COMBINATORIAL IDENTITIES ASSOCIATED WITH A BIVARIATE GENERATING FUNCTION FOR OVERPARTITION PAIRS

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Abstract. We obtain a three-parameter q-series identity that generalizes two results of Chan and Mao. By specializing our identity, we derive new results of combinatorial significance in connection with $N(r,s,m,n)$, a function counting certain overpartition pairs recently introduced by Bringmann, Lovejoy and Osburn. For example, one of our identities gives a closed-form evaluation of a double series in terms of Chebyshev polynomials of the second kind, thereby resulting in an analogue of Euler's pentagonal number theorem. Another of our results expresses a multi-sum involving $N(r,s,m,n)$ in terms of just the partition function $p(n)$. Using a result of Shimura we also relate a certain double series with a weight 7/2 theta series.

1. Introduction and main results

A *partition* of a natural number n is the number of ways of writing n as a sum of natural numbers in a non-increasing order. The partition function $p(n)$ enumerates the number of partitions of n. Euler showed that the generating function of $p(n)$ is

$$1+\sum_{n=1}^{\infty}p(n)q^n=\frac{1}{(q;q)_\infty}, \tag{1.1}$$

where we assume throughout that $|q|<1$, and for $A\in\mathbb{C}$, use the notation

$$(A)_n=(A;q)_n:=\prod_{j=1}^{n}(1-Aq^{j-1}),\quad (A)_\infty=(A;q)_\infty=\lim_{n\to\infty}(A;q)_n. \tag{1.2}$$

The function $p(n)$ satisfies amazing congruences discovered by Ramanujan, namely,

$$p(5n+4)\equiv 0 \pmod 5,\quad p(7n+5)\equiv 0 \pmod 7,\quad p(11n+6)\equiv 0 \pmod{11}. \tag{1.3}$$

One of the various ways to prove (1.3) is to note that the generating function on the right-hand side of (1.1) is essentially a half-integral weight (meromorphic) modular form. The congruences in (1.3) then follow easily using the well-known theory of modular forms; see, for example, [11]. For more information on this topic, we refer the reader to [6, Chapter 2] and [14, Chapter 5].

In [2], Andrews discovered the remarkable smallest parts partition function $\mathrm{spt}(n)$. It counts the total number of smallest parts in all partitions of n. The generating function of $\mathrm{spt}(n)$ is given by [2, p. 138]

$$\sum_{n=1}^{\infty}\mathrm{spt}(n)q^n=\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2(q^{n+1};q)_\infty}. \tag{1.4}$$

The generating function on the right-hand side of (1.4) is essentially a mock modular form as shown by Folsom and Ono [9, Lemma 2.1] but not a modular form. Nevertheless, it is surprising to note that

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$\operatorname{spt}(n)$ satisfies the following remarkable congruences found by Andrews [2] which are reminiscent of Ramanujan's congruences for $p(n)$ given in (1.3):

$$\operatorname{spt}(5n+4) \equiv 0 \pmod 5, \quad \operatorname{spt}(7n+5) \equiv 0 \pmod 7, \quad \operatorname{spt}(13n+6) \equiv 0 \pmod{13}. \tag{1.5}$$

To prove (1.5), Andrews first establishes an identity connecting $p(n)$, $\operatorname{spt}(n)$ and $N_2(n)$, where $N_2(n)$ is the Atkin-Garvan second rank moment [5]. His identity is [2, Theorem 3]

$$\operatorname{spt}(n) = np(n) - \frac{1}{2}N_2(n). \tag{1.6}$$

This identity is, in turn, proven by him by appropriately specializing Watson's q-analogue of Whipple's theorem thereby resulting in [2, p. 138]

$$\sum_{n=0}^{\infty} \frac{(z)_n(z^{-1})_n q^n}{(q)_n} = \frac{(zq)_\infty(z^{-1}q)_\infty}{(q)_\infty^2}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)(z)_n(z^{-1})_n}{(zq)_n(z^{-1}q)_n}\right). \tag{1.7}$$

The rest of the proof proceeds in a magical fashion and requires the following "differentiation identity" of Andrews [2, Equation (2.4)]:

$$-\frac{1}{2}\left[\frac{d^2}{dz^2}(zq;q)_\infty(z^{-1}q;q)_\infty\right]_{z=1} = (q;q)_\infty^2 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \tag{1.8}$$

with the help of which he obtains

$$\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2(q^{n+1};q)_\infty} = \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-q^n)^2}, \tag{1.9}$$

which is nothing but the generating function version of (1.6).

Notice that the expression $(zq;q)_\infty(z^{-1}q;q)_\infty$ appearing in (1.7) is essentially the product involving the variable z occurring in the Jacobi triple product identity. Our present work was realized from our quest to seek analogues of (1.7) and (1.8) together wherein the expression $(zq;q)_\infty(z^{-1}q;q)_\infty$ is replaced by the analogous expression

$$D(z,q) := (zq;q)_\infty(z^{-1}q;q)_\infty(z^2q;q^2)_\infty(z^{-2}q;q^2)_\infty = \frac{(z^2q;q)_\infty(z^{-2}q;q)_\infty}{(-zq;q)_\infty(-z^{-1}q;q)_\infty}, \tag{1.10}$$

which arises in the quintuple product identity (see (3.4) below). One of the reasons this is important is because (1.7) and (1.8) were instrumental in obtaining (1.6). The desired analogue of (1.7) is stated in the theorem below, that is,

Theorem 1.1. *For $z \in \mathbb{C}$ and $z \notin \{0, e^{\pm\pi i/3}, -q^j, j \in \mathbb{Z}\setminus\{0\}\}$ and $|q| < 1$,*

$$\sum_{n\geq 0} \frac{(z^2;q)_n(z^{-2};q)_n q^n}{(-zq;q)_n(-z^{-1}q;q)_n} = (1-z)(1-z^{-1})\left[\frac{-1}{z(1-z^{-1}+z^{-2})}\cdot\frac{(z^{-2}q, z^2q)_\infty}{(-z^{-1}q,-zq)_\infty}\right] + \frac{(1+z^{-1})}{z(1+z^{-3})}. \tag{1.11}$$

The corresponding analogue of (1.8) is

Theorem 1.2. *We have*

$$\begin{aligned}&\left[\frac{d^2}{dz^2}(zq;q)_\infty(z^{-1}q;q)_\infty(z^2q;q^2)_\infty(z^{-2}q;q^2)_\infty\right]_{z=1}\\ &= -2(q;q)_\infty^2(q;q^2)_\infty^2\left\{3\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + 2\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}\right\}.\end{aligned}$$

Observe that the expression $D(z,q)$ in (1.10) occurs in Theorems 1.1 and 1.2 in a way similar to how $(zq;q)_\infty(z^{-1}q;q)_\infty$ appears in (1.7) and (1.8). Differentiating both sides of (1.11) with respect to z twice and letting $z=1$ leads to an analogue of (1.9) given below:

$$4\sum_{n=1}^{\infty}\frac{(q)_{n-1}^2q^n}{(-q)_n^2} \;=\; -\frac{(q)_\infty^2}{(-q)_\infty^2}+1=-\frac{\eta(\tau)^4}{\eta(2\tau)^2}+1. \tag{1.12}$$

It is easy to see that the expression on the left-hand side is (essentially) a modular form. The identity (1.12) is already obtained in [7, Remark 1.4]. However, one of our goals in this paper was to obtain the intermediate identity (1.11) which is not given in [7], for, it gives, as special cases, some new results of combinatorial significance.

Before we discuss these new results, we show that Theorem 1.1 follows as a special case of a more general identity which we establish in the following theorem.

Theorem 1.3. *For $\alpha,\gamma\in\mathbb{C}$, and $\beta\in\mathbb{C}$ except possibly in the set $\{0,\alpha q,\gamma q,q^{-j},\alpha\gamma q^{j+2}: j\geq 0\}$, we have*

$$\sum_{n=0}^{\infty}\frac{(\alpha,\gamma)_n}{(\beta,\alpha\gamma q^2/\beta)_n}q^n=\frac{\beta^{-1}q}{(1-\alpha q/\beta)(1-\gamma q/\beta)}\cdot\frac{(\alpha,\gamma)_\infty}{(\beta,\alpha\gamma q^2/\beta)_\infty}+\frac{(1-q/\beta)(1-\alpha\gamma q/\beta)}{(1-\gamma q/\beta)(1-\alpha q/\beta)}. \tag{1.13}$$

Chan and Mao [8, Theorem 1.2] recently established the following two q-series identities.

$$\sum_{n=0}^{\infty}\frac{(x,1/x;q)_nq^n}{(zq,q/z;q)_n}=\frac{(1-z)^2}{(1-z/x)(1-xz)}+\frac{z(x,1/x;q)_\infty}{(1-z/x)(1-xz)(zq,q/z;q)_\infty}, \tag{1.14}$$

$$\sum_{n=0}^{\infty}\frac{(x,q/x;q)_nq^n}{(z,q/z;q)_{n+1}}=\frac{1}{x(1-z/x)(1-q/(xz))}+\frac{(x,q/x;q)_\infty}{z(1-x/z)(1-q/(xz))(z,q/z;q)_\infty}. \tag{1.15}$$

We obtain these identities as special cases of Theorem 1.3.

Theorem 1.1, in turn, gives the closed-form evaluations of certain bibasic sums such as

$$4\sum_{n=0}^{\infty}\frac{(-q)_{n-1}^2q^n}{(-q^2;q^2)_n}=2\frac{(-q)_\infty^2}{(-q^2;q^2)_\infty}-1=2\frac{(q^2;q^2)_\infty^3}{(q)_\infty^2(q^4;q^4)_\infty}-1=2\frac{(q^2;q^4)_\infty}{(q;q^2)_\infty^2}-1, \tag{1.16}$$

and

$$1+3\sum_{n=1}^{\infty}\frac{(-q)_n(q^3;q^3)_{n-1}q^n}{(q)_{n-1}(-q^3;q^3)_n}=\frac{3}{2}\cdot\frac{(q^3;q^3)_\infty(-q)_\infty}{(-q^3;q^3)_\infty(q)_\infty}-\frac{1}{2}. \tag{1.17}$$

We note that (1.16) is obtainable from (1.14) by letting $x=-1$ and $z=i$. Also, (1.17) follows by letting $z=\pm\omega=\pm e^{2\pi i/3}$ in Theorem 1.1.

We now return to Theorem 1.1. The coefficients of the bivariate generating series in this theorem are connected to certain overpartition pairs considered by Bringmann, Lovejoy and Osburn [7]. This is now explained.

An overpartition λ of a positive integer n is a partition in which the first (or the last) occurrence of a number may be overlined. An overpartition pair (λ,μ) of n is a pair of overpartitions where the sum of all of the parts of λ as well as μ is n. Let $\ell((\lambda,\mu))$ denote the largest part of the overpartition pair (λ,μ), that is, the maximum of the largest parts of λ and μ. Also, let $n(\pi)$ denote the number of parts of the partition π. Then the rank of an overpartition pair (λ,μ) is defined by

$$\ell((\lambda,\mu))-n(\lambda)-n(\mu)-\chi((\lambda,\mu))$$

where $\chi((\lambda,\mu))$ is defined to be 1 if the largest part of (λ,μ) is non-overlined and is in μ, and 0 otherwise.

Let $N(r,s,m,n)$ denote the number of overpartition pairs of n having rank m such that r is the number of overlined parts in λ plus the number of non-overlined parts in μ, and s is the number of parts in μ. By specializing a result in [13], it was shown in [7] that

$$N(d,e,z;q):=\sum_{\substack{r,s,n\geq 0\\ m\in\mathbb{Z}}} N(r,s,m,n)d^re^sz^mq^n=\sum_{n\geq 0}\frac{(-1/d,-1/e)_n(deq)^n}{(zq,q/z)_n}. \tag{1.18}$$

This leads to

$$\sum_{n\geq 0}\frac{(z^2;q)_n(z^{-2};q)_nq^n}{(-zq;q)_n(-z^{-1}q;q)_n} = \sum_{\substack{r,s,n\geq 0\\ m\in\mathbb{Z}}} N(r,s,m-2s+2r,n)(-1)^{r+s+m}z^mq^n. \tag{1.19}$$

More generally, letting $d=-x=e^{-1}$ in (1.18), one can represent the left-hand side of (1.14) in terms of the function $N(r,s,m,n)$.

Using Theorem 1.1 and (1.19), we obtain a closed-form evaluation of a double series involving $N(r,s,m,n)$ in terms of Chebyshev polynomials of the second kind $U_n(x)$ (see Section 2 for the definition and properties of $U_n(x)$). Before stating this result, we discuss the necessary setup. We define the following subsets of integers:

$$\begin{aligned}
&I_1:=(-\infty,-3n)\cap\mathbb{Z},\quad I_2:=\{-3n\},\quad I_3:=(-3n,1)\cap\mathbb{Z},\\
&I_4:=[1,3n+1)\cap\mathbb{Z},\quad I_5:=\{3n+1\},\quad I_6:=(3n+1,\infty)\cap\mathbb{Z},
\end{aligned} \tag{1.20}$$

and

$$\begin{aligned}
&I_1':=(-\infty,-3n]\cap\mathbb{Z},\quad I_2':=\{-3n+1\},\quad I_3':=(-3n+1,1)\cap\mathbb{Z},\\
&I_4':=[1,3n)\cap\mathbb{Z},\quad I_5':=\{3n\},\quad I_6':=[3n+1,\infty)\cap\mathbb{Z}.
\end{aligned} \tag{1.21}$$

Then we have the following result.

Theorem 1.4. *For $\ell\in\mathbb{Z}$, let $\omega_\ell:=\frac{3\ell^2+\ell}{2}$ be a pentagonal number. Then the coefficient of z^mq^n in $(q)_\infty\sum_{n\geq 0}\frac{(z^2;q)_n(z^{-2};q)_nq^n}{(-zq;q)_n(-z^{-1}q;q)_n}$ is $\sum_{\substack{r,s\geq 0\\ 0\leq\omega_k\leq n}} N\left(r,s,m-2s+2r,n-\omega_k\right)(-1)^{r+s+m+k}$, and is given by*

$$\begin{cases}
0, & m\in I_1\\
U_1(1/2), & m\in I_2\\
U_{m+3n+1}(1/2), & m\in I_3\\
U_{m+3n+1}(1/2)+(-1)^nU_{m-1}(1/2), & m\in I_4\\
-U_1(1/2)+U_{6n+2}(1/2)+(-1)^nU_{3n}(1/2), & m\in I_5\\
-U_{m-3n}(1/2)+U_{m+3n+1}(1/2)+(-1)^nU_{m-1}(1/2), & m\in I_6
\end{cases}$$

when $n=\omega_\ell$, $(\ell\geq 0)$, and by

$$\begin{cases}
0, & m\in I_1'\\
-U_1(1/2), & m\in I_2'\\
-U_{m+3n}(1/2), & m\in I_3'\\
-U_{m+3n}(1/2)+(-1)^nU_{m-1}(1/2), & m\in I_4'\\
-U_{6n}(1/2)+U_1(1/2)+(-1)^nU_{3n-1}(1/2), & m\in I_5'\\
-U_{m+3n}(1/2)+U_{m-3n+1}(1/2)+(-1)^nU_{m-1}(1/2), & m\in I_6'
\end{cases}$$

when $n=\omega_{-\ell}$, $(\ell\geq 1)$.

We also obtain a combinatorial identity expressing a multi-sum involving $N(r,s,m,n)$ explicitly in terms of just the partition function $p(n)$.

Theorem 1.5. *Let*

$$a(n) := \sum_{k=0}^{2n} (-1)^k p(k) p(2n-k). \tag{1.22}$$

Then

$$-\frac{1}{24} \sum_{\substack{r,s\geq 0\\ m\in\mathbb{Z}}} \sum_{1\leq j\leq n} (-1)^{r+s+m} m^2(m^2-11) a(j) N(r,s,m-2s+2r,n-j).$$
$$= \sum_{k=-\infty}^{\infty} (-1)^k \left(3(n-k^2)p(n-k^2) - 2n(-1)^n p(n-2k^2)\right).$$

Remark 1.6. Observe that both sides of the above identity are finite sums.

Next, using Theorem 1.1 and Theorem 1.2, we express a double series in terms of a linear combination of single series as follows:

Theorem 1.7. *We have*

$$5\sum_{n=1}^{\infty} \frac{(q)_{n-1}^2 q^n}{(-q)_n^2} - 4\sum_{n=1}^{\infty} \frac{(q)_{n-1}^2 q^n}{(-q)_n^2} \left(\sum_{k=1}^{n-1} \frac{q^k(5+6q^k+5q^{2k})}{(1-q^{2k})^2} + \frac{q^n}{(1+q^n)^2} \right)$$
$$= \frac{(q)_\infty^2}{(-q)_\infty^2} \left\{ 3\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} + 2\sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} \right\}.$$

Using a result of Shimura, the double series in Theorem 1.7 can be expressed in terms of a linear combination of modular forms involving a unary theta series of weight $7/2$.

Theorem 1.8. *We have*

$$4\sum_{n=1}^{\infty} \frac{(q)_{n-1}^2 q^n}{(-q)_n^2} \left(\sum_{k=1}^{n-1} \frac{q^k(5+6q^k+5q^{2k})}{(1-q^{2k})^2} + \frac{q^n}{(1+q^n)^2} \right) = \frac{5}{4} - \frac{31}{24}\cdot\frac{\eta(\tau)^4}{\eta(2\tau)^2} + \frac{1}{24}\cdot\frac{\theta(\tau)}{\eta(\tau)}$$

where

$$\theta(\tau) := \sum_{n=1}^{\infty} \left(\frac{n}{12}\right) n^3 q^{\frac{n^2}{24}}, \qquad \left(\frac{n}{12}\right) := \begin{cases} 1, & n\equiv 1 \pmod 6\\ -1, & n\equiv 5 \pmod 6\\ 0, & \text{otherwise}\end{cases}$$

and $\eta(\tau) = q^{\frac{1}{24}}(q;q)_\infty$ *is the Dedekind eta-function. Also,* $\theta(2\tau)$ *satisfies*

$$\theta\left(2\cdot\frac{a\tau+b}{c\tau+d}\right) = e^{\frac{i\pi ab}{6}} \left(\frac{3c}{d}\right) \varepsilon_d^{-1} (c\tau+d)^{7/2} \theta(2\tau), \ \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma_1(12). \tag{1.23}$$

Here $\left(\frac{\cdot}{\cdot}\right)$ *is the extended Jacobi symbol and* $\varepsilon_d = 1$ *or* i *according as* $d \equiv 1$ *or* $3 \pmod 4$ *and* $\Gamma_1(N)$ *is a congruence subgroup of* $SL_2(\mathbb{Z})$ *consisting of all* 2×2 *matrices with diagonal entries* $\equiv 1 \pmod N$ *and the lower left-entry* $\equiv 0 \pmod N$.

2. Notations and preliminaries

In addition to (1.2), we adopt the following notations:

$$(\alpha_1,\alpha_2,\alpha_3,\cdots,\alpha_k;q)_n=\prod_{i=1}^{k}(\alpha_i;q)_n,\qquad (\alpha_1,\alpha_2,\alpha_3,\cdots,\alpha_k;q)_\infty=\lim_{n\to\infty}(\alpha_1,\alpha_2,\alpha_3,\cdots,\alpha_k;q)_n. \tag{2.1}$$

We also require the following *unilateral basic hypergeometric series*:

$${}_{k+1}\phi_k\left(\begin{matrix}a_1 & a_2 & \dots & a_k & a_{k+1}\\ & b_1 & b_2 & \dots & b_k\end{matrix};q,z\right)=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\dots,a_{k+1};q)_n}{(b_1,b_2,\dots,b_k,q;q)_n}z^n. \tag{2.2}$$

The Chebyshev polynomials of the second kind $U_n(x)$ are defined by the recurrence relation [15, p. 9, Equation (1.2.15 (a)-(b))]

$$U_0(x)=1,\qquad U_1(x)=2x,\qquad U_{n+1}(x)=2x\,U_n(x)-U_{n-1}(x). \tag{2.3}$$

The ordinary generating function of $U_n(x)$ is [17, p. 155, Equation (6.45) with $\gamma=1$]

$$\sum_{n=0}^{\infty}U_n(x)t^n=\frac{1}{1-2tx+t^2}. \tag{2.4}$$

Also we have [15, p. 7, Equation (1.23)]

$$U_n(x)=\frac{1}{n+1}T'_{n+1}(x)=\frac{\sin((n+1)\cos^{-1}(x))}{\sin(\cos^{-1}(x))} \tag{2.5}$$

where $T_k(x)$ is a Chebyshev polynomial of the first kind with the generating function [15, p. 36, Equation (1.105)]

$$\sum_{n=0}^{\infty}T_n(x)t^n=\frac{1-tx}{1-2tx+t^2}.$$

From (2.4), it follows that for $n>1$

$$U_{-n}(x)=-U_{n-2}(x),\qquad\qquad U_{-1}(x)=0. \tag{2.6}$$

Next, we require a result due to Agarwal [1, Equation (3.1)].

Theorem 2.1. *We have*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\gamma)_n}{(\beta)_n(\delta)_n}t^n &= \frac{(q/(\alpha t),\gamma,\alpha t,\beta/\alpha,q;q)_\infty}{(\beta/(\alpha t),\delta,t,q/\alpha,\beta;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}\delta/\gamma, & t\\ & q\alpha t/\beta\end{matrix};q,\ \gamma q/\beta\right)\\ &+\frac{(\gamma)_\infty}{(\delta)_\infty}\left(1-\frac{q}{\beta}\right)\sum_{m=0}^{\infty}\frac{(\delta/\gamma)_m(t)_m}{(q)_m(\alpha t/\beta)_{m+1}}(q\gamma/\beta)^m\left({}_2\phi_1\left(\begin{matrix}q, & q/t\\ & q\beta/(\alpha t)\end{matrix};q,\ q/\alpha\right)-1\right)\\ &+\frac{(\gamma)_\infty}{(\delta)_\infty}\left(1-\frac{q}{\beta}\right)\sum_{p=0}^{\infty}\frac{\gamma^p(\delta/\gamma)_p}{(q)_p}\sum_{m=0}^{\infty}\frac{(\delta q^p/\gamma)_m(tq^p)_m}{(q^{1+p})_m(\alpha tq^p/\beta)_{m+1}}(q\gamma/\beta)^m.\end{aligned} \tag{2.7}$$

The next result generalizes an identity of Andrews [2] in the case $k=2$. It is required later in our proofs.

Lemma 2.2. *For any C^∞-function f and $2\le k\in\mathbb{N}$, we have*

$$-\frac{(-1)^k}{k!}\left[\frac{d^k}{dz^k}(1-z)(1-z^{-1})f(z)\right]_{z=1}=\sum_{\ell=2}^{k}\frac{(-1)^\ell}{(\ell-2)!}\left[\frac{d^{\ell-2}}{dz^{\ell-2}}f(z)\right]_{z=1}. \tag{2.8}$$

Proof. The proof follows by Taylor's expansion and successive differentiation. □

Lemma 2.3. *We have*

$$\frac{(-q;q)_\infty}{(q;q)_\infty^2} =: \sum_{n=0}^\infty a(n)q^n$$

where $a(n)$ is defined in (1.22).

Proof. Observe that

$$\frac{1}{(q;q)_\infty}\prod_{n=1}^\infty \frac{1}{1-(-q)^n} = \frac{1}{(q;q)_\infty(q^2;q^2)_\infty(-q;q^2)_\infty} = \frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty^3} =: \sum_{n\geq 0} a(n)q^{2n}, \tag{2.9}$$

say. On the other hand,

$$\frac{1}{(q;q)_\infty}\prod_{n=1}^\infty \frac{1}{1-(-q)^n} = \left(\sum_{\ell=0}^\infty p(\ell)q^\ell\right)\left(\sum_{k=0}^\infty (-1)^k p(k)q^k\right) = \sum_{n\geq 0}\left(\sum_{k=0}^n (-1)^k p(k)p(n-k)\right)q^n. \tag{2.10}$$

Comparing coefficients of q^{2n} on both sides of (2.9) and (2.10) yields the result. Additionally, we obtain

$$\sum_{k=0}^{2n+1} (-1)^k p(k)p(2n+1-k) = 0, \tag{2.11}$$

which also follows immediately by rearranging the sum. □

3. Proofs of the main results

3.1. **Proof of Theorem 1.3.** We first prove the result for $|\gamma| < \min(1, |\beta/q|)$. Let $t = q$ and $\delta = \alpha\gamma q^2/\beta$ in (2.1) to obtain

$$\begin{aligned}
\sum_{n=0}^\infty \frac{(\alpha,\gamma)_n}{(\beta,\alpha\gamma q^2/\beta)_n}q^n &= \frac{(\alpha^{-1},\gamma,\alpha q,\alpha^{-1}\beta,q)_\infty}{(\alpha^{-1}q^{-1}\beta,\alpha\beta^{-1}\gamma q^2,q,\alpha^{-1}q,\beta)_\infty}\sum_{k=0}^\infty\left(\frac{\gamma q}{\beta}\right)^k \\
&\quad + \frac{(\gamma)_\infty}{(\alpha\beta^{-1}\gamma q^2)_\infty}\left(1-\frac{q}{\beta}\right)\sum_{p=0}^\infty \frac{\gamma^p(\alpha q^2/\beta)_p}{(q)_p}\sum_{m=0}^\infty \frac{(\alpha q^{p+2}/\beta)_m}{(\alpha q^{p+1}/\beta)_{m+1}}\left(\frac{\gamma q}{\beta}\right)^m \\
&= \frac{(1-\alpha^{-1})}{(1-\alpha^{-1}\beta q^{-1})(1-\beta^{-1}\gamma q)}\cdot\frac{(\alpha q,\gamma)_\infty}{(\beta,\alpha\gamma q^2/\beta)_\infty} \\
&\quad + \frac{(\gamma)_\infty}{(\alpha\beta^{-1}\gamma q^2)_\infty}\left(1-\frac{q}{\beta}\right)\sum_{p=0}^\infty \frac{\gamma^p(\alpha q^2/\beta)_p}{(1-\alpha q^{p+1}/\beta)(q)_p}\sum_{m=0}^\infty\left(\frac{\gamma q}{\beta}\right)^m \\
&= \frac{\beta^{-1}q}{(1-\alpha\beta^{-1}q)(1-\beta^{-1}\gamma q)}\cdot\frac{(\alpha,\gamma)_\infty}{(\beta,\alpha\gamma q^2/\beta)_\infty} \\
&\quad + \frac{(\gamma)_\infty}{(\alpha\beta^{-1}\gamma q^2)_\infty}\cdot\frac{(1-\beta^{-1}q)}{(1-\beta^{-1}\gamma q)}\sum_{p=0}^\infty \frac{\gamma^p(\alpha q^2/\beta)_p}{(1-\alpha q^{p+1}/\beta)(q)_p},
\end{aligned} \tag{3.1}$$

where in the penultimate step we used the condition $|\gamma q/\beta| < 1$.

Next, we notice that the sum in the right-hand side of (3.1) can be rewritten as

$$\sum_{p=0}^\infty \frac{\gamma^p(\alpha q^2/\beta)_p}{(1-\alpha q^{p+1}/\beta)(q)_p} = \frac{1}{(1-\alpha\beta^{-1}q)} + \sum_{p=1}^\infty \frac{(\alpha q^2/\beta)_{p-1}}{(q)_p}\gamma^p$$

$$= \frac{1}{(1-\alpha\beta^{-1}q)} + \frac{1}{(1-\alpha\beta^{-1}q)}\sum_{p=1}^{\infty}\frac{(\alpha q/\beta)_p}{(q)_p}\gamma^p$$
$$= \frac{1}{(1-\alpha\beta^{-1}q)}\sum_{p=0}^{\infty}\frac{(\alpha q/\beta)_p}{(q)_p}\gamma^p$$
$$= \frac{1}{(1-\alpha\beta^{-1}q)}\cdot\frac{(\alpha\gamma q/\beta)_\infty}{(\gamma)_\infty} \tag{3.2}$$

where the last step follows by q-binomial theorem [6, p. 8, Theorem 1.3.1] since $|\gamma| < 1$. Identity (1.13) now follows for $|\gamma| < \min(1, |\beta/q|)$ from (3.1) and (3.2). By analytic continuation, the result is easily seen to be extended to the said values in the hypotheses. □

3.2. **Proofs of (1.14) and (1.15).** Equation (1.14) readily follows from Theorem 1.3 by letting $\alpha = x = \gamma^{-1}$ and $\beta = zq$. Similarly, letting $\alpha = x, \gamma = q/x$ and $\beta = zq$ results in (1.15). □

3.3. **Proof of Theorem 1.1.** Theorem 1.1 follows from (1.14) by first replacing z by $-z$ and then letting $x = z^2$. □

3.4. **Proof of Theorem 1.2.** The quintuple product identity is given by [6, p. 18, Theorem 1.3.17]

$$\sum_{n=-\infty}^{\infty} q^{3n^2+n}\left(z^{3n}q^{-3n} - z^{-3n-1}q^{3n+1}\right) = (q^2;q^2)_\infty(qz;q^2)_\infty(q/z;q^2)_\infty(z^2;q^4)_\infty(q^4/z^2;q^4)_\infty. \tag{3.3}$$

Replacing q by $\sqrt{q}$ and then z by $z\sqrt{q}$ in (3.3) gives

$$\sum_{n=-\infty}^{\infty} q^{(3n^2+n)/2}\left(z^{3n} - z^{-3n-1}\right) = (q;q)_\infty(zq;q)_\infty(z^{-1};q)_\infty(z^2q;q^2)_\infty(z^{-2}q;q^2)_\infty. \tag{3.4}$$

Hence

$$\left[\frac{d^2}{dz^2}(zq;q)_\infty(z^{-1}q;q)_\infty(z^2q;q^2)_\infty(z^{-2}q;q^2)_\infty\right]_{z=1} = \frac{1}{(q;q)_\infty}\left[\frac{d^2}{dz^2}\frac{\sum_{n=-\infty}^{\infty} q^{(3n^2+n)/2}(z^{3n} - z^{-3n-1})}{1-z^{-1}}\right]_{z=1}$$
$$= \frac{1}{(q;q)_\infty}\left[\frac{d^2}{dz^2}\sum_{n=-\infty}^{\infty} q^{(3n^2+n)/2}\left(\frac{1-z^{6n+1}}{1-z}\right)\right]_{z=1}$$
$$= \frac{1}{(q;q)_\infty}\left[\frac{d^2}{dz^2}\sum_{n=-\infty}^{\infty} q^{(3n^2+n)/2}\sum_{j=0}^{6n} z^{-3n+j}\right]_{z=1}$$
$$= \frac{1}{(q;q)_\infty}\left[\frac{d^2}{dz^2}\sum_{n=-\infty}^{\infty} q^{(3n^2+n)/2}\sum_{j=0}^{6n}(-3n+j)(-3n+j-1)z^{-3n+j-2}\right]_{z=1}$$
$$= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty} q^{(3n^2+n)/2}n(3n+1)(6n+1)$$
$$= \frac{2q}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(6n+1)\frac{n(3n+1)}{2}q^{(3n^2+n)/2-1}$$
$$= \frac{2q}{(q;q)_\infty}\frac{d}{dq}\sum_{n=-\infty}^{\infty}(6n+1)q^{(3n^2+n)/2}. \tag{3.5}$$

Now employing an identity of Ramanujan and proved by Gordon [6, p. 20, Corollary 1.3.21],

$$\sum_{n=-\infty}^{\infty}(6n+1)q^{(3n^2+n)/2}=(q;q)_\infty^3(q;q^2)_\infty^2. \tag{3.6}$$

Thus, (3.5) and (3.6) give

$$\begin{aligned}\left[\frac{d^2}{dz^2}(zq;q)_\infty(z^{-1}q;q)_\infty(z^2q;q^2)_\infty(z^{-2}q;q^2)_\infty\right]_{z=1}&=\frac{2q}{(q;q)_\infty}\frac{d}{dq}(q;q)_\infty^3(q;q^2)_\infty^2\\&=-2(q;q)_\infty^2(q;q^2)_\infty^2\left\{3\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}+2\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}\right\}.\end{aligned}$$

□

3.5. **Proof of Theorem 1.4.** Using the Quintuple product identity (3.4) on the right-hand side of Theorem 1.1, we obtain

$$\begin{aligned}&(1-z)(1-z^{-1})\left[\frac{-1}{z(1-z^{-1}+z^{-2})}\cdot\frac{(z^{-2}q,z^2q)_\infty}{(-z^{-1}q,-zq)_\infty}\right]+\frac{(1+z^{-1})}{z(1+z^{-3})}\\=&-\frac{(1-z)(1-z^{-1})}{z(1-z^{-1}+z^{-2})}\cdot\frac{\sum_{n=-\infty}^{\infty}q^{(3n^2+n)/2}(z^{3n}-z^{-3n-1})}{(1-z^{-1})(q)_\infty}+\frac{1}{z(1-z^{-1}+z^{-2})}\\=&-\frac{1}{z(1-z^{-1}+z^{-2})}\left[\frac{(1-z)}{(q)_\infty}\sum_{n=-\infty}^{\infty}q^{(3n^2+n)/2}(z^{3n}-z^{-3n-1})-1\right].\end{aligned}\tag{3.7}$$

Substituting $x=1/2$ and $t=z$ in (2.4), we obtain

$$\sum_{n=0}^{\infty}U_n\left(\frac{1}{2}\right)z^n=\frac{1}{1-z+z^2}. \tag{3.8}$$

Thus, (2.3), (3.7) and (3.8) yield

$$\begin{aligned}&(1-z)(1-z^{-1})\left[\frac{-1}{z(1-z^{-1}+z^{-2})}\cdot\frac{(z^{-2}q,z^2q)_\infty}{(-z^{-1}q,-zq)_\infty}\right]+\frac{(1+z^{-1})}{z(1+z^{-3})}\\=&\left(-z\sum_{m=0}^{\infty}U_m\left(\frac{1}{2}\right)z^m\right)\left[\frac{(1-z)}{(q)_\infty}\sum_{n=-\infty}^{\infty}q^{(3n^2+n)/2}(z^{3n}-z^{-3n-1})-1\right]\\=&-\frac{(1-z)}{(q)_\infty}\sum_{m=1}^{\infty}U_{m-1}(1/2)z^m\sum_{n=-\infty}^{\infty}q^{(3n^2+n)/2}(z^{3n}-z^{-3n-1})+\sum_{m=1}^{\infty}U_{m-1}(1/2)z^m\\=&-\frac{1}{(q)_\infty}\left(\sum_{m=1}^{\infty}U_m(1/2)z^m\right)\sum_{n=-\infty}^{\infty}q^{(3n^2+n)/2}(z^{3n}-z^{-3n-1})+\sum_{m=1}^{\infty}U_{m-1}(1/2)z^m\\=&-\frac{1}{(q)_\infty}\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}U_m(1/2)q^{(3n^2+n)/2}(z^{m+3n}-z^{m-3n-1})+\sum_{m=1}^{\infty}U_{m-1}(1/2)z^m.\\=&-\frac{1}{(q)_\infty}\left(\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}U_m(1/2)q^{(3n^2+n)/2}z^{m+3n}-\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}U_m(1/2)q^{(3n^2+n)/2}z^{m-3n-1}\right)\\&+\sum_{m=1}^{\infty}U_{m-1}(1/2)z^m.\end{aligned}\tag{3.9}$$

Making the change of variables $m\to m-3n$ and $m\to m+3n+1$ respectively on the two double series in the right-hand side of (3.9), we obtain

$$(1-z)(1-z^{-1})\left[\frac{-1}{z(1-z^{-1}+z^{-2})}\cdot\frac{(z^{-2}q,z^2q)_\infty}{(-z^{-1}q,-zq)_\infty}\right]+\frac{(1+z^{-1})}{z(1+z^{-3})}$$
$$=-\frac{1}{(q)_\infty}\left(\sum_{n=-\infty}^{\infty}\sum_{m=3n+1}^{\infty}U_{m-3n}(1/2)q^{(3n^2+n)/2}z^m-\sum_{n=-\infty}^{\infty}\sum_{m=-3n}^{\infty}U_{m+3n+1}(1/2)q^{(3n^2+n)/2}z^m\right)$$
$$+\sum_{m=1}^{\infty}U_{m-1}(1/2)z^m. \tag{3.10}$$

Using Theorem 1.1 and (1.19), we immediately see that the left-hand side of (3.10) can be rewritten in the following way and we have:

$$\sum_{\substack{r,s,n\geq 0\\ m\in\mathbb{Z}}}N(r,s,m-2s+2r,n)(-1)^{r+s+m}z^mq^n$$
$$=-\frac{1}{(q)_\infty}\left(\sum_{n=-\infty}^{\infty}\sum_{m=3n+1}^{\infty}U_{m-3n}(1/2)q^{(3n^2+n)/2}z^m-\sum_{n=-\infty}^{\infty}\sum_{m=-3n}^{\infty}U_{m+3n+1}(1/2)q^{(3n^2+n)/2}z^m\right)$$
$$+\sum_{m=1}^{\infty}U_{m-1}(1/2)z^m. \tag{3.11}$$

Multiplying both sides of (3.11) by $(q)_\infty$, employing Euler's pentagonal number theorem $(q)_\infty=\sum_{k=-\infty}^{\infty}(-1)^kq^{k(3k+1)/2}$, and simplifying, we get

$$\sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}}\left(\sum_{\substack{r,s\geq 0\\ 0\leq\omega_k\leq n}}N\left(r,s,m-2s+2r,n-\omega_k\right)(-1)^{r+s+m+k}\right)z^mq^n$$
$$=-\sum_{n=-\infty}^{\infty}\sum_{m=3n+1}^{\infty}U_{m-3n}(1/2)q^{(3n^2+n)/2}z^m+\sum_{n=-\infty}^{\infty}\sum_{m=-3n}^{\infty}U_{m+3n+1}(1/2)q^{(3n^2+n)/2}z^m$$
$$+\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}(-1)^nU_{m-1}(1/2)z^mq^{(3n^2+n)/2}. \tag{3.12}$$

We define the following functions:

$$f_1(m,n):=\begin{cases}U_{m-3n}(1/2), & m>3n\\ 0, & \text{otherwise,}\end{cases}\qquad f_2(m,n):=\begin{cases}U_{m+3n+1}(1/2), & m\geq -3n\\ 0, & \text{otherwise,}\end{cases}$$
$$f_3(m,n):=\begin{cases}(-1)^nU_{m-1}(1/2), & m\geq 1\\ 0, & \text{otherwise}\end{cases} \tag{3.13}$$

$$\tilde{f}_1(m,n):=\begin{cases}U_{m+3n}(1/2), & m>-3n\\ 0, & \text{otherwise,}\end{cases}\qquad \tilde{f}_2(m,n):=\begin{cases}U_{m-3n+1}(1/2), & m\geq 3n\\ 0, & \text{otherwise,}\end{cases}$$
$$\tilde{f}_3(m,n):=\begin{cases}(-1)^nU_{m-1}(1/2), & m\geq 1\\ 0, & \text{otherwise.}\end{cases} \tag{3.14}$$

Consider the first double series

$$\begin{aligned}
&-\sum_{n=-\infty}^{\infty}\sum_{m=3n+1}^{\infty}U_{m-3n}(1/2)q^{(3n^2+n)/2}z^m\\
&=-\sum_{n=1}^{\infty}\sum_{m=-3n+1}^{\infty}U_{m+3n}(1/2)q^{(3n^2-n)/2}z^m-\sum_{n=0}^{\infty}\sum_{m=3n+1}^{\infty}U_{m-3n}(1/2)q^{(3n^2+n)/2}z^m\\
&=-\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\tilde{f}_1(m,n)z^mq^{\omega_{-n}}-\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}f_1(m,n)z^mq^{\omega_n}. \qquad (3.15)
\end{aligned}$$

Similarly the remaining two double series in (3.12) can be written using $f_2(m,n)$, $\tilde{f}_2(m,n)$, $f_3(m,n)$ and $\tilde{f}_3(m,n)$ thereby leading to

$$\begin{aligned}
&\sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}}\left(\sum_{\substack{r,s\geq 0\\ 0\leq\omega_k\leq n}}N\left(r,s,m-2s+2r,n-\omega_k\right)(-1)^{r+s+m+k}\right)z^mq^n\\
&=\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\left(-f_1(m,n)+f_2(m,n)+f_3(m,n)\right)z^mq^{\omega_n}\\
&\quad+\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\left(-\tilde{f}_1(m,n)+\tilde{f}_2(m,n)+\tilde{f}_3(m,n)\right)z^mq^{\omega_{-n}}. \qquad (3.16)
\end{aligned}$$

This establishes the result. □

Before proving Theorem 1.5, we establish a crucial lemma which also appears to be new. One of the ideas employed in its proof resulted through a personal communication with George Andrews [3].

Lemma 3.1. *Let $p_{sc}(n)$ denote the number of self-conjugate partitions of n. Then*

$$p_{sc}(n)=p(n)+2\sum_{j\geq 1}(-1)^jp(n-2j^2). \qquad (3.17)$$

Proof. To prove (3.17), we need the two identities below

$$\sum_{n=0}^{\infty}p(2n)q^n=\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q)_\infty^2},\qquad \sum_{n=0}^{\infty}p(2n+1)q^n=\frac{(-q,-q^7,q^8;q^8)_\infty}{(q)_\infty^2}, \qquad (3.18)$$

which follow from the 2-dissection of Gauss' triangular series identity:

$$\psi(q):=\sum_{n=-\infty}^{\infty}q^{2n^2-n}=\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \qquad (3.19)$$

At this point, we note that [4, p. 5430]

$$\sum_{n\geq 0}p_{\mathrm{sc}}(n)q^n=\sum_{n\geq 0}\frac{q^{n^2}}{(q^2;q^2)_n}, \qquad (3.20)$$

which yields the following identities upon 2-dissections:

$$\sum_{n\geq 0}p_{\mathrm{sc}}(2n)q^n=\sum_{n\geq 0}\frac{q^{2n^2}}{(q;q)_{2n}},\qquad \sum_{n\geq 0}p_{\mathrm{sc}}(2n+1)q^n=\sum_{n\geq 0}\frac{q^{2n^2+2n}}{(q;q)_{2n+1}}. \qquad (3.21)$$

Using identities (38) and (39) from Slater's list [18], (3.18) and (3.21) yield

$$\begin{aligned}\sum_{n\geq 0}p_{\mathrm{sc}}(2n)q^n&=\sum_{n\geq 0}\frac{q^{2n^2}}{(q;q)_{2n}}=\frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}=\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n\geq 0}p(2n)q^n\\ \sum_{n\geq 0}p_{\mathrm{sc}}(2n+1)q^n&=\sum_{n\geq 0}\frac{q^{2n^2+2n}}{(q;q)_{2n+1}}=\frac{(-q,-q^7,q^8;q^8)_\infty}{(q^2;q^2)_\infty}=\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n\geq 0}p(2n+1)q^n.\end{aligned}\tag{3.22}$$

Using (3.29) on the extreme right-hand sides of each identity in (3.22) and comparing coefficients of q^n on both sides of the identities together yield (3.17). □

3.6. **Proof of Theorem 1.5.** From Theorem 1.1 and (1.19),

$$\begin{aligned}&\sum_{\substack{r,s,n\geq 0\\ m\in\mathbb{Z}}}N(r,s,m-2s+2r,n)(-1)^{r+s+m}z^mq^n\\ &\qquad=(1-z)(1-z^{-1})\left[\frac{-1}{z(1-z^{-1}+z^{-2})}\cdot\frac{(z^{-2}q,z^2q)_\infty}{(-z^{-1}q,-zq)_\infty}\right]+\frac{(1+z^{-1})}{z(1+z^{-3})}.\end{aligned}\tag{3.23}$$

The idea is to take the fourth derivative on both sides of the above identity with respect to z, let $z=1$, and then equate the coefficients of q^n on both sides of the resulting identity. We first concentrate on the right-hand side.

Invoking Lemma 2.2 and using the definition of $D(z,q)$ in (1.10), it is seen using routine simplification that

$$\begin{aligned}&-\frac{1}{24}\left[\frac{d^4}{dz^4}\left\{(1-z)(1-z^{-1})\left(\frac{-1}{z(1-z^{-1}+z^{-2})}\cdot D(z,q)\right)+\frac{(1+z^{-1})}{z(1+z^{-3})}\right\}\right]_{z=1}\\ &=-\frac{1}{2}D''(1,q)\\ &=(q;q)_\infty^2(q;q^2)_\infty^2\left\{3\sum_{n=1}^\infty\frac{nq^n}{1-q^n}+2\sum_{n=1}^\infty\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}\right\},\end{aligned}\tag{3.24}$$

where in the last step we invoked Theorem 1.2. Note that differentiating Euler's generating function for $p(n)$ leads to [2, Equation (3.3)]

$$\sum_{n=1}^\infty np(n)q^n=\frac{1}{(q;q)_\infty}\sum_{n=1}^\infty\frac{nq^n}{1-q^n}.\tag{3.25}$$

Since $p_{\mathrm{sc}}(n)$ equals the number of partitions of n into distinct odd parts, we have

$$\sum_{n=1}^\infty p_{\mathrm{sc}}(n)q^n=(-q;q^2)_\infty.$$

Replacing q by $-q$ in the above identity and then differentiating both sides with respect to q leads us to

$$\sum_{n=1}^\infty(-1)^nnp_{\mathrm{sc}}(n)q^n=-(q;q^2)_\infty\sum_{n=1}^\infty\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}.\tag{3.26}$$

Therefore from (3.25) and (3.26), we deduce that

$$\begin{aligned}3\sum_{n=1}^\infty\frac{nq^n}{1-q^n}+2\sum_{n=1}^\infty\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}&=3(q;q)_\infty\sum_{n\geq 1}np(n)q^n-\frac{2}{(q;q^2)_\infty}\sum_{n\geq 1}(-1)^nnp_{\mathrm{sc}}(n)q^n\\ &=3(q;q)_\infty\sum_{n\geq 1}np(n)q^n-2(-q;q)_\infty\sum_{n\geq 1}(-1)^nnp_{\mathrm{sc}}(n)q^n,\end{aligned}\tag{3.27}$$

where in the last step, we employed the elementary result $1/(q;q^2)_\infty = (-q;q)_\infty$. Thus taking the fourth derivative on both sides of (3.23) with respect to z and then letting $z=1$, employing (3.24) and (3.27), and then multiplying both sides by $(-q;q)_\infty/(q;q)_\infty^2$, we obtain

$$\begin{aligned}
&-\frac{1}{24}\frac{(-q;q)_\infty}{(q;q)_\infty^2}\sum_{n=1}^{\infty} q^n \sum_{\substack{r,s\geq 0\\ m\in\mathbb{Z}}} m(m-1)(m-2)(m-3)N(r,s,m-2s+2r,n)(-1)^{r+s+m}\\
&\qquad = 3\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n\geq 1} np(n)q^n - 2\sum_{n\geq 1}(-1)^n np_{\mathrm{sc}}(n)q^n. \qquad (3.28)
\end{aligned}$$

By an application of the Jacobi triple product identity [6, Theorem 1.3.3],

$$1+2\sum_{j=1}^{\infty}(-1)^j q^{j^2} = \sum_{j\in\mathbb{Z}}(-1)^j q^{j^2} = \frac{(q;q)_\infty}{(-q;q)_\infty} \qquad (3.29)$$

so that the right-hand side of (3.28) can be rewritten as

$$\begin{aligned}
&3\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n\geq 1} np(n)q^n - 2\sum_{n\geq 1}(-1)^n np_{\mathrm{sc}}(n)q^n\\
&= \sum_{n\geq 1}\left[3np(n) + 6\sum_{j\geq 1}(-1)^j(n-j^2)p(n-j^2) - 2(-1)^n np_{\mathrm{sc}}(n)\right]q^n\\
&= \sum_{n\geq 1}\left[(3-2(-1)^n)np(n) + 6\sum_{j\geq 1}(-1)^j\left\{(n-j^2)p(n-j^2) - 2(-1)^n np(n-2j^2)\right\}\right]q^n, \qquad (3.30)
\end{aligned}$$

where in the last step, we invoked Lemma 3.1. Lastly, observe that replacing z by z^{-1} in (1.19) results in

$$N(r,s,m-2s+2r,n) = N(r,s,-m-2s+2r,n), \qquad (3.31)$$

which, in turn, implies that for fixed r,s and n,

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}} m(m-1)(m-2)(m-3)N(r,s,m-2s+2r,n)(-1)^m\\
&= \sum_{m\in\mathbb{Z}}(-1)^m m^2(m^2-11)N(r,s,m-2s+2r,n). \qquad (3.32)
\end{aligned}$$

Hence invoking Lemma 2.3, (3.28), (3.30) and (3.32), the result now follows by comparing the coefficients of q^n on both sides and by expressing the resulting right-hand side as a bilateral series. $\square$

3.7. **Proof of Theorem 1.7.** We compute the fourth derivative of the identity in Theorem 1.1 with respect to z and then let $z=1$. Using Lemma 2.2, we obtain

$$\begin{aligned}
-\frac{1}{4!}\left[\frac{d^4}{dz^4}(1-z)(1-z^{-1})f(z)\right]_{z=1} &= \sum_{\ell=2}^{4}\frac{(-1)^\ell}{(\ell-2)!}\left[\frac{d^{\ell-2}}{dz^{\ell-2}}f(z)\right]_{z=1}\\
&= f(1)-f'(1)+\frac{f''(1)}{2}. \qquad (3.33)
\end{aligned}$$

First, let $f(z)$ be defined by

$$f(z) := (1+z)(1+z^{-1})\sum_{n=1}^{\infty}\frac{(z^2q)_{n-1}(z^{-2}q)_{n-1}q^n}{(-zq)_n(-z^{-1}q)_n} \qquad (3.34)$$

so that

$$1+(1-z)(1-z^{-1})f(z)=\sum_{n\geq 0}\frac{(z^2;q)_n(z^{-2};q)_nq^n}{(-zq;q)_n(-z^{-1}q;q)_n}. \tag{3.35}$$

Clearly, we have

$$f(1)=4\sum_{n=1}^{\infty}\frac{(q)_{n-1}^2q^n}{(-q)_n^2}. \tag{3.36}$$

By logarithmic differentiation, it follows that

$$f'(z)=f(z)\left(\frac{1}{1+z}-\frac{z^{-2}}{1+z^{-1}}+\frac{\left(\sum_{n=1}^{\infty}\frac{(z^2q)_{n-1}(z^{-2}q)_{n-1}q^n}{(-zq)_n(-z^{-1}q)_n}\right)'}{\sum_{n=1}^{\infty}\frac{(z^2q)_{n-1}(z^{-2}q)_{n-1}q^n}{(-zq)_n(-z^{-1}q)_n}}\right). \tag{3.37}$$

Next, we have

$$\begin{aligned}\left(\sum_{n=1}^{\infty}\frac{(z^2q)_{n-1}(z^{-2}q)_{n-1}q^n}{(-zq)_n(-z^{-1}q)_n}\right)' &= \sum_{n=1}^{\infty}\frac{(z^2q)_{n-1}(z^{-2}q)_{n-1}q^n}{(-zq)_n(-z^{-1}q)_n}\left\{\sum_{k=1}^{n-1}\frac{-2zq^k}{1-z^2q^k}+\sum_{k=1}^{n-1}\frac{2z^{-3}q^k}{1-z^{-2}q^k}\right.\\ &\qquad\left.-\sum_{k=1}^{n}\frac{q^k}{1+zq^k}+\sum_{k=1}^{n}\frac{z^{-2}q^k}{1+z^{-1}q^k}\right\}.\end{aligned} \tag{3.38}$$

Thus (3.38) yields

$$\left[\left(\sum_{n=1}^{\infty}\frac{(z^2q)_{n-1}(z^{-2}q)_{n-1}q^n}{(-zq)_n(-z^{-1}q)_n}\right)'\right]_{z=1}=0. \tag{3.39}$$

Hence (3.37) and (3.39) yield

$$f'(1) = 0. \tag{3.40}$$

Next, using (3.37) we compute the second derivative of $f(z)$ to get

$$f''(z)=f'(z)\cdot S(z)+f(z)\cdot S'(z), \tag{3.41}$$

where

$$S(z):=\frac{1}{1+z}-\frac{z^{-2}}{1+z^{-1}}+\frac{\left(\sum_{n=1}^{\infty}\frac{(z^2q)_{n-1}(z^{-2}q)_{n-1}q^n}{(-zq)_n(-z^{-1}q)_n}\right)'}{\sum_{n=1}^{\infty}\frac{(z^2q)_{n-1}(z^{-2}q)_{n-1}q^n}{(-zq)_n(-z^{-1}q)_n}}. \tag{3.42}$$

We compute $S'(z)$ first. Let us further put

$$S_1(z):=\left(\sum_{n=1}^{\infty}\frac{(z^2q)_{n-1}(z^{-2}q)_{n-1}q^n}{(-zq)_n(-z^{-1}q)_n}\right)', \qquad S_2(z):=\sum_{n=1}^{\infty}\frac{(z^2q)_{n-1}(z^{-2}q)_{n-1}q^n}{(-zq)_n(-z^{-1}q)_n}. \tag{3.43}$$

Then (3.42) and (3.43) yield

$$S'(z)=-\frac{1}{(1+z)^2}+\frac{1+2z}{(z+z^2)^2}+\frac{S_2(z)\cdot S_1'(z)-S_1(z)\cdot S_2'(z)}{S_2^2(z)}. \tag{3.44}$$

Next, we note from (3.34) and (3.43) that $f(z) = (1+z)(1+z^{-1})S_2(z)$ and from (3.39) that $S_1(1) = 0$. Then from (3.40), (3.41) and (3.44) that

$$f''(1) = 4 \cdot S_2(1) \cdot \left(-\frac{1}{4} + \frac{3}{4} + \frac{S_1'(1)}{S_2(1)} \right) = 2 \sum_{n=1}^{\infty} \frac{(q)_{n-1}^2 q^n}{(-q)_n^2} + 4 \cdot S_1'(1). \tag{3.45}$$

Thus, it remains to calculate $S_1'(1)$. Before we do that, we note that $S_1(1) = 0$ and this precisely happens since the quantity inside curly braces in the right-hand side of (3.38) is zero. Let us call this quantity $C(z)$. Thus, in order to calculate $S_1(1)'$, we need only calculate $C'(1)$.

$$\begin{aligned} C'(z) &= \sum_{k=1}^{n-1} \frac{(1 - z^2 q^k) \cdot (-2q^k) - (-2zq^k) \cdot (-2zq^k)}{(1 - z^2 q^k)^2} + \sum_{k=1}^{n-1} \frac{-2q^k(3z^2 - q^k)}{(z^3 - zq^k)^2} \\ &- \sum_{k=1}^{n} \frac{-q^{2k}}{(1 + zq^k)^2} + \sum_{k=1}^{n} \frac{-q^k(2z + q^k)}{(z^2 + zq^k)^2}. \end{aligned} \tag{3.46}$$

Thus (3.46) implies

$$\begin{aligned} C'(1) &= -8 \sum_{k=1}^{n-1} \frac{q^k}{(1 - q^k)^2} - 2 \sum_{k=1}^{n} \frac{q^k}{(1 + q^k)^2} \\ &= -2 \sum_{k=1}^{n-1} \frac{q^k(5 + 6q^k + 5q^{2k})}{(1 - q^{2k})^2} - \frac{2q^n}{(1 + q^n)^2}. \end{aligned} \tag{3.47}$$

Hence (3.38),(3.43), (3.45) and (3.47) yield

$$f''(1) = 2 \sum_{n=1}^{\infty} \frac{(q)_{n-1}^2 q^n}{(-q)_n^2} - 8 \sum_{n=1}^{\infty} \frac{(q)_{n-1}^2 q^n}{(-q)_n^2} \left(\sum_{k=1}^{n-1} \frac{q^k(5 + 6q^k + 5q^{2k})}{(1 - q^{2k})^2} + \frac{q^n}{(1 + q^n)^2} \right). \tag{3.48}$$

Thus (3.33), (3.35), (3.36), (3.40) and (3.48) yield

$$\begin{aligned} -\frac{1}{24} \left[\frac{d^4}{dz^4} \sum_{n \geq 0} \frac{(z^2; q)_n (z^{-2}; q)_n q^n}{(-zq; q)_n (-z^{-1}q; q)_n} \right]_{z=1} &= 5 \sum_{n=1}^{\infty} \frac{(q)_{n-1}^2 q^n}{(-q)_n^2} \\ &-4 \sum_{n=1}^{\infty} \frac{(q)_{n-1}^2 q^n}{(-q)_n^2} \left(\sum_{k=1}^{n-1} \frac{q^k(5 + 6q^k + 5q^{2k})}{(1 - q^{2k})^2} + \frac{q^n}{(1 + q^n)^2} \right). \end{aligned} \tag{3.49}$$

Along with (1.11) and (3.24), this implies the result. □

3.8. **Proof of Theorem 1.8.** Using Theorem 1.7 and (1.12), we have

$$\begin{aligned} 4 \sum_{n=1}^{\infty} \frac{(q)_{n-1}^2 q^n}{(-q)_n^2} \left(\sum_{k=1}^{n-1} \frac{q^k(5 + 6q^k + 5q^{2k})}{(1 - q^{2k})^2} + \frac{q^n}{(1 + q^n)^2} \right) &= -\frac{5}{4} \left(\frac{(q)_\infty^2}{(-q)_\infty^2} - 1 \right) \\ &- \frac{(q)_\infty^2}{(-q)_\infty^2} \left\{ 5 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} - 4 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} \right\}. \end{aligned} \tag{3.50}$$

Note that $\dfrac{(q)_\infty^2}{(-q)_\infty^2} = \dfrac{(q)_\infty^4}{(q^2; q^2)_\infty^2}$ and by logarithmic differentiation we have

$$\frac{d}{dq} \left(\frac{(q)_\infty^5}{(q^2; q^2)_\infty^2} \right) = \frac{(q)_\infty^5}{(q^2; q^2)_\infty^2} \left\{ -5 \sum_{n=1}^{\infty} \frac{nq^{n-1}}{1 - q^n} + 4 \sum_{n=1}^{\infty} \frac{nq^{2n-1}}{1 - q^{2n}} \right\}. \tag{3.51}$$

Thus, (3.51) gives

$$\frac{(q)_\infty^2}{(-q)_\infty^2}\left\{5\sum_{n=1}^\infty \frac{nq^n}{1-q^n}-4\sum_{n=1}^\infty \frac{nq^{2n}}{1-q^{2n}}\right\}=-\frac{q^{25/24}}{\eta(\tau)}\cdot\frac{d}{dq}\left(\frac{(q)_\infty^5}{(q^2;q^2)_\infty^2}\right)=-\frac{q^{25/24}}{\eta(\tau)}\cdot\frac{d}{dq}\left(q^{-1/24}\frac{\eta(\tau)^5}{\eta(2\tau)^2}\right) \tag{3.52}$$

and from [12, Theorem 1.1], it follows that

$$\frac{\eta(\tau)^5}{\eta(2\tau)^2}=\sum_{n=1}^\infty\left(\frac{n}{12}\right)nq^{\frac{n^2}{24}}. \tag{3.53}$$

Thus

$$\frac{d}{dq}\left(q^{-1/24}\frac{\eta(\tau)^5}{\eta(2\tau)^2}\right)=\frac{1}{24}\sum_{n=1}^\infty\left(\frac{n}{12}\right)n(n^2-1)q^{\frac{n^2-1}{24}-1}. \tag{3.54}$$

Combining (3.50), (3.52) and (3.54), we get

$$\begin{aligned}4\sum_{n=1}^\infty\frac{(q)_{n-1}^2q^n}{(-q)_n^2}&\left(\sum_{k=1}^{n-1}\frac{q^k(5+6q^k+5q^{2k})}{(1-q^{2k})^2}+\frac{q^n}{(1+q^n)^2}\right)\\&=-\frac{5}{4}\left(\frac{(q)_\infty^2}{(-q)_\infty^2}-1\right)+\frac{1}{24\eta(\tau)}\sum_{n=1}^\infty\left(\frac{n}{12}\right)n(n^2-1)q^{\frac{n^2}{24}}\\&=-\frac{5}{4}\left(\frac{(q)_\infty^2}{(-q)_\infty^2}-1\right)-\frac{1}{24}\frac{\eta(\tau)^4}{\eta(2\tau)^2}+\frac{1}{24}\cdot\frac{\theta(\tau)}{\eta(\tau)}\end{aligned} \tag{3.55}$$

where we have used (3.53) in the last step and where $\theta(\tau)$ is defined as in the theorem. The result now follows. It remains to show that $\theta(2\tau)$ satisfies the transformation in (1.23). This follows by choosing $N=6,\ h=1,\ P(m)=m^3,\ A=[6]$ in [16, Proposition 2.1]. □

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