \mathcal{I}(H)$.

By Proposition [13, the following corollary is obtained.
Corollary 14. For any cover $\mathcal{H}=(L, H)$ of $G$ with $|L(v)| \geq|V(G)|$ for each $v \in V(G)$, if $\mathcal{H}^{\prime}=\left(L, H^{\prime}\right)$ is a cover of $G$, where $H^{\prime}$ is obtained from $H$ by removing any edge in some set $E_{H}\left(L\left(v_{1}\right), L\left(v_{2}\right)\right)$, where $v_{1} \neq v_{2}$, then $P_{D P}\left(G, \mathcal{H}^{\prime}\right)>P_{D P}(G, \mathcal{H})$.

For any $u \in V(G)$ and an $m$-fold cover $\mathcal{H}=(L, H)$ of $G$, let $\mathcal{H}_{u}^{\prime}=\left(L^{\prime}, H^{\prime}\right)$ be the cover of $G-u$, where $H^{\prime}=H-L(u)$ and $L^{\prime}(v)=L(v)$ for each $v \in V(G) \backslash\{u\}$. For any $I^{\prime} \in \mathcal{I}\left(H^{\prime}\right)$, let

$$
\mathcal{I}_{H}\left(I^{\prime}\right)=\left\{I^{\prime} \cup\{(u, i)\} \in \mathcal{I}(H):(u, i) \in L(u)\right\}
$$

Obviously, for $m \geq d(u)$ and $I^{\prime} \in \mathcal{I}\left(H^{\prime}\right),\left|\mathcal{I}_{H}\left(I^{\prime}\right)\right| \geq(m-d(u))$ holds, implying that for $m>d(u)$,

$$
\begin{align*}
P_{D P}(G, \mathcal{H}) & =|\mathcal{I}(H)|=\sum_{I^{\prime} \in \mathcal{I}\left(H^{\prime}\right)}\left|\mathcal{I}_{H}\left(I^{\prime}\right)\right| \geq \sum_{I^{\prime} \in \mathcal{I}\left(H^{\prime}\right)}(m-d(u)) \\
& =(m-d(u))\left|\mathcal{I}\left(H^{\prime}\right)\right|=(m-d(u)) P_{D P}\left(G-u, \mathcal{H}_{u}^{\prime}\right), \tag{34}
\end{align*}
$$

where $P_{D P}(G, \mathcal{H})>(m-d(u)) P_{D P}\left(G-u, \mathcal{H}_{u}^{\prime}\right)$ if $\left|\mathcal{I}_{H}\left(I^{\prime}\right)\right|>m-d(u)$ for some $I^{\prime} \in \mathcal{I}\left(H^{\prime}\right)$.
Proposition 15. Let $\mathcal{H}=(L, H)$ be an $m$-fold cover of $G$, where $m \geq|V(G)|$, and $u \in V(G)$. Then $P_{D P}(G, \mathcal{H}) \geq(m-d(u)) P_{D P}\left(G-u, \mathcal{H}_{u}^{\prime}\right)$, where the inequality is strict under each of the following conditions:
(i). $\left|E_{H}(L(u), L(v))\right| \leq m-1$ for some $v \in N_{G}(u)$; or
(ii). $N_{H}((u, i)) \backslash L(u)$ is not a clique of $H$ for some vertex $(u, i) \in L(u)$.

Proof. By (34), $P_{D P}(G, \mathcal{H}) \geq(m-d(u)) P_{D P}\left(G-u, \mathcal{H}_{u}^{\prime}\right)$ holds. We need to prove that $P_{D P}(G, \mathcal{H})>(m-d(u)) P_{D P}\left(G-u, \mathcal{H}_{u}^{\prime}\right)$ if either condition (i) or (ii) is satisfied.

Assume that condition (i) holds, i.e., $\left|E_{H}(L(u), L(v))\right| \leq m-1$ for some $v \in N_{G}(u)$. Then, there exists a $m$-fold cover $\mathcal{H}^{*}=\left(L, H^{*}\right)$ of $G$, where $H^{*}$ is obtained from $H$ by adding a new edge joining some vertex in $L(u)$ to some vertex in $L(v)$. By Corollary 14 ,

$$
\begin{equation*}
P_{D P}(G, \mathcal{H})>P_{D P}\left(G, \mathcal{H}^{*}\right) \geq(m-d(u)) P_{D P}\left(G-u, \mathcal{H}_{u}^{\prime}\right) . \tag{35}
\end{equation*}
$$

Now assume that condition (ii) holds. Without loss of generality, assume that $N_{H}((u, 1)) \backslash L(u)$ is not a clique of $H$. Let $\left(v_{1}, i_{1}\right)$ and $\left(v_{2}, i_{2}\right)$ be non-adjacent vertices in $N_{H}((u, 1)) \backslash L(u)$. Clearly, $v_{1} \neq v_{2}$.

As $\mathcal{H}_{u}^{\prime}=\left(L^{\prime}, H^{\prime}\right)$ is an $m$-fold cover of $G-u$ and $m \geq|V(G)|$, by Proposition 13, there exists $I^{\prime} \in \mathcal{I}\left(H^{\prime}\right)$ such that $\left\{\left(v_{1}, i_{1}\right),\left(v_{2}, i_{2}\right)\right\} \subseteq I^{\prime}$.

Note that $\left|I^{\prime} \cap L(v)\right|=1$ for each $v \in N_{G}(u)$ and $\left\{\left(v_{1}, i_{1}\right),\left(v_{2}, i_{2}\right)\right\} \subseteq I^{\prime} \cap N_{H}((u, 1))$. Assume that $N_{G}(u)=\left\{v_{1}, v_{2}, \cdots, v_{r}\right\}$, where $r=d(u)$, and $I^{\prime} \cap L\left(v_{j}\right)=\left\{\left(v_{j}, \pi_{j}\right)\right\}$ for all $j \in[r]$. Then

$$
\begin{align*}
\left|L(u) \cap \bigcup_{j \in[r]} N_{H}\left(\left(v_{j}, \pi_{j}\right)\right)\right| & \leq\left|L(u) \cap \bigcup_{j \in \llbracket 2 \rrbracket} N_{H}\left(\left(v_{j}, \pi_{j}\right)\right)\right|+\left|L(u) \cap \bigcup_{3 \leq j \leq r} N_{H}\left(\left(v_{j}, \pi_{j}\right)\right)\right| \\
& \leq|\{(u, 1)\}|+(r-2)=d(u)-1, \tag{36}
\end{align*}
$$

implying that

$$
\left|\mathcal{I}_{H}\left(I^{\prime}\right)\right|=m-\left|L(u) \cap \bigcup_{j \in[r]} N_{H}\left(\left(v_{j}, \pi_{j}\right)\right)\right| \geq m-d(u)+1
$$

By (34),$P_{D P}(G, \mathcal{H})>(m-d(u)) P_{D P}\left(G-u, \mathcal{H}_{u}^{\prime}\right)$ holds. The result is proven.

We are now ready to prove Theorem 5 by applying (8) and Propositions 12 and 15 .
Proof of Theorem [55 If $P_{D P}(G-u) \approx P(G-u)$, then $P_{D P}(G) \approx P(G)$ due to Proposition 12.
Now assume that $G-u \in \mathcal{D} \mathcal{P}^{*}$. Then, there exists $M \in \mathbb{N}$ such that $P_{D P}\left(G-u, \mathcal{H}^{\prime}\right)>$ $P(G-u, m)$ for each integer $m \geq M$ and every $m$-fold cover $\mathcal{H}^{\prime}=\left(L^{\prime}, H^{\prime}\right)$ of $G-u$ with $H^{\prime} \not \not 二 H_{0}(G-u, m)$.

Let $\mathcal{H}=(L, H)$ be any $m$-fold cover of $G$ such that $H \not \approx H_{0}(G, m)$. We may assume that $L(v)=\{(v, i): i \in \llbracket m \rrbracket\}$ for each $v \in V(G)$. If $\left|E_{H}\left(L\left(v_{1}\right), L\left(v_{2}\right)\right)\right|<m$ for some edge $v_{1} v_{2} \in E(G)$, then, by Corollary 14, $P_{D P}(G, \mathcal{H})>P_{D P}\left(G, \mathcal{H}^{*}\right)$ for $m \geq|V(G)|$, where $\mathcal{H}^{*}$ is the $m$-fold cover $\left(L, H^{*}\right)$ obtained from $\mathcal{H}$ by adding a new edge joining a vertex in $L\left(v_{1}\right)$ to a vertex in $L\left(v_{2}\right)$. Therefore, we can assume that $\left|E_{H}\left(L\left(v_{1}\right), L\left(v_{2}\right)\right)\right|=m$ for each edge $v_{1} v_{2} \in E(G)$ and $H \not \neq H_{0}(G, m)$.

Consider the $m$-fold cover $\mathcal{H}_{u}^{\prime}=\left(L^{\prime}, H^{\prime}\right)$ of $G-u$.
Case 1: $H^{\prime} \not \neq H_{0}(G-u, m)$.
By the assumption in the beginning of the proof, $P_{D P}\left(G-u, \mathcal{H}_{u}^{\prime}\right)>P(G-u, m)$ for each integer $m \geq M$. By (8) and Proposition [15, for $m \geq \max \{M,|V(G)|\}$,

$$
\begin{equation*}
P_{D P}(G, \mathcal{H}) \geq(m-d(u)) P_{D P}\left(G-u, \mathcal{H}^{\prime}\right)>(m-d(u)) P(G-u, m)=P(G, m) . \tag{37}
\end{equation*}
$$

Case 2: $H^{\prime} \cong H_{0}(G-u, m)$.
We can assume that $H^{\prime}=H_{0}(G-u, m)$. Since $H \not \approx H_{0}(G, m)$, there must be some vertex $(u, i) \in L(u)$ that is adjacent to two vertices $\left(v_{1}, i_{1}\right)$ and $\left(v_{2}, i_{2}\right)$ with $v_{1} \neq v_{2}$ and $i_{1} \neq i_{2}$. Since $H^{\prime}=H_{0}(G-u, m)$ and $i_{1} \neq i_{2},\left(v_{1}, i_{1}\right)$ and $\left(v_{2}, i_{2}\right)$ are not adjacent in $H$, implying that $N_{H}((u, i)) \backslash L(u)$ is not a clique of $H$. By Proposition 15 again, $P_{D P}(G, \mathcal{H})>(m-$ $d(u)) P(G, m)$ for $m \geq|V(G)|$.

Thus Theorem 5 holds.
By Theorem 5, we have the following consequence, which generalizes the known conclusion that $P_{D P}(G) \approx P(G)$ holds for every chordal graph $G$.

Corollary 16. Let $G_{1}$ and $G_{2}$ be vertex-disjoint graphs and $k \in \mathbb{N}$, where $k \leq \min \left\{\omega\left(G_{i}\right)\right.$ : $i=1,2\}$. Assume that $G_{1}$ is chordal and $G \in \mathscr{G}\left(G_{1} \cup_{k} G_{2}\right)$. If $P_{D P}\left(G_{2}\right) \approx P\left(G_{2}\right)$, then $P_{D P}(G) \approx P(G)$; also, if $G_{2} \in \mathcal{D P}^{*}$, then $G \in \mathcal{D P}^{*}$.

Proof. As $G_{1}$ is chordal, there must be an ordering $v_{1}, v_{2}, \cdots, v_{r}$ of vertices in $V(G) \backslash V\left(G_{2}\right)$, where $r=\left|V\left(G_{1}\right)\right|-k$, such that $v_{i}$ is a simplicial vertex in $G-\left\{v_{j}: j \in \llbracket i-1 \rrbracket\right\}$ for each
$i \in \llbracket r \rrbracket$. Then, the result follows from Theorem 5,
We wonder if Corollary 16 holds without the condition that $G_{1}$ is chordal.
Problem 6. For any vertex-disjoint graphs $G_{1}$ and $G_{2}$ and $k \in \mathbb{N}$, where $k \leq \min \left\{\omega\left(G_{i}\right)\right.$ : $i=1,2\}$, is it true that if $P_{D P}\left(G_{i}\right) \approx P\left(G_{i}\right)$ for $i=1,2$, then $P_{D P}(G) \approx P(G)$ for every graph $G \in \mathscr{G}\left(G_{1} \cup_{k} G_{2}\right)$; also, if $G_{1}, G_{2} \in \mathcal{D P}^{*}$, then $\mathscr{G}\left(G_{1} \cup_{k} G_{2}\right) \subseteq \mathcal{D} \mathcal{P}^{*}$ ?

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