# ON THE NUMBER OF WORDS WITH RESTRICTIONS ON THE NUMBER OF SYMBOLS 

VERÓNICA BECHER AND EDA CESARATTO


#### Abstract

We show that, in an alphabet of $n$ symbols, the number of words of length $n$ whose number of different symbols is away from $(1-1 / e) n$, which is the value expected by the Poisson distribution, has exponential decay in $n$. We use Laplace's method for sums and known bounds of Stirling numbers of the second kind. We express our result in terms of inequalities.


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## 1. Introduction and statement of results

Consider an alphabet of $n$ symbols and let $\chi^{(i)}$ be the number of symbols that appear exactly $i$ times in a word of length $m$. This can be seen as the allocation of $m$ balls (the positions in a word of length $m$ ) in $n$ bins (the $n$ symbols of the alphabet), which determines a total of $n^{m}$ allocations. When $m / n$ is a fixed constant $\lambda$,

$$
\frac{1}{n} \chi^{(i)} \text { converges in probability to } e^{-\lambda} \frac{\lambda^{i}}{i!},
$$

which is the Poisson formula, the proof can be read from [6, Example III. 10 and Proposition V.11].

We are interested in the case when the alphabet size $n$ equals the word length $m$, hence $\lambda=m / n=1$. The number of symbols that do not appear in a word of length $n$ is $\chi^{(0)}$ and its expected value is $n / e$. Hence, the expected number of different symbols in a word of length $n$ is $n-n / e=(1-1 / e) n$. The probability that $\chi^{(0)}$ is equal to $j$ for $j=0,1, \ldots, n$ is expressible in terms of Stirling numbers of the second kind: the number $a(n, j)$ of words of length $n$ with exactly $j$ different symbols is the number of ways to choose $j$ out of $n$ elements times the number of surjective maps from a set of $n$ positions to a set of $j$ symbols. To make such a surjective map, first partition the set of $n$ elements into $j$ nonempty subsets and, in one of the $j$ ! many ways, assign one of these subsets to each element in the set of $j$ elements,

$$
a(n, j)=\binom{n}{j} j!S_{n}^{(j)}
$$

where

$$
S_{n}^{(j)}=\frac{1}{j!} \sum_{i=0}^{j}(-1)^{i}\binom{j}{i}(j-i)^{n}
$$

Notice that

$$
\sum_{j=0}^{n} a(n, j)=n^{n}
$$

Theorem 1 is the main result of this note and shows that in an alphabet of $n$ symbols, the number of words of length $n$ with exactly $j$ symbols, has exponential decay in $n$ when $j$ is away from the value expected by the Poisson distribution. Precisely, Theorem 1 proves that $a(n, j)$, has exponential decay in $n$ when $j$ is away from $(1-1 / e) n$. And this implies that for every positive $\varepsilon<1$,

$$
\sum_{n \geq 1} n^{-n}\left(\sum_{j=1}^{(1-1 / e-\varepsilon) n} a(n, j)+\sum_{j=(1-1 / e+\varepsilon) n}^{n} a(n, j)\right)<\infty
$$

Theorem 1. There is a function $\phi:(0,1) \mapsto \mathbb{R}$ such that $\phi(x)<1$ for every $x \neq 1-1 / e$, positive reals $r$ and $\Lambda$ both less than 1 , and positive constants $c$ and $C$ satisfying the following condition: For every pair $n, j$ of integers with $1 \leq j \leq n$,

$$
\begin{aligned}
& a(n, j) \leq \begin{cases}C \sqrt{n} \Lambda^{n} n^{n} & , \text { if } j / n \in[0, r] \cup[1-r, 1] \\
C \phi(j / n)^{n} n^{n} & , \text { if } j / n \in[r, 1-r]\end{cases} \\
& a(n, j) \geq(c / \sqrt{n}) \phi(j / n)^{n} n^{n}, \text { if } j / n \in[r, 1-r]
\end{aligned}
$$

Precisely,

$$
\begin{aligned}
& \phi:(0,1) \mapsto \mathbb{R}, \quad \phi(x)=\left(e \ln \left(1+e^{-\delta(x)}\right)^{-1} \varphi(x) e^{-x \delta(x)}\right. \\
& \varphi:[0,1] \mapsto \mathbb{R}, \quad \varphi(x)=x^{-x}(1-x)^{-(1-x)}, \varphi(0)=\varphi(1)=1 \\
& \delta:(0,1) \mapsto \mathbb{R}, \quad \delta^{-1}(y)=\frac{1}{\left(1+e^{y}\right) \ln \left(1+e^{-y}\right)}
\end{aligned}
$$

Each of the values $c, C, \Lambda$ and $r$ in the statement of Theorem 1 can be effectively computed. Figure 1 plots the upper bound of $\sqrt[n]{a(n, j)} n^{-1}$ with the function $\phi(j / n)$ given in Theorem 1.

As a straightforward application of Theorem 1 we obtain the following.
Corollary 2. For any positive real number $\varepsilon$ there exist positive constants $c$ and $C$ and a positive real number $\Lambda$ strictly less than 1 such that for every positive integers $n, \ell$,

$$
\text { if } \quad|\ell / n-(1-1 / e)| \geq \varepsilon \quad \text { then } \quad(c / \sqrt{n}) \Lambda^{n} \leq n^{-n} \sum_{j=1}^{\ell} a(n, j) \leq C n \sqrt{n} \Lambda^{n} .
$$

A tail estimate is a quantification of the rate of decrease of probabilities away from the central part of a distribution. It is known that the tail of a given arbitrary discrete distribution has exponential decay if its probability generating function is analytic on a disk centered on zero and of radius greater than 1 [6, Theorem IX.3, page 627]. Theorem 1 gives, indeed, a tail estimate with exponential decay, but our methods are not analytic.



Figure 1. On the left, the graph of $\phi(x)$. On the right, the points are $\sqrt[n]{a(n, j)} n^{-1}$ for $n=200$ and $j=0,5,10, \ldots, 195,200$ and the solid line is $\phi(j / n)$ with $r=0.1$ and $\Lambda \approx 0.701$.

Our proof of Theorem 1 is elementary except for the estimates for Stirling numbers of the second kind that we use as a black box. We follow the principles of Laplace's method for sums, which is useful for sums of positive terms which increase to a certain point and then decrease. For a general explanation with examples we refer to Flajolet and Sedgewick's book [6, p.761], see [10] for a rigorous application to an hypergeometric-type series. However, we do not use the exp-log transformation to build the approximation function.

Specifically, to prove Theorem 1 we give a smooth function $\phi$ so that $\phi(j / n)^{n}$ bounds $a(n, j) n^{-n}$ from above and below (up to multiplicative sequences that increase or decrease slowly). We consider the ratio between $j$ and $n$. When $j$ is near to 0 or near to $n$ we use the classical upper bound of Stirling numbers of the second kind given by Rennie and Dobson [11]. When $j$ is not near to 0 nor near to $n$ we use Bender's approximation of Stirling numbers of the second kind [2] as a black box. This approximation comes from analytic combinatorics methods and it was initially devised by Laplace, then proved by Moser and Wyman [9] and later sharpened by Bender, see also [8]. Our two choices are motivated by the comparison of bounds on Stirling numbers by Rennie-Dobson [11], Arratia and DeSalvo [1], and also a trivial bound, given in Section 2 ,

The approach we use in the proof of Theorem 1 was previously used by one of the authors in two different problems. In [3] it is used to estimate $n!\prod_{i=1}^{k} p_{i}^{j_{i} s} / j_{i}$ ! where each $p_{i}$ is the probability of the symbol $i$ in an alphabet of $k$ elements, $s$ is a real number in $(0,1)$ and the integers $j_{i}$ sum up $n$ and $\sum_{i=1}^{k} i j_{i} \leq M n$ for a fixed $M>1$. In [4, Remark 4.3] the same approach is used to obtain an upper bound for $\binom{n}{j} / j$ ! when $n$ is fixed and $j$ varies. Besides, the asymptotic behavior of these quantities when $n$ tends to infinity was studied using a similar technique in [7].

We crossed the problem solved in the present note when studying the set $\mathcal{S}$ of infinite binary sequences with too many or too few, with respect to the expected by the Poisson distribution, different words of length $\lfloor\log n\rfloor$, counted with no
overlapping in their initial segment of length $n\lfloor\log n\rfloor$, for infinitely many $n \mathrm{~s}$. Corollary 2 allows us to prove that the Lebesgue measure of this set $\mathcal{S}$ is zero, as follows. For simplicity, let $n$ be a power of 2 and let log be the logarithm in base 2. Identify the binary words of length $\log n$ with integers from 0 to $n-1$. Thus, each binary word of length $n \log n$ is identified with a with a word of $n$ integers from 0 to $n-1$. Notice that there are $2^{n \log n}=n^{n}$ many of these binary words. Corollary 2 assumes an alphabet of $n$ symbols and gives an upper bound for the proportion of words of length $n$ having a number of different symbols away from $(1-1 / e) n$, which is the quantity expected by the Poisson distribution. By the identification we made, this yields an upper bound of the proportion of binary words of length $n \log n$ having too many or too few different binary blocks with respect to what is expected by the Poisson distribution. Since this upper bound has exponential decay in $n$, we can apply Borel-Cantelli lemma to show that the sum, for every $n$, of these bounds is finite. Consequently, the Lebesgue measure of the set $\mathcal{S}$ is zero. A different proof of this result follows from the metric theorem given by Benjamin Weiss and Yuval Peres in [13] where they show that the set of Poisson generic sequences on a finite alphabet has Lebesgue measure 1. Their proof is probabilistic, with a randomized part and a concentration part.

## 2. On different bounds on Stirling numbers of SECOND Kind

We compare four estimates on Stirling numbers of the second kind $S_{n}^{(j)}$. When $j / n$ belongs to $(0,1)$, we consider a trivial bound, Rennie and Dobson's bound [11] and Arratia and DeSalvo's bounds [1]. When $j / n$ belongs to a closed interval included in $(0,1)$, we consider Bender's estimate [2]. We start by giving bounds for the binomial coefficients.
2.1. Binomial coefficients. Consider the following bounds for the factorial which are consequence of the classical Stirling's formula for the factorial, see [12],

$$
n!=\sqrt{2 \pi} n^{n+1 / 2} e^{-n+r_{n}}, \quad \frac{1}{12 n+1} \leq r_{n} \leq \frac{1}{12 n}
$$

Then, for any $n \geq 1$,

$$
\begin{equation*}
\sqrt{2 \pi} n^{n+1 / 2} e^{-n} \leq n!\leq \sqrt{2 \pi} e^{1 / 12} n^{n+1 / 2} e^{-n} \tag{1}
\end{equation*}
$$

In the sequel we write $a \approx b$ to indicate that the two numbers $a$ and $b$ coincide up to the precision explicitly indicated, but they may differ in the fractional part that is not exhibited. For example, $\pi \approx 3.14159$. From this approximation of the factorial, we obtain bounds for the binomial coefficient that involve the following functions,

$$
\begin{align*}
\varphi:[0,1] \mapsto \mathbb{R}, & \varphi(x)=x^{-x}(1-x)^{-(1-x)}, \varphi(0)=\varphi(1)=1  \tag{2}\\
\gamma:(0,1) \mapsto \mathbb{R}, & \gamma(x)=\left(x-x^{2}\right)^{1 / 2}
\end{align*}
$$

There exist constants $c_{0}$ and $C_{0}$ such that for any pair of integers $n, j$ where $n \geq 2$ and $1 \leq j \leq n-1$,

$$
\frac{c_{0}}{\sqrt{n} \gamma(j / n)} \varphi(j / n)^{n} \leq\binom{ n}{j} \leq \frac{C_{0}}{\sqrt{n} \gamma(j / n)} \varphi(j / n)^{n}
$$

The constants $c_{0}$ and $C_{0}$ can be chosen as $c_{0}=\left(\sqrt{2 \pi} e^{1 / 6}\right)^{-1} \approx 0.33$ and $C_{0}=e^{1 / 12}(\sqrt{2 \pi})^{-1} \approx 0.43$. From (1), it follows that

$$
\begin{equation*}
\binom{n}{j} \leq e^{1 / 12}(\sqrt{2 \pi})^{-1}\left(\frac{n}{j(n-j)}\right)^{1 / 2} \frac{n^{n}}{j^{j}(n-j)^{n-j}} \tag{3}
\end{equation*}
$$

First, notice that

$$
\left(\frac{n}{j(n-j)}\right)^{1 / 2}=\frac{n^{1 / 2}}{n(j / n(1-j / n))^{1 / 2}}=\frac{1}{\sqrt{n} \gamma(j / n)} .
$$

Now, we deal with the last factor of (3). The following holds:

$$
\begin{aligned}
\frac{n^{n}}{j^{j}(n-j)^{n-j}} & =\frac{n^{n}}{n^{n}(j / n)^{j}(1-j / n)^{n-j}} \\
& =\left((j / n)^{-j / n}(1-j / n)^{-(1-j / n)}\right)^{n} \\
& =\varphi(j / n)^{n}
\end{aligned}
$$

This proves the upper bound on the binomial coefficient. The proof of the lower bound is similar, except that the factor $e^{1 / 12}$ appears twice in the denominator.

Finally, we remark that for any pair of positive integers $n, j$ such that $n \geq 2$ and $1 \leq j \leq n-1$, we have $\min \{j(n-j): 1 \leq j \leq n-1\}=n-1$ (this value is attained at $j=1$ or $j=n-1$ ). Also $n-1 \geq n / 2$ for $n \geq 2$. Hence,

$$
\begin{aligned}
& \gamma(j / n)=\left(\frac{j(n-j)}{n^{2}}\right)^{1 / 2} \geq\left(\frac{n}{2 n^{2}}\right)^{1 / 2}=\frac{\sqrt{2}}{2} n^{-1 / 2}, \text { and } \\
& \gamma(j / n) \leq \max \{\gamma(x): x \in[0,1]\}=\max \left\{\left(x-x^{2}\right)^{1 / 2}: x \in[0,1]\right\} \leq 1 / 2
\end{aligned}
$$

Thus, multiplying by $\sqrt{n}$,

$$
\sqrt{2} / 2 \leq \sqrt{n} \gamma(j / n) \leq(1 / 2) \sqrt{n}
$$

which implies

$$
\frac{2}{\sqrt{n}} \leq \frac{1}{\sqrt{n} \gamma(j / n)} \leq \sqrt{2}
$$

We have that $2 c_{0} \approx 0.67>1 / 2$ and $\sqrt{2} C_{0} \approx 0.61<1$. This shows the following inequalities, for every positive $n \geq 2$ and every $j$ such that $1 \leq j \leq n-1$,

$$
\begin{equation*}
\frac{1}{2 \sqrt{n}} \varphi(j / n)^{n} \leq \frac{2 c_{0}}{\sqrt{n}} \varphi(j / n)^{n} \leq\binom{ n}{j} \leq \sqrt{2} C_{0} \varphi(j / n)^{n} \leq \varphi(j / n)^{n} \tag{4}
\end{equation*}
$$

2.2. A trivial bound on Stirling numbers the second kind. The simplest upper bound takes just the first term of the alternating sum that defines $S_{n}^{(j)}$,

$$
S_{n}^{(j)} \leq j^{n} / j!
$$

This upper bound appears explicitly taking just one term in Bonferroni inequalities, see [5, Section 4.7]. First remark that the upper bound given in (1) for the factorial yields
(5) $\frac{j^{n}}{j!} \leq \frac{j^{n}}{\sqrt{2 \pi j} j^{j} e^{-j}}=\frac{1}{\sqrt{2 \pi j}} \frac{n^{n-j}(j / n)^{n} e^{j}}{(j / n)^{j}}=\frac{1}{\sqrt{2 \pi j}}\left(n^{1-j / n}(j / n)^{1-j / n} e^{j / n}\right)^{n}$.

The same lines together with the lower bound for (1) give a lower bound for $j^{n} / j$ !.
Let $\theta:[0,1] \mapsto \mathbb{R}$,

$$
\begin{equation*}
\theta(x)=x^{1-x} e^{x} \tag{6}
\end{equation*}
$$

It follows that

$$
\frac{1}{e^{1 / 12} \sqrt{2 \pi j}}\left(n^{1-j / n} \theta(j / n)\right)^{n} \leq j^{n} / j!\leq \frac{1}{\sqrt{2 \pi j}}\left(n^{1-j / n} \theta(j / n)\right)^{n}
$$

Consequently,

$$
\begin{equation*}
S_{n}^{(j)} \leq \frac{1}{\sqrt{2 \pi j}}\left(n^{1-j / n} \theta(j / n)\right)^{n} \tag{7}
\end{equation*}
$$

2.3. Rennie and Dobson's bound. The following is the classsical upper bound of Stirling numbers of the second kind given by Rennie and Dobson [11], which holds for every positive $n$ and every $j$ such that $1 \leq j \leq n-1$,

$$
\begin{equation*}
S_{n}^{(j)} \leq \frac{1}{2}\binom{n}{j} j^{n-j} \tag{8}
\end{equation*}
$$

Let $\eta:[0,1] \mapsto \mathbb{R}$,

$$
\begin{equation*}
\eta(x)=x^{1-x} \varphi(x) \tag{9}
\end{equation*}
$$

where $\varphi$ is defined in (2). Since $j^{n-j}=\left(n^{1-j / n}(j / n)^{1-j / n}\right)^{n}$, the bounds on the binomial given in (4) imply

$$
\begin{equation*}
\frac{1}{2 \sqrt{n}}\left(n^{1-j / n} \eta(j / n)\right)^{n} \leq\binom{ n}{j} j^{n-j} \leq\left(n^{1-j / n} \eta(j / n)\right)^{n} \tag{10}
\end{equation*}
$$

2.4. Arratia and DeSalvo's bound. Arratia and DeSalvo 11, Theorems 5 and 6 ] give these bounds for $n \geq 3$ and $1 \leq j \leq n-2$,

$$
\begin{aligned}
& S_{n}^{(j)} \leq A_{5}(n, j) \\
& S_{n}^{(j)} \leq A_{6}(n, j)
\end{aligned}
$$

where

$$
\begin{aligned}
& A_{5}(n, j):=\binom{N}{n-j} e^{-2 \mu_{5}(n, j)}\left(1+e^{2 \mu_{5}(n, j)} D_{5}(n, j)\right) \\
& A_{6}(n, j):: \frac{N^{n-j}}{(n-j)!} e^{-\mu_{6}(n, j)}\left(1+e^{\mu_{6}(n, j)} D_{6}(n, j)\right) \\
& N::\binom{n}{2} \\
& \mu_{5}(n, j):=\binom{(n-j)}{2}\binom{n}{3} /\binom{N}{2} \\
& \mu_{6}(n, j):=\binom{(n-j)}{2} \frac{n(n-1)(4 n-5)}{6 N^{2}} \\
& d_{5}(n, j):=P+Q+(1-Q)((n-j)-2)(R+S+T) \text { where } \\
& P:=\frac{2\binom{n}{3}}{\binom{N}{2}} \\
& Q:=\frac{13-12(n-j)+3(n-j)^{2}}{\binom{N}{2}} \\
& R:=\frac{8\binom{n}{3}}{\binom{N}{2}}
\end{aligned}
$$

$$
\begin{aligned}
& S:=\frac{6\binom{n}{4}}{\binom{n}{3}(N-2)} \\
& T:=\frac{1}{(N-2)}\left(\frac{5 n-11}{4}\right) \\
& d_{6}(n, j):=U+2(V+W+X) \text { where } \\
& U:=\frac{n(n-1)(4 n-5)}{6 N^{2}} \\
& V:=4((n-j)-2) \frac{n(n-1)(2 n-1)}{6 N^{2}} \\
& W:=\frac{3((n-j)-2) n(n-1)}{(4 n-5) N} \\
& Z:=\frac{2((n-j)-2)(2 n-1)(n+1)}{(4 n-5) N} \\
& D_{5}(n, j):=\min \left\{d_{5}(n, j), 2 \mu_{5}(n, j) d_{5}(n, j), 1\right\} \\
& D_{6}(n, j):=\min \left\{\left(d_{6}(n, j), 2 \mu_{6}(n, j) d_{6}(n, j), 1\right\} .\right.
\end{aligned}
$$

The goal of this section is to give bounds on $A_{5}(n, j)$ and $A_{6}(n, j)$ from below and above. They are displayed in Proposition 5, and the proofs of these bounds rely on Lemma 3 and Lemma 4. In the sequel when we write $A_{5,6}$ we denote two statements, one about $A_{5}$ and one about $A_{6}$. Similarly for $D_{5,6}$ and $\mu_{5,6}$

Lemma 3. For any $n \geq 3$ and $1 \leq j \leq n-2$,

$$
\frac{1}{2 n^{2}} \leq e^{-\mu_{5,6}(n, j)}\left(1+e^{\mu_{5,6}(n, j)} D_{5,6}(n, j)\right) \leq 2
$$

Proof. By definition, $D_{5,6}(n, j) \leq 1$, and clearly $\mu_{5,6}(n, j) \geq 0$, then

$$
e^{-\mu_{5,6}(n, j)}\left(1+e^{\mu_{5,6}(n, j)} D_{5,6}(n, j)\right)=e^{-\mu_{5,6}(n, j)}+D_{5,6}(n, j) \leq 2
$$

To obtain a lower bound for $e^{-\mu_{5,6}(n, j)}+D_{5,6}(n, j)$, it suffices to bound the quantities $D_{5,6}(n, j)$.

Lower bound for $D_{5}(n, j)$. First we consider $\mu_{5}(n, j)$. The equality

$$
\begin{equation*}
\binom{N}{2}=\frac{1}{2} \frac{n(n-1)}{2}\left(\frac{n(n-1)}{2}-1\right)=\frac{(n+1) n(n-1)(n-2)}{8} \tag{11}
\end{equation*}
$$

yields

$$
\mu_{5}(n, j)=\frac{2}{3} \frac{(n-j)(n-j-1)}{n+1}
$$

This quantity, $\mu_{5}(n, j)$, takes its minimum when $j=n-2$. It follows that

$$
\mu_{5}(n, j) \geq \frac{1}{n}
$$

We claim that the quantity $d_{5}(n, j)=P+Q+(1-Q)((n-j)-2)(R+S+T)$ satisfies that

$$
d_{5}(n, j) \geq P \quad \text { for any } \quad 1 \leq j \leq n-2 .
$$

It is clear that $Q$ and $((n-j)-2)(R+S+T)$ are nonnegative. It only remains to prove that $1-Q$ is nonnegative. In fact, $Q \leq 1 / 2$ for the values of $n$ and $j$ under
consideration. To prove that, first, we complete squares and apply 11; then we take $j=1$, and finally, we maximize over over $n$ to obtain the last inequality,

$$
Q=8 \frac{3((n-j)-2)^{2}+1}{(n+1) n(n-1)(n-2)} \leq 8 \frac{3(n-3)^{2}+1}{(n+1) n(n-1)(n-2)} \leq \frac{1}{2}
$$

Finally,

$$
d_{5}(n, j) \geq P=2 \frac{\binom{n}{3}}{\binom{N}{2}}=\frac{8}{3(n+1)} \geq \frac{1}{n}
$$

From the last lower bound and the bound $\mu_{5}(n, j) \geq 1 / n$, we get the following:

$$
D_{5}(n, j):=\min \left(d_{5}(n, j), 2 \mu_{5}(n, j) d_{5}(n, j), 1\right) \geq \min \left(\frac{1}{n}, \frac{2}{n^{2}}, 1\right) \geq \frac{1}{n^{2}}
$$

Lower bound for $D_{6}(n, j)$. First we consider $\mu_{6}(n, j)$. By definition, $N=\binom{n}{2}$. It turns out that for every $n \geq 3$ and $j$ such that $1 \leq j \leq n-2$,

$$
\mu_{6}(n, j)=\binom{(n-j)}{2} \frac{n(n-1)(4 n-5)}{6 N^{2}}=\frac{1}{3} \frac{(n-j)(n-j-1)(4 n-5)}{n(n-1)} \geq \frac{1}{n}
$$

All the terms involved in the sum defining $d_{6}(n, j)$ are non-negative. Hence,

$$
d_{6}(n, j) \geq U=\frac{n(n-1)(4 n-5)}{6 N^{2}}=\frac{2}{3} \frac{4 n-5}{n(n-1)} \geq \frac{2}{3 n} .
$$

Finally, the following holds and completes the proof of this lemma.

$$
D_{6}(n, j):=\min \left(d_{6}(n, j), 2 \mu_{6}(n, j) d_{6}(n, j), 1\right) \geq \min \left(\frac{2}{3 n}, \frac{4}{3 n^{2}}, 1\right) \geq \frac{1}{2 n^{2}}
$$

Let $\kappa$ be the map from $[0,1]$ to $\mathbb{R}$ given by

$$
\begin{equation*}
\kappa(x)=(e / 2)^{1-x}(1-x)^{-(1-x)}, \quad \kappa(1)=1 \tag{12}
\end{equation*}
$$

Lemma 4. For any $n \geq 3$ and $j$ with $1 \leq j \leq n-2$, the following holds

$$
\begin{align*}
\frac{e^{-2}}{2 \sqrt{n(n-1)}}\left(n^{1-j / n} \kappa(j / n)\right)^{n} & \leq\binom{ N}{n-j} \leq\left(n^{1-j / n} \kappa(j / n)\right)^{n}  \tag{13}\\
\frac{1}{4 \sqrt{2 \pi} \sqrt{n}}\left(n^{1-j / n} \kappa(j / n)\right)^{n} & \leq \frac{N^{n-j}}{(n-j)!} \leq \frac{1}{\sqrt{2 \pi}}\left(n^{1-j / n} \kappa(j / n)\right)^{n} \tag{14}
\end{align*}
$$

Proof. We start by proving Inequality (13). With the bounds given for the binomial coefficients in (4), the following holds

$$
\begin{equation*}
\frac{1}{2 \sqrt{N}} \varphi((n-j) / N)^{N} \leq\binom{ N}{n-j} \leq \varphi((n-j) / N)^{N} \tag{15}
\end{equation*}
$$

with $\varphi(x)=x^{-x}(1-x)^{-(1-x)}$, for any $n \geq 3$ and $1 \leq j \leq n-2$. The expression $\varphi((n-j) / N)^{N}$ has two factors, the first one corresponds to $x^{-x}$ and the second one corresponds to $(1-x)^{1-x}$. We replace $N$ by $n(n-1) / 2$ only in the first factor. The exponent of the second factor is multiplied and divided by $N /(n-j)$. This leads to the following equality

$$
\varphi\left(\frac{(n-j)}{N}\right)^{N}=n^{n-j}\left(1-\frac{1}{n}\right)^{n-j}\left(2\left(1-\frac{j}{n}\right)\right)^{-(n-j)} b(n, j)
$$

with

$$
\begin{equation*}
b(n, j)=\left(\left(1-\frac{n-j}{N}\right)^{\frac{N}{n-j}}\right)^{-(n-j)+\frac{(n-j)^{2}}{N}} \tag{16}
\end{equation*}
$$

Let $c(n, j)$ be defined as

$$
c(n, j)=\left(1-\frac{1}{n}\right)^{n-j} b(n, j)
$$

The right hand side of the equality before $\sqrt{16}$ is the product of four factors. We leave the first and the third as they are. We deal with the second and the fourth. The factor $(1-1 / n)^{n-j}$ satisfies

$$
\begin{equation*}
e^{-1} \leq\left(1-\frac{1}{n}\right)^{n-1} \leq\left(1-\frac{1}{n}\right)^{n-j} \leq 1 \tag{17}
\end{equation*}
$$

The right hand side inequality is due to the fact that $1-1 / n \leq 1$. The left hand side inequality is due to the fact that $(1-1 / n)^{n-1}$ decreases towards its limit as $n \rightarrow \infty$.

We study $b(n, j)$, defined in 16 . First, we use the classical inequality

$$
-x-x^{2} \leq \ln (1-x) \leq-x \quad(0<x \leq 2 / 3)
$$

After multiplying by $1 / x$ and taking powers, we get

$$
\begin{equation*}
e^{-1-x} \leq(1-x)^{1 / x} \leq e^{-1} \quad(0<x \leq 2 / 3) \tag{18}
\end{equation*}
$$

Observe that, for $j \geq 1$,

$$
\frac{n-j}{N}=\frac{2(n-j)}{n(n-1)} \leq \frac{2}{n}
$$

Notice that $0<(n-j) / N \leq 2 / 3$ since $n \geq 3$. This allows us to replace $x$ by $(n-j) / N$ in 18). It turns out that

$$
e^{-1-\left(\frac{n-j}{N}\right)} \leq\left(1-\frac{n-j}{N}\right)^{\frac{N}{n-j}} \leq e^{-1}
$$

To obtain $b(n, j)$, consider the previous expressions to the power $-(n-j)+(n-j)^{2} / N$. With our bound on $(n-j) / N$, the exponent of the left hand side satisfies

$$
\left(-1-\frac{n-j}{N}\right)\left(-(n-j)+\frac{(n-j)^{2}}{N}\right)=(n-j)-\frac{(n-j)^{3}}{N^{2}} \geq(n-j)-1
$$

Finally,

$$
\begin{equation*}
e^{-1} e^{(n-j)} \leq b(n, j) \leq e^{(n-j)-\frac{(n-j)^{2}}{N}} \leq e^{(n-j)} \tag{19}
\end{equation*}
$$

From Inequalities 17$)$ and 19$)$, it follows that $c(n, j) e^{-(n-j)}$ takes values in $\left[e^{-2}, 1\right]$ and the following holds

$$
\begin{aligned}
\varphi\left(\frac{(n-j)}{N}\right)^{N} & =c(n, j) n^{n-j}\left(2\left(1-\frac{j}{n}\right)\right)^{-(n-j)} \\
& =c(n, j) e^{-(n-j)}\left(n^{1-j / n} \kappa(j / n)\right)^{n}
\end{aligned}
$$

To end the proof of Inequality (13) consider Inequality 15 together with the fact that $N=n(n-1) / 2$.

Proof of Inequality (14). Approximating the factorial by (1); extracting $n$ as a common factor in $(n-j)^{n-j}$, in $(n-1)^{n-j}$, and in $(n-1)^{n-j}$; and writing the final expression as an $n$-th power (similarly to what it is done in (5), we get

$$
\begin{aligned}
\frac{N^{n-j}}{(n-j)!} & \leq \frac{n^{n-j}(n-1)^{n-j}}{2^{n-j}} \frac{e^{n-j}}{\sqrt{2 \pi(n-j)}(n-j)^{n-j}} \\
& =\frac{(1-1 / n)^{n-j}}{\sqrt{2 \pi(n-j)}}\left(n^{1-j / n} \kappa(j / n)\right)^{n}
\end{aligned}
$$

We obtain the lower bound similarly,

$$
\frac{N^{n-j}}{(n-j)!} \geq e^{-1 / 12} \frac{(1-1 / n)^{n-j}}{\sqrt{2 \pi(n-j)}}\left(n^{1-j / n} \kappa(j / n)\right)^{n}
$$

Finally, with Inequality (17), and since $1 \leq n-j \leq n$, we obtain the bounds

$$
\frac{1}{4 \sqrt{2 \pi n}} \leq \frac{e^{-1-1 / 12}}{\sqrt{2 \pi n}} \leq \frac{(1-1 / n)^{n-j}}{\sqrt{2 \pi(n-j)}} \leq \frac{1}{\sqrt{2 \pi}}
$$

that prove the estimates on $N^{n-j} /(n-j)$ !.
The next Proposition 5 is a direct consequence of Lemmas 3 and 4 Recall that $\kappa:[0,1] \rightarrow \mathbb{R}$ defined in $12, \kappa(x)=(e / 2)^{1-x}(1-x)^{-(1-x)}, \quad \kappa(1)=1$.

Proposition 5. For any $n \geq 3$ and $1 \leq j \leq n-2$,

$$
\begin{equation*}
\frac{e^{-2}}{4 n^{3}}\left(n^{1-j / n} \kappa(j / n)\right)^{n} \leq A_{5,6}(n, j) \leq 2\left(n^{1-j / n} \kappa(j / n)\right)^{n} \tag{20}
\end{equation*}
$$

Proof. Lemma 3 proves that, for any $n \geq 3$ and $1 \leq j \leq n-2$,

$$
\frac{1}{2 n^{2}} \leq e^{-\mu_{5,6}(n, j)}\left(1+e^{\mu_{5,6}(n, j)} D_{5,6}(n, j)\right) \leq 2
$$

Then,

$$
\begin{aligned}
\frac{1}{2 n^{2}}\binom{N}{n-j} & \leq A_{5}(n, j)
\end{aligned} \leq 2\binom{N}{n-j}, ~=\frac{N^{n-j}}{(n-j)!},
$$

Lemma 4 provides us bounds on the terms involving combinatorials and factorials and gives

$$
\begin{aligned}
\frac{e^{-2}}{2 \sqrt{n(n-1)}}\left(n^{1-j / n} \kappa(j / n)\right)^{n} & \leq\binom{ N}{n-j} \leq\left(n^{1-j / n} \kappa(j / n)\right)^{n} \\
\frac{1}{4 \sqrt{2 \pi} \sqrt{n}}\left(n^{1-j / n} \kappa(j / n)\right)^{n} & \leq \frac{N^{n-j}}{(n-j)!} \leq \frac{1}{\sqrt{2 \pi}}\left(n^{1-j / n} \kappa(j / n)\right)^{n}
\end{aligned}
$$

Finally,

$$
\left.\begin{array}{rl}
\frac{e^{-2}}{4 n^{3}}\left(n^{1-j / n} \kappa(j / n)\right)^{n} & \leq A_{5}(n, j)
\end{array}\right) 2\left(n^{1-j / n} \kappa(j / n)\right)^{n}, ~=\frac{1}{\sqrt{2 \pi}}\left(n^{1-j / n} \kappa(j / n)\right)^{n} .
$$

Combining both inequalities, Proposition 5 follows.

ON THE NUMBER OF WORDS WITH RESTRICTIONS ON THE NUMBER OF SYMBOLS11
2.5. Bender's estimate. The notation $r_{n} \sim s_{n}$ indicates that $\lim _{n \rightarrow \infty} r_{n} / s_{n}=1$ when $n \rightarrow \infty$. Bender [2] establishes that for any real number $r$ such that $0<r<1 / 2$, then

$$
S_{n}^{(j)} \sim \frac{n!e^{-\alpha j}}{j!\rho^{n+1}\left(1+e^{\alpha}\right) \sigma \sqrt{2 \pi n}}
$$

uniformly for $j / n \in[r, 1-r]$, where $\alpha$ is such that

$$
\frac{n}{j}=\left(1+e^{\alpha}\right) \ln \left(1+e^{-\alpha}\right)
$$

and

$$
\begin{aligned}
\rho & =\ln \left(1+e^{-\alpha}\right) \\
\sigma^{2} & =\left(\frac{j}{n}\right)^{2}\left(1-e^{\alpha} \ln \left(1+e^{-\alpha}\right)\right)
\end{aligned}
$$

We introduce two functions to describe the behavior of $S_{n}^{(j)}$ in terms of $j / n$ (see Fig. 22,

$$
\begin{array}{ll}
\psi:(0,1) \mapsto \mathbb{R}, & \psi(x)=\frac{e^{-((1-x)+x \delta(x))}}{x^{x} \ln \left(1+e^{-\delta(x)}\right)}  \tag{21}\\
\mu:(0,1) \mapsto \mathbb{R}, & \mu(x)=\left(x\left(1-e^{\delta(x)} \ln \left(1+e^{-\delta(x)}\right)\right)\right)^{1 / 2}
\end{array}
$$

where $\delta:(0,1) \mapsto \mathbb{R}$ is defined by

$$
\begin{equation*}
\delta^{-1}(y)=\frac{1}{\left(1+e^{y}\right) \ln \left(1+e^{-y}\right)} \tag{22}
\end{equation*}
$$

The next lemma rephrases Bender's estimate using $\psi(j / n)$ and $\mu(j / n)$.
Lemma 6. For any positive real number $r$ such that $0<r<1 / 2$ and for any real number $C>1$ there exists an integer $n_{0}=n_{0}(r, C) \geq 2$ such that for every integer $n \geq n_{0}$ and for every integer $j$ with $1 \leq j \leq n-1$ and $j / n \in[r, 1-r]$.

$$
\frac{e^{-1 / 12}}{C \sqrt{2 \pi n} \mu(j / n)}\left(n^{1-j / n} \psi(j / n)\right)^{n} \leq S_{n}^{(j)} \leq \frac{e^{1 / 12} C}{\sqrt{2 \pi n} \mu(j / n)}\left(n^{1-j / n} \psi(j / n)\right)^{n}
$$

Proof. Observe that

$$
\left(1+e^{\alpha}\right) \rho \sigma=\left(1-e^{\alpha} \ln \left(1+e^{-\alpha}\right)\right)^{1 / 2}
$$

Thus, Bender's estimate implies that for any $r$ with $0<r<1 / 2$ and for any $C>1$ there exists $n_{0}=n_{0}(r, C)$ such that for any pair of positive integers $n, j$, with $n \geq n_{0}$ and $j / n \in[r, 1-r]$,

$$
\begin{equation*}
\frac{1}{C} T_{\alpha}(n, j) \leq S_{n}^{(j)} \leq C T_{\alpha}(n, j) \tag{23}
\end{equation*}
$$

where

$$
T_{\alpha}(n, j)=\frac{n!}{j!} \frac{e^{-\alpha j}}{\rho^{n}\left(1-e^{\alpha} \ln \left(1+e^{-\alpha}\right)\right)^{1 / 2} \sqrt{2 \pi n}}
$$

Using (1) we have

$$
e^{-1 / 12} e^{j-n} \frac{\sqrt{n}}{\sqrt{j}} \frac{n^{n}}{j^{j}} \leq \frac{n!}{j!} \leq e^{1 / 12} e^{j-n} \frac{\sqrt{n}}{\sqrt{j}} \frac{n^{n}}{j^{j}}
$$




Figure 2. Graphs of $\psi(x)$ and $\mu(x)$.

We remark that

$$
e^{j-n} \frac{n^{n}}{j^{j}}=\left(e^{-(1-j / n)}(j / n)^{-j / n}\right)^{n} .
$$

Then, using the expressions for $\psi(n / j)$ and $\mu(j / n)$,

$$
\frac{e^{-1 / 12}}{\sqrt{2 \pi n} \mu(j / n)}\left(n^{\frac{(n-j)}{n}} \psi(j / n)\right)^{n} \leq T_{\alpha}(n, j) \leq \frac{e^{1 / 12}}{\sqrt{2 \pi n} \mu(j / n)}\left(n^{\frac{(n-j)}{n}} \psi(j / n)\right)^{n} .
$$

Combining these inequalities with (23) we obtain the wanted result.
The functions $\psi(x)$ and $\mu(x)$ are smooth and concave in the open interval $(0,1)$. The function $\delta^{-1}(y)$ is increasing and

$$
\lim _{x \rightarrow 0^{+}} \delta(x)=-\infty \text { and } \lim _{x \rightarrow 1^{-}} \delta(x)=+\infty .
$$

From this, it is clear that $\lim _{x \rightarrow 0^{+}} \psi(x)=0, \lim _{x \rightarrow 1^{-}} \psi(x)=1$, and $\lim _{x \rightarrow 0^{+}} \mu(x)=$ $\lim _{x \rightarrow 1^{-}} \mu(x)=0$. Then, the bounds given in Lemma 6 become indeterminate when $j / n$ is near 0 or 1 . This is why $j / n$ must be in a central interval in $(0,1)$.

The next corollary is a straightforward consequence of Lemma 6 and the fact that $\mu(x)$ is uniformly bounded on any closed interval included in $(0,1)$. The constants $c_{1}$ and $C_{1}$ in the statement of Corollary 7 can be chosen as the minimum and maximum values of $\{\mu(x): x \in[r, 1-r]\}$.

Corollary 7. For any positive real number $r$ such that $0<r<1 / 2$, there exist $c_{1}$ and $C_{1}$ such that for every pair of positive integers $n, j$ with $j / n \in[r, 1-r]$ we have

$$
\begin{equation*}
e^{-1 / 12} \frac{c_{1}}{\sqrt{2 \pi n}}\left(n^{1-j / n} \psi(j / n)\right)^{n} \leq S_{n}^{(j)} \leq e^{1 / 12} \frac{C_{1}}{\sqrt{2 \pi n}}\left(n^{1-j / n} \psi(j / n)\right)^{n} . \tag{24}
\end{equation*}
$$

2.6. A plot. The four upper bounds given in (7), (10), (20) and (24) are of the form

$$
S_{n}^{(j)} \leq n^{n-j} \text { bound }
$$

In order to visualize them we divide both sides by $n^{n-j}$ and we take $n$-th root in both sides.

$$
\left(S_{n}^{(j)} / n^{n-j}\right)^{1 / n} \leq \text { bound }^{1 / n}
$$

In the four cases bound ${ }^{1 / n}$ is of the form

$$
\text { expression }^{1 / n}\left(f^{n}\right)^{1 / n}
$$

where expression ${ }^{1 / n}$ goes to 1 as $n$ goes to infinity and $f$ is either $\theta, \eta, \kappa$ or $\psi$. Thus, we ignore expression ${ }^{1 / n}$. Figure 3 plots the following:

In dotted line, the exact value

$$
\widehat{S}_{n}^{(j)}=\left(S_{n}^{(j)} / n^{n-j}\right)^{1 / n}
$$

The graphic of the function $\theta$ involved in the trivial bound $\widehat{S}_{n}^{(j)} \leq \frac{1}{(\sqrt{2 \pi j})^{1 / n}} \theta(j / n)$, where $\theta(x)$ is given in (6).

The graphic of the function $\eta$ involved in Rennie and Dobson's bound $\widehat{S}_{n}^{(j)} \leq \frac{1}{2^{1 / n}} \eta(j / n)$, where $\eta(x)$ is given in (9).

The graphic of the function $\kappa$ involved in Arratia and DeSalvo's bound $\widehat{S}_{n}^{(j)} \leq 2^{1 / n} \kappa(j / n)$, where $\kappa(x)$ is given in 12).

In stroke gray line, Bender's estimate
$\widehat{S}_{n}^{(j)} \leq\left(e^{1 / 12} \frac{C_{1}}{\sqrt{2 \pi n}}\right)^{1 / n} \psi(j / n)$,
where $\psi(x)$ is given in 21 and $C_{1}$ in Corollary 7
with $j / n \in[r, 1-r]$ for any real $r$ such that $0<r<1 / 2$.
The constant $C_{1}$ depends on $r$. In the plot of Figure $3, r=0.1$.


Figure 3. Comparison of the normalized Stirling numbers of the second kind $\widehat{S}_{n}^{(j)}=\left(S_{n}^{(j)} / n^{n-j}\right)^{1 / n}$ (in dotted line), for $n=100$ and $j=1, . ., 100$, with the four estimates (in solid lines).

## 3. Application to our problem

For the proof of Theorem 1 we must give an upper bounds of $a(n, j)$, which is always a positive term. Since $a(n, j)=\binom{n}{j} j!S_{n}^{(j)}$, we can use upper bounds for the Stirling numbers of the second kind. We choose Rennie and Dobson's bound in the case $j / n$ is near 0 or 1 , and the bound originated in Bender's estimate when $j / n$ is in $[1 / r, 1-1 / r]$, for $r>0$.
3.1. When the ratio $j / n$ is near 0 or 1 . The next lemma expresses this bound in terms of the ratio $j / n$ with the help of the function

$$
\begin{equation*}
\nu:[0,1] \rightarrow \mathbb{R}, \quad \nu(x)=x e^{-x} \varphi(x)^{2} \tag{25}
\end{equation*}
$$

where $\varphi(x)$ is defined in 22 .
Lemma 8. For any pair of positive integers $n, j$ such that $n \geq 1$ and $1 \leq j \leq n-1$,

$$
a(n, j) n^{-n} \leq \sqrt{j} \nu(j / n)^{n} .
$$

Proof. Recall that $a(n, j)=\binom{n}{j} j!S_{n}^{(j)}$. Rennie and Dobson's upper bound (8) for $S_{n}^{(j)}$ yields

$$
a(n, j) \leq \frac{1}{2}\binom{n}{j}^{2} j!j^{n-j}
$$

We apply the estimates (1) for the factorial. Then we use the upper bound for the binomial coefficient given in (4) that involves the constant $C_{0}=e^{1 / 12}(\sqrt{2 \pi})^{-1}$, which yields

$$
\begin{aligned}
\frac{1}{2}\binom{n}{j}^{2} j!j^{n-j} & \leq \frac{1}{2}\left(\sqrt{2} C_{0}\right)^{2} e^{1 / 12} \sqrt{2 \pi} \sqrt{j} \varphi(j / n)^{2} e^{-j} j^{n} \\
& \leq \frac{e^{1 / 4}}{\sqrt{2 \pi}} \sqrt{j} \nu(j / n)^{n} \\
& \leq \sqrt{j} \nu(j / n)^{n}
\end{aligned}
$$

The function $\nu(x)$ is smooth and concave, $\nu(0)=0$, and $\nu(1)=e^{-1}$. The bound given in Lemma 8 is tight when $j / n$ is near 0 or 1 . However, it is not good when $j / n$ takes values in middle of the interval $[0,1]$. In fact, this bound satisfies $\sqrt{j}(\nu(1 / 2))^{n} \geq \sqrt{j}(1.1)^{n}>1$ but we know that $n^{-n} a(n, j) \leq 1$ for any choice of $j$ and $n$. This leads us to consider the only two real numbers $x_{0}$ and $x_{1}$ in $[0,1]$ for which $\nu\left(x_{0}\right)=\nu\left(x_{1}\right)=1$ and $x_{0}<x_{1}$. These numbers are $x_{0} \approx 0.387$ and $x_{1} \approx 0.790$. Figure 4 displays the graphs of $\nu(x)$ and $\varphi(x)$.

Lemma 9. Let $x_{0}$ and $x_{1}$ be such that $0<x_{0}<x_{1}<1$ and $\nu\left(x_{0}\right)=\nu\left(x_{1}\right)=1$. For any pair of real numbers $r_{0}$ and $r_{1}$ such that $0<r_{0}<x_{0}$ and $x_{1}<r_{1}<1$ there exists a real number $\Lambda$ less than 1 , such that for every positive integer $n$,

$$
n^{-n} a(n, j) \leq \sqrt{n} \Lambda^{n}, \text { if } j / n \in\left[0, r_{0}\right] \cup\left[r_{1}, 1\right]
$$

Proof. Lemma 8 says that $a(n, j) n^{-n} \leq \sqrt{j} \nu(j / n)^{n}$. The function $\nu(x)$ is smooth and concave with $\nu(0)=0$, and $\nu(1)=e^{-1}$. This implies the existence of unique points $x_{0}$ and $x_{1}$ such that $0<x_{0}<x_{1}<1$ and $\nu\left(x_{0}\right)=\nu\left(x_{1}\right)=1$. Fix $r_{0}$ and $r_{1}$


Figure 4. Graphs of functions $\nu(x)$ and $\varphi(x)$.
such that $0<r_{0}<x_{0}$ and $x_{1}<r_{1}<1$. Necessarily, $\nu\left(r_{0}\right)<1$ and $\nu\left(r_{1}\right)<1$. Let $\Lambda_{0}=\nu\left(r_{0}\right)$ and $\Lambda_{1}=\nu\left(r_{1}\right)$. If $j / n \in\left[0, r_{0}\right]$ then

$$
\nu(j / n) \leq \max \left\{\nu(x): x \in\left[0, r_{0}\right]\right\} \leq \Lambda_{0} .
$$

Similarly, if $j / n \in\left[r_{1}, 1\right]$, we have $\nu(j / n) \leq \Lambda_{1}$. Taking $\Lambda=\max \left\{\Lambda_{0}, \Lambda_{1}\right\}$, the lemma is proved.

Example: The choice $r_{0}=0.1$ yields $\Lambda_{0} \approx 0.173$, and $r_{1}=0.9$ yields $\Lambda_{1} \approx 0.701$. In Figure 1, the value of $\Lambda$ equals the maximum between the approximations of $\Lambda_{0}$ and $\Lambda_{1}$.
3.2. When the ratio $j / n$ is not near 0 nor 1 . We introduce the function

$$
\begin{equation*}
\phi:(0,1) \mapsto \mathbb{R}, \quad \phi(x)=\left(e \ln \left(1+e^{-\delta(x)}\right)\right)^{-1} \varphi(x) e^{-x \delta(x)} \tag{26}
\end{equation*}
$$

where $\varphi(x)$ is defined in (22 and $\delta(x)$ is defined in (22).
Lemma 10. Consider the constants $c_{1}$ and $C_{1}$ in Corollary 7 , For any real number $r$ such that $0<r<1 / 2$, and for any pair of positive integers $n, j$ such that $j / n \in[r, 1-r]$

$$
\frac{e^{-1 / 6} c_{1}}{\sqrt{2 \pi(n-j)}} \phi(j / n)^{n} \leq n^{-n} a(n, j) \leq \frac{e^{1 / 6} C_{1}}{\sqrt{2 \pi(n-j)}} \phi(j / n)^{n} .
$$

Proof. Write $a(n, j)=S_{n}^{(j)} n!/(n-j)$ !, then use Stirling estimates (1) for the factorial, apply Corollary 7 and use the definition of $\varphi(x)$ given in (22).

The function $\phi(x)$ is displayed in Figure 1 It is smooth, concave, $\phi(0)=0$ and $\phi(1)=e^{-1}$. The auxiliary function $\delta(x)$ takes the value $-\ln (e-1)$ at $x=1-1 / e$ and then, $\phi(1-1 / e)=1$. This value is the maximum of $\phi(x)$ because the lower bound given in Lemma 10 implies that $\phi(x) \leq 1$ for $x \in(0,1)$.
3.3. Proofs of Theorem 1 and Corollary 2. Theorem 1 considers the ratio between $j$ and $n$. The proof combines the two cases we just studied: when $j / n$ is near 0 or 1 , and when $j / n$ is in a central interval away from 0 and 1 .
Proof of Theorem 1. Consider the function $\nu$ given in 25). Pick numbers $x_{0}$ and $x_{1}$ such that $0<x_{0}<x_{1}<1$ and $\nu\left(x_{0}\right)=\nu\left(x_{1}\right)=1$. Take any $r \in(0,1 / 2)$ so that $r \leq \max \left\{x_{0}, 1-x_{1}\right\}$. If $j / n \in[r, 1-r]$ apply Lemma 10 Otherwise, apply Lemma 9 ,

The proof of Corollary 2 is immediate from the statement of Theorem 1 .
Proof of Corollary 2. The result is a direct application of Theorem 1 because

$$
\max \left\{n^{-n} a(n, j), 1 \leq j \leq \ell\right\} \leq n^{-n} \sum_{j=1}^{\ell} a(n, j) \leq n \max \left\{n^{-n} a(n, j), 1 \leq j \leq \ell\right\}
$$

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V. Becher, Departamento de Computación, Facultad de Ciencias Exactas y Naturales \& ICC, Universidad de Buenos Aires \& CONiCET, Argentina

Email address: vbecher@dc.uba.ar
E. Cesaratto, Universidad Nac. de Gral. Sarmiento \& CONiCEt, Argentina

Email address: ecesaratto@campus.ungs.edu.ar

