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ON MOTZKIN NUMBERS AND CENTRAL TRINOMIAL COEFFICIENTS

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ABSTRACT. The Motzkin numbers $M_n = \sum_{k=0}^n \binom{n}{2k} \binom{2k}{k} / (k+1)$ ($n = 0, 1, 2, \dots$) and the central trinomial coefficients T_n ($n = 0, 1, 2, \dots$) given by the constant term of $(1+x+x^{-1})^n$, have many combinatorial interpretations. In this paper we establish the following surprising arithmetic properties of them with n any positive integer:

$$\frac{2}{n} \sum_{k=1}^n (2k+1) M_k^2 \in \mathbb{Z},$$

$$\frac{n^2(n^2-1)}{6} \mid \sum_{k=0}^{n-1} k(k+1)(8k+9) T_k T_{k+1},$$

and also

$$\sum_{k=0}^{n-1} (k+1)(k+2)(2k+3) M_k^2 3^{n-1-k} = n(n+1)(n+2) M_n M_{n-1}.$$

1. INTRODUCTION

In combinatorics, the Motzkin number M_n with $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ is the number of lattice paths from the point $(0, 0)$ to the point $(n, 0)$ which never dip below the line $y = 0$ and are made up only of the allowed steps $(1, 0)$ (east), $(1, 1)$ (northeast) and $(1, -1)$ (southeast). It is well known that

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k$$

where C_k denotes the Catalan number $\binom{2k}{k} - \binom{2k}{k+1} = \binom{2k}{k} / (k+1)$.

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For $n \in \mathbb{N}$, the central trinomial coefficient T_n is the constant term in the expansion of $(1 + x + x^{-1})^n$. By the multi-nomial theorem, we see that

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} = \sum_{k=0}^n \binom{n}{k} \binom{n-k}{k}.$$

It is known that T_n coincides with the number of lattice paths from the point $(0, 0)$ to $(n, 0)$ with only allowed steps $(1, 0)$ (east), $(1, 1)$ (northeast) and $(1, -1)$ (southeast).

The Motzkin numbers, the Catalan numbers and the central trinomial coefficients arise naturally in enumerative combinatorics. As the Fibonacci numbers arising from combinatorics have rich number-theoretic properties, we think that important combinatorial quantities like M_n and T_n with $n \in \mathbb{N}$ should also have nice arithmetic properties. For example, in [S14a] we conjectured that for any $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ the arithmetic mean of the n numbers $(8k + 5)T_k^2$ ($k = 0, \dots, n-1$) is always an integer, and this was later confirmed by Y.-P. Mu and the author [MS] via symbolic computation. Motivated by congruence properties of such numbers, we found in [S14b, S20] many series for $1/\pi$ involving central trinomial coefficients or their extensions. For example, in [S20, Section 10] we conjectured the combinatorial identity

$$\sum_{k=1}^{\infty} \frac{(105k - 44)T_{k-1}}{k^2 \binom{2k}{k}^2 3^{k-1}} = \frac{5\pi}{\sqrt{3}} + 6 \log 3$$

based on the conjectural congruence

$$p^2 \sum_{k=1}^{p-1} \frac{(105k - 44)T_{k-1}}{k^2 \binom{2k}{k}^2 3^{k-1}} \equiv 11 \left(\frac{p}{3} \right) + \frac{p}{2} \left(13 - 35 \left(\frac{p}{3} \right) \right) \pmod{p^2},$$

where p is a prime greater than 3 and $(-)$ is the Legendre symbol. Thus it is interesting to investigate congruence properties of combinatorial quantities like M_n and T_n with $n \in \mathbb{N}$, and the study in turn may stimulate us to find some new combinatorial identities.

Let $p > 3$ be a prime. In [S14a, Conjecture 1.1(ii)] we conjectured

$$\sum_{k=0}^{p-1} M_k^2 \equiv (2 - 6p) \left(\frac{p}{3} \right) \pmod{p^2}, \quad \sum_{k=0}^{p-1} k M_k^2 \equiv (9p - 1) \left(\frac{p}{3} \right) \pmod{p^2},$$

and

$$\sum_{k=0}^{p-1} T_k M_k \equiv \frac{4}{3} \left(\frac{p}{3} \right) + \frac{p}{6} \left(1 - 9 \left(\frac{p}{3} \right) \right) \pmod{p^2}.$$

The three supercongruences look curious and challenging.

Motivated by the above conjectures, we establish the following new results.

Theorem 1.1. (i) For any $n \in \mathbb{Z}^+$, we have

$$s(n) := \frac{2}{n} \sum_{k=1}^n (2k+1)M_k^2 \in \mathbb{Z}. \quad (1.1)$$

(ii) For any prime $p > 3$, we have

$$\sum_{k=0}^{p-1} (2k+1)M_k^2 \equiv 12p \left(\frac{p}{3}\right) \pmod{p^2}. \quad (1.2)$$

Remark 1.1. The values of $s(1), \dots, s(10)$ are as follows:

$$6, 23, 90, 432, 2286, 13176, 80418, 513764, 3400518, 23167311.$$

Theorem 1.2. For any integer $n \geq 2$, we have

$$\frac{n^2(n^2-1)}{6} \left| \sum_{k=0}^{n-1} k(k+1)(8k+9)T_k T_{k+1} \right|. \quad (1.3)$$

Remark 1.2. If we define

$$t(n) := \frac{6}{n^2(n^2-1)} \sum_{k=0}^{n-1} k(k+1)(8k+9)T_k T_{k+1} \quad (n = 2, 3, \dots),$$

then the values of $t(2), t(3), \dots, t(10)$ are as follows:

$$51, 271, 1398, 8505, 54387, 367551, 2570931, 18510739, 136282347.$$

Let $b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$. The generalized central trinomial coefficient $T_n(b, c)$ denotes the coefficient of x^n in the expansion of $(x^2 + bx + c)^n$ (cf. [S14a] and [S14b]). By the multi-nomial theorem, we see that

$$T_n(b, c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} b^{n-2k} c^k.$$

The generalized Motzkin number $M_n(b, c)$ introduced in [S14a] is given by

$$M_n(b, c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k b^{n-2k} c^k.$$

Note that $T_n(1, 1) = T_n$, $M_n(1, 1) = M_n$, $T_n(2, 1) = \binom{2n}{n}$ and $M_n(2, 1) = C_{n+1}$. Also, $T_n(3, 2)$ coincides with the (central) Delannoy number

$$D_n = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} = \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k},$$

which counts lattice paths from $(0, 0)$ to (n, n) in which only east $(1, 0)$, north $(0, 1)$, and northeast $(1, 1)$ steps are allowed (cf. R. P. Stanley [St99, p. 185]). And $M_n(3, 2)$ equals the little Schröder number

$$s_{n+1} = \sum_{k=1}^{n+1} N(n+1, k) 2^{n+1-k}$$

with the Narayana number $N(m, k)$ ($m \geq k \geq 1$) given by

$$N(m, k) := \frac{1}{m} \binom{m}{k} \binom{m}{k-1} \in \mathbb{Z}.$$

The little Schröder numbers and the Narayana numbers also have many combinatorial interpretations (cf. [St97] and [Gr, pp. 268–281]). See also [S11, S18b] for some congruences involving the Delannoy numbers or the little Schröder numbers.

Theorem 1.3. *Let $b, c \in \mathbb{Z}$ with $b \neq 0$ and $d = b^2 - 4c \neq 0$, and let $n \in \mathbb{Z}^+$. Then*

$$b \frac{n(n+1)}{2} \left| \sum_{k=1}^n k T_k(b, c) T_{k-1}(b, c) d^{n-k} \right| \quad (1.4)$$

and

$$b \frac{n^2(n+1)^2}{4} \left| 3 \sum_{k=1}^n k^3 T_k(b, c) T_{k-1}(b, c) d^{n-k} \right|. \quad (1.5)$$

Also,

$$\frac{(2, n)}{n(n+1)(n+2)} \sum_{k=0}^{n-1} (k+1)(k+2)(2k+3) M_k(b, c)^2 d^{n-1-k} \in \mathbb{Z} \quad (1.6)$$

and

$$\sum_{k=0}^{n-1} \frac{(k+1)(k+2)(2k+3)}{n(n+1)(n+2)} M_k(b, c)^2 (-d)^{n-1-k} = \frac{M_n(b, c) M_{n-1}(b, c)}{b} \in \mathbb{Z}, \quad (1.7)$$

where (m, n) denotes the greatest common divisor of two integers m and n .

Remark 1.3. For each $n \in \mathbb{Z}^+$, (1.7) with $b = c = 1$ gives the curious identity

$$\sum_{k=0}^{n-1} (k+1)(k+2)(2k+3) M_k^2 3^{n-1-k} = n(n+1)(n+2) M_n M_{n-1}. \quad (1.8)$$

In the case $b = 3$ and $c = 2$, Theorem 1.3 yields the following consequence.

Corollary 1.1. *For any $n \in \mathbb{Z}^+$ we have*

$$3 \frac{n(n+1)}{2} \left| \sum_{k=1}^n k D_k D_{k-1}, \frac{n^2(n+1)^2}{4} \left| \sum_{k=1}^n k^3 D_k D_{k-1}, \right. \right. \quad (1.9)$$

$$\frac{n(n+1)(n+2)}{(2, n)} \left| \sum_{k=1}^n k(k+1)(2k+1) s_k^2, \right. \quad (1.10)$$

and

$$\frac{1}{n(n+1)(n+2)} \sum_{k=1}^n k(k+1)(2k+1)(-1)^{n-k} s_k^2 = \frac{s_n s_{n+1}}{3} \in \mathbb{Z}. \quad (1.11)$$

Theorems 1.1-1.3 are quite sophisticated and their proofs need various techniques. We will prove Theorems 1.1-1.3 in Sections 2-4 respectively. In Section 5 we are going to pose some related conjectures for further research.

2. PROOF OF THEOREM 1.1

For $n \in \mathbb{Z}^+$, in [S18b] we introduced the polynomial

$$s_n(x) := \sum_{k=1}^n N(n, k) x^{k-1} (x+1)^{n-k} \quad (2.1)$$

for which $s_n(1)$ is just the little Schröder number s_n . For $n \in \mathbb{N}$, define

$$S_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \frac{x^k}{k+1} = \sum_{k=0}^n \binom{n+k}{2k} C_k x^k. \quad (2.2)$$

Then $S_n(1)$ equals the large Schröder number S_n which counts the lattice paths from the point $(0, 0)$ to (n, n) with steps $(1, 0)$, $(0, 1)$ and $(1, 1)$ that never rise above the line $y = x$. As proved in [S18b], we have

$$S_n(x) = (x+1)s_n(x) \quad \text{for all } n \in \mathbb{Z}^+. \quad (2.3)$$

Lemma 2.1. (i) *For any $n \in \mathbb{Z}^+$ we have*

$$n(n+1)s_n(x)^2 = \sum_{k=1}^n \binom{n+k}{2k} \binom{2k}{k} \binom{2k}{k+1} (x(x+1))^{k-1}. \quad (2.4)$$

(ii) *Let $b, c \in \mathbb{Z}$ with $d = b^2 - 4c \neq 0$. For any $n \in \mathbb{N}$ we have*

$$M_n(b, c) = (\sqrt{d})^n s_{n+1} \left(\frac{b/\sqrt{d} - 1}{2} \right). \quad (2.5)$$

Proof. As $(x+1)s_n(x) = S_n(x)$ by (2.3), the identity (2.4) has the equivalent version

$$n(n+1)S_n(x)^2 = \sum_{k=1}^n \binom{n+k}{2k} \binom{2k}{k} \binom{2k}{k+1} x^{k-1} (x+1)^{k+1}$$

which appeared as [S12a, (2.1)]. So (2.4) holds. The identity (2.5) was proved in [S18b, Lemma 3.1]. \square

Remark 2.1. For $n \in \mathbb{N}$ and $b, c \in \mathbb{Z}$ with $b^2 \neq 4c$, by combining the two parts of Lemma 2.1 we obtain that

$$M_n(b, c)^2 = \frac{1}{(n+1)(n+2)} \sum_{k=1}^{n+1} \binom{n+k+1}{2k} \binom{2k}{k} \binom{2k}{k+1} c^{k-1} (b^2 - 4c)^{n+1-k}. \quad (2.6)$$

Lemma 2.2. *For any $n \in \mathbb{Z}^+$ we have*

$$\begin{aligned} & \sum_{k=1}^n (2k+1) M_k^2 \\ &= \sum_{k=0}^{n+1} \frac{(4n-2k+3)(n+k+2)}{n+2} \binom{n+k+1}{2k} \binom{2k}{k} \binom{2k+1}{k} (-3)^{n+1-k}. \end{aligned} \quad (2.7)$$

Proof. In view of (2.6), we have

$$\begin{aligned} \sum_{k=0}^n (2k+1) M_k^2 &= \sum_{k=0}^n \frac{2k+1}{(k+1)(k+2)} \sum_{j=1}^{k+1} \binom{k+j+1}{2j} \binom{2j}{j} \binom{2j}{j+1} (-3)^{k+1-j} \\ &= \sum_{k=0}^n \frac{2k+1}{(k+1)(k+2)} \sum_{l=0}^k \binom{k+l+2}{2l+2} \binom{2l+2}{l+1} \binom{2l+2}{l} (-3)^{k-l} \\ &= \sum_{k=0}^n \sum_{l=0}^n F(k, l), \end{aligned}$$

where

$$F(k, l) := \frac{2k+1}{(k+1)(k+2)} \binom{k+l+2}{2l+2} \binom{2l+2}{l+1} \binom{2l+2}{l} (-3)^{k-l}.$$

By the telescoping method developed by Chen, Hou and Mu [CHM] and applied by Mu and Sun [MS], the double sum can be reduced to a single sum:

$$\sum_{k=0}^n \sum_{l=0}^n F(k, l) = 1 + (4n+3)(-3)^{n+1} + \sum_{j=0}^n (-3)^{n-j} \frac{(4n-2j+1)(n+j+3)!(2j+3)!}{(n+2)(n-j)!(j+2)(j+1)!^4}. \quad (2.8)$$

Therefore

$$\begin{aligned}
& \sum_{k=1}^n (2k+1)M_k^2 \\
&= \sum_{j=-1}^n (-3)^{n-j} \frac{(4n-2j+1)(n+j+3)!(2j+3)!}{(n+2)(n-j)!(j+2)(j+1)!^4} \\
&= \sum_{k=0}^{n+1} (-3)^{n+1-k} \frac{(4n-2k+3)(n+k+2)!(2k+1)!}{(n+2)(n+1-k)!(k+1)!^4} \\
&= \sum_{k=0}^{n+1} \frac{(4n-2k+3)(n+k+2)}{n+2} \binom{n+k+1}{2k} \binom{2k}{k} \binom{2k+1}{k} (-3)^{n+1-k}
\end{aligned}$$

and this concludes the proof. \square

For each integer n we set

$$[n]_q = \frac{1-q^n}{1-q},$$

which is the usual q -analogue of n . For any $n \in \mathbb{Z}$, we define

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1 \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\prod_{j=0}^{k-1} [n-j]_q}{\prod_{j=1}^k [j]_q} \quad \text{for } k = 1, 2, 3, \dots$$

Obviously $\lim_{q \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$ for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}$. It is easy to see that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \quad \text{for all } k, n = 1, 2, 3, \dots$$

By this recursion, $\begin{bmatrix} n \\ k \end{bmatrix}_q \in \mathbb{Z}[q]$ for all $k, n \in \mathbb{N}$. For any integers a, b and $n > 0$, clearly

$$a \equiv b \pmod{n} \implies [a]_q \equiv [b]_q \pmod{[n]_q}.$$

Let n be a positive integer. The cyclotomic polynomial

$$\Phi_n(q) := \prod_{\substack{a=1 \\ (a,n)=1}}^n \left(q - e^{2\pi i a/n} \right) \in \mathbb{Z}[q]$$

is irreducible in the ring $\mathbb{Z}[q]$. It is well-known that

$$q^n - 1 = \prod_{d|n} \Phi_d(q).$$

Note that $\Phi_1(q) = q - 1$.

Lemma 2.3. *For any $a, b \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, we have*

$$\sum_{k=0}^{n-1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q^a \begin{bmatrix} n+k \\ k \end{bmatrix}_q^b \begin{bmatrix} 2k \\ k \end{bmatrix}_q [k+2]_q (-[3]_q)^{n-1-k} \equiv 0 \pmod{[n]_q}. \quad (2.9)$$

Proof. (2.9) is trivial in the case $n = 1$. Below we assume $n > 1$. As

$$[n]_q = \prod_{1 < d|n} \Phi_d(q)$$

and $\Phi_2(q), \Phi_3(q), \dots$ are pairwise coprime, it suffices to show that the sum in (2.9) is divisible by $\Phi_d(q)$ for any given divisor $d > 1$ of n .

A well-known q -Lucas theorem (see, e.g., [O]) states that if $a, b, d, s, t \in \mathbb{N}$ with $s < d$ and $t < d$ then

$$\begin{bmatrix} ad+s \\ bd+t \end{bmatrix}_q \equiv \begin{pmatrix} a \\ b \end{pmatrix} \begin{bmatrix} s \\ t \end{bmatrix}_q \pmod{\Phi_d(q)}.$$

Let S denote the sum in (2.9) and write $n = dm$ with $m \in \mathbb{Z}^+$. Then

$$\begin{aligned} S &= \sum_{j=0}^{m-1} \sum_{r=0}^{d-1} \begin{bmatrix} md+1 \\ jd+r \end{bmatrix}_q^a \begin{bmatrix} md+jd+r \\ jd+r \end{bmatrix}_q^b \begin{bmatrix} 2jd+2r \\ jd+r \end{bmatrix}_q [jd+r+2]_q (-[3]_q)^{md-1-(jd+r)} \\ &\equiv \sum_{j=0}^{m-1} \sum_{r=0}^{d-1} \begin{pmatrix} m \\ j \end{pmatrix} \begin{bmatrix} 1 \\ r \end{bmatrix}_q^a \begin{pmatrix} m+j \\ j \end{pmatrix} \begin{bmatrix} r \\ r \end{bmatrix}_q^b \begin{bmatrix} 2jd+2r \\ jd+r \end{bmatrix}_q [r+2]_q (-[3]_q)^{(m-j)d-(r+1)} \\ &\equiv \sum_{j=0}^{m-1} \begin{pmatrix} m \\ j \end{pmatrix} \begin{pmatrix} m+j \\ j \end{pmatrix} \sum_{r=0}^1 \begin{bmatrix} 2jd+2r \\ jd+r \end{bmatrix}_q [r+2]_q (-[3]_q)^{(m-j)d-(r+1)} \\ &\equiv \sum_{j=0}^{m-1} \begin{pmatrix} m \\ j \end{pmatrix} \begin{pmatrix} m+j \\ j \end{pmatrix} \begin{pmatrix} 2j \\ j \end{pmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q [2]_q (-[3]_q)^{(m-j)d-1} \\ &\quad + \sum_{j=0}^{m-1} \begin{pmatrix} m \\ j \end{pmatrix} \begin{pmatrix} m+j \\ j \end{pmatrix} [1+2]_q (-[3]_q)^{(m-j)d-2} \times \begin{cases} \begin{pmatrix} 2j+1 \\ j \end{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_q & \text{if } d = 2, \\ \begin{pmatrix} 2j \\ j \end{pmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q & \text{if } d > 2, \end{cases} \\ &\equiv 0 \pmod{\Phi_d(q)}. \end{aligned}$$

(Note that $[2]_q = 1 + q = \Phi_2(q)$.) This concludes the proof. \square

Lemma 2.4. *For any prime $p > 3$ we have*

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k3^k} \equiv \frac{3^{p-1} - 1}{p} \pmod{p}. \quad (2.10)$$

Proof. Let $u_n = \left(\frac{n}{3}\right)$ for $n \in \mathbb{N}$. Then $u_0 = 0$, $u_1 = 1$ and $u_{n+1} = -u_n - u_{n-1}$ for all $n = 1, 2, 3, \dots$. Applying [S12b, Lemma 3.5] with $m = 1$, we obtain

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k3^k} \equiv \frac{(-3)^{p-1} - 1}{p} - \frac{1}{2} \left(\frac{-3}{p}\right) \frac{u_{p-(\frac{-3}{p})}}{p} \pmod{p}.$$

Note that $u_{p-(\frac{-3}{p})} = 0$ since $p \equiv (\frac{-3}{p}) \pmod{3}$. So (2.10) holds. \square

Proof of Theorem 1.1. (i) Observe that

$$\frac{4}{n+2} \equiv \begin{cases} 4/2 = 2 \pmod{n} & \text{if } 2 \nmid n, \\ 2/(n/2+1) \equiv 2 \pmod{n} & \text{if } 2 \mid n. \end{cases}$$

Thus, for each $k \in \{1, \dots, n+1\}$, we have

$$2 \times \frac{\binom{2k}{k}}{n+2} = \frac{4}{n+2} \binom{2k-1}{k} \equiv 2 \binom{2k-1}{k} = \binom{2k}{k} \pmod{n}.$$

Combining this with (2.7) we see that

$$\begin{aligned} & 2 \sum_{k=1}^n (2k+1) M_k^2 \\ & \equiv 2(4n+3)(-3)^{n+1} \\ & \quad + \sum_{k=1}^{n+1} (4n-2k+3)(n+k+2) \binom{n+k+1}{2k} \binom{2k}{k} \binom{2k+1}{k} (-3)^{n+1-k} \\ & \equiv - \sum_{k=0}^{n+1} (2k-3)(k+2) \binom{n+k+1}{n+1} \binom{n+1}{k} \binom{2k+1}{k} (-3)^{n+1-k} \\ & \equiv - \sum_{k=0}^{n+1} (2k-3)(k+2) \frac{n+k+1}{n+1} \binom{n+k}{k} \binom{n+1}{k} (2k+1) C_k (-3)^{n+1-k} \\ & \equiv - \sum_{k=0}^{n+1} (2k-3)(k+2)(k+1) \binom{n+k}{k} \binom{n+1}{k} (2k+1) C_k (-3)^{n+1-k} \pmod{n}. \end{aligned}$$

For each $k = 0, \dots, n+1$, clearly

$$k(k-1) \binom{n+1}{k} = n(n+1) \binom{n-1}{n+1-k} \equiv 0 \pmod{n}.$$

Since $(2k-3)(2k+1) = 4k(k-1) - 3$, by the above we have

$$2 \sum_{k=1}^n (2k+1) M_k^2 \equiv - \sum_{k=0}^{n+1} \binom{n+1}{k} \binom{n+k}{k} \binom{2k}{k} (k+2) (-3)^{n+2-k} \pmod{n}.$$

Note that

$$\begin{aligned}
& \sum_{k=n}^{n+1} \binom{n+1}{k} \binom{n+k}{k} \binom{2k}{k} (k+2)(-3)^{n+2-k} \\
&= \binom{n+1}{n} \binom{2n}{n}^2 (n+2)(-3)^2 + \binom{2n+1}{n+1} \binom{2n+2}{n+1} (n+3)(-3) \\
&\equiv 18 \binom{2n}{n}^2 - 18 \left(\frac{2n+1}{n+1} \binom{2n}{n} \right)^2 \equiv 0 \pmod{n}.
\end{aligned}$$

Therefore

$$2 \sum_{k=1}^n (2k+1) M_k^2 \equiv 27 \sum_{k=0}^{n-1} \binom{n+1}{k} \binom{n+k}{k} \binom{2k}{k} (k+2)(-3)^{n-1-k} \pmod{n}. \quad (2.11)$$

By (2.9) with $a = b = 1$ and $q = 1$, we have

$$\sum_{k=0}^{n-1} \binom{n+1}{k} \binom{n+k}{k} \binom{2k}{k} (k+2)(-3)^{n-1-k} \equiv 0 \pmod{n}.$$

Combining this with (2.11) we immediately obtain the desired (1.1).

(ii) Applying (2.7) with $n = p - 1$, we get

$$\begin{aligned}
\sum_{k=1}^{p-1} (2k+1) M_k^2 &= \sum_{k=0}^p \frac{(4p-2k-1)(p+k+1)}{p+1} \binom{p+k}{2k} \binom{2k}{k} \binom{2k+1}{k} (-3)^{p-k} \\
&= \sum_{k=1}^{p-1} \frac{(4p-2k-1)(p+k+1)}{p+1} \binom{p}{k} \binom{p+k}{k} \frac{2k+1}{k+1} \binom{2k}{k} (-3)^{p-k} \\
&\quad + (4p-1)(-3)^p + \frac{(2p-1)(2p+1)}{p+1} \binom{2p}{p} \frac{2p+1}{p+1} \binom{2p}{p} \\
&\equiv 3 \sum_{k=1}^{p-1} \frac{p}{k} \binom{p-1}{k-1} (2k+1)^2 \frac{\binom{2k}{k}}{(-3)^k} + (3-12p)3^{p-1} - \left(2 \binom{2p-1}{p-1} \right)^2 \\
&\equiv -3p \sum_{k=1}^{p-1} \left(4k+4+\frac{1}{k} \right) \frac{\binom{2k}{k}}{3^k} + 3^p - 12p - 4 \pmod{p^2}
\end{aligned}$$

with the aid of Wolstenholme's congruence $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$ (cf. [W]). Com-

binning this with (2.10) and noting that $\binom{-1/2}{k} = \binom{2k}{k}/(-4)^k$ for $k \in \mathbb{N}$, we obtain

$$\begin{aligned}
-\frac{1}{12p} \sum_{k=0}^{p-1} (2k+1)M_k^2 &\equiv 1 + \sum_{k=1}^{p-1} (k+1) \binom{-1/2}{k} \frac{(-4)^k}{3^k} \\
&\equiv \sum_{k=0}^{(p-1)/2} \binom{(p-1)/2}{k} \left(-\frac{4}{3}\right)^k + \sum_{k=1}^{(p-1)/2} k \binom{(p-1)/2}{k} \left(-\frac{4}{3}\right)^k \\
&\equiv \left(1 - \frac{4}{3}\right)^{(p-1)/2} - \frac{4}{3} \cdot \frac{p-1}{2} \sum_{k=1}^{(p-1)/2} \binom{(p-3)/2}{k-1} \left(-\frac{4}{3}\right)^{k-1} \\
&\equiv \left(\frac{-3}{p}\right) + \frac{2}{3} \left(1 - \frac{4}{3}\right)^{(p-3)/2} \\
&\equiv \left(\frac{-3}{p}\right) - 2 \left(\frac{-3}{p}\right) = -\left(\frac{p}{3}\right) \pmod{p}.
\end{aligned}$$

This proves (1.2).

The proof of Theorem 1.1 is now complete. \square

3. PROOF OF THEOREM 1.2

Lemma 3.1. *Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. Then*

$$b \sum_{k=0}^{n-1} (2k+1)T_k(b, c)^2 (-d)^{n-1-k} = nT_n(b, c)T_{n-1}(b, c) \quad \text{for any } n \in \mathbb{Z}^+, \quad (3.1)$$

and

$$T_k(b, c)^2 = \sum_{j=0}^k \binom{k+j}{2j} \binom{2j}{j}^2 c^j d^{k-j} \quad \text{for all } k \in \mathbb{N}. \quad (3.2)$$

Remark 3.1. For (3.1) and (3.2), see [S14a, (1.19) and (4.1)].

Lemma 3.2. *For any $n \in \mathbb{Z}^+$, we have*

$$\sum_{k=0}^{n-1} k(k+1)(8k+9)T_k T_{k+1} = \frac{(-1)^n n}{6} \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{-n-1}{k} C_k 3^{n-1-k} a(n, k), \quad (3.3)$$

where

$$a(n, k) = 4k^2 n^2 - 8kn^3 - 14k^2 n - 14kn^2 - 4n^3 + 13k^2 - 11kn - 26n^2 + 39k + 4n + 26.$$

Proof. In light of (3.1) with $b = c = 1$,

$$\begin{aligned}
& \sum_{k=0}^{n-1} k(k+1)(8k+9)T_k T_{k+1} \\
&= \sum_{k=1}^n k(k-1)(8k+1)T_k T_{k-1} \\
&= \sum_{k=1}^n (k-1)(8k+1) \sum_{j=0}^{k-1} (2j+1)T_j^2 3^{k-1-j} \\
&= \sum_{j=0}^{n-1} (2j+1)T_j^2 \sum_{k=j+1}^n (k-1)(8k+1)3^{k-1-j}.
\end{aligned}$$

By induction, for each $j \in \mathbb{N}$ we have

$$\sum_{k=j+1}^m (k-1)(8k+1)3^{k-1-j} = \frac{1}{4} (3^{m-j}(16m^2 - 30m + 21) - (16j^2 - 30j + 21))$$

for all $m = j+1, j+2, \dots$. Thus, in view of the above and (3.2) with $b = c = 1$, we get

$$\begin{aligned}
& 4 \sum_{k=0}^{n-1} k(k+1)(8k+9)T_k T_{k+1} \\
&= \sum_{k=0}^n (2k+1)T_k^2 (3^{n-k}(16n^2 - 30n + 21) - (16k^2 - 30k + 21)) = \sum_{k=0}^n \sum_{l=0}^n F(k, l),
\end{aligned}$$

where $F(k, l)$ denotes

$$(2k+1) \binom{k+l}{2l} \binom{2l}{l}^2 (-3)^{k-l} (3^{n-k}(16n^2 - 30n + 21) - (16k^2 - 30k + 21)).$$

Via the telescoping method stated in [CHM, MS], the double sum can be reduced to a single sum:

$$\sum_{k=0}^n \sum_{l=0}^n F(k, l) = \frac{2}{9} \sum_{k=0}^{n-1} \frac{a(n, k)(-3)^{n-k}(n+k)!(2k)!}{(n-k-1)!k!^4(k+1)}. \quad (3.4)$$

Therefore

$$\begin{aligned}
& \sum_{k=0}^{n-1} k(k+1)(8k+9)T_k T_{k+1} \\
&= \frac{1}{18} \sum_{k=0}^{n-1} \binom{n}{k+1} \binom{n+k}{k} \binom{2k}{k} (-3)^{n-k} a(n, k) \\
&= \frac{(-1)^n}{6} \sum_{k=0}^{n-1} \frac{n}{k+1} \binom{n-1}{k} \binom{-n-1}{k} \binom{2k}{k} 3^{n-1-k} a(n, k)
\end{aligned}$$

and hence (3.3) holds. \square

Lemma 3.3. *For any $n \in \mathbb{Z}^+$, we have*

$$n^2 - 1 \mid \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{-n-1}{k} C_k 3^{n-1-k} a(n, k) \quad (3.5)$$

with $a(n, k)$ given in Lemma 3.2.

Proof. It suffices to show that $n^2 - 1$ divides $\binom{n-1}{k} \binom{-n-1}{k} a(n, k)$ for any fixed $k \in \{0, \dots, n-1\}$. Clearly,

$$\begin{aligned} a(n, k) &\equiv 4k^2 - 8kn - 14k^2n - 14k - 4n + 13k^2 - 11kn - 26 + 39k + 4n + 26 \\ &= k^2(17 - 14n) + k(25 - 19n) \pmod{n^2 - 1}, \end{aligned}$$

and $(\pm n - 1) \mid k \binom{\pm n - 1}{k}$ since $k \binom{\pm n - 1}{k} = (\pm n - 1) \binom{\pm n - 2}{k-1}$ if $k > 0$. So

$$\binom{n-1}{k} \binom{-n-1}{k} a(n, k) \equiv \binom{n-1}{k} \binom{-n-1}{k} k(25 - 19n) \pmod{n^2 - 1}.$$

If $2 \nmid n$, then $n \pm 1$ and $25 - 19n$ are all even, hence both $2(n-1)$ and $2(n+1)$ divide $\binom{n-1}{k} \binom{-n-1}{k} a(n, k)$. If n is even, then $(n-1, n+1) = (n-1, 2) = 1$ and hence $n^2 - 1$ coincides with the least common multiple $[n-1, n+1]$ of $n-1$ and $n+1$. Note that when n is odd we have $(2, n-1) = 2$ and

$$[2(n-1), 2(n+1)] = \frac{2(n-1)2(n+1)}{(2(n-1), 2(n+1))} = \frac{4(n^2 - 1)}{2(n-1, 2)} = n^2 - 1.$$

Therefore $n^2 - 1 \mid \binom{n-1}{k} \binom{-n-1}{k} a(n, k)$ no matter n is odd or even. This concludes the proof. \square

Lemma 3.4. *Let $a, b \in \mathbb{N}$ with $a + b$ even, and let $n \in \mathbb{Z}^+$. Then*

$$2n \mid \sum_{k=0}^{n-1} \binom{n-1}{k}^a \binom{-n-1}{k}^b \binom{2k}{k} (k+2) 3^{n-1-k}. \quad (3.6)$$

Proof. Let $f(k) = k \binom{2k-1}{k} 3^{n-k}$ for $k = 0, \dots, n$. For each $k = 0, \dots, n-1$, we clearly have

$$\begin{aligned} \Delta f(k) &= f(k+1) - f(k) = (k+1) \binom{2k+1}{k+1} 3^{n-k-1} - k \binom{2k-1}{k} 3^{n-k} \\ &= (2k+1) \binom{2k}{k} 3^{n-k-1} - 3k \binom{2k-1}{k} 3^{n-1-k} = \frac{k+2}{2} \binom{2k}{k} 3^{n-1-k}. \end{aligned}$$

Thus, by [S18a, Theorem 4.1] we get

$$\begin{aligned} &\sum_{k=0}^{n-1} \binom{n-1}{k}^a \binom{-n-1}{k}^b \frac{k+2}{2} \binom{2k}{k} 3^{n-1-k} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k}^a \binom{-n-1}{k}^b \Delta f(k) \equiv 0 \pmod{n} \end{aligned}$$

and hence (3.6) holds. \square

Proof of Theorem 1.2. Since $(n, n^2 - 1) = 1$, by Lemmas 3.2 and 3.3 it suffices to show that

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \binom{-n-1}{k} C_k 3^{n-1-k} a(n, k) \equiv 0 \pmod{n}.$$

For each $k = 0, \dots, n-1$, clearly

$$a(n, k) \equiv 13k^2 + 39k + 26 = 13(k+1)(k+2) \pmod{n}.$$

So

$$\begin{aligned} & \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{-n-1}{k} C_k 3^{n-1-k} a(n, k) \\ & \equiv 13 \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{-n-1}{k} \binom{2k}{k} (k+2) 3^{n-1-k} \equiv 0 \pmod{n}. \end{aligned}$$

with the help of Lemma 3.4. This completes the proof. \square

4. PROOF OF THEOREM 1.3

Lemma 4.1. *Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. For any $n \in \mathbb{Z}^+$ we have*

$$nT_n(b, c)T_{n-1}(b, c) = b \sum_{j=0}^{n-1} (n-j) \binom{n+j}{2j} \binom{2j}{j}^2 c^j d^{n-1-j}. \quad (4.1)$$

Proof. In view of Lemma 3.1,

$$\begin{aligned} nT_n(b, c)T_{n-1}(b, c) &= b \sum_{k=0}^{n-1} (2k+1) \sum_{j=0}^k \binom{k+j}{2j} \binom{2j}{j}^2 c^j d^{k-j} (-d)^{n-1-k} \\ &= b \sum_{j=0}^{n-1} \binom{2j}{j}^2 c^j d^{n-1-j} \sum_{k=j}^{n-1} (-1)^{n-1-k} (2k+1) \binom{k+j}{2j}. \end{aligned}$$

For each $j \in \mathbb{N}$, by induction we have

$$\sum_{k=j}^{m-1} (-1)^{m-1-k} (2k+1) \binom{k+j}{2j} = (m-j) \binom{m+j}{2j} \quad \text{for all } m = j+1, j+2, \dots. \quad (4.2)$$

Thus

$$nT_n(b, c)T_{n-1}(b, c) = b \sum_{j=0}^{n-1} \binom{2j}{j}^2 c^j d^{n-1-j} (n-j) \binom{n+j}{2j}$$

and hence (4.1) holds. \square

Lemma 4.2. *For any $k, n \in \mathbb{Z}^+$ with $k \leq n$, we have*

$$\frac{n(n+1)(n+2)}{(2, n)} \mid (n+k+1) \binom{n+k}{k} \binom{n+1}{k+1} \binom{2k}{k+1}. \quad (4.3)$$

Proof. Clearly,

$$\begin{aligned} & (n+k+1) \binom{n+k}{k} \binom{n+1}{k+1} \binom{2k}{k+1} \\ &= (n+k+1) \binom{n+k}{k} \frac{n+1}{k+1} \binom{n}{k} k C_k \\ &= (n+1) \binom{n+k+1}{k+1} n \binom{n-1}{n-k} C_k, \end{aligned}$$

and also

$$(n+k+1) \binom{n+k}{k} \binom{2k}{k+1} \equiv (k-1)(-1)^k \binom{-n-1}{k} k C_k \equiv 0 \pmod{n+2}$$

since

$$k(k-1) \binom{-n-1}{k} = (-n-1)(-n-2) \binom{-n-3}{k-2} \quad \text{if } k > 1.$$

Thus

$$[n(n+1), n+2] \mid (n+k+1) \binom{n+k}{k} \binom{n+1}{k+1} \binom{2k}{k+1}.$$

Note that

$$[n(n+1), n+2] = \frac{n(n+1)(n+2)}{(n(n+1), n+2)} = \frac{n(n+1)(n+2)}{(2, n)}.$$

So we have (4.3). \square

Lemma 4.3. *For any $n \in \mathbb{N}$ we have*

$$6 \binom{2n}{n} \equiv 0 \pmod{n+2}. \quad (4.4)$$

Proof. Observe that

$$\binom{2n+2}{n+1} = 2 \binom{2n+1}{n} = \frac{2(2n+1)}{n+1} \binom{2n}{n}$$

and hence

$$2(2n+1) \binom{2n}{n} = (n+1) \binom{2n+2}{n+1} = (n+1)(n+2) C_{n+1}.$$

Thus

$$\frac{n+2}{(n+2, 2n+1)} \mid \frac{2n+1}{(n+2, 2n+1)} 2 \binom{2n}{n}$$

and hence

$$\frac{n+2}{(n+2, 2n+1)} \mid 2 \binom{2n}{n}. \quad (4.5)$$

Since $(n+2, 2n+1) = (n+2, 2(n+2)-3) = (n+2, 3)$ divides 3, we obtain (4.4) from (4.5). \square

As in [S18b], for $n \in \mathbb{Z}^+$ we define

$$w_n(x) := \sum_{k=1}^n w(n, k) x^{k-1} \text{ with } w(n, k) = \frac{1}{k} \binom{n-1}{k-1} \binom{n+k}{k-1} \in \mathbb{Z}.$$

Lemma 4.4. *For any integers $n \geq k \geq 1$, we have*

$$w(n, k) = \sum_{j=1}^k \binom{n-j}{k-j} N(n, j) \quad (4.6)$$

and

$$N(n, k) = \sum_{j=1}^k \binom{n-j}{k-j} (-1)^{k-j} w(n, j). \quad (4.7)$$

Proof. We first prove (4.7). Observe that

$$\begin{aligned} \sum_{j=1}^k \binom{n-j}{k-1} (-1)^{k-j} w(n, j) &= \sum_{j=1}^k \binom{n-j}{k-j} \frac{(-1)^{k-j}}{n} \binom{n}{j} \binom{n+j}{j-1} \\ &= \frac{(-1)^{k-1}}{n} \binom{n}{k} \sum_{j=1}^k \binom{k}{k-j} \binom{-n-2}{j-1}. \end{aligned}$$

Thus, with the help of the Chu-Vandermonde identity (cf. [G, (3.1)]), we get

$$\sum_{j=1}^k \binom{n-j}{k-1} (-1)^{k-j} w(n, j) = \frac{(-1)^{k-1}}{n} \binom{n}{k} \binom{k-n-2}{k-1} = N(n, k).$$

This proves (4.7).

In view of (4.7), we have

$$\begin{aligned} \sum_{j=1}^k \binom{n-j}{k-j} N(n, j) &= \sum_{j=1}^k \binom{n-j}{k-j} \sum_{i=1}^j \binom{n-i}{j-i} (-1)^{j-i} w(n, i) \\ &= \sum_{i=1}^k w(n, i) \binom{n-i}{k-i} \sum_{j=i}^k \binom{k-i}{j-i} (-1)^{j-i} = w(n, k). \end{aligned}$$

So (4.6) also holds. This ends the proof. \square

Lemma 4.5. *For any $n \in \mathbb{Z}^+$ we have*

$$w_n(x) = s_n(x). \quad (4.8)$$

Proof. With the aid of (4.7), we get

$$\begin{aligned} s_n(x) &= \sum_{k=1}^n N(n, k) x^{k-1} (x+1)^{n-k} \\ &= \sum_{k=1}^n \sum_{j=1}^k \binom{n-j}{k-j} (-1)^{k-j} w(n, j) x^{k-1} (x+1)^{n-k} \\ &= \sum_{j=1}^n w(n, j) x^{n-1} \sum_{k=j}^n \binom{n-j}{k-j} (-1)^{k-j} \left(1 + \frac{1}{x}\right)^{n-j-(k-j)} \\ &= \sum_{j=1}^n w(n, j) x^{n-1} \left(1 + \frac{1}{x} - 1\right)^{n-j} = w_n(x). \end{aligned}$$

This concludes the proof. \square

Lemma 4.6. *For any $n \in \mathbb{Z}^+$ we have the new identity*

$$(2x+1) \sum_{k=1}^n k(k+1)(2k+1)(-1)^{n-k} w_k(x)^2 = n(n+1)(n+2)w_n(x)w_{n+1}(x). \quad (4.9)$$

Proof. In the case $n = 1$, both sides of (4.9) are equal to $6(2x+1)$.

Now assume that (4.9) holds for a fixed positive integer n . Applying the Zeilberger algorithm (cf. [PWZ, pp. 101-119]) via **Mathematica 9** we find that

$$(n+3)w_{n+2}(x) = (2x+1)(2n+3)w_{n+1}(x) - nw_n(x).$$

Thus

$$\begin{aligned} &(2x+1) \sum_{k=1}^{n+1} k(k+1)(2k+1)(-1)^{n+1-k} w_k(x)^2 \\ &= (2x+1)(n+1)(n+2)(2n+3)w_{n+1}(x)^2 - (2x+1) \sum_{k=1}^n k(k+1)(2k+1)(-1)^{n-k} w_k(x)^2 \\ &= (2x+1)(n+1)(n+2)(2n+3)w_{n+1}(x)^2 - n(n+1)(n+2)w_n(x)w_{n+1}(x) \\ &= (n+1)(n+2)w_{n+1}(x)((2x+1)(2n+3)w_{n+1}(x) - nw_n(x)) \\ &= (n+1)(n+2)(n+3)w_{n+1}(x)w_{n+2}(x). \end{aligned}$$

In view of the above, by induction, (4.9) holds for each $n \in \mathbb{Z}^+$. \square

Proof of Theorem 1.3. (i) Let $\delta \in \{0, 1\}$. In light of Lemma 4.1,

$$\begin{aligned}
& \sum_{k=1}^n k^{2\delta+1} T_k(b, c) T_{k-1}(b, c) d^{n-k} \\
&= \sum_{k=1}^n k^{2\delta} b \sum_{j=0}^{k-1} (k-j) \binom{k+j}{2j} \binom{2j}{j}^2 c^j d^{k-1-j} d^{n-k} \\
&= b \sum_{j=0}^{n-1} \binom{2j}{j}^2 c^j d^{n-1-j} \sum_{k=j+1}^n k^{2\delta} (k-j) \binom{k+j}{2j}.
\end{aligned}$$

By induction, for each $j \in \mathbb{N}$, we have

$$\sum_{k=j+1}^m k^{2\delta} (k-j) \binom{k+j}{2j} = \frac{m^\delta (m+1)^\delta}{2} \cdot \frac{(m-j)(m+j+1)}{j+\delta+1} \binom{m+j}{2j} \quad (4.10)$$

for every $m = j+1, j+2, \dots$. Therefore,

$$\begin{aligned}
& \sum_{k=1}^n k^{2\delta+1} T_k(b, c) T_{k-1}(b, c) d^{n-k} \\
&= b \frac{n^\delta (n+1)^\delta}{2} \sum_{j=0}^{n-1} \binom{2j}{j}^2 c^j d^{n-1-j} \frac{(n-j)(n+j+1)}{j+\delta+1} \binom{n+j}{2j} \\
&= \frac{b}{2} (n(n+1))^\delta \sum_{j=0}^{n-1} \frac{\binom{2j}{j}}{j+\delta+1} c^j d^{n-1-j} (n-j)(n+j+1) \binom{n}{j} \binom{n+j}{j}
\end{aligned}$$

and hence

$$\begin{aligned}
& \sum_{k=1}^n k^{2\delta+1} T_k(b, c) T_{k-1}(b, c) d^{n-k} \\
&= \frac{b}{2} (n(n+1))^{\delta+1} \sum_{j=0}^{n-1} \binom{n-1}{j} \binom{n+j+1}{j} \frac{\binom{2j}{j}}{j+\delta+1} c^j d^{n-1-j}. \quad (4.11)
\end{aligned}$$

In the case $\delta = 0$, (4.11) yields (1.4) since $\binom{2j}{j}/(j+1) = C_j \in \mathbb{Z}$. By Lemma 4.3 and (4.11) with $\delta = 1$, we immediately obtain (1.5).

(ii) By induction, for each $j \in \mathbb{N}$ we have

$$\sum_{k=j}^m (2k+1) \binom{k+j}{2j} = \frac{(m+1)(m+j+1)}{j+1} \binom{m+j}{2j} \quad \text{for all } m = j, j+1, \dots \quad (4.12)$$

In view of this and (2.4), we have

$$\begin{aligned}
& \sum_{k=1}^n k(k+1)(2k+1)s_k(x)^2 \\
&= \sum_{k=1}^n (2k+1) \sum_{j=1}^k \binom{k+j}{2j} \binom{2j}{j} \binom{2j}{j+1} (x(x+1))^{j-1} \\
&= \sum_{j=1}^n \binom{2j}{j} \binom{2j}{j+1} (x(x+1))^{j-1} \sum_{k=j}^n (2k+1) \binom{k+j}{2j} \\
&= \sum_{j=1}^n \binom{2j}{j} \binom{2j}{j+1} (x(x+1))^{j-1} \frac{(n+1)(n+j+1)}{j+1} \binom{n+j}{2j} \\
&= \sum_{j=1}^n \binom{2j}{j+1} (x(x+1))^{j-1} \frac{(n+1)(n+j+1)}{j+1} \binom{n}{j} \binom{n+j}{j}
\end{aligned}$$

and hence

$$\sum_{k=1}^n k(k+1)(2k+1)s_k(x)^2 = \sum_{k=1}^n (n+k+1) \binom{n+1}{k+1} \binom{n+k}{k} \binom{2k}{k+1} (x(x+1))^{k-1}. \quad (4.13)$$

Let $x = (b/\sqrt{d} - 1)/2$. Then $x(x+1) = c/d$. In view of Lemma 2.1(ii) and (4.13), we have

$$\begin{aligned}
& \sum_{k=0}^{n-1} (k+1)(k+2)(2k+3)M_k(b, c)^2 d^{n-1-k} \\
&= \sum_{k=0}^{n-1} (k+1)(k+2)(2k+3)d^k s_{k+1}(x)^2 d^{n-1-k} \\
&= d^{n-1} \sum_{k=1}^n k(k+1)(2k+1)s_k(x)^2 \\
&= \sum_{k=1}^n (n+k+1) \binom{n+1}{k+1} \binom{n+k}{k} \binom{2k}{k+1} c^{k-1} d^{n-k}.
\end{aligned}$$

Combining this with Lemma 4.2, we get the desired (1.6).

In light of Lemma 2.1(ii) and Lemmas 4.5-4.6, we have

$$\begin{aligned}
& \sum_{k=0}^{n-1} (k+1)(k+2)(2k+3)M_k(b, c)^2(-d)^{n-1-k} \\
&= \sum_{k=0}^{n-1} (k+1)(k+2)(2k+3)d^k s_{k+1}(x)^2(-d)^{n-1-k} \\
&= d^{n-1} \sum_{k=1}^n k(k+1)(2k+1)(-1)^{n-k} w_k(x)^2 \\
&= n(n+1)(n+2)d^{n-1} \frac{s_n(x)s_{n+1}(x)}{2x+1} \\
&= n(n+1)(n+2)d^{n-1} \frac{M_{n-1}(b, c)}{\sqrt{d}^{n-1}} \cdot \frac{M_n(b, c)}{\sqrt{d}^n} \cdot \frac{\sqrt{d}}{b} \\
&= n(n+1)(n+2) \frac{M_n(b, c)M_{n-1}(b, c)}{b}.
\end{aligned}$$

If $2 \nmid n$ then $b \mid M_n(b, c)$; if $2 \mid n$ then $2 \nmid n-1$ and $b \mid M_{n-1}(b, c)$. So b divides $M_n(b, c)M_{n-1}(b, c)$. Therefore (1.7) holds.

The proof of Theorem 1.3 is now complete. \square

5. SOME OPEN PROBLEMS

Clearly,

$$\frac{\binom{2k}{k}}{2k-1} = \frac{2}{2k-1} \binom{2k-1}{k} = \frac{2}{k} \binom{2k-2}{k-1} = 2C_{k-1} \text{ for } k \in \mathbb{Z}^+,$$

and thus $2k-1 \mid \binom{2k}{k}$ for all $k \in \mathbb{N}$. Motivated by this we introduce a new kind of numbers

$$W_n := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{\binom{2k}{k}}{2k-1} \quad (n = 0, 1, 2, \dots) \quad (5.1)$$

which are analogues of the Motzkin numbers. The values of W_0, W_1, \dots, W_{12} are as follows:

$$-1, -1, 1, 5, 13, 29, 63, 139, 317, 749, 1827, 4575, 11699.$$

Applying the Zeilberger algorithm (cf. [PWZ, pp.101-119]) via **Mathematica 9**, we obtain the recurrence

$$(n+3)W_{n+3} = (3n+7)W_{n+2} + (n-5)W_{n+1} - 3(n+1)W_n \quad (n = 0, 1, 2, \dots).$$

For this new kind of numbers, we have the following conjecture similar to Theorem 1.1.

Conjecture 5.1. (i) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} (8k+9)W_k^2 \equiv n \pmod{2n}. \quad (5.2)$$

Also, for any odd prime p we have

$$\frac{1}{p} \sum_{k=0}^{p-1} (8k+9)W_k^2 \equiv 24 + 10 \left(\frac{-1}{p} \right) - 9 \left(\frac{p}{3} \right) - 18 \left(\frac{3}{p} \right) \pmod{p}. \quad (5.3)$$

(ii) For any prime $p > 3$ and positive integer n , the number

$$\frac{\sum_{k=0}^{pn-1} W_k^2 - 2(\sum_{k=0}^{n-1} T_k)^2}{pn}$$

is always a p -adic integer.

Remark 5.1. We also guess that the sequence $(W_{n+1}/W_n)_{n \geq 5}$ is strictly increasing to the limit 3 and the sequence $(\sqrt[n+1]{W_{n+1}}/\sqrt[n]{W_n})_{n \geq 9}$ is strictly decreasing to the limit 1.

For $h, n \in \mathbb{Z}^+$, we define

$$w_n^{(h)}(x) := \sum_{k=1}^n w(n, k)^h x^{k-1}.$$

Conjecture 5.2. Let $h, m, n \in \mathbb{Z}^+$. Then

$$\frac{(2, n)}{n(n+1)(n+2)} \sum_{k=1}^n k(k+1)(2k+1)w_k^{(h)}(x)^m \in \mathbb{Z}[x]. \quad (5.4)$$

Also,

$$\frac{(2, m-1, n)}{n(n+1)(n+2)} \sum_{k=1}^n (-1)^k k(k+1)(2k+1)w_k(x)^m \in \mathbb{Z}[x], \quad (5.5)$$

and

$$\frac{1}{n(n+1)(n+2)} \sum_{k=1}^n (-1)^k k(k+1)(2k+1)w_k^{(h)}(x)^m \in \mathbb{Z}[x] \text{ for } h > 1. \quad (5.6)$$

Remark 5.2. Fix $n \in \mathbb{Z}^+$. By combining (4.13) with Lemma 4.2, we obtain

$$\frac{(2, n)}{n(n+1)(n+2)} \sum_{k=1}^n k(k+1)(2k+1)s_k(x)^2 \in \mathbb{Z}[x(x+1)]. \quad (5.7)$$

As $s_k(x) = w_k(x)$ for all $k \in \mathbb{Z}^+$ (by Lemma 4.5), this implies (5.4) with $h = 1$ and $m = 2$. Since $w_{2j}(x)/(2x+1) \in \mathbb{Z}[x]$ for all $j \in \mathbb{Z}^+$ (cf. [S18b, Section 4]), (5.5) with $m = 2$ follows from (4.9).

For $h \in \mathbb{Z}^+$ and $n \in \mathbb{N}$, we define

$$D_n^{(h)}(x) := \sum_{k=0}^n \binom{n}{k}^h \binom{n+k}{k}^h x^k \text{ and } S_n^{(h)}(x) := \sum_{k=0}^n \binom{n+k}{2k}^h C_k^h x^k.$$

Note that $S_n^{(1)}(x) = S_n(x)$ for all $n \in \mathbb{N}$.

Conjecture 5.3. *Let $h, m, n \in \mathbb{Z}^+$.*

(i) *We have*

$$\frac{(2, n)}{n(n+1)(n+2)} \sum_{k=1}^n k(k+1)(2k+1)S_k^{(h)}(x)^m \in \mathbb{Z}[x] \quad (5.8)$$

and

$$\frac{(2, m-1, n)}{n(n+1)(n+2)} \sum_{k=1}^n (-1)^k k(k+1)(2k+1)S_k^{(h)}(x)^m \in \mathbb{Z}[x]. \quad (5.9)$$

(ii) *We have*

$$\frac{(2, n)}{n(n+1)(n+2)} \sum_{k=1}^n k(k+1)(2k+1)D_k^{(h)}(x)^m \in \mathbb{Z}[x]$$

and

$$\frac{(2, hm-1, n)}{n(n+1)(n+2)} \sum_{k=1}^n (-1)^k k(k+1)(2k+1)D_k^{(h)}(x)^m \in \mathbb{Z}[x].$$

Remark 5.3. Fix $n \in \mathbb{Z}^+$. As $S_k(x) = (x+1)s_k(x) = (x+1)w_k(x)$ for all $k \in \mathbb{Z}^+$, (5.8) and (5.9) with $h = 1$ and $m = 2$ do hold in view of Remark 5.2. We also conjecture that

$$\frac{2}{3n(n+1)} \sum_{k=1}^n (-1)^{n-k} k^2 D_k D_{k-1} \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^n (-1)^{n-k} (4k^2 + 2k - 1) D_{k-1} s_k$$

are positive odd integers.

Conjecture 5.4. (i) *For any $h, m, n \in \mathbb{Z}^+$ we have*

$$\frac{2(2, n)}{n(n+1)(n+2)} \sum_{k=1}^n k(k+1)(k+2)(w_k^{(h)}(x)w_{k+1}^{(h)}(x))^m \in \mathbb{Z}[x] \quad (5.10)$$

(ii) *For any $m, n \in \mathbb{Z}^+$ we have*

$$\frac{2(2, n)}{n(n+1)(n+2)(2x+1)^m} \sum_{k=1}^n k(k+1)(k+2)(w_k(x)w_{k+1}(x))^m \in \mathbb{Z}[x]. \quad (5.11)$$

If $n \in \mathbb{Z}^+$ is even, then

$$\frac{4}{n(n+1)(n+2)(2x+1)^3} \sum_{k=1}^n k(k+1)(k+2)w_k(x)w_{k+1}(x) \in \mathbb{Z}[x]. \quad (5.12)$$

Remark 5.4. Recall that $w_{2j}(x)/(2x+1) \in \mathbb{Z}[x]$ for all $j \in \mathbb{Z}^+$ (by [S18b, Section 4]).

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