

THE POP-STACK-SORTING OPERATOR ON TAMARI LATTICES

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ABSTRACT. Motivated by the pop-stack-sorting map on the symmetric groups, Defant defined an operator $\text{Pop}_M : M \rightarrow M$ for each complete meet-semilattice M by

$$\text{Pop}_M(x) = \bigwedge (\{y \in M : y \prec x\} \cup \{x\}).$$

This paper concerns the dynamics of $\text{Pop}_{\text{Tam}_n}$, where Tam_n is the n -th Tamari lattice.

We say an element $x \in \text{Tam}_n$ is t -**Pop**-sortable if $\text{Pop}_M^t(x)$ is the minimal element and we let $h_t(n)$ denote the number of t -**Pop**-sortable elements in Tam_n . We find an explicit formula for the generating function $\sum_{n \geq 1} h_t(n)z^n$ and verify Defant's conjecture that it is rational. We furthermore prove that the size of the image of $\text{Pop}_{\text{Tam}_n}$ is the Motzkin number M_n , settling a conjecture of Defant and Williams.

1. INTRODUCTION

Building on Knuth's stack-sorting algorithm [16], West's ground-breaking work on stack-sorting map on symmetric groups [24] inspired subsequent studies, including the reverse-stack-sorting map [12] and the pop-stack-sorting map [3]. Recently, there has been considerable attention by combinatorialists on the pop-stack sorting map [1, 2, 7, 13, 20]. For each complete meet-semilattice M , Defant defined an operator Pop_M that agrees with the pop-stack-sorting map when M is the weak order on S_n [8]. It is defined so that Pop_M sends an element to the meet of itself and all elements that it covers. By definition, M 's minimal element $\hat{0}$ stays the same when Pop_M is applied. We say an element x is t -**Pop**-sortable if $\text{Pop}_M^t(x) = \hat{0}$.

Pudwell and Smith [20] enumerated the number of 2-**Pop**-sortable elements in S_n under the weak order. Claesson and Guðmundsson [7] proved that for each fixed nonnegative integer t , the generating function that counts t -**Pop**-sortable elements in S_n is rational. Defant [9] established the analogous rationality result for the generating functions of t -**Pop**-sortable elements of type B and type \tilde{A} weak orders.

Introduced in 1962, the n -th Tamari lattice Tam_n consists of semilength- n Dyck paths (lattice paths from $(0, 0)$ to (n, n) above the diagonal $y = x$) [23]; its partial order will be defined in Section 2. There are generalizations of the definition, most notably the m -Tamari lattices by Bergeron and Préville-Ratelle [4] and the ν -Tamari lattices introduced

by Préville-Ratelle and Viennot [19]. Fundamental in algebraic combinatorics [17], the n -th Tamari lattice Tam_n is also isomorphic to $\text{Av}_n(312)$, the lattice of 312-pattern-avoiding permutations under the weak order of S_n [5].

In this paper, we study the **Pop** operator on Tamari lattices. Let $h_t(n)$ be the number of t -**Pop**-sortable elements in Tam_n . A part of a conjecture by Defant [8] is that for every fixed t , the generating function $\sum_{n \geq 1} h_t(n)z^n$ is rational. We confirm this statement by giving the exact formula of the generating function:

Theorem 1.1. *Let $h_t(n)$ denote the number of t -**Pop**-sortable Dyck paths in the n -th Tamari lattice Tam_n . Then*

$$\sum_{n \geq 1} h_t(n)z^n = \frac{z}{1 - 2z - \sum_{j=2}^t C_{j-1}z^j},$$

where C_j are the Catalan numbers.

Moreover, settling a conjecture in Defant and Williams's paper (Conjecture 11.2 (2) in [11]), we have the following theorem:

Theorem 1.2. *Define $\text{Pop}(L; q) = \sum_{b \in \text{Pop}_L(L)} q^{|\mathcal{U}_L(b)|}$, where $\mathcal{U}_L(b)$ is the set of elements of L that cover b . Then we have*

$$\text{Pop}(\text{Tam}_{n+1}; q) = \sum_{k=0}^n \frac{1}{k+1} \binom{2k}{k} \binom{n}{2k} q^{n-k},$$

where the coefficients form OEIS sequence [22] A055151.

In particular, when $q = 1$, we have that

$$|\text{Pop}_{\text{Tam}_n}(\text{Tam}_n)| = M_{n-1},$$

where M_n is the n -th Motzkin number (OEIS sequence [22] A001006).

Additional motivation for studying the size of the image of $\text{Pop}_{\text{Tam}_n}$ comes from a theorem by Defant and Williams (Theorem 9.13 in [11]). In that theorem, they proved that $|X_n| = \{y \in \text{Tam}_n \mid \text{Row}(y) \leq y\}$, where Row is the rowmotion operator on Tam_n (which is equivalent to the Kreweras complement operator on noncrossing partitions [10]). They also showed that $|X_n|$ is the number of independent dominating sets in a certain graph associated with Tam_n called its *Galois graph*.

The paper is organized as follows. In Section 2 we give the necessary definitions. In Section 3 and Section 4 we prove Theorem 1.1 and Theorem 1.2.

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2. DEFINITIONS

2.1. Lattice basics and the Pop operator.

Definition 2.1. A *meet-semilattice* is a poset M such that any two elements $x, y \in M$ have a greatest lower bound (which is called their *meet*, denoted by $x \wedge y$). A *lattice* L is a meet-semilattice such that any two elements $x, y \in L$ also have a least upper bound (which is called their *join*, denoted by $x \vee y$). A meet-semilattice is *complete* if every nonempty subset $A \subset M$ has a meet.

Given $x, y \in M$, we say that y is *covered* by x (denoted $y \triangleleft x$) if $y < x$ and no $z \in M$ satisfies $y < z < x$.

In this paper we only consider finite meet-semilattices, each of which has a unique minimal element $\hat{0}$. They are automatically complete.

Definition 2.2 ([8]). Let M be a complete meet-semilattice. Define the *semilattice pop-stack-sorting operator* $\text{Pop}_M : M \rightarrow M$ by

$$\text{Pop}_M(x) = \bigwedge (\{y \in M : y \triangleleft x\} \cup \{x\}).$$

Definition 2.3. We say an element x of a complete meet-semilattice M is *t-Pop-sortable* if $\text{Pop}^t(x) = \hat{0}$.

2.2. Generalized Tamari lattices. In this paper, a lattice path is a finite planar path that starts from the origin and at each step travels either up/N : $(0, 1)$ or right/E : $(1, 0)$.

Definition 2.4. The *horizontal distance* of a point p with respect to a lattice path ν is the maximum number of east steps one can take starting from p before being strictly to the right of ν .

Definition 2.5 ([19]). Let ν be a lattice path from $(0, 0)$ to $(\ell - n, n)$. The *generalized ν -Tamari lattice* $\text{Tam}(\nu)$ is defined as follows:

- (1) elements of $\text{Tam}(\nu)$ are lattice paths μ from $(0,0)$ to $(\ell - n, n)$ that are weakly above ν ;
- (2) the partial order of $\text{Tam}(\nu)$ is given by the covering relation: $\mu \prec \mu'$ if μ' is obtained by shifting a subpath D of μ by 1 unit to the left, where D satisfies (i) it is preceded by E; (ii) its first step is N; (iii) its endpoints p, p' are of the same horizontal distance to ν and there is no point between them with the same horizontal distance to ν as p . In other words, $\mu \prec \mu'$ if for such subpath D , $\mu = XEDY$ and $\mu' = XDEY$.

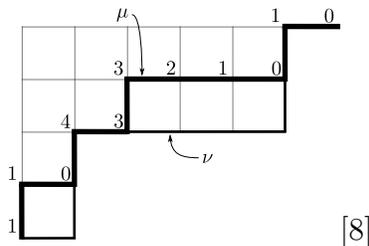


FIGURE 1. Lattice path $\mu = \text{NENENEENE}$ is in $\text{Tam}(\nu)$ where $\nu = \text{ENNEEEENNE}$. Each point on μ is labeled with its horizontal distance.

Definition 2.6. When $\nu = (\text{NE})^n$, the lattice $\text{Tam}(\nu)$ is the n -th *Tamari lattice* Tam_n consisting of the *Dyck paths*. It is well-known that $|\text{Tam}_n|$ is the n -th Catalan number C_n .

3. PROOF OF THEOREM 1.1

3.1. Preliminaries: the ν -bracket vector.

Definition 3.1. Let $\mathbf{b}(\nu) = (b_0(\nu), b_1(\nu), \dots, b_\ell(\nu))$ be the vector denoting the heights at each step of the lattice path ν . Let the *fixed position* f_k denote the largest index such that $b_{f_k}(\nu) = k$. We say that an integer vector $\vec{\mathbf{b}} = (\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_\ell)$ is a ν -bracket vector, denoted as $\vec{\mathbf{b}} \in \text{Vec}(\nu)$, if

- (1) $\mathbf{b}_{f_k} = k$ for all $k = 0, \dots, n$.
- (2) $b_i(\nu) \leq \mathbf{b}_i \leq n$ for all $0 \leq i \leq \ell$.
- (3) If $\mathbf{b}_i = k$, then $\mathbf{b}_j \leq k$ for all $i + 1 \leq j \leq f_k$.

The partial order of $\text{Vec}(\nu)$ is defined as follows: we say $(\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_\ell) \leq (\mathbf{b}'_0, \mathbf{b}'_1, \dots, \mathbf{b}'_\ell)$ if $\mathbf{b}_i \leq \mathbf{b}'_i$ for all i .

Remark. An equivalent interpretation of (3) is that $\vec{\mathbf{b}}$ is 121-pattern-avoiding. These conditions also imply the sequence $\{\mathbf{b}_i\}_{f_{k-1}+1}^{f_k}$ is non-increasing for all $k = 0, \dots, n$.

Definition 3.2. Let $\mu \in \text{Tam}(\nu)$ be a path from $(0,0)$ to $(\ell - n, n)$. We define $\mathbf{b}(\mu) = (b_0(\mu), b_1(\mu), \dots, b_\ell(\mu))$ its *associated vector* as follows: make $(\ell + 1)$ empty slots; traverse μ , and when arriving at a new grid point, write its height k at the rightmost available slot among those that are weakly to the left of index f_k .

Remark. We alert the readers that the notation of the vector $\mathbf{b}(\mu)$ does not reflect its dependence on the fixed lattice path ν .

Example 3.3. We use $\mu = \text{NENENEEENE}$ and $\nu = \text{ENNEEEENNE}$ as in Figure 1. The fixed positions are $f_0 = 1$, $f_1 = 2$, $f_2 = 7$, $f_3 = 8$, and $f_4 = 10$. Then we create 11 empty slots and construct the associated vector $\mathbf{b}(\mu)$ as follows:

$$\begin{aligned} & (_, \underline{0}, _, _, _, _, _, _, _, _, _, _) \rightarrow (\underline{1}, \underline{0}, \underline{1}, _, _, _, _, _, _, _, _) \\ \rightarrow & (\underline{1}, \underline{0}, \underline{1}, _, _, _, \underline{2}, \underline{2}, _, _, _) \rightarrow (\underline{1}, \underline{0}, \underline{1}, \underline{3}, \underline{3}, \underline{3}, \underline{2}, \underline{2}, \underline{3}, _, _) \\ \rightarrow & (\underline{1}, \underline{0}, \underline{1}, \underline{3}, \underline{3}, \underline{3}, \underline{2}, \underline{2}, \underline{3}, \underline{4}, \underline{4}). \end{aligned}$$

Theorem 3.4 ([6]). *The map $\mathbf{b} : \text{Tam}(\nu) \rightarrow \text{Vec}(\nu)$ is an order-preserving bijection. Furthermore, for any paths $\mu, \mu' \in \text{Tam}(\nu)$, we have $\mathbf{b}(\mu \wedge \mu') = \min(\mathbf{b}(\mu), \mathbf{b}(\mu'))$ the term-wise minimum vector.*

Notation 3.5. We define the followings.

- (1) $\Delta(\mu) := \{i \mid i < \ell \text{ and } b_i(\mu) > b_{i+1}(\mu)\}$.
- (2) $\eta_i(\mu) := \begin{cases} \max\{x \in [b_i(\nu), b_i(\mu) - 1] \mid b_j(\mu) \leq x, \forall j \in [i + 1, f_x]\} & \text{if } i \in \Delta(\mu), \\ b_i(\mu) & \text{if } i \notin \Delta(\mu). \end{cases}$
- (3) $\mathbf{b}_\downarrow^i(\mu) := (b_0(\mu), \dots, b_{i-1}(\mu), \eta_i(\mu), \dots, b_\ell(\mu))$.

Example 3.6. Again we use $\mu = \text{NENENEEENE}$ as in Figure 1 and by Example 3.3 we have that $\mathbf{b}(\mu) = (1, 0, 1, 3, 3, 3, 2, 2, 3, 4, 4)$. Hence, $\Delta(\mu) = \{0, 5\}$, $\eta_0(\mu) = 0$, and $\eta_5(\mu) = 2$.

Proposition 3.7 ([8]). We have that

$$\mathbf{b}(\text{Pop}_{\text{Tam}(\nu)}(\mu)) = (\eta_0(\mu), \eta_1(\mu), \dots, \eta_\ell(\mu)).$$

Corollary 3.8 ([8]). Suppose $\mu \in \text{Tam}(\nu)$ and $f_{k-1} < i < f_k$ ($0 \leq k \leq n$). Then $b_i(\text{Pop}_{\text{Tam}(\nu)}(\mu)) \geq b_{i+1}(\mu)$.

We use the assumptions for a lattice path ν from above. Let $\nu^\#$ be the path obtained from ν by deleting its first $f_0 + 1$ steps. Let $\mathbf{b}^\#$ be the vector obtained from \mathbf{b} by deleting its first $f_0 + 1$ entries and subtracting 1 from all remaining entries. We call this action the *hash* map. Let $\mu^\#$ be the unique element in $\text{Tam}(\nu^\#)$ whose associated vector is $\mathbf{b}(\mu)^\#$.

Corollary 3.9. If $\mu \in \text{Tam}(\nu)$ is t -Pop-sortable, then so is $\mu^\# \in \text{Tam}(\nu^\#)$.

Proof. This directly follows from the fact that $\eta_i(\mu)$ is determined only by $b_j(\mu)$ for $j \geq i$. \square

3.2. Proof of the result. Let $H_t(z) = \sum_{n \geq 1} h_t(n)z^n$, the generating function in Theorem 1.1. Let $\tilde{H}_t(z)$ be the truncated polynomial $\sum_{n=1}^{t-1} h_t(n)z^n$. Let $G_t(z) = \sum_{n \geq 1} g_t(n)z^n$, where $g_t(n)$ denotes the t -Pop-sortable irreducible elements in $\text{Vec}(\nu)$ for $\nu = \mathbb{E}(\text{NE})^{n-1}$. In this case, using the notations from Definition 3.1, we have $f_k = 2k+1$, and $b_i(\nu) = \lfloor i/2 \rfloor$. Therefore, the restrictions are $\mathbf{b}_{2k+1} = k$, $\mathbf{b}_{2k} \in \{k, k+1, \dots, n\}$, and that if $\mathbf{b}_i = k$, then $\mathbf{b}_j \leq k$ for all $j = i+1, \dots, 2k+1$, i.e., no 121-pattern can appear. Finally, we note that $\text{Vec}(\mathbb{E}(\text{NE})^{n-1}) \cong \text{Vec}((\text{NE})^n) \cong \text{Tam}_n$.

Definition 3.10. We say $\vec{\mathbf{b}} = (\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_\ell) \in \text{Vec}(\nu)$ for some fixed ν is *irreducible* if $\mathbf{b}_0 = \mathbf{b}_\ell$.

Lemma 3.11. *Every ν -bracket vector can be decomposed into irreducible ν_i -bracket vectors, where ν and each ν_i are of the form $\mathbb{E}(\text{NE})^{k-1}$. A vector is t -Pop-sortable if and only if all its irreducible components are.*

Proof. We first define the addition of two irreducible vectors $\vec{\mathbf{b}} \in \text{Vec}(\mathbb{E}(\text{NE})^{n_1-1})$ and $\vec{\mathbf{b}}' \in \text{Vec}(\mathbb{E}(\text{NE})^{n_2-1})$ as follows:

$$\vec{\mathbf{b}} + \vec{\mathbf{b}}' := (\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_{2n_1-1}, \mathbf{b}'_0 + n_1, \mathbf{b}'_1 + n_1, \dots, \mathbf{b}'_{2n_2-1} + n_1) \in \text{Vec}(\mathbb{E}(\text{NE})^{n_1+n_2-1}).$$

To prove the first claim we induct on the length of the vector and note that it suffices to show that every bracket vector can be decomposed as the sum of an irreducible vector $\vec{\mathbf{b}}_{irr}$ and a shorter vector. Simply take $\vec{\mathbf{b}}_{irr} := (\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_{f_{\mathbf{b}_0}})$. The second claim is clear. \square

Lemma 3.12. *Assume the notations above. Then we have*

$$1 + H_t(z) = \frac{1}{1 - G_t(z)}.$$

Proof. The formula is a direct corollary of Lemma 3.11. \square

Lemma 3.13. *The hash map is a one-to-one correspondence between irreducible vectors in $\text{Vec}(\mathbb{E}(\text{NE})^{n-1})$ and bracket vectors in $\text{Vec}(\mathbb{E}(\text{NE})^{n-2})$. An irreducible vector $\vec{\mathbf{b}}$ is t -Pop-sortable if and only if $\vec{\mathbf{b}}^\#$ is t -Pop-sortable and $t \geq n - x_r + 1$, where $2x_r$ is the length of the last irreducible vector component of $\vec{\mathbf{b}}^\#$.*

Proof. Let the irreducible vector $\vec{\mathbf{b}} \in \text{Vec}(\mathbb{E}(\text{NE})^{n-1})$ be $(n, 0, u_0, u_1, \dots, u_{2n-3})$ and $\vec{\mathbf{b}}^\# = (u_0 - 1, u_1 - 1, \dots, u_{2n-3} - 1) \in \text{Vec}(\mathbb{E}(\text{NE})^{n-2})$. First, it is clear that from $\vec{\mathbf{b}}^\#$ we can recover $\vec{\mathbf{b}}$, so the hash map is a bijection. Next, if we decompose $\vec{\mathbf{b}}^\#$ as the sum of some (say r) irreducible vectors of lengths $2x_1, \dots, 2x_r$, respectively (corresponding to elements in $\text{Vec}(\nu)$ for $\nu = (\mathbb{E}(\text{NE})^{x_i-1})$, $1 \leq i \leq r$), then we can write

$$\vec{\mathbf{b}} = (n, 0, u_0, u_1, \dots, u_{2n-3}) = (n, 0, u_0, \dots, u_0, \dots, n - x_r, \dots, n - x_r, n, \dots, n).$$

The irreducible vector $\vec{\mathbf{b}}$ being t -Pop-sortable is equivalent to $\vec{\mathbf{b}}^\#$ being t -Pop-sortable and the first entry of $\vec{\mathbf{b}}$ turning 0 after t Pop's. Applying $\text{Pop}_{\text{Vec}(\mathbb{E}(\text{NE})^{n-1})}$ once changes the first

entry from n to $n - x_r$, and each subsequent $\text{Pop}_{\text{Vec}(\text{E}(\text{NE})^{n-1})}$ decreases it by 1, hence this is then equivalent to $t \geq n - x_r + 1$. \square

Lemma 3.14. *Assume the notations above. Then we have*

$$G_t(z) = z \left((1 + \tilde{H}_t(z))G_t(z) + 1 \right).$$

Proof. This is a corollary of Lemma 3.13. Since the hash map's image of the middle sub-vector $(u_0 - 1, \dots, u_0 - 1, \dots, n - x_r - 1, \dots, n - x_r - 1) \in \text{Vec}(\text{E}(\text{NE})^{n-x_r-1})$ is t -Pop-sortable when $n - x_r \leq t - 1$ and the last irreducible component starts and ends with n as well, we have justified the desired expression (adding 1 to $\tilde{H}_t(z)$ is to account for the $r = 0$ case). \square

Lemma 3.15. *When $n \leq t$, every path in Tam_n is t -Pop-sortable.*

Proof. Consider the path's associated vector $\vec{\mathbf{b}} \in \text{Vec}(\text{E}(\text{NE})^{n-1})$. For each $0 \leq i \leq n - 1$, \mathbf{b}_{2i} decreases by at least 1 each time unless $\mathbf{b}_{2i} = \mathbf{b}_{2i+1}$. Since $n \leq t$, during the t applications of $\text{Pop}_{\text{Vec}(\text{E}(\text{NE})^{n-1})}$ this equality will be reached. This applies to all i , so we obtain the minimum element's associated vector. \square

We are now ready to prove our first main result.

Proof of Theorem 1.1. By Lemma 3.15, $\tilde{H}_t(z) = \sum_{n=1}^{t-1} C_n z^n$. By Lemma 3.14, we have that

$$G_t(z) = \frac{z}{1 - \sum_{n=1}^t C_{n-1} z^n},$$

and substituting this into Lemma 3.12, we obtain that

$$H_t(z) = \frac{G_t(z)}{1 - G_t(z)} = \frac{\frac{z}{1 - \sum_{n=1}^t C_{n-1} z^n}}{1 - \frac{z}{1 - \sum_{n=1}^t C_{n-1} z^n}} = \frac{z}{1 - 2z - \sum_{j=2}^t C_{j-1} z^j},$$

as desired. \square

4. PROOF OF THEOREM 1.2

4.1. Preliminaries: congruence and Pop on subsemilattices.

Definition 4.1. A *lattice congruence* on a lattice L is an equivalence relation \equiv on L such that if $x_1 \equiv x_2$ and $y_1 \equiv y_2$, then $x_1 \wedge y_1 \equiv x_2 \wedge y_2$ and $x_1 \vee y_1 \equiv x_2 \vee y_2$.

For each $x \in L$, we denote by $\pi_{\downarrow}(x)$ the minimal element of the congruence class of x .

Definition 4.2. A *subsemilattice* of a lattice L is a subset $M \subset L$ such that $x \wedge y \in M$ for all $x, y \in M$.

Theorem 4.3. ([8]) *Let L be a finite lattice. Let \equiv be a lattice congruence on L such that the set $M = \{\pi_{\downarrow}(x) \mid x \in L\}$ is a subsemilattice of L . Then for all $x \in M$,*

$$\text{Pop}_M(x) = \pi_{\downarrow}(\text{Pop}_L(x)).$$

We now provide an example that shows how the Tamari lattice can be realized as a sublattice of S_n .

Definition 4.4. A *descent* of a permutation $x = x_1 \cdots x_n$ is a pair of adjacent entries $x_i > x_{i+1}$. A *descending run* is a maximal decreasing subsequence of x . The *pop-stack-sorting map* is the operator on S_n that reverses each descending run.

Definition 4.5. The partial order of S_n defined by the following covering relation is the *right weak order*: a permutation y is covered by permutation x if y is obtained by swapping one of x 's descents.

Definition 4.6. ([15]) Two words u, v are *sylvester-adjacent* if there exist $a < b < c$ and words X, Y, Z such that $u = XacYbZ$ and $v = XcaYbZ$. We write $u \triangleleft v$.

Two words u, v are *sylvester-congruent* if there is a chain of words $u = w_0, w_1, \dots, w_m = v$ such that w_i and w_{i+1} are sylvester-adjacent for all i ($w_i \triangleleft w_{i+1}$ or $w_i \triangleright w_{i+1}$).

We say that a permutation π is *312-avoiding* if it has no $i < j < k$ such that $x_j < x_k < x_i$, and is $\overline{312}$ -*avoiding* if it has no $i < j$ such that $x_i < x_j < x_{i-1}$.

Let $L = S_n$, and let $M = \text{Av}_n(312)$ be the set of 312-avoiding permutations, both under the right weak order. It is established by Björner and Wachs [5] in their Theorem 9.6 (i) that $\text{Av}_n(312)$ is a sublattice of S_n and is isomorphic to the Tamari lattice Tam_n . Reading [21] observes that the sylvester-congruence is a lattice congruence for S_n under the right weak order (note that $u \triangleleft v$ also implies $u \leq v$), and, furthermore, if we divide S_n into sylvester-congruence classes, then each class has a unique 312-avoiding element. More precisely, $\text{Av}_n(312) = \{\pi_{\downarrow}(x) \mid x \in S_n\}$.

A concrete description of π_{\downarrow} is that we can compute a chain $x = y_0 \triangleright y_1 \triangleright \cdots \triangleright y_m = \pi_{\downarrow}(x)$ until we must stop (one can easily show that no $XcaYbZ$ (i.e., $\overline{312}$) pattern implies no 312 pattern), and we remark that the exact construction of the chain does not matter, that is, regardless of the order of swapping one obtains the same eventual outcome.

Therefore, Theorem 4.3 tells us that

$$\text{Pop}_{\text{Av}_n(312)}(x) = \pi_{\downarrow}(\text{Pop}_{S_n}(x)).$$

This is especially helpful, given that Pop_{S_n} on the right hand side is equal to the easily characterized pop-stack-sorting map.

4.2. Proof of the result.

Theorem 4.7. *We have that $x \in X_n = \{\text{Pop}_{\text{Av}_n(312)}(\text{Av}_n(312))\}$ if and only if $x = x_1x_2 \cdots x_n$ has no consecutive double descents and ends with n .*

Proof. In this proof we interpret Pop as reversing all descending runs of a string (not required to be a permutation of 1 to m), e.g., $\text{Pop}(74513) = 47153$, though we specify by using a subscript when it is indeed Pop_{S_m} . We also recall the identity $\text{Pop}_{\text{Av}_n(312)}(y) = \pi_{\downarrow}(\text{Pop}_{S_n}(y))$ which will be used extensively.

For the “only if” direction, we first suppose that $x = \pi_{\downarrow}(\text{Pop}_{S_n}(y))$ and we want to show that x ends with n and has no consecutive double descents.

It is known that every permutation in the image of π_{\downarrow} must be 312-avoiding. We first prove that the last entry must be n . Wherever n is located for a permutation y , in order for it to be 312-avoiding we must have that the segment after n is decreasing. Then after the effect of Pop_{S_n} , n is put at the end of the permutation and continues to stay there when we apply π_{\downarrow} because it is never involved as a, b , or c in any $XcaYbZ$ pattern.

Next we prove that there are no consecutive double descents. We use induction on the permutation length, and, with the base case being clear, we assume this claim holds for length $n - 1$. Write $y = y_1y_2 \cdots y_n$ and let $y_r = n$.

Suppose $y_n = n$. We thus know that $\text{Pop}_{S_n}(y)$ ends with n and it stays at the same place under the effect of π_{\downarrow} . Using the induction hypothesis, we have that $\pi_{\downarrow}(\text{Pop}_{S_n}(y))$ will end with $(n - 1)n$ with no double descents.

Suppose $y_{n-1} = n$. Let $y_n = k$. Let $\text{Pop}_{S_n}(y) = z_1 \cdots z_n$. Then $(z_{n-1}, z_n) = (k, n)$ and n stays at the same place throughout. We prove the following two claims: there is no 312 pattern involving k after Pop_{S_n} , and there is no 312 pattern involving k at any stage in the chain of pairwise sylvester-adjacent permutations that we use to compute π_{\downarrow} . For the first claim, if there is a 312 pattern then there must be some z_i, z_j such that $z_i > k > z_j$ and $i < j < n - 1$. Since Pop_{S_n} does not change the relative position of entries in different descending runs, it must be that z_i is before z_j in preimage y . However, there is no 312 pattern initially in y , which is a contradiction. For the second claim, we know that $z_1 \cdots z_n$ has no z_i, z_j such that $z_i > k > z_j$ and $i < j < n - 1$, and any swap $(XcaYbZ \rightarrow XacYbZ)$ in the chain would not create such a pair as it moves a smaller element to the front of a larger element.

Therefore, we can delete k and n from y and lower the entries of values $k + 1, \dots, n - 1$ by 1 respectively in $y_1 \cdots y_{n-2}$. We then have an element in S_{n-2} , say, $y'_1 \cdots y'_{n-2}$, and can apply the induction hypothesis to it. Therefore, $\pi_{\downarrow}(\text{Pop}_{S_{n-2}}(y'_1 \cdots y'_{n-2}))$ ends with $n - 2$ and has no double descents. Now we take this image and add 1 to entries of values

$k, \dots, n-2$ and denote it as $x'_1 \cdots x'_{n-2}$. Because of the previous paragraph we have shown that $\pi_{\downarrow}(\text{Pop}_{S_n}(y)) = x'_1 \cdots x'_{n-2} \cdot kn$, and the entire string has no double descents.

Now suppose $r \leq n-2$. First we consider the case $y_{r-1} < y_{r+1}$. We have $\text{Pop}_{S_n}(y) = \text{Pop}_{S_{n-1}}(y_1 \cdots y_{r-1} y_{r+1} \cdots y_n) n$. Therefore,

$$\begin{aligned} \pi_{\downarrow}(\text{Pop}_{S_n}(y)) &= \pi_{\downarrow}(\text{Pop}_{S_{n-1}}(y_1 \cdots y_{r-1} y_{r+1} \cdots y_n) \cdot n) \\ &= \pi_{\downarrow}(\text{Pop}_{S_{n-1}}(y_1 \cdots y_{r-1} y_{r+1} \cdots y_n)) \cdot n, \end{aligned}$$

where \cdot stands for concatenation. We apply the induction hypothesis to $y_1 \cdots y_{r-1} y_{r+1} \cdots y_n$, an element of S_{n-1} , and obtain that the first $n-1$ places of x must not have consecutive double descents. Concatenating with n will not change this statement, and we conclude this case.

Now we suppose $y_{r-1} > y_{r+1}$. Let $y_q y_{q+1} \cdots y_{r-1}$ be the longest descending run that ends with y_{r-1} . On one hand,

$$\text{Pop}_{S_n}(y_1 \cdots y_{r-1} n y_{r+1} \cdots y_n) = \text{Pop}(y_1 \cdots y_{q-1}) \cdot y_{r-1} \cdots y_q y_n \cdots y_{r+1} n,$$

where $y_n < \cdots < y_{r+1} < y_{r-1} < \cdots < y_q$.

Now we start applying the series of swaps to apply π_{\downarrow} . Notice that every swap removes a $\overline{312}$ pattern and $y_q y_n y_{r+1}$ is one such pattern. Thus, first y_q is swapped with y_n . Then, $y_q y_{n-1} y_{r+1}$ should also be removed, so y_q is again swapped with y_{n-1} . We repeat the process, and after $n-r$ swaps involving y_q as the c in $XcaYbZ$, the permutation becomes

$$\text{Pop}(y_1 \cdots y_{q-1}) \cdot y_{r-1} \cdots y_{q+1} y_n \cdots y_{r+1} y_q n.$$

Similarly, y_{q+1} is moved to the end of $y_n \cdots y_{r+1}$, right before $y_q n$, and so is y_{q+2}, \dots, y_{r-1} . We arrive at

$$\text{Pop}(y_1 \cdots y_{q-1}) \cdot y_n \cdots y_{r+1} y_{r-1} \cdots y_q n.$$

We should clarify that the process of swapping is not finished yet; what we claim is that since π_{\downarrow} is the same for sylvester-adjacent elements, we have

$$\pi_{\downarrow}(\text{Pop}_{S_n}(y)) = \pi_{\downarrow}(\text{Pop}(y_1 \cdots y_{q-1}) \cdot y_n \cdots y_{r+1} y_{r-1} \cdots y_q n).$$

On the other hand,

$$\text{Pop}_{S_n}(y_1 \cdots y_{r-1} y_{r+1} \cdots y_n \cdot n) = \text{Pop}(y_1 \cdots y_{q-1}) \cdot y_n \cdots y_{r+1} y_{r-1} \cdots y_q \cdot n.$$

Combining these observations we obtain that

$$\begin{aligned} \pi_{\downarrow}(\text{Pop}_{S_n}(y)) &= \pi_{\downarrow}(\text{Pop}_{S_n}(y_1 \cdots y_{r-1} y_{r+1} \cdots y_n \cdot n)) \\ &= \pi_{\downarrow}(\text{Pop}_{S_{n-1}}(y_1 \cdots y_{r-1} y_{r+1} \cdots y_n)) \cdot n. \end{aligned}$$

We apply the induction hypothesis to $y_1 \cdots y_{r-1} y_{r+1} \cdots y_n$, an element of S_{n-1} , and obtain that the first $n-1$ places of x must not have consecutive double descents. Concatenating with n will not change this statement, and we conclude this case as well.

For the “if” direction, we suppose that $x = x_1 \cdots x_n \in S_n$ with $x_n = n$ and x has no consecutive double descents. We want to show that there is some 312-avoiding permutation y such that $\pi_{\downarrow}(\mathbf{Pop}_{S_n}(y)) = x$. We use strong induction on x 's length.

We consider the position of 1, say $x_k = 1$. Then there are two immediate observations. Firstly, all entries x_1, \dots, x_{k-1} are smaller than all of x_{k+1}, \dots, x_n to avoid a 312 pattern $x_j x_k x_{\ell}$ where $j < k < \ell$. Hence, it is clear that $\{x_1, \dots, x_{k-1}\} = \{2, \dots, k\}$ and $\{x_{k+1}, \dots, x_n\} = \{k+1, \dots, n\}$. Secondly, if $k \geq 2$, then $x_{k-1} = k$. Otherwise, if $x_j = k$ for some other $j \leq k-2$, then $x_j x_{j+1} x_{j+2}$ forms either a double descents or a 312-pattern, which is impossible.

We let $x'_i = x_i - 1$ if $1 \leq i \leq k-1$ and let $x'_i = x_i - k$ if $k+1 \leq i \leq n$. Then $x'_1 x'_2 \cdots x'_{k-1} \in S_{k-1}$ and $x'_{k+1} x'_{k+2} \cdots x'_n \in S_{n-k}$ are two strings with no double descents, and $x'_{k-1} = k-1$, $x'_n = n-k$. Both of them satisfy the induction hypothesis, so we can find $z = z_1 \cdots z_{k-1} \in S_{k-1}$ and $w = w_1 \cdots w_{n-k} \in S_{n-k}$ such that $\pi_{\downarrow}(\mathbf{Pop}_{S_{k-1}}(z)) = x'_1 x'_2 \cdots x'_{k-1}$ and $\pi_{\downarrow}(\mathbf{Pop}_{S_{n-k}}(w)) = x'_{k+1} x'_{k+2} \cdots x'_n$.

Let $z' = z'_1 \cdots z'_{k-1}$ where $z'_i = z_i + 1$. Suppose $w_t = k+1$. Let $w' = w'_1 \cdots w'_t \cdot 1 \cdot w'_{t+1} \cdots w'_{n-k}$, where we let $w'_i = w_i + k$. Consider $y = z' \cdot w'$. It is clear that y is 312-avoiding. Indeed, z' and w' are both 312-avoiding, and no pattern can be formed by entries from both segments because no entry of z' can be larger than any entry of w' except 1. It suffices to show that $\pi_{\downarrow}(\mathbf{Pop}_{S_n}(y)) = x$.

We carefully investigate $\pi_{\downarrow}(\mathbf{Pop}(w'))$ as follows. After \mathbf{Pop} , $w_t = k+1$ will be after 1, and thus for π_{\downarrow} we can perform a series of $XcaYbZ \rightarrow XacYbZ$ swaps with $a = 1$ and $b = k+1$, until 1 is perturbed to the start of this string. In other words, due to sylvester-adjacent elements have the same π_{\downarrow} image,

$$\pi_{\downarrow}(\mathbf{Pop}(w')) = \pi_{\downarrow}(1 \cdot \mathbf{Pop}(w'_1 \cdots w'_{n-k})) = 1 \cdot \pi_{\downarrow}(\mathbf{Pop}(w'_1 \cdots w'_{n-k})).$$

Since no pattern can be cross-composed by entries from both z' and w' , we have that

$$\begin{aligned} \pi_{\downarrow}(\mathbf{Pop}_{S_n}(y)) &= \pi_{\downarrow}(\mathbf{Pop}(z')) \cdot \pi_{\downarrow}(\mathbf{Pop}(w')) \\ &= \pi_{\downarrow}(\mathbf{Pop}(z')) \cdot 1 \cdot \pi_{\downarrow}(\mathbf{Pop}(w'_1 \cdots w'_{n-k})) \\ &= x_1 \cdots x_{k-1} \cdot 1 \cdot x_{k+1} \cdots x_n, \end{aligned}$$

which is exactly x . This concludes the proof. □

The last ingredient that we will need in the proof of Theorem 1.2 is the following enumerative result.

Theorem 4.8. ([18]) *The number of 231-avoiding permutations $\pi \in S_{n+1}$ with exactly k descents and k peaks is $\frac{1}{k+1} \binom{2k}{k} \binom{n}{2k}$.*

Proof of Theorem 1.2. Define the bijective map $r(\pi) = \pi' = \pi'_1 \cdots \pi'_{n+1}$ where $\pi'_i = n + 2 - \pi_{n+2-i}$. We claim that the effect of r preserves the number of ascents (descents) of the permutation. Indeed, place i being an ascent (descent) in π' is equivalent to place $n + 1 - i$ being an ascent (descent) in π , respectively. Furthermore, if in π the descending runs are of lengths ℓ_1, \dots, ℓ_m , then in π' the descending runs are of lengths ℓ_m, \dots, ℓ_1 .

By Theorem 4.8 it suffices for us to establish a bijection between 231-avoiding permutations $\pi \in S_{n+1}$ with exactly k descents and k peaks and $\{r(\pi) \mid \pi \in \text{Pop}_{\text{Av}_n(312)}(\text{Av}_n(312)), \mathcal{U}_L(\pi) = n - k\}$. On one hand, take π from the former set and we have $\mathcal{U}_L(\pi') = n - k$, as having k descents is equivalent to having $n - k$ ascents for elements in S_{n+1} . Here, we use the well-known fact that $\mathcal{U}_L(\pi)$ equals to the number of ascents in π .

On the other hand, we will show that if $\mathcal{U}_L(\pi) = n - k$, then $r(\pi) = \pi'$ is 231-avoiding and has exactly k descents and k peaks. Being 231-avoiding and having k descents are clear. Moreover, Theorem 4.7 establishes that π has no double descents and ends with $n + 1$. Therefore, π' has no double descents either. This implies that the number of peaks of π' is either equal to or is smaller by 1 than the number of its descents, depending on whether the first index is a descent. Since $\pi'_{n+1} = n + 2 - \pi_{n+1} = 1$, we know that π' has k peaks. This concludes the proof. \square

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