

Triangular recurrences, generalized Eulerian numbers, and related number triangles

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Abstract

Many combinatorial and other number triangles are solutions of recurrences of the Graham–Knuth–Patashnik (GKP) type. Such triangles and their defining recurrences are investigated analytically. They are acted upon by a transformation group generated by two involutions: a left–right reflection and an upper binomial transformation, acting row-wise. The group also acts on the bivariate exponential generating function (EGF) of the triangle. By the method of characteristics, the EGF of any GKP triangle has an implicit representation in terms of the Gauss hypergeometric function. There are several parametric cases when this EGF can be obtained in closed form. One is when the triangle elements are the generalized Stirling numbers of Hsu and Shiue. Another is when they are generalized Eulerian numbers of a newly defined kind. These numbers are related to the Hsu–Shiue ones by an upper binomial transformation, and can be viewed as coefficients of connection between polynomial bases, in a manner that generalizes the classical Worpitzky identity. Many identities involving these generalized Eulerian numbers and related generalized Narayana numbers are derived, including closed-form evaluations in combinatorially significant cases.

Keywords: Eulerian number, Stirling number, triangular recurrence, number triangle, combinatorial triangle, Narayana number, Worpitzky identity

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1. Introduction

1.1. Notation

Recurrences of the form

$$\begin{matrix} n+1 \\ k+1 \end{matrix} = [\alpha n + \beta(k+1) + \gamma] \begin{matrix} n \\ k+1 \end{matrix} + [\alpha' n + \beta' k + \gamma'] \begin{matrix} n \\ k \end{matrix}, \quad (1.1)$$

satisfied by an infinite triangle of numbers $\begin{matrix} n \\ k \end{matrix}$, $0 \leq k \leq n < \infty$, with $\begin{matrix} n \\ k \end{matrix}$ equal to zero by convention if $k < 0$ or $k > n$, and normalized so that the apex element

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$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ equals unity, occur in pure and applied combinatorics, the analysis of discrete algorithms, and elsewhere in mathematics. Graham, Knuth, and Patashnik [33, Chapter 6, Problem 94] have indicated the need for a general theory of such triangular recurrences, which are now said to be of P94 or GKP type [57].

This would include the construction of explicit solutions $\begin{Bmatrix} n \\ k \end{Bmatrix}$ of minimal rank, for the broadest choices of the parameter vectors $\alpha, \beta; \gamma$ and $\alpha', \beta'; \gamma'$, and the identification of especially simple ‘fundamental’ solutions in terms of which other solutions can be expressed. Here, ‘rank’ refers to the depth to which summations are nested in any explicit formula, the summand(s) being products and quotients of factorials and powers [22, §5.7]. The strongest results to date in these directions include those of Spivey [59] and Barbero G. et al. [4], who employed both series manipulations and generating functions. The present work builds on theirs.

Many familiar numerical or combinatorial triangles are solutions of GKP recurrences. The new, explicitly parametric notations

$$\begin{Bmatrix} n \\ k \end{Bmatrix} =: \left[\begin{array}{c|c} \alpha, & \beta \\ \alpha', & \beta' \end{array} \middle| \begin{array}{c} \gamma \\ \gamma' \end{array} \right]_{n,k}, \quad (1.2a)$$

$$\sum_{k=0}^n \begin{Bmatrix} n \\ k \end{Bmatrix} t^k =: \left[\begin{array}{c|c} \alpha, & \beta \\ \alpha', & \beta' \end{array} \middle| \begin{array}{c} \gamma \\ \gamma' \end{array} \right]_n (t), \quad (1.2b)$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^n \begin{Bmatrix} n \\ k \end{Bmatrix} t^k \frac{z^n}{n!} =: \left[\begin{array}{c|c} \alpha, & \beta \\ \alpha', & \beta' \end{array} \middle| \begin{array}{c} \gamma \\ \gamma' \end{array} \right] (t, z) \quad (1.2c)$$

will be employed here. The first symbolizes the infinite triangle derived from the recurrence (1.1), and the second is its n 'th row polynomial, the univariate ordinary generating function of its n 'th row. The third is the bivariate exponential generating function (EGF) of the triangle as a whole. The six GKP parameters will be allowed to be complex, like the generating function arguments t, z . Transformed or modified versions of a GKP triangle $\begin{Bmatrix} n \\ k \end{Bmatrix}$ or a GKP parameter array $\left[\begin{array}{c|c} \alpha & \beta \\ \alpha' & \beta' \end{array} \middle| \begin{array}{c} \gamma \\ \gamma' \end{array} \right]$ will be indicated by an asterisk, as $\begin{Bmatrix} n \\ k \end{Bmatrix}^*$ or $\left[\begin{array}{c|c} \alpha & \beta \\ \alpha' & \beta' \end{array} \middle| \begin{array}{c} \gamma \\ \gamma' \end{array} \right]^*$. An alternative six-parameter notation will be introduced in Section 3.1.

Partial lists of GKP recurrences and solutions that have appeared in the literature can be found in [4, 63, 64]. In particular,

$$\begin{aligned} \left\{ \begin{array}{c} n \\ k \end{array} \right\} &= \left[\begin{array}{c|c} 0, & 1 \\ 0, & 0 \end{array} \middle| \begin{array}{c} 0 \\ 1 \end{array} \right]_{n,k}, & \left[\begin{array}{c} n \\ k \end{array} \right] &= \left[\begin{array}{c|c} 1, & 0 \\ 0, & 0 \end{array} \middle| \begin{array}{c} 0 \\ 1 \end{array} \right]_{n,k}, \\ \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle &= \left[\begin{array}{c|c} 0, & 1 \\ 1, & -1 \end{array} \middle| \begin{array}{c} 1 \\ 0 \end{array} \right]_{n,k}, & \binom{n}{k} &= \left[\begin{array}{c|c} 0, & 0 \\ 0, & 0 \end{array} \middle| \begin{array}{c} 1 \\ 1 \end{array} \right]_{n,k}, \end{aligned} \quad (1.3)$$

where $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ are the Stirling subset numbers (also denoted by $S(n, k)$ and called the Stirling numbers of the second kind), $\left[\begin{array}{c} n \\ k \end{array} \right]$ are the Stirling cycle numbers (also denoted by $(-1)^{n-k} s(n, k)$ and called the unsigned Stirling numbers of the first kind), and $\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle$ are the Eulerian numbers (also denoted in the traditional indexing by $A_{n,k+1}$). Each of these triangles has a combinatorial interpretation. For any

n -set, $\{k^n\}$ counts the number of its partitions into k blocks, and $[k^n]$ counts the number of its permutations that have k cycles. If the n -set is totally ordered, $\langle k^n \rangle$ counts the number of its permutations that have k descents. The Stirling and Eulerian numbers are introduced in [22, 33, 55]. The Eulerian numbers are reviewed in [28] (see also [62]) and are treated more abstractly in [29, 54].

1.2. Context and overview

This paper introduces a new, GKP-type generalization of the Eulerian numbers $\langle k^n \rangle$, in addition to developing further the analytic theory of GKP recurrences and their solutions. Two other parametric GKP triangles are also studied. The following remarks place the new Eulerian numbers in context.

Hsu and Shiue [40] introduced a parametric, GKP-type generalization of the Stirling numbers $\{k^n\}$ and $[k^n]$, which subsumes various previously treated ones. In a slight modification of their notation, it is

$$S_{n,k}(a, b; r) := \left[\begin{array}{c|c} -a, & b \\ 0, & 0 \end{array} \middle| \begin{array}{c} r \\ 1 \end{array} \right]_{n,k}. \quad (1.4)$$

(Examples are listed in [7, 37, 40].) They originally defined the $S_{n,k}(a, b; r)$ numbers as coefficients of connection between graded polynomial bases of factorial type, i.e.,

$$(x)^{\overline{n,a}} = \sum_{k=0}^n S_{n,k}(a, b; r) (x-r)^{\underline{k,b}}. \quad (1.5)$$

When $(a, b; r) = (0, 1; 0)$, this reduces to the original definition of $\{k^n\}$,

$$x^n = \sum_{k=0}^n \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^{\underline{k}}, \quad (1.6)$$

which is the Newton–Gregory expansion of x^n , and similarly when $(a, b; r) = (-1, 0; 0)$, it reduces to the original definition of $[k^n]$. (Here $(x)^{\overline{n,a}}$ and $x^{\underline{k}}$ denote the falling factorials $(x)(x-a)\cdots[x-(n-1)a]$ and $x(x-1)\cdots[x-(k-1)]$; rising factorials will be indicated by an overbar.) The explicit general formula

$$S_{n,k}(a, b; r) = \frac{1}{b^k k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (bj+r)^{\overline{n,a}} \quad (1.7)$$

was pointed out by Corcino [23]. Equation (1.7) is a rank-1 formula that subsumes the well-known formula for the Stirling subset numbers [22, 33], which is

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n. \quad (1.8)$$

But (1.7) clearly does not apply when $b = 0$, as is the case for the Stirling cycle numbers $[k^n]$, for which a rank-2 formula must be used [22]. For a dis-

cussion of the related definitions (1.4), (1.5), and (1.7), see [23] and [47, §4.2].¹ When restricted to integer parameter values, $S_{n,k}(a, b; r)$ has been interpreted combinatorially [7, 18, 24, 46, 48].

The GKP recurrences solved in the present paper are largely ones with $\beta\beta' \neq 0$ (they are of ‘type I’ in the classification of [4]). They would seem unrelated to the generalized Stirling numbers of (1.4). But in fact, if $\beta' \neq 0$ then

$$\left[\begin{array}{c|c} \alpha, & \beta \\ 0, & \beta' \end{array} \middle| \begin{array}{c} \gamma \\ \gamma' \end{array} \right]_{n,k} = \left(\frac{\gamma'}{\beta'} \right)^{\bar{k}} (\beta')^k \left[\begin{array}{c|c} \alpha, & \beta \\ 0, & 0 \end{array} \middle| \begin{array}{c} \gamma \\ 1 \end{array} \right]_{n,k} = (\gamma')^{\bar{k}, \beta'} S_{n,k}(-\alpha, \beta; \gamma). \quad (1.9)$$

This is because if the lower parameter vector $(\alpha', \beta'; \gamma')$ of a GKP recurrence is equal to $(0, 0; c_1)$, replacing it by $(0, c_1; sc_1)$ will multiply the solution $\left| \begin{array}{c} n \\ k \end{array} \right|$ by $s^{\bar{k}}$ [59]. Also, multiplying $(\alpha', \beta'; \gamma')$ by any common constant factor A clearly multiplies $\left| \begin{array}{c} n \\ k \end{array} \right|$ by A^k . (If the upper vector $(\alpha, \beta; \gamma)$ were multiplied by A , the solution would be multiplied instead by A^{n-k} .) Thus solutions of the Hsu–Shiue type (1.4) are fundamental ones, in terms of which the solution of any GKP recurrence with $\alpha' = 0$ and $\beta' \neq 0$ can be expressed.

Formulas for the GKP solution $\left| \begin{array}{c} n \\ k \end{array} \right|$ when $\beta\beta' \neq 0$ have been systematically derived in three cases: (A I) $\alpha' = 0$, (A II) $\alpha + \beta = 0$, and (A III) $\frac{\alpha}{\beta} = 1 + \frac{\alpha'}{\beta'}$. In each case, $\left| \begin{array}{c} n \\ k \end{array} \right|$ can be expressed as a double sum involving subset and cycle Stirling numbers, binomial coefficients, and generalized factorials. (See [59], and [27, Proposition 2.5] for a compact restatement.) The derived expressions for $\left| \begin{array}{c} n \\ k \end{array} \right|$ are therefore of high rank: up to 5, though the nested summations can sometimes be simplified. In case (A I), with $\alpha' = 0$, it is better to compute the triangle $\left| \begin{array}{c} n \\ k \end{array} \right|$ by applying (1.9) to reduce it to the generalized Stirling triangle $S_{n,k}(-\alpha, \beta; \gamma)$, which can be computed from formula (1.7). The resulting formula for $\left| \begin{array}{c} n \\ k \end{array} \right|$ is of rank 1. The handling of case (A II), when $\alpha + \beta = 0$, can be similarly improved, because one can show that there is an involution

$$\left[\begin{array}{c|c} \alpha, & \beta \\ 0, & \beta' \end{array} \middle| \begin{array}{c} \gamma \\ \gamma' \end{array} \right]_n(t) = t^n \left[\begin{array}{c|c} \beta', & -\beta' \\ \alpha + \beta, & -\beta \end{array} \middle| \begin{array}{c} \gamma' \\ \gamma \end{array} \right]_n \left(\frac{1}{t} \right). \quad (1.10)$$

That is, row polynomials in cases (A I), (A II) are reversed or ‘reflected’ versions of each other.

A major theme of the present paper is that the analysis of case (A III) leads naturally to a new generalization of the Eulerian numbers $\langle \begin{array}{c} n \\ k \end{array} \rangle$, combinatorial interpretations of which remain to be explored. When $\beta\beta' \neq 0$, one can assume without loss of generality that $\beta' = -\beta$, which is an innocuous normalization. Case (A III) is then the case when $\alpha + \alpha' = \beta$, which suggests defining the

¹It should be noted that numbers equivalent to the $S_{n,k}(a, b; r)$ had been introduced previously by Singh Chandel [58] and Charalambides and Koutras [18]. Their starting point was not (1.4) or (1.5), but (1.7).

parametric generalized Eulerian triangle

$$E_{n,k}(a, b; c_0, c_\infty) := \left[\begin{array}{c|c} -a, & b \\ a+b, & -b \end{array} \middle| \begin{array}{c} c_0 \\ c_\infty \end{array} \right]_{n,k}, \quad (1.11)$$

which reduces to $\langle n \rangle_k$ when $(a, b; c_0, c_\infty)$ is $(0, 1; 1, 0)$ and to the traditionally indexed numbers $A_{n,k}$ when it is $(0, 1; 0, 1)$.

Equation (1.11) will be shown to imply that for all $n \geq 0$,

$$(c_0 + c_\infty)^{\bar{n},b} (x)^{\underline{2},a} = \sum_{k=0}^n E_{n,k}(a, b; c_0, c_\infty) (x - c_0)^{\underline{k},b} (x + c_\infty)^{\overline{n-k},b}, \quad (1.12)$$

which if $(c_0 + c_\infty)^{\bar{n},b} \neq 0$, defines the generalized numbers $E_{n,k}(a, b; c_0, c_\infty)$, $0 \leq k \leq n$, as expansion coefficients. (For a proof that $(x - c_0)^{\underline{k},b} (x + c_\infty)^{\overline{n-k},b}$, $0 \leq k \leq n$, are a basis for the space of polynomials of degree $\leq n$ in the indeterminate x , see [13].) When $(a, b; c_0, c_\infty) = (0, 1; 1, 0)$, eq. (1.12) reduces to the celebrated identity of Worpitzky,

$$n! x^n = \sum_{k=0}^n \langle n \rangle_k (x - k)^{\bar{n}}. \quad (1.13)$$

Thus (1.12) is a generalized Worpitzky identity. Just as it is clear from (1.5) that the generalized Stirling numbers $S_{n,k}(a, b; r)$ are coefficients that connect a factorial basis of the ring of polynomials (depending on a) to another one (depending on b), so it is clear from (1.12) that for all $n \geq 0$, the generalized Eulerian numbers $E_{n,k}(a, b; c_0, c_\infty)$, $0 \leq k \leq n$, relate the factorial element $(x)^{\underline{2},a}$ of the $(n+1)$ -dimensional space of polynomials of degree $\leq n$ to a b -dependent ‘bifactorial’ basis of this space.

A rank-1 formula for these numbers will also be derived, applying when $b \neq 0$, namely

$$E_{n,k}(a, b; c_0, c_\infty) = \frac{1}{b^k k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (bn + c_0 + c_\infty)^{\underline{k-j},b} (c_0 + c_\infty)^{\bar{j},b} (bj + c_0)^{\underline{2},a}. \quad (1.14)$$

When $(a, b; c_0, c_\infty) = (0, 1; 1, 0)$ this reduces to the classical formula [22, 33]

$$\langle n \rangle_k = \sum_{j=0}^k (-1)^{k-j} \binom{n+1}{k-j} (j+1)^n. \quad (1.15)$$

The reader should notice that the five equations (1.11)–(1.15), dealing with the new generalized Eulerian numbers, are bijective (by design) with (1.4)–(1.8), which dealt with the Stirling numbers of Hsu and Shiue.

There are two subcases of the parametric $E_{n,k}(a, b; c_0, c_\infty)$, extending the standard numbers $E_{n,k}(0, 1; 1, 0) = \langle n \rangle_k$, which have been treated in the literature. The first is when $c_0 + c_\infty = b$. It could be called the single-progression

subcase, because the expansion functions in (1.12) then simplify:

$$(x - c_0)^{k \cdot b} (x + c_\infty)^{\overline{n-k, b}} = [x - c_0 - (k - 1)b]^{\overline{n, b}}, \quad (1.16)$$

with the two arithmetic progressions combining into one, as seen in (1.13). This subcase includes the ‘degenerate’ Eulerian numbers of Carlitz [14, §8], which are of the form $E_{n,k}(\lambda, 1; c_0, 1 - c_0)$ and have recently been combinatorially interpreted (when $c_0 = 1$) by Herscovici [39]. It has been extensively treated in [16] (see also [41]).

The second subcase is when $a = 0$, causing the function $(x)^{\underline{n, a}}$ expanded in (1.12) to reduce to the monomial x^n , as seen too in (1.13). This could be called the Carlitz–Scoville subcase. It can be traced to [15], in an equivalent symmetric formulation, and also to the solution of a stochastic model of habitat selection by certain insect larvae [17, 42].

Generalized Eulerian numbers of the $a = 0$ type have appeared in many applications. The numbers $E_{n,k}(0, 1; u, v)$ are the (u, v) -Eulerian numbers, which when $u, v \in \mathbb{N}$ have a combinatorial interpretation [5]. When $u = 0$ they reduce to the order- v Eulerian numbers $A_{n,k}^{(v)}$ of [25]. When $v = 0$ they are related by scaling to the $1/K$ -Eulerian numbers [45], which are of the form $E_{n,k}(0, K; 1, 0)$.

The numbers $E_{n,k}(0, 1; r, 1 - r)$ count the permutations of an ordered n -set that have k r -descents [29], and are identical to the numbers $E_{n,k}(0, 1; \delta, 1 - \delta)$ studied in [36]. They satisfy both of the preceding conditions: $c_0 + c_\infty = b$ as well as $a = 0$. The numbers $E_{n,k}(0, 2; 1, 1) =: \langle \binom{n}{k} \rangle_B$ count the number of signed permutations of an ordered n -set which have k ‘signed descents.’ (See [1, 10] and [52, A060187].) They are called the type- B Eulerian numbers or the MacMahon numbers, and when $(a, b; c_0, c_\infty) = (0, 2; 1, 1)$, eq. (1.12) accordingly reduces to the Worpitzky identity of type B [2, 10]. These numbers also satisfy both conditions, as do the single-progression numbers of [66].

Restricted to the subcases $c_0 + c_\infty = b$ and/or $a = 0$, the identities (1.12) and (1.14) are known. (For instance, see [41, eq. (31)], [42, eq. (3.1)], and [5, eq. 18].) But in their full generality, (1.12) and (1.14) appear to be new.

Many additional identities involving the new numbers $E_{n,k}(a, b; c_0, c_\infty)$ and the Hsu–Shiue numbers $S_{n,k}(a, b; r)$ are derived below. They include explicit formulas holding for certain choices of parameter, including ones of combinatorial significance. A fundamental tool is the method of characteristics, applied as a solution technique to the partial differential equation (PDE) satisfied by the bivariate generating function of any GKP triangle $\left| \begin{smallmatrix} n \\ k \end{smallmatrix} \right|$.

This parametric PDE is acted upon by an order-6 transformation group isomorphic to S_3 , the group of permutations of 3 letters. On the triangle level, this group is generated by two involutive transformations $\left| \begin{smallmatrix} n \\ k \end{smallmatrix} \right| \mapsto \left| \begin{smallmatrix} n \\ k \end{smallmatrix} \right|^*$ that act row-wise: the reflection transformation (RT) $k \leftarrow n - k$, and a so-called upper binomial transformation (UBT). They map between (respectively) case-(AI) and case-(AII) triangles, and case-(AII) and case-(AIII) ones. The two trans-

formations are illustrated by (see Theorem 4.7)

$$E_{n,k}(a, b; c_0, c_\infty) = \sum_{j=k}^n (-1)^{j-k} \binom{j}{k} (c_0 + c_\infty)^{\overline{n-j}, b} S_{n, n-j}(-a, b; c_\infty). \quad (1.17)$$

This relates the new numbers $E_{n,k}$ to those of Hsu and Shiue, and is an alternative to (1.14). One sees in (1.17) an initial reflection $j \leftarrow n - j$, performed on a case-(A I) triangle $(c_0 + c_\infty)^{\overline{j}, b} S_{n, j}(-a, b; c_\infty)$ and yielding a case-(A II) one; and a subsequent binomial transformation, yielding the triangle $E_{n,k}(a, b; c_0, c_\infty)$, which belongs to case (A III).

By applying the method of characteristics and well-known facts about the Gauss hypergeometric function ${}_2F_1$, it is possible to derive an explicit formula for the bivariate exponential generating function $G(t, z)$ of a GKP triangle not merely in the generalized Stirling–Eulerian case (A), which subsumes cases (A I), (A II), (A III), but also in two others: (B), called here the generalized Narayana case, and (C), called the generalized secant–tangent case. If in a GKP recurrence $\beta\beta' \neq 0$ and the pair (β, β') is adjusted to equal $(2, -2)$, which can be done without loss of generality, then the corresponding cases (B I), (B II), (B III) become $(\alpha, \alpha') = (1, -2), (-4, 3), (1, 3)$, and cases (C I), (C II), (C III) become $(\alpha, \alpha') = (-1, 2), (0, 1), (-1, 1)$. Many explicit formulas for the triangle elements $\binom{n}{k}$ in these cases are derived; especially, in case (B). More than a dozen GKP triangles of the generalized Narayana kind for which $\binom{n}{k}$ can be expressed as a hypergeometric term have been identified in the OEIS [52], and are tabulated below. (See Tables 2 and 3.)

By following a context-free grammar approach to exponential structures [19, 26], some formulas for case-(C) GKP triangles are also derived, which justify the ‘secant–tangent’ description. In fact, the approach leads to interesting formulas in all the just-mentioned cases. It must be said that although cases (A), (B), (C) are treated in isolation here, they are related: by quadratic changes of variable, certain case-(A) generating functions can be reduced to case-(C) ones, and certain case-(B) ones to case-(A) ones. This is illustrated by the combinatorics of polytopes [30]. The f -vectors of n -dimensional permutohedra (of either type A_n or type B_n) are the rows of a certain case-(A) GKP triangle, and the corresponding γ -vectors (quadratically reduced) are the rows of a case-(C) one. For associahedra, there is a similar reduction from case (B) to case (A). But quadratic transformations of GKP triangles are left to another paper.

The body of this paper is structured as follows. GKP triangle EGF’s, and the S_3 transformation group acting on EGF’s or row-wise on triangles, are introduced in Section 2. In Section 3 the method of characteristics is applied to the EGF PDE, and a new GKP parametrization adapted to the S_3 -group and the construction of ${}_2F_1$ -based solutions is introduced. The generalized Stirling–Eulerian case (A) is treated in the multi-part Section 4. Many identities, including contiguous function relations and explicit formulas, are derived. Results on the generalized Narayana and secant–tangent triangles (cases (B), (C)) are in Sections 5 and 6, the latter including some grammar-based identities.

2. Generating functions

Suppose that a number triangle $\begin{bmatrix} n \\ k \end{bmatrix}$, $0 \leq k \leq n < \infty$, is a GKP triangle²: it satisfies a GKP-type recurrence (1.1) with parameters $\alpha, \beta, \gamma; \alpha', \beta', \gamma' \in \mathbb{C}$, and the initial condition $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$. (By convention, $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ if $k < 0$ or $k > n$.)

The EGF (exponential generating function) $G(t, z) = \left[\begin{smallmatrix} \alpha & \beta \\ \alpha' & \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right] (t, z)$ defined in (1.2c) equals unity at $(0, 0)$ and is defined and analytic in a neighborhood of $(0, 0)$ in $\mathbb{C} \times \mathbb{C}$; in fact, in a neighborhood of $\{t = 0\} \cup \{z = 0\}$. It may be possible to compute the EGF in closed form. The computation may involve solving from scratch the PDE (partial differential equation) satisfied by the EGF, or showing that the EGF is related to that of another GKP triangle. The latter relationship may follow from the existence of a PDE-to-PDE transformation. ‘First degree’ transformation among the PDE’s satisfied by the EGF’s of GKP triangles form a group, as is summarized in Theorem 2.8 below; quadratic transformations will be explored elsewhere.

The following triangle-to-triangle transformations are elementary but useful. They ‘trim’ a GKP triangle by removing its left or right edge, provided that the edge consists only of zeroes (with the exception of the apex element $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$). This phenomenon occurs when $\gamma = 0$, resp. $\gamma' = 0$.

Theorem 2.1. *Suppose that a number triangle $\begin{bmatrix} n \\ k \end{bmatrix}$, $0 \leq k \leq n < \infty$, satisfies a GKP recurrence with parameter array $\left[\begin{smallmatrix} \alpha & \beta \\ \alpha' & \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]$ and EGF $G(t, z)$. Then,*

- (i) *If $\gamma = 0$, $\gamma' \neq 0$, the left-trimmed triangle $\begin{bmatrix} n \\ k \end{bmatrix}^* := (\gamma')^{-1} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$, $0 \leq k \leq n$, is a GKP triangle with parameter array $\left[\begin{smallmatrix} \alpha & \beta \\ \alpha' & \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \alpha + \beta \\ \alpha' + \beta' + \gamma' \end{smallmatrix} \right]$. Its EGF equals $(\gamma')^{-1} (\partial/\partial z) G(t, z)/t$.*
- (ii) *If $\gamma' = 0$, $\gamma \neq 0$, the right-trimmed triangle $\begin{bmatrix} n \\ k \end{bmatrix}^* := (\gamma)^{-1} \begin{bmatrix} n+1 \\ k \end{bmatrix}$, $0 \leq k \leq n$, is a GKP triangle with parameter array $\left[\begin{smallmatrix} \alpha & \beta \\ \alpha' & \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \alpha + \gamma \\ \alpha' + \gamma' \end{smallmatrix} \right]$. Its EGF equals $(\gamma)^{-1} (\partial/\partial z) G(t, z)$.*

Proof. Both statements follow by elementary series manipulations. The factors $(\gamma')^{-1}$, $(\gamma)^{-1}$ are included to satisfy the normalization $\begin{bmatrix} 0 \\ 0 \end{bmatrix}^* = 1$. \square

Remark 2.2. A third type of trimming will be encountered below, in the proof of Theorem 4.17 and elsewhere. Let the row polynomials $\left[\begin{smallmatrix} \alpha & \beta \\ \alpha' & \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]_n (t)$ of a GKP triangle be denoted by $G_n(t)$, $n \geq 0$. The first two row polynomials are $G_0(t) = 1$ and $G_1(t) = \gamma + \gamma't$, and it can be shown by induction that if $\gamma + \gamma't$ equals $A(\beta + \beta't)$ for some constant A , each $G_{n+1}(t)$, $n \geq 0$, will be a multiple of $\beta + \beta't$; and if $A \neq 0$ and $\beta\beta' \neq 0$, the quotients $G_{n+1}(t)/[A(\beta + \beta't)]$, $n \geq 0$, will be the row polynomials of a new, ‘mid-trimmed’ GKP triangle, with parameter array $\left[\begin{smallmatrix} \alpha & \beta \\ \alpha' & \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \alpha + \gamma \\ \alpha' + \beta' + \gamma' \end{smallmatrix} \right]$.

²As is clear from (1.1), the convention of Spivey [59] on the definition of the GKP parameters γ, γ' is adhered to here, rather than that of Barbero G. et al. [4]. In present notation, the parameters γ, γ' used in [4] would be written respectively as $\gamma - \alpha$ and $\gamma' - \alpha' - \beta'$.

Example 2.3. Three distinct EGF's for the standard Eulerian numbers appear in the literature [22], but Theorem 2.1 relates them.³ They are

$$\sum_{n=0}^{\infty} \sum_{k=0}^n \left\langle \begin{matrix} n \\ k-1 \end{matrix} \right\rangle t^k \frac{z^n}{n!} = \left[\begin{matrix} 0, & 1 \\ 1, & -1 \end{matrix} \middle| \begin{matrix} 0 \\ 1 \end{matrix} \right] (t, z) = \frac{1-t}{1-te^{(1-t)z}}, \quad (2.1a)$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle t^k \frac{z^n}{n!} = \left[\begin{matrix} 0, & 1 \\ 1, & -1 \end{matrix} \middle| \begin{matrix} 1 \\ 0 \end{matrix} \right] (t, z) = \frac{1-t}{e^{(t-1)z} - t} \quad (2.1b)$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^n \left\langle \begin{matrix} n+1 \\ k \end{matrix} \right\rangle t^k \frac{z^n}{n!} = \left[\begin{matrix} 0, & 1 \\ 1, & -1 \end{matrix} \middle| \begin{matrix} 1 \\ 1 \end{matrix} \right] (t, z) = \frac{(1-t)^2 e^{(1+t)z}}{(e^{tz} - te^z)^2}. \quad (2.1c)$$

Left-trimming the triangle in (2.1a) and right-trimming the triangle in (2.1b) both yield the triangle in (2.1c). From an analytic rather than a combinatorial point of view, there is little to choose between the classical definition $A_{n,k} = \left\langle \begin{matrix} n \\ k-1 \end{matrix} \right\rangle$ and the modern shifted definition $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ of these numbers.

The following theorem was mentioned in the introduction and is also easily proved.

Theorem 2.4. (i) Let $\left| \begin{matrix} n \\ k \end{matrix} \right|$ denote the GKP triangle with parameters $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$, and let $G(t, z)$ denote its EGF. Then, the GKP triangle with parameters $(A\alpha, A\beta, A\gamma; B\alpha', B\beta', B\gamma')$ will be $A^{n-k} B^k \left| \begin{matrix} n \\ k \end{matrix} \right|$, and if $A \neq 0$ the EGF of this triangle will be $G(Bt/A, Az)$.

(ii) Let $\left| \begin{matrix} n \\ k \end{matrix} \right|$ denote the GKP triangle with parameters $(\alpha, \beta, \gamma; 0, 0, \gamma')$. Then, the GKP triangle with parameters $(\alpha, \beta, \gamma; 0, \gamma', s\gamma')$ will be $s^{\overline{k}} \left| \begin{matrix} n \\ k \end{matrix} \right|$.

Example 2.5. An illustration of part (ii) of the theorem, with $s = 1$, is provided by the De Morgan numbers $\text{Surj}(n, k)$, $0 \leq k \leq n$, which count the number of maps from an n -set onto a k -set. By examination, they satisfy a recurrence of GKP type and equal $\left[\begin{matrix} 0, & 1 \\ 0, & 1 \end{matrix} \middle| \begin{matrix} 0 \\ 1 \end{matrix} \right]_{n,k}$. As $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left[\begin{matrix} 0, & 1 \\ 0, & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 1 \end{matrix} \right]_{n,k}$, one has $\text{Surj}(n, k) = (1)^{\overline{k}} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$.

Many sophisticated transformations of GKP triangles or their EGF's are based on transformations of the parametric PDE satisfied by the latter.

Theorem 2.6. (i) The EGF $G(t, z) = \left[\begin{matrix} \alpha, & \beta \\ \alpha', & \beta' \end{matrix} \middle| \begin{matrix} \gamma \\ \gamma' \end{matrix} \right] (t, z)$ satisfies the first-order PDE

$$[\mathcal{A}(t)z - 1] \frac{\partial G}{\partial z} + \mathcal{B}(t) \frac{\partial G}{\partial t} + \mathcal{C}(t)G = 0, \quad (2.2)$$

where $\mathcal{A}(t) = \alpha + \alpha't$, $\mathcal{B}(t) = (\beta + \beta't)t$, and $\mathcal{C}(t) = \gamma + \gamma't$, with the initial condition $G(t, 0) \equiv 1$.

³Traditionally the Eulerian numbers were denoted by $A_{n,k}$, and were defined and nonzero for $1 \leq k \leq n$, with $A_{n,k} = 0$ if $k > n$ by convention. To fit them into a GKP framework, one must also set $A_{0,0} = 1$ and $A_{n,0} = 0$, $n \geq 1$. In the modern indexing $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ signifies $A_{n,k+1}$, except that $\left\langle \begin{matrix} 0 \\ 0 \end{matrix} \right\rangle = 1$; note also that when $0 \leq k \leq n$, $\left\langle \begin{matrix} n+1 \\ k \end{matrix} \right\rangle$ equals $A_{n+1,k+1}$ without exception. The occasionally used notation $\left\langle \begin{matrix} n \\ k-1 \end{matrix} \right\rangle$ should be understood as signifying $A_{n,k}$.

(ii) The row polynomials $G_n(t) = \left[\begin{smallmatrix} \alpha, \beta \\ \alpha', \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]_n(t)$ satisfy the differential recurrence

$$G_{n+1} = [\mathcal{A}(t)n + \mathcal{C}(t)]G_n + \mathcal{B}(t)G'_n \quad (2.3)$$

and the initial condition $G_0(t) \equiv 1$.

(iii) If $\beta\beta' \neq 0$, the row polynomials $G_n(t)$ can be computed from

$$[t^{1+\hat{\alpha}}(\beta + \beta't)^{1-\hat{\alpha}-\hat{\alpha}'}D_t]^n \frac{t^{\hat{\gamma}}}{(\beta + \beta't)^{\hat{\gamma}+\hat{\gamma}'}} = \frac{t^{\hat{\alpha}n+\hat{\gamma}}G_n(t)}{(\beta + \beta't)^{(\hat{\alpha}+\hat{\alpha}')n+\hat{\gamma}+\hat{\gamma}'}} \quad (2.4)$$

where $\hat{\alpha}, \hat{\alpha}', \hat{\gamma}, \hat{\gamma}'$ signify $\alpha/\beta, -\alpha'/\beta', \gamma/\beta, -\gamma'/\beta'$, and $D_t = d/dt$.

Proof. Substitute the definitions (1.2c) and (1.2b) of the EGF and row polynomials into (2.2) and (2.3); and in both, use the triangular recurrence (1.1). By examination, the formula for $G_n(t)$ provided by (2.4) satisfies (2.3). \square

Consider the transformation induced by a lifting map or change of variables $(t^*, z^*) \mapsto (t, z)$, which is of the form $(t, z) = (R(t^*), S(t^*)z^*)$ where R, S are rational functions of their argument. (R will be taken to be nonconstant.) Substitution into (2.2) yields the following.

Theorem 2.7. *The transformed EGF*

$$G^*(t^*, z^*) := G(R(t^*), S(t^*)z^*) = G(t, z),$$

lifted from the EGF $G(t, z)$ by the map $(t^*, z^*) \mapsto (t, z)$ specified by (R, S) , satisfies the first-order PDE

$$[\mathcal{A}^*(t^*)z^* - 1] \frac{\partial G^*}{\partial z^*} + \mathcal{B}^*(t^*) \frac{\partial G^*}{\partial t^*} + \mathcal{C}^*(t^*)G^* = 0, \quad (2.5)$$

where

$$\mathcal{A}^*(t^*) = (\alpha + \alpha'R)S - (\beta + \beta'R)R\dot{S}/\dot{R}, \quad (2.6a)$$

$$\mathcal{B}^*(t^*) = (\beta + \beta'R)RS/\dot{R}, \quad (2.6b)$$

$$\mathcal{C}^*(t^*) = (\gamma + \gamma'R)S, \quad (2.6c)$$

or more compactly, $\mathcal{A}^* = (\mathcal{A} \circ R)S - (\mathcal{B} \circ R)\dot{S}/\dot{R}$, $\mathcal{B}^* = (\mathcal{B} \circ R)S/\dot{R}$, $\mathcal{C}^* = (\mathcal{C} \circ R)S$. An overdot indicates differentiation with respect to t^* .

For the lifted PDE (2.5) to be of the GKP type, like the original PDE (2.2), its coefficient functions $\mathcal{A}^*(t^*)$, $\mathcal{B}^*(t^*)/t^*$, $\mathcal{C}^*(t^*)$, must be degree-1 polynomials in t^* . It should be noted that any two liftings can be composed. If the pairs (R, S) , (R^*, S^*) specify successive liftings, i.e., $(t, z) = (R(t^*), S(t^*)z^*)$ and $(t^*, z^*) = (R^*(t^{**}), S^*(t^{**})z^{**})$, their composition $(R, S) \circ (R^*, S^*)$ is the pair

$$(\mathbf{R}, \mathbf{S}) := (R \circ R^*, (S \circ R^*)S^*), \quad (2.7)$$

which specifies the composite map $(t^*, z^*) \mapsto (t, z) = (\mathbf{R}(t^*), \mathbf{S}(t^*)z^*)$. Also, liftings (R, S) in which R has a compositional inverse \bar{R} (which will be the case if $R(t^*) = \frac{\lambda + \mu t^*}{\rho + \sigma t^*}$ with $\lambda\sigma \neq \mu\rho$, i.e., if $t^* \mapsto t$ is a degree-1 rational map) have compositional inverses of the same form, i.e.,

$$(R, S)^{-1} = \left(\bar{R}, \frac{1}{S \circ \bar{R}} \right), \quad (2.8)$$

which specifies the inverse map $(t, z) \mapsto (t^*, z^*)$. Hence such liftings form a group under composition.

Suppose that in a GKP recurrence, $\beta' = -\beta$, which if $\beta\beta' \neq 0$ is a mere matter of normalization. Such a restriction facilitates the study of the pairs (R, S) that yield a lifted PDE which is of the GKP type, like the original. The case when R, S are rational of at most degree 1 in t^* is especially easy to treat. By direct calculation, one finds that if (R, S) equals $(\frac{1}{t^*}, t^*)$, resp. $(1 - t^*, -1)$, the lifted PDE will indeed be of the GKP type, having a lifted or transformed parameter array $\left[\begin{smallmatrix} \alpha' & \beta \\ \alpha' & \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]^*$ equal to $\left[\begin{smallmatrix} \alpha' - \beta & \beta \\ \alpha' + \beta & -\beta \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]$, resp. $\left[\begin{smallmatrix} -\alpha - \alpha' & \beta \\ \alpha' & -\beta \end{smallmatrix} \middle| \begin{smallmatrix} -\gamma - \gamma' \\ \gamma' \end{smallmatrix} \right]$. Both these liftings preserve the property $\beta' = -\beta$ and in fact leave β, β' unchanged, though they transform in an affine-linear way the vector comprising the other four parameters. Both liftings are involutions, as follows from either (2.7) or (2.8). By (2.7), the composition of either $(\frac{1}{t^*}, t^*)$ or $(1 - t^*, -1)$ with itself is the pair $(t^*, 1)$, which specifies the identity transformation.

When acting on a GKP-type EGF, row polynomial, and number triangle, these two liftings yield the involutive transformation identities

$$\left[\begin{smallmatrix} \alpha' - \beta & \beta \\ \alpha' + \beta & -\beta \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right] (t^*, z^*) = \left[\begin{smallmatrix} \alpha & \beta \\ \alpha' & -\beta \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right] \left(\frac{1}{t^*}, t^* z^* \right), \quad (2.9a)$$

$$\left[\begin{smallmatrix} \alpha' - \beta & \beta \\ \alpha' + \beta & -\beta \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]_n (t^*) = (t^*)^n \left[\begin{smallmatrix} \alpha & \beta \\ \alpha' & -\beta \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]_n \left(\frac{1}{t^*} \right), \quad (2.9b)$$

$$\left[\begin{smallmatrix} \alpha' - \beta & \beta \\ \alpha' + \beta & -\beta \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]_{n,k} = \left[\begin{smallmatrix} \alpha & \beta \\ \alpha' & -\beta \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]_{n,n-k}, \quad (2.9c)$$

resp.

$$\left[\begin{smallmatrix} -\alpha - \alpha' & \beta \\ \alpha' & -\beta \end{smallmatrix} \middle| \begin{smallmatrix} -\gamma - \gamma' \\ \gamma' \end{smallmatrix} \right] (t^*, z^*) = \left[\begin{smallmatrix} \alpha & \beta \\ \alpha' & -\beta \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right] (1 - t^*, -z^*), \quad (2.10a)$$

$$\left[\begin{smallmatrix} -\alpha - \alpha' & \beta \\ \alpha' & -\beta \end{smallmatrix} \middle| \begin{smallmatrix} -\gamma - \gamma' \\ \gamma' \end{smallmatrix} \right]_n (t^*) = (-1)^n \left[\begin{smallmatrix} \alpha & \beta \\ \alpha' & -\beta \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]_n (1 - t^*), \quad (2.10b)$$

$$\left[\begin{smallmatrix} -\alpha - \alpha' & \beta \\ \alpha' & -\beta \end{smallmatrix} \middle| \begin{smallmatrix} -\gamma - \gamma' \\ \gamma' \end{smallmatrix} \right]_{n,k} = (-1)^{n-k} \sum_{j=k}^n \binom{j}{k} \left[\begin{smallmatrix} \alpha & \beta \\ \alpha' & -\beta \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]_{n,j}. \quad (2.10c)$$

Each of (2.9a) and (2.10a) expresses a transformed EGF $G^*(t^*, z^*)$, on the left, in terms of an original EGF $G(t, z)$, on the right; and is valid in a neighborhood of $(t^*, z^*) = (0, 0)$ in $\mathbb{C} \times \mathbb{C}$, the trivial case $t^* = 0$ not being covered by (2.9a).

$g \in S_3$	$(R(t^*), S(t^*))$	$\left[\begin{array}{c} \alpha, \beta \\ \alpha', \beta' \end{array} \middle \gamma \right]_{n,k}^*$	$ n _k^*$
(0)(1)(∞)	$(t^*, 1)$	$\left[\begin{array}{c} \alpha, \beta \\ \alpha', -\beta \end{array} \middle \gamma \right]_{n,k}$	$ n _k$
(0 ∞)(1)	$(\frac{1}{t^*}, t^*)$	$\left[\begin{array}{c} \alpha' - \beta, \beta \\ \alpha + \beta, -\beta \end{array} \middle \gamma \right]_{n,k}$	$ n _{n-k}$
(01)(∞)	$(1 - t^*, -1)$	$\left[\begin{array}{c} -\alpha - \alpha', \beta \\ \alpha', -\beta \end{array} \middle -\gamma - \gamma' \right]_{n,k}$	$(-1)^{n-k} \sum_{j=k}^n \binom{j}{k} n _j$
(1 ∞)(0)	$(\frac{-t^*}{1-t^*}, 1 - t^*)$	$\left[\begin{array}{c} -\alpha, \beta \\ \beta - \alpha - \alpha', -\beta \end{array} \middle -\gamma - \gamma' \right]_{n,k}$	$(-1)^k \sum_{j=0}^k \binom{n-j}{n-k} n _j$
(0 ∞ 1)	$(\frac{-(1-t^*)}{t^*}, -t^*)$	$\left[\begin{array}{c} \alpha' - \beta, \beta \\ \beta - \alpha - \alpha', -\beta \end{array} \middle -\gamma - \gamma' \right]_{n,k}$	$(-1)^k \sum_{j=n-k}^n \binom{j}{n-k} n _j$
(01 ∞)	$(\frac{1}{1-t^*}, -(1-t^*))$	$\left[\begin{array}{c} -\alpha - \alpha', \beta \\ \alpha + \beta, -\beta \end{array} \middle -\gamma - \gamma' \right]_{n,k}$	$(-1)^{n-k} \sum_{j=0}^{n-k} \binom{n-j}{k} n _j$

Table 1: Six degree-1 transformations of GKP triangles with $\beta' = -\beta$, which preserve β and β' . The first four are the identity transformation and the involutions RT, UBT, and $\text{RT} \circ \text{UBT} \circ \text{RT}$. The last two are the order-3 transformations $\text{UBT} \circ \text{RT}$ and $\text{RT} \circ \text{UBT}$.

The row-polynomial identities (2.9b) and (2.10b) come by expanding (2.9a) and (2.10a). In general, if $|n|_{\cdot}(t)$ denotes the n 'th row polynomial and $|n|_{\cdot}^*(t^*)$ its transform, one can write

$$\left| \begin{array}{c} n \\ \cdot \end{array} \right|_{\cdot}^*(t^*) = S(t^*) \left| \begin{array}{c} n \\ \cdot \end{array} \right|_{\cdot}(R(t^*)), \quad (2.11)$$

which (2.9b) and (2.10b) exemplify. Identity (2.10c) comes by expanding binomially the factor $(1 - t^*)^k$ in the summation that defines the row polynomial on the right-hand side of (2.10b).

Equation (2.9c) defines a *reflection* transformation (RT): it reverses each row of a GKP triangle, yielding a triangle with altered GKP parameters. (A special case was mentioned in the introduction.) Equation (2.10c) defines a (signed, involutive) *upper binomial* transformation (UBT), which also acts row-wise on any GKP triangle with $\beta' = -\beta$. The appearance of a UBT in this context was first pointed out in [59]. There is a literature on binomial transforms of finite or infinite sequences, but most of it deals with lower rather than upper transforms [9]. A lower binomial transform would be based not on the operator $\sum_{j=k}^n \binom{j}{k} \times$ but on $\sum_{j=0}^k \binom{k}{j} \times$, as in (1.7) and (1.8).

The RT and UBT generate a group of transformations, all of which come from liftings. By examination, this group is of order 6 and is isomorphic to S_3 , the symmetric group on three letters. The action of the six transformations, including the identity, RT, and UBT, can be summarized as follows.

Theorem 2.8. *For each of the six rows in Table 1, there is a transformation of the GKP triangle $|n|_k = \left[\begin{array}{c} \alpha, \beta \\ \alpha', \beta' \end{array} \middle| \gamma \right]_{n,k}$ with $\beta' = -\beta$ to a new GKP triangle $|n|_k^* = \left[\begin{array}{c} \alpha, \beta \\ \alpha', \beta' \end{array} \middle| \gamma \right]_{n,k}^*$ with $\beta^* = \beta$ and $\beta'^* = \beta'$, performed thus: $|n|_k^*$, the*

parameter array of which is given in the third column, equals the expression given in the fourth. For each $n \geq 0$, the new n 'th row polynomial $\left| \begin{smallmatrix} n \\ \cdot \end{smallmatrix} \right|^*(t^*)$ equals $S(t^*) \left| \begin{smallmatrix} n \\ \cdot \end{smallmatrix} \right|(R(t^*))$, and the new EGF $G^*(t^*, z^*)$ comes from the old EGF $G(t, z)$ as $G(R(t^*), S(t^*)z^*)$, in a neighborhood of $(t^*, z^*) = (0, 0)$ in $\mathbb{C} \times \mathbb{C}$.

For each of these six maps $(t^*, z^*) \mapsto (t, z) = (R(t^*), S(t^*)z^*)$, the map $t^* \mapsto R(t^*)$ is a degree-1 rational map that stabilizes the subset $\{0, 1, \infty\}$ of the projective t -line, or equivalently permutes the points $0, 1, \infty$, which makes concrete the isomorphism to S_3 . These permutations (elements $g \in S_3$) are given in cycle notation in the first column of the table. The permutation $(0\infty)(1)$ specifies the RT, and $(01)(\infty)$ the UBT. In all cases the function $S(t^*)$ is equal to the denominator of $R(t^*)$, up to sign. This order-6 group is quite different from the known group of lower binomial transforms [31].

The transformation $\left| \begin{smallmatrix} n \\ k \end{smallmatrix} \right| \mapsto \left| \begin{smallmatrix} n \\ k \end{smallmatrix} \right|^*$ specified by the third involution $(1\infty)(0)$ is conjugated to the UBT by the RT: it is the composition $\text{RT} \circ \text{UBT} \circ \text{RT}$. It is a variant form of a sequence transformation of Stanton and Sprott [61]. The transformations specified by the cyclic permutations $(0\infty 1)$ and (01∞) , when acting on (the rows of) any GKP triangle with $\beta' = -\beta$, are not involutive: they are sequence transformations of order 3, each being both the inverse and the square of the other. They are the compositions $\text{UBT} \circ \text{RT}$ and $\text{RT} \circ \text{UBT}$.

This S_3 transformation group can be extended in various ways. (Compare Salas and Sokal [57].) One can append 6 additional elements, in each of which $S(t^*)$ is negated, relative to what appears in the table. This negation (in effect, a negation of z^*) will multiply $\left| \begin{smallmatrix} n \\ k \end{smallmatrix} \right|^*$ by $(-1)^n$, and by Theorem 2.4(i), negate the entire array $\left[\begin{smallmatrix} \alpha, \beta \\ \alpha', \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]^*$ of transformed parameters. So, although the additional 6 elements will preserve the property that $\beta' = -\beta$, they will negate both β and β' . The extended group $S_3 \times \mathbb{Z}_2$ is isomorphic to the dihedral group with 12 elements.

One could also relax or alter the condition that the GKP triangle being transformed satisfy $\beta' = -\beta$. (The convention adopted here that the condition $\beta'/\beta = -1$ is fundamental is largely due to its holding for the Eulerian triangles; recall Example 2.3.) For any specified β, β' with $\beta\beta' \neq 0$, there is a transformation group isomorphic to S_3 which depends only on the ratio $\beta : \beta'$ and leaves β, β' invariant. Each of its elements comes from a pair (R, S) in which $t^* \mapsto R(t^*)$ permutes the points $0, -\beta'/\beta, \infty$ of the projective line.

An example of this is the original Stanton–Sprott transformation [61, Theorem 3], which in present notation is the involution

$$\left| \begin{smallmatrix} n \\ k \end{smallmatrix} \right|^* = \sum_{j=0}^k \binom{n-j}{n-k} (-1)^j \left| \begin{smallmatrix} n \\ j \end{smallmatrix} \right|. \quad (2.12)$$

It is similar but not identical to the transformation specified by $(1\infty)(0)$. By examination, it can be viewed as acting on (the rows of) any GKP triangle with $\beta' = \beta$, and preserves both β and β' . Its effect is summarized by $\left[\begin{smallmatrix} \alpha, \beta \\ \alpha', \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]^* = \left[\begin{smallmatrix} -\beta + \alpha - \alpha', \beta \\ \beta, \gamma - \gamma' \end{smallmatrix} \right]$, and it comes from the pair $(R, S) = \left(\frac{-t^*}{1+t^*}, 1+t^* \right)$. The

map $t^* \mapsto \frac{-t^*}{1+t^*}$ stabilizes not $\{0, 1, \infty\}$ but $\{0, -1, \infty\}$: in cycle notation, it is the permutation $(-1, \infty)(0)$.

3. The method of characteristics

The first-order PDE satisfied by the EGF $G(t, z)$ of any GKP triangle having been derived (see Theorem 2.6), it will be shown how in several interesting cases, the PDE can be solved in closed form. The method of characteristics, which has been applied previously to the problem of GKP triangles [4, 65], will be exploited to the full. The key result is Theorem 3.2 below.

It was noted by Wilf [65] that this method leads to special functions, in particular the Gauss hypergeometric function ${}_2F_1$. It will be seen that in three cases, the EGF is nonetheless an elementary function of its arguments. The three could be called (A) the generalized Stirling–Eulerian case, (B) the generalized Narayana case, and (C) the generalized secant–tangent case. Case (A) was introduced in Section 1 and is relatively familiar. Like (A), cases (B) and (C) have three subcases: (I), (II), and (III), which are related by transformations that belong to the S_3 -group of the last section. (Recall Table 1 and Theorem 2.8; also see Theorem 3.1 below.) In the present section only the EGF for subcase (I) of each is computed, in Section 3.3. Cases (A), (B), and (C) are treated in greater generality in the respective Sections 4, 5, and 6.

3.1. A new GKP parametrization

The six transformations of the S_3 transformation group, in particular the GKP parameter maps $\left[\begin{smallmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right] \mapsto \left[\begin{smallmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]^*$, can be written in a unified and manifestly symmetric form, given in Theorem 3.1.

For this a new notation is needed, which as a matter of convention will be centered on the putatively fundamental case when $(\beta, \beta') = (1, -1)$. The parameter array $\left[\begin{smallmatrix} \alpha, & 1 \\ \alpha', & -1 \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]$ can be written alternatively as the tableau

$$\left[\begin{array}{ccc} 0, & 1, & \infty \\ r_0, & r_1, & r_\infty \\ g_0, & g_1, & g_\infty \end{array} \right], \quad (3.1)$$

where

$$\begin{aligned} r_0 &= -\alpha, & r_1 &= \alpha + \alpha', & r_\infty &= 1 - \alpha', \\ g_0 &= \gamma, & g_1 &= -\gamma - \gamma', & g_\infty &= \gamma', \end{aligned} \quad (3.2)$$

so that $r_0 + r_1 + r_\infty = 1$ and $g_0 + g_1 + g_\infty = 0$. (The ordering of the columns is arbitrary: if the parameter-pair r_0, g_0 lie in that order below 0, etc., an array of this kind has an unambiguous meaning.) Such new-style parameter arrays can be used as specifications of GKP triangles, row polynomials, and EGF's, much as parameter arrays $\left[\begin{smallmatrix} \alpha, & 1 \\ \alpha', & -1 \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]$ or $\left[\begin{smallmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right]$ are used. (Recall eqs. (1.2).)

By Theorem 2.4(i), whenever $\beta\beta' \neq 0$, one can write

$$\begin{aligned} \left[\begin{array}{c|c} \alpha, & \beta \\ \alpha', & \beta' \end{array} \middle| \begin{array}{c} \gamma \\ \gamma' \end{array} \right]_{n,k} &= \beta^{n-k} (-\beta')^k \left[\begin{array}{c|c} \alpha/\beta, & 1 \\ -\alpha'/\beta', & -1 \end{array} \middle| \begin{array}{c} \gamma/\beta \\ -\gamma'/\beta' \end{array} \right]_{n,k} \\ &= \beta^{n-k} (-\beta')^k \left[\begin{array}{c|c} 0, & 1, & \infty \\ r_0, & r_1, & r_\infty \\ g_0, & g_1, & g_\infty \end{array} \right]_{n,k}, \end{aligned} \quad (3.3)$$

where (in an extension of (3.2) to arbitrary nonzero β, β')

$$\begin{aligned} r_0 &= -\alpha/\beta, & r_1 &= \alpha/\beta - \alpha'/\beta', & r_\infty &= 1 + \alpha'/\beta' = (\alpha' + \beta')/\beta', \\ g_0 &= \gamma/\beta, & g_1 &= -\gamma/\beta + \gamma'/\beta', & g_\infty &= -\gamma'/\beta'. \end{aligned} \quad (3.4)$$

Inverting these, one has that for any (r_0, r_1, r_∞) and (g_0, g_1, g_∞) satisfying the conditions $r_0 + r_1 + r_\infty = 1$ and $g_0 + g_1 + g_\infty = 0$, and β, β' satisfying $\beta\beta' \neq 0$, it is the case that

$$\left[\begin{array}{c|c} 0, & 1, & \infty \\ r_0, & r_1, & r_\infty \\ g_0, & g_1, & g_\infty \end{array} \right]_{n,k} = \beta^{k-n} (-\beta')^{-k} \left[\begin{array}{c|c} \alpha, & \beta \\ \alpha', & \beta' \end{array} \middle| \begin{array}{c} \gamma \\ \gamma' \end{array} \right]_{n,k}, \quad (3.5)$$

and therefore

$$\left[\begin{array}{c|c} 0, & 1, & \infty \\ r_0, & r_1, & r_\infty \\ g_0, & g_1, & g_\infty \end{array} \right] (t, z) = \left[\begin{array}{c|c} \alpha, & \beta \\ \alpha', & \beta' \end{array} \middle| \begin{array}{c} \gamma \\ \gamma' \end{array} \right] (-\beta t/\beta', z/\beta), \quad (3.6)$$

where

$$\begin{aligned} \alpha &= -\beta r_0, & \alpha' &= -\beta'(r_0 + r_1) = \beta'(r_\infty - 1), \\ \gamma &= \beta g_0, & \gamma' &= \beta'(g_0 + g_1) = -\beta' g_\infty. \end{aligned} \quad (3.7)$$

Formula (3.6) expresses any new-style parametric EGF in terms of an old-style one. The following theorem employs the new notation but is equivalent to Theorem 2.8. That it does away with the intricate parameter transformations of Table 1 justifies the new notation.

Theorem 3.1. *For each of the six rows in Table 1, the corresponding lifting-based transformation of GKP triangles acts as follows on EGF's: the equality $G^*(t^*, z^*) = G(t, z)$, where $(t, z) = (R(t^*), S(t^*)z^*)$, can be written as*

$$\left[\begin{array}{c|c} 0, & 1, & \infty \\ r_0, & r_1, & r_\infty \\ g_0, & g_1, & g_\infty \end{array} \right] (R(t^*), S(t^*)z^*) = \left[\begin{array}{c|c} R^{-1}(0), & R^{-1}(1), & R^{-1}(\infty) \\ r_0, & r_1, & r_\infty \\ g_0, & g_1, & g_\infty \end{array} \right] (t^*, z^*), \quad (3.8)$$

where $R^{-1}(0), R^{-1}(1), R^{-1}(\infty)$ is a permutation of $0, 1, \infty$. That is, in the new notation, each element of the S_3 -group acts as a permutation of the parameter-pairs $(r_0, g_0), (r_1, g_1), (r_\infty, g_\infty)$.

This is proved by rewriting the $(\beta, \beta') = (1, -1)$ case of each of the parameter maps $\left[\begin{smallmatrix} \alpha' & \beta \\ \alpha' & \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \gamma' \\ \gamma' \end{smallmatrix} \right] \mapsto \left[\begin{smallmatrix} \alpha' & \beta \\ \alpha' & \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \gamma' \\ \gamma' \end{smallmatrix} \right]^*$ of Table 1 as a map from original parameter-pairs $(r_0, g_0), (r_1, g_1), (r_\infty, g_\infty)$, to transformed (in effect, lifted) parameter-pairs $(r_0, g_0)^*, (r_1, g_1)^*, (r_\infty, g_\infty)^*$, with the aid of (3.2).

For instance, the RT (reflection transformation), specified by $(0\infty)(1) \in S_3$ and previously written as (2.9a), can be rewritten as

$$\left[\begin{array}{ccc} 0, & 1, & \infty \\ r_0, & r_1, & r_\infty \\ g_0, & g_1, & g_\infty \end{array} \right] \left(\frac{1}{t^*}, t^* z^* \right) = \left[\begin{array}{ccc} 0, & 1, & \infty \\ r_\infty, & r_1, & r_0 \\ g_\infty, & g_1, & g_0 \end{array} \right] (t^*, z^*). \quad (3.9)$$

The UBT (upper binomial transformation), specified by $(01)(\infty) \in S_3$ and previously written as (2.10a), can be rewritten as

$$\left[\begin{array}{ccc} 0, & 1, & \infty \\ r_0, & r_1, & r_\infty \\ g_0, & g_1, & g_\infty \end{array} \right] (1 - t^*, -z^*) = \left[\begin{array}{ccc} 0, & 1, & \infty \\ r_1, & r_0, & r_\infty \\ g_1, & g_0, & g_\infty \end{array} \right] (t^*, z^*). \quad (3.10)$$

As before, these identities are valid in a neighborhood of $(t^*, z^*) = (0, 0)$ in $\mathbb{C} \times \mathbb{C}$, the trivial case $t^* = 0$ not being covered by (3.9). Both (3.9) and (3.10) are consistent with the theorem, and because $(0\infty)(1)$ and $(01)(\infty)$ generate S_3 , the theorem follows.

3.2. Integrating the PDE

The following theorem applies to any GKP triangle, parametrized as explained in Section 3.1 by r_0, r_1, r_∞ and g_0, g_1, g_∞ satisfying $r_0 + r_1 + r_\infty = 1$ and $g_0 + g_1 + g_\infty = 0$. It supplies a formula for the EGF which is implicit rather than explicit, and is based upon a special function: the Gauss hypergeometric function ${}_2F_1(w)$. But in several cases (see Section 3.3), the EGF can nonetheless be computed in closed form.

The function ${}_2F_1(w)$ is parametric, with one lower and two upper parameters. Its Maclaurin series is

$${}_2F_1 \left(\begin{array}{c} A, B \\ C \end{array} \middle| w \right) = \sum_{k=0}^{\infty} \frac{A^{\overline{k}} B^{\overline{k}}}{1^{\overline{k}} C^{\overline{k}}} w^k, \quad (3.11)$$

and it is defined and analytic in a neighborhood of $w = 0$, provided that C is not a non-positive integer. In this and the following subsection the alternative in-line notation ${}_2F_1(A, B; C | w)$ will be used, for compactness of expressions.

Theorem 3.2. *In a neighborhood of $(t, z) = (0, 0)$, at which it is analytic and equals unity, the EGF*

$$G(t, z) := \left[\begin{array}{ccc} 0, & 1, & \infty \\ r_0, & r_1, & r_\infty \\ g_0, & g_1, & g_\infty \end{array} \right] (t, z) \quad (3.12)$$

of a GKP triangle is given by the formula

$$G(t, z) = \left(\frac{s}{t}\right)^{g_0} \left(\frac{1-s}{1-t}\right)^{g_1}, \quad (3.13)$$

where $s = s(t, z) = t(1 + zO(t, z))$, with $s(t, 0) = t$, is defined implicitly by

$$\left(\frac{s}{t}\right)^{r_0} \left(\frac{1-s}{1-t}\right)^{r_1} = \frac{{}_2F_1(r_0 + r_1, 1; 1 + r_0 \mid t)}{{}_2F_1(r_0 + r_1, 1; 1 + r_0 \mid s)}. \quad (3.14)$$

This formula applies when r_0 is not a non-positive integer.

Remark 3.3. Though the parametric function ${}_2F_1(A, B; C; \cdot)$ is not defined when C is a non-positive integer, the EGF of any triangle with $r_0 = 0, -1, -2, \dots$ can be computed by taking a limit. Alternatively, owing to the fact at least one of r_0, r_1, r_∞ must not be a non-positive integer (as $r_0 + r_1 + r_\infty = 1$), one can handle any case when $r_0 = 0, -1, -2, \dots$ by permuting the parameter-pairs $(r_0, g_0), (r_1, g_1), (r_\infty, g_\infty)$ with the aid of Theorem 3.1, to obtain a triangle EGF which is covered by Theorem 3.2.

Proof. By (3.6),

$$G(t, z) = \left[\begin{array}{c} 0, 1, \infty \\ r_0, r_1, r_\infty \\ g_0, g_1, g_\infty \end{array} \right] (t, z) = \left[\begin{array}{c} -r_0, 1 \\ 1 - r_\infty, -1 \end{array} \middle| \begin{array}{c} g_0 \\ g_\infty \end{array} \right] (t, z), \quad (3.15)$$

and by Theorem 2.6, $G(t, z)$ satisfies the PDE

$$\left\{ [-r_0 + (1 - r_\infty)t]z - 1 \right\} \frac{\partial G}{\partial z} + (1 - t)t \frac{\partial G}{\partial t} + (g_0 + g_\infty t)G = 0, \quad (3.16)$$

with the initial condition $G(t, 0) = 1$.

To this first-order PDE, the method of characteristics can be applied. For all (t, z) in a neighborhood of $(0, 0)$ in $\mathbb{C} \times \mathbb{C}$, $G(t, z)$ can be computed by flowing the initial condition $G = 1$ from the $z = 0$ line to the point (t, z) , along the characteristic curve extending to (t, z) . The Lagrange–Charpit equations coming from (3.16) are

$$\left[\frac{1}{t(t-1)} - \left(\frac{r_0}{t} + \frac{r_1}{t-1} \right) z \right]^{-1} dz = dt = - \left(\frac{g_0}{t} + \frac{g_1}{t-1} \right)^{-1} \frac{dG}{G}, \quad (3.17)$$

and it is convenient to parametrize each characteristic by t . By the equality between the second and third members, G satisfies $G \propto t^{-g_0}(1-t)^{-g_1}$ along each characteristic. If $s = s(t, z)$ denotes the value of t at which the characteristic extending to (t, z) leaves the $z = 0$ line, the initial condition becomes $G(s(t, z), 0) = 1$ and can be imposed by expressing $G(t, z)$ as in (3.13).

It remains to find $s = s(t, z)$. It will turn out that $u := s/t = 1 + zO(t, z)$, by an application of the implicit function theorem. By the equality between the

first and second members of (3.17), z as a function of t along any characteristic satisfies the inhomogeneous first-order ODE

$$\frac{dz}{dt} + \left(\frac{r_0}{t} + \frac{r_1}{t-1} \right) z = \frac{1}{t(t-1)}. \quad (3.18)$$

The homogeneous solutions of (3.18) are of the form $Kt^{-r_0}(1-t)^{-r_1}$, where K is arbitrary. A particular solution of (3.18) can be found by differentiating (after multiplying by $t(t-1)$). This yields the homogeneous second-order ODE

$$\frac{d^2z}{dt^2} + \left[\frac{1+r_0}{t} + \frac{1+r_1}{t-1} \right] \frac{dz}{dt} + \frac{r_0+r_1}{t(t-1)} z = 0, \quad (3.19)$$

which is a version of the Gauss hypergeometric equation. Provided that r_0 is not a negative integer, which holds by hypothesis, its solutions in a neighborhood of $t = 0$ include the analytic functions

$$C {}_2F_1(1-r_0, 1; 1+r_0 | t), \quad (3.20)$$

C being arbitrary. Substitution reveals that C must equal $-1/r_0$ for (3.20) to be a solution of (3.18). Combining the particular solution (3.20) with the homogeneous solution, one finds that any characteristic must be of the form

$$z(t) = Kt^{-r_0}(1-t)^{-r_1} + (-1/r_0) {}_2F_1(r_0+r_1, 1; 1+r_0 | t) \quad (3.21)$$

where K specifies the characteristic. If the characteristic is to extend to (t, z) from $(s, 0)$, it must be the case that

$$\begin{cases} z = Kt^{-r_0}(1-t)^{-r_1} + (-1/r_0) {}_2F_1(r_0+r_1, 1; 1+r_0 | t), \\ 0 = Ks^{-r_0}(1-s)^{-r_1} + (-1/r_0) {}_2F_1(r_0+r_1, 1; 1+r_0 | s). \end{cases} \quad (3.22)$$

This system determines the function $s = s(t, z)$. Eliminating K yields

$$\begin{aligned} s^{r_0}(1-s)^{r_1} {}_2F_1(r_0+r_1, 1; 1+r_0 | s) = \\ t^{r_0}(1-t)^{r_1} [r_0 z + {}_2F_1(r_0+r_1, 1; 1+r_0 | t)], \end{aligned} \quad (3.23)$$

which as an equation for $u := s/t$ can be written as $\mathcal{F}(t, z; u) = 0$, where

$$\begin{aligned} \mathcal{F}(t, z; u) = u^{r_0}(1-tu)^{r_1} {}_2F_1(r_0+r_1, 1; 1+r_0 | tu) \\ - (1-t)^{r_1} [r_0 z + {}_2F_1(r_0+r_1, 1; 1+r_0 | t)]. \end{aligned} \quad (3.24)$$

Clearly $\mathcal{F}(0, 0; 1) = 0$, and the question is whether (3.24) defines a function $u = u(t, z)$ which is analytic in a neighborhood of $(t, z) = (0, 0)$, at which point it equals unity. By direct computation $(\partial\mathcal{F}/\partial u)$ equals r_0 when $(t, z; u) = (0, 0; 1)$, and $r_0 \neq 0$ by hypothesis; so this follows by the analytic version of the implicit function theorem. Equation (3.14) is a rewritten version of (3.23). \square

Remark 3.4. The Gauss hypergeometric ODE (3.19) would be written in Riemann's P-symbol notation as

$$P \left[\begin{array}{ccc|c} 0, & 1, & \infty & \\ \hline 0, & 0, & 1 & t \\ -r_0, & -r_1, & 1 - r_\infty & \end{array} \right], \quad (3.25)$$

which lists the two characteristic exponents of each of its singular points (the points $0, 1, \infty$ on the projective t -line). (The exponent *differences*, important in the construction of local series solutions, are $-r_0, -r_1, -r_\infty$.) By Fuchs's relation, the sum of the six exponents equals unity. The new GKP parametrization introduced in Section 3.1 was suggested by (3.25). However, it also lists the parameters g_0, g_1, g_∞ , which do not appear in the ODE though they appear in the PDE (3.16), from which the ODE was derived.

3.3. Some special cases

The EGF of a GKP triangle, parametrized by r_0, r_1, r_∞ and g_0, g_1, g_∞ with respective sums 1 and 0, can in some interesting cases be computed in closed form from Theorem 3.2. These include (A) when one of r_0, r_1, r_∞ equals 1; (B) when $\{r_0, r_1, r_\infty\} = \{-\frac{1}{2}, -\frac{1}{2}, 2\}$; and (C) when $\{r_0, r_1, r_\infty\} = \{\frac{1}{2}, \frac{1}{2}, 0\}$. In the following sections these are related to the generalized Stirling–Eulerian, Narayana, and secant–tangent triangles. In each of (A),(B),(C), r_0, r_1, r_∞ can be permuted with the aid of Theorem 3.1. Hence without loss of generality, it suffices to examine the cases (AI) when r_∞ equals 1; (BI) when $(r_0, r_1, r_\infty) = (-\frac{1}{2}, -\frac{1}{2}, 2)$; and (CI) when $(r_0, r_1, r_\infty) = (\frac{1}{2}, \frac{1}{2}, 0)$.

The following three theorems evaluate

$$G(t, z) = \left[\begin{array}{ccc} 0, & 1, & \infty \\ r_0, & r_1, & r_\infty \\ g_0, & g_1, & g_\infty \end{array} \right] (t, z) \quad (3.26)$$

in cases (AI), (BI), and (CI). The first two are especially easy to treat because when an upper parameter of the hypergeometric function ${}_2F_1$ equals a non-positive integer $-N$, the power series defining the function terminates and becomes a degree- N polynomial.

Theorem 3.5. (AI) *If $r_\infty = 1$ (so that $r_0 + r_1 = 0$), the EGF is given in a neighborhood of $(0, 0)$ by*

$$\begin{aligned} G(t, z) &= \left[t + (1-t)(1+r_0z)^{-1/r_0} \right]^{-g_0} \left[(1-t) + t(1+r_0z)^{1/r_0} \right]^{-g_1} \\ &= \left[(1+r_0z)^{1/r_0} \right]^{g_0} \left[(1-t) + t(1+r_0z)^{1/r_0} \right]^{g_\infty} \end{aligned} \quad (3.27)$$

when $r_0 \neq 0$.

Proof. If r_0 is not a non-positive integer, formula (3.14) of Theorem 3.2 applies, and if $r_\infty = 1$, each ${}_2F_1$ in the formula degenerates to the unit (constant)

function. Some algebra then yields

$$s = s(t, z) = \frac{t(1 + r_0 z)^{1/r_0}}{(1 - t) + t(1 + r_0 z)^{1/r_0}}, \quad (3.28)$$

and (3.27) follows from (3.13). If r_0 is a negative integer (though not if $r_0 = 0$), (3.27) holds by a limit argument. \square

The value $r_0 = 0$ not covered by the theorem is handled thus: If $(r_0, r_1, r_\infty) = (0, 0, 1)$ then

$$\begin{aligned} G(t, z) &= [t + (1 - t)e^{-z}]^{-g_0} [(1 - t) + te^z]^{-g_1} \\ &= e^{g_0 z} [(1 - t) + te^z]^{g_\infty}, \end{aligned} \quad (3.29)$$

by taking $r_0 \rightarrow 0$.

Theorem 3.6. (BI) *If $(r_0, r_1, r_\infty) = (-\frac{1}{2}, -\frac{1}{2}, 2)$, the EGF is given in a neighborhood of $(0, 0)$ by*

$$G(t, z) = \left(\frac{s_+}{t_+}\right)^{g_0} \left(\frac{s_-}{t_-}\right)^{g_1}, \quad (3.30)$$

where

$$s_\pm = \frac{1}{2} \pm \frac{4(t - \frac{1}{2}) + z}{2\sqrt{4 + 8(t - \frac{1}{2})z + z^2}},$$

with $s_+ + s_- = 1$, and $t_+ = t$ and $t_- = 1 - t$.

Proof. As $r_0 + r_1 = -1$, each ${}_2F_1$ in (3.14) is a degree-1 polynomial function of its argument, and (3.14) becomes a quadratic equation for s , the solutions of which are s_+ and s_- . The one with the correct behavior as $t, z \rightarrow 0$, satisfying $s = t(1 + zO(t, z))$, is s_+ , so in (3.13), s and $1 - s$ are respectively equal to s_+ and s_- . One can confirm that (3.30) satisfies the PDE (3.16). \square

It is clear from formula (3.14) that if r_∞ is a positive integer and r_0, r_1 are nonzero rational numbers, $s = s(t, z)$ will be an algebraic function, and if moreover $g_0, g_1, g_\infty \in \mathbb{Q}$, the same will be true of $G(t, z)$. But the polynomial of which s is a root will typically be of higher degree than quadratic. The following theorem deals with an inherently non-algebraic case ($r_\infty = 0$).

Theorem 3.7. (CI) *If $(r_0, r_1, r_\infty) = (\frac{1}{2}, \frac{1}{2}, 0)$, the EGF is given in a neighborhood of $(0, 0)$ by*

$$G(t, z) = \left(\frac{s_+}{t_+}\right)^{g_0} \left(\frac{s_-}{t_-}\right)^{g_1}, \quad (3.31)$$

where

$$s_\pm = \left[\sqrt{t_\pm} \cos\left(\frac{z}{2}\sqrt{t_+ t_-}\right) \pm \sqrt{t_\mp} \sin\left(\frac{z}{2}\sqrt{t_+ t_-}\right) \right]^2,$$

with $s_+ + s_- = 1$, and $t_+ = t$ and $t_- = 1 - t$.

Proof. This comes from the known fact that in a neighborhood of $t = 0$, ${}_2F_1\left(1, 1; \frac{3}{2}; t\right)$ equals $\sin^{-1}(\sqrt{t})/\sqrt{t(1-t)}$. Squaring both sides of (3.14), one sees that $s = s(t, z)$ is defined implicitly by

$$\frac{s(1-s)}{t(1-t)} = \left[\frac{(z/2) + \sin^{-1}(\sqrt{t})/\sqrt{t(1-t)}}{\sin^{-1}(\sqrt{s})/\sqrt{s(1-s)}} \right]^2, \quad (3.32)$$

which simplifies to the statement that

$$s = \sin^2 \left(\frac{z}{2} \sqrt{t(1-t)} + \sin^{-1}(\sqrt{t}) \right), \quad (3.33)$$

or equivalently to $s = s_+$. By examination, $1-s$ equals s_- , and (3.31) follows from (3.13). One can confirm that (3.31) satisfies the PDE (3.16). \square

Cases (A I),(B I),(C I) can be converted to what will be called (A II),(B II), (C II) and (A III),(B III),(C III) by applying respectively the elements $(0\infty)(1)$ and $(1\infty)(0)$ of the S_3 -group, i.e., the sequence transformations RT and $\text{RT} \circ \text{UBT} \circ \text{RT}$. The resulting EGF formulas will appear in the following three sections but can be summarized as follows.

Theorem 3.8. *The statements of Theorems 3.5, 3.6, and 3.7 remain valid if the parameter pairs (r_0, g_0) and (r_∞, g_∞) are interchanged, with t replaced by $\frac{1}{t}$ and z by tz ; and similarly if (r_1, g_1) and (r_∞, g_∞) are interchanged, with t replaced by $\frac{-t}{1-t}$ and z by $(1-t)z$.*

However, each of the three theorems is unchanged (or is unchanged up to parametrization, in the case of Theorem 3.5) by the remaining involution $(01)(\infty)$, i.e., the UBT.

Barbero G. et al. [4], besides computing the bivariate EGF $G(t, z)$ in the three subcases of case (A), have treated the case when (in present notation) the unordered set $\{r_0, r_1, r_\infty\}$ equals $\{N, 1-N, 0\}$, for some $N \in \mathbb{Z} \setminus \{0, 1\}$. (See [4], §§A.1.3, A.1.5, A.1.6.) This case also has three subcases, which are related by the S_3 -group. But in each, the EGF turns out not to be an elementary function, but rather to be expressible in terms of an implicitly defined tree function (of combinatorial significance). Analytically, this can be attributed in part to the ${}_2F_1$'s in (3.14) not being elementary functions.

4. Generalized Stirling–Eulerian triangles

In GKP case (A I) of the last section, when $r_\infty = 1$ or equivalently $\alpha' = 0$, Theorem 3.5 supplies a closed-form expression for the EGF $G(t, z)$. This case leads naturally to the definition of the generalized Stirling and generalized Eulerian numbers, $S_{n,k}(a, b; r)$ and $E_{n,k}(a, b; c_0, c_\infty)$. The latter are characterized by the similar condition $r_1 = 1$, and in §4.2, they are alternatively interpreted as connection coefficients. (The key result is Theorem 4.8, which also includes a rank-1 formula for $E_{n,k}(a, b; c_0, c_\infty)$.) In §§4.3 and 4.4, some important parameter choices when ‘rank-0’ formulas for $S_{n,k}(a, b; r)$ and $E_{n,k}(a, b; c_0, c_\infty)$ exist are examined. These may contain binomial coefficients and generalized factorials, without much summation.

4.1. Basic formulas

Applying the transformations RT and $\text{RT} \circ \text{UBT} \circ \text{RT}$ to case (A I), as summarized in Theorem 3.8, yields the two additional EGF formulas that appear in the following three theorems. (Case (A II) is when $\alpha = -\beta$ and case (A III) is when $\frac{\alpha}{\beta} = \frac{\alpha'}{\beta'} + 1$; as above, $\beta\beta' \neq 0$ is assumed.) The parameters $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$ of each EGF have been computed from the new parameters $(r_0, r_1, r_\infty; g_0, g_1, g_\infty)$ with the aid of (3.6) and (3.7).

These three EGF formulas have been derived previously. (See [4, (A.8), (A.4),(A.2)], and also [50, 63, 64].) But the present derivation, making explicit use of the S_3 -group of sequence transformations to derive the latter two from the first, seems the most efficient.

Theorem 4.1 ((A I), $r_\infty = 1, \alpha' = 0$: generalized Stirling). *If $\alpha' = 0$ then $G(t, z)$ equals (when $\alpha \neq 0$)*

$$(1 - \alpha z)^{-\gamma/\alpha} \left\{ 1 + (\beta'/\beta)t \left[1 - (1 - \alpha z)^{-\beta/\alpha} \right] \right\}^{-\gamma'/\beta'},$$

which in the $\alpha \rightarrow 0$ limit becomes

$$e^{\gamma z} \left\{ 1 + (\beta'/\beta)t \left[1 - e^{\beta z} \right] \right\}^{-\gamma'/\beta'}.$$

Theorem 4.2 ((A II), $r_0 = 1, \alpha = -\beta$: generalized Stirling, reflected). *If $\alpha = -\beta$ then $G(t, z)$ equals (when $\alpha' + \beta' \neq 0$)*

$$\left[1 - (\alpha' + \beta')zt \right]^{-\gamma'/(\alpha' + \beta')} \left\{ 1 + (\beta'/\beta')t^{-1} \left[1 - (1 - (\alpha' + \beta')zt)^{\beta'/(\alpha' + \beta')} \right] \right\}^{\gamma'/\beta'},$$

which in the $\alpha' \rightarrow -\beta'$ limit becomes

$$e^{\gamma'zt} \left\{ 1 + (\beta'/\beta')t^{-1} \left[1 - e^{-\beta'zt} \right] \right\}^{\gamma'/\beta'}.$$

Theorem 4.3 ((A III), $r_1 = 1, \frac{\alpha}{\beta} = \frac{\alpha'}{\beta'} + 1$: generalized Eulerian). *If $\frac{\alpha}{\beta} = \frac{\alpha'}{\beta'} + 1$ then $G(t, z)$ equals (when $\alpha \neq 0$, i.e., $\alpha' + \beta' \neq 0$)*

$$\begin{aligned} & \left\{ 1 - [\beta/(\beta + \beta't)] \left[1 - (1 - \alpha z / [\beta/(\beta + \beta't)])^{\beta/\alpha} \right] \right\}^{-\gamma/\beta} \\ & \times \left\{ 1 - [\beta't/(\beta + \beta't)] \left[1 - (1 - \alpha z / [\beta/(\beta + \beta't)])^{-\beta/\alpha} \right] \right\}^{\gamma'/\beta'}, \end{aligned}$$

which in the $\alpha \rightarrow 0$ or equivalently $\alpha' \rightarrow -\beta'$ limit becomes

$$\begin{aligned} & \left\{ 1 - [\beta/(\beta + \beta't)] \left[1 - e^{-z(\beta + \beta't)} \right] \right\}^{-\gamma/\beta} \\ & \times \left\{ 1 - [\beta't/(\beta + \beta't)] \left[1 - e^{z(\beta + \beta't)} \right] \right\}^{\gamma'/\beta'}. \end{aligned}$$

Cases (A I) and (A II) are related by RT, the action of which is straightforward. (It reverses each row of a GKP triangle; see the second row in Table 1.) The focus will therefore be on (A I) and (A III). In both, it is natural to reduce the number of free parameters by defining a specialized or normalized version of the GKP triangle. Case (A I) will be treated first.

Definition 4.4. The 3-parameter generalized Stirling triangle $S_{n,k}(a, b; r)$ is defined by

$$S_{n,k} = S_{n,k}(a, b; r) := \left[\begin{array}{c|c} -a, & b \\ 0, & 0 \end{array} \middle| \begin{array}{c} r \\ 1 \end{array} \right]_{n,k} \quad (4.1)$$

and satisfies $S_{n+1,k+1} = [-an + b(k+1) + r]S_{n,k+1} + S_{n,k}$.

The corresponding denormalization is

$$\left[\begin{array}{c|c} \alpha, & \beta \\ 0, & \beta' \end{array} \middle| \begin{array}{c} \gamma \\ \gamma' \end{array} \right]_{n,k} = \left(\frac{\gamma'}{\beta'} \right)^{\bar{k}} (\beta')^k \left[\begin{array}{c|c} \alpha, & \beta \\ 0, & 0 \end{array} \middle| \begin{array}{c} \gamma \\ 1 \end{array} \right]_{n,k} = (\gamma')^{\bar{k}, \beta'} S_{n,k}(-\alpha, \beta; \gamma) \quad (4.2)$$

and a homogeneity property is

$$S_{n,k}(\lambda a, \lambda b; \lambda r) = \lambda^{n-k} S_{n,k}(a, b; r). \quad (4.3)$$

The numbers $S_{n,k}(a, b; r)$ are of course the generalized Stirling numbers of Hsu and Shiue, which reduce to $\{n\}_k$ and $[n]_k$ when $(a, b; r) = (0, 1; 0)$, resp. $(-1, 0; 0)$. When restricted to integer parameter values, they have been interpreted combinatorially [24, 46]. If $b \neq 0$ and $a \neq 0$, these numbers have the bivariate EGF

$$\sum_{n=0}^{\infty} \sum_{k=0}^n k! S_{n,k}(a, b; r) t^k \frac{z^n}{n!} = (1 + az)^{r/a} \left\{ 1 - \frac{t}{b} \left[(1 + az)^{b/a} - 1 \right] \right\}^{-1}, \quad (4.4)$$

which follows from (4.2) and Theorem 4.1 by choosing $\beta' = \gamma' = 1$, and the equivalent but perhaps less useful EGF

$$\sum_{n=0}^{\infty} \sum_{k=0}^n S_{n,k}(a, b; r) t^k \frac{z^n}{n!} = (1 + az)^{r/a} \exp \left\{ \frac{t}{b} \left[(1 + az)^{b/a} - 1 \right] \right\}, \quad (4.5)$$

which follows from Theorem 4.1 by taking $\beta' \rightarrow 0$. Taking account of the elementary t -dependence in (4.4), one has for all $k \geq 0$ the ‘vertical’ univariate EGF

$$\sum_{n=0}^{\infty} S_{n,k}(a, b; r) \frac{k!}{n!} z^n = (1 + az)^{r/a} \left[\frac{(1 + az)^{b/a} - 1}{b} \right]^k, \quad (4.6)$$

which is of the simple form $d(z)h(z)^k$; so irrespective of the choice of parameters, $\frac{k!}{n!} S_{n,k}$ is a Riordan array, and $S_{n,k}$ itself is a so-called exponential Riordan array [6]. Taking the $a \rightarrow 0$ limit in any of the three preceding formulas is straightforward (the $b \rightarrow 0$ limit is not considered here).

The following additional facts are well known but are proved for completeness. In this, Δ_x is the forward first difference operator with respect to x , defined by $\Delta_x f(x) = f(x+1) - f(x)$.

Theorem 4.5. (i) When $b \neq 0$, $S_{n,k}(a, b; r)$ is given by the rank-1 formula

$$S_{n,k}(a, b; r) = \frac{1}{b^k k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (bj + r)^{\underline{n}, a} \quad (4.7a)$$

$$= \frac{1}{b^k k!} \Delta_x^k [(bx + r)^{\underline{n}, a}] \Big|_{x=0}. \quad (4.7b)$$

(ii) In general the numbers $S_{n,k}(a, b; r)$, $0 \leq k \leq n < \infty$, satisfy

$$(x)^{\underline{n}, a} = \sum_{k=0}^n S_{n,k}(a, b; r) (x - r)^{\underline{k}, b}, \quad (4.8)$$

defining them as coefficients of connection between certain graded bases of the space of polynomials in an indeterminate x .

Proof. The exponential Riordan array $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ specified by a pair of formal power series d, h in z of respective orders $0, 1$, i.e., $d(z) = d_0 + d_1 z + \dots$ with $d_0 \neq 0$ and $h(z) = h_1 z + h_2 z^2 + \dots$ with $h_1 \neq 0$, denoted by $e\mathcal{R}(d, h)$ or $[d, h]$, is the infinite lower-triangular matrix defined by

$$[d, h]_{n,k} = \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = \frac{n!}{k!} [z^n] d(z) h(z)^k, \quad (4.9)$$

where $[z^n]$ extracts the coefficient of z^n . It is a fundamental fact [6] that if a column vector u_k has EGF $u(t) = \sum_{k=0}^{\infty} u_k t^k / k!$, the matrix-vector product $v_n = \sum_{k=0}^{\infty} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] u_k$ will have EGF $v(z) = \sum_{n=0}^{\infty} v_n z^n / n!$ equal to $d(z)u(h(z))$. In (4.8), the column vector $u_k = (x - r)^{\underline{k}, b}$ has EGF $u(t) = (1 + bt)^{(x-r)/b}$ and $v_n = (x)^{\underline{n}, a}$ has EGF $v(z) = (1 + az)^{x/a}$, consistent with this fact and the functions $d(z), h(z)$ appearing in (4.6). This proves (4.8).

From (4.8) with x replaced by $bx + r$, it follows by Newton's interpolation formula [49] that when $b \neq 0$, (4.7b) holds. Equation (4.7a) is an expanded version of (4.7b). \square

Case (AIII), when $\frac{\alpha}{\beta} = \frac{\alpha'}{\beta'} + 1$, can also be specialized or normalized to reduce the number of free parameters.

Definition 4.6. The 4-parameter generalized Eulerian triangle $E_{n,k}(a, b; c_0, c_\infty)$ is defined by

$$E_{n,k} = E_{n,k}(a, b; c_0, c_\infty) := \left[\begin{array}{cc|c} -a, & b & c_0 \\ a + b, & -b & c_\infty \end{array} \right]_{n,k}, \quad (4.10)$$

which if $b \neq 0$ comes equivalently from a tableau with $r_1 = 1$, as

$$b^n \left[\begin{array}{ccc} 0, & 1, & \infty \\ \hline a/b, & 1, & -a/b \\ c_0/b, & -(c_0 + c_\infty)/b, & c_\infty/b \end{array} \right]_{n,k}. \quad (4.11)$$

It satisfies $E_{n+1,k+1} = [-an + b(k+1) + c_0]E_{n,k+1} + [(a+b)n - bk + c_\infty]E_{n,k}$.

The corresponding denormalization is

$$\left[\begin{array}{c} \alpha, \quad \beta \\ \left(\frac{\alpha}{\beta} - 1\right) \beta', \quad \beta' \end{array} \middle| \begin{array}{c} \gamma \\ \gamma' \end{array} \right]_{n,k} = \left(\frac{-\beta'}{\beta}\right)^k E_{n,k}(-\alpha, \beta; \gamma, -\beta\gamma'/\beta') \quad (4.12)$$

and a homogeneity property is

$$E_{n,k}(\lambda a, \lambda b; \lambda c_0, \lambda c_\infty) = \lambda^n E_{n,k}(a, b; c_0, c_\infty). \quad (4.13)$$

The generalized Eulerian numbers $E_{n,k}(a, b; c_0, c_\infty)$ reduce to the standard numbers $\langle \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \rangle := A_{n,k}$, $\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle$, and $\langle \begin{smallmatrix} n+1 \\ k \end{smallmatrix} \rangle = A_{n+1,k+1}$ when $(a, b; c_0, c_\infty)$ is respectively equal to $(0, 1; 0, 1)$, $(0, 1; 1, 0)$, and $(0, 1; 1, 1)$; recall Example 2.3.

It is noteworthy that when $b \neq 0$, the $E_{n,k}(a, b; c_0, c_\infty)$ or the corresponding row polynomials $G_n(t) = \sum_{k=0}^n E_{n,k}(a, b; c_0, c_\infty) t^k$, $n \geq 0$, can be computed by repeated differentiation. According to the formula of Theorem 2.6(iii),

$$(bt^{1-\hat{a}} D_t)^n \left\{ \frac{t^{\hat{c}_0}}{(1-t)^{\hat{c}_0 + \hat{c}_\infty}} \right\} = \frac{t^{\hat{c}_0 - \hat{a}n} G_n(t)}{(1-t)^{\hat{c}_0 + \hat{c}_\infty + n}}, \quad (4.14)$$

or equivalently

$$b^n \sum_{k=0}^{\infty} (\hat{c}_0 + k)^{\underline{n, \hat{a}}} (\hat{c}_0 + \hat{c}_\infty)^{\bar{k}} \frac{t^k}{k!} = \frac{G_n(t)}{(1-t)^{\hat{c}_0 + \hat{c}_\infty + n}}, \quad (4.15)$$

where $\hat{a} = a/b$, $\hat{c}_0 = c_0/b$, $\hat{c}_\infty = c_\infty/b$. These formulas subsume Euler's ones for the numbers $A_{n,k}$, to which they reduce when $(a, b; c_0, c_\infty)$ equals $(0, 1; 0, 1)$.

Combinatorial interpretations of the numbers $E_{n,k}(a, b; c_0, c_\infty)$ for general parameter values remain to be explored, though as mentioned in the introduction, some special cases (such as the Carlitz–Scoville $a = 0$ case) have appeared in the literature. When $b \neq 0$ and $a \neq 0$ these numbers have the bivariate EGF

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^n E_{n,k}(a, b; c_0, c_\infty) t^k \frac{z^n}{n!} = \\ & \left\{ 1 - (1-t)^{-1} \left[1 - (1+az - atz)^{-b/a} \right] \right\}^{-c_0/b} \\ & \times \left\{ 1 + t(1-t)^{-1} \left[1 - (1+az - atz)^{b/a} \right] \right\}^{-c_\infty/b}, \end{aligned} \quad (4.16)$$

which becomes

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^n E_{n,k}(0, b; c_0, c_\infty) t^k \frac{z^n}{n!} = \\ & \left\{ 1 - (1-t)^{-1} \left[1 - e^{-b(1-t)z} \right] \right\}^{-c_0/b} \left\{ 1 + t(1-t)^{-1} \left[1 - e^{b(1-t)z} \right] \right\}^{-c_\infty/b} \end{aligned} \quad (4.17)$$

when $a \rightarrow 0$ (the $b \rightarrow 0$ limit is not considered here). The EGF's (4.16),(4.17) are specializations of the EGF's of Theorem 4.3. Note that by setting $b = 1$ with $(c_0, c_\infty) = (0, 1)$, $(1, 0)$, and $(1, 1)$, one recovers from (4.17) the classical Eulerian EGF's of Example 2.3. More generally, by setting $b = 1$ with $(c_0, c_\infty) = (u, v)$, one obtains an EGF for the (u, v) -Eulerian numbers of Barbero G. et al. [5]. These include the traditional order- v Eulerian numbers $A_{n,k}^{(v)}$ defined in [25], for which $(c_0, c_\infty) = (0, v)$.

The analogue of Theorem 4.5 for the new numbers $E_{n,k}(a, b; c_0, c_\infty)$, including a rank-1 formula, will appear as Theorem 4.8 below. These numbers can also be expressed in terms of the Hsu-Shiue numbers $S_{n,k}(a, b; r)$, and vice versa, by applying appropriate elements of the S_3 -group of sequence transformations acting row-wise:

Theorem 4.7. *For all $n \geq 0$ and choices of parameters $(a, b; c_0, c_\infty)$, one has the UBT (upper binomial transform) pair*

$$E_{n,k}(a, b; c_0, c_\infty) = \sum_{j=k}^n (-1)^{j-k} \binom{j}{k} (c_0 + c_\infty)^{\overline{n-j}, b} S_{n, n-j}(-a, b; c_\infty), \quad (4.18a)$$

$$(c_0 + c_\infty)^{\overline{n-k}, b} S_{n, n-k}(-a, b; c_\infty) = \sum_{j=k}^n \binom{j}{k} E_{n,j}(a, b; c_0, c_\infty). \quad (4.18b)$$

Proof. The first identity is an application of the composite sequence transformation $\text{UBT} \circ \text{RT}$, which is the order-3 transformation listed on the fifth line of Table 1. Equation (4.18a) can be written as

$$\left[\begin{array}{cc|c} -a, & b & c_0 \\ a+b, & -b & c_\infty \end{array} \right]_{n,k} = (-1)^{n-k} \sum_{j=k}^n \binom{j}{k} \hat{S}_{n,j}, \quad (4.19)$$

where

$$\begin{aligned} \hat{S}_{n,j} &= \left\{ (-1)^j (c_0 + c_\infty)^{\overline{j}, b} \left[\begin{array}{cc|c} a, & b & c_\infty \\ 0, & 0 & 1 \end{array} \right]_{n,j} \right\}_{j \leftarrow n-j} \\ &= \left\{ \left[\begin{array}{cc|c} a, & b & c_\infty \\ 0, & -b & -c_0 - c_\infty \end{array} \right]_{n,j} \right\}_{j \leftarrow n-j} = \left[\begin{array}{cc|c} -b, & b & -c_0 - c_\infty \\ a+b, & -b & c_\infty \end{array} \right]_{n,j}, \end{aligned} \quad (4.20)$$

in which the final equality comes from the RT formula (2.9c). The identity now follows by applying the UBT formula (2.10c). The second identity is inverse to the first, and comes from the inverted group element $\text{RT} \circ \text{UBT}$. \square

When $(a, b; c_0, c_\infty) = (0, 1; 1, 0)$, the UBT pair in this theorem (incorporating an initial $j \leftarrow n - j$ reflection, resp. a final $k \leftarrow n - k$ reflection) reduces to the classical UBT pair [33] relating the Eulerian numbers $E_{n,k}(0, 1; 1, 0) = \langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle$ and

the Stirling subset numbers $S_{n,k}(0, 1; 0) = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$:

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \sum_{j=k}^n (-1)^{j-k} \binom{j}{k} (n-j)! \left\{ \begin{matrix} n \\ n-j \end{matrix} \right\}, \quad (4.21a)$$

$$(n-k)! \left\{ \begin{matrix} n \\ n-k \end{matrix} \right\} = \sum_{j=k}^n \binom{j}{k} \left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle. \quad (4.21b)$$

The sequence transformation performed in (4.21b) is an (unsigned and non-involutive) UBT, and the transformation in (4.21a) is its inverse.

Slightly modified versions of the UBT pair (4.21) can be obtained by setting $(a, b; c_0, c_\infty)$ equal to $(0, 1; 0, 1)$ and $(0, 1; 1, 1)$ in the theorem; again, recall Example 2.3. Also, setting $(a, b; c_0, c_\infty)$ equal to $(0, 2; 1, 1)$ yields an additional known UBT pair with a combinatorial interpretation, which relates the type- B Eulerian numbers $E_{n,k}(0, 2; 1, 1) =: \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_B$ (see [52, A060187]) and the type- B Stirling subset numbers $S_{n,k}(0, 2; 1) =: \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_B$ (see [1] and [52, A039755]).

For each n , sequences $(2)^{\bar{k}} S_{n,k}(0, 1; 1) = (k+1)! \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\}$ and $E_{n,k}(0, 1; 1, 1) = \left\langle \begin{matrix} n+1 \\ k \end{matrix} \right\rangle$, resp. $2^k k! S_{n,k}(0, 2; 1)$ and $E_{n,k}(0, 2; 1, 1) = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_B$, where $0 \leq k \leq n$, arise combinatorially as the f -vector and h -vector of a simplicial complex dual to the permutohedron of type A_n , resp. B_n [30]. In that context, the composite transformation $\text{UBT} \circ \text{RT}$ implicit in (4.18a) can be identified with the ‘reverse Pascal’s triangle’ construction that maps the f -vector of a dual simplicial complex to its h -vector [30, Example 5.6].

4.2. Connection coefficient interpretation

Now that the generalized Eulerian numbers $E_{n,k}(a, b; c_0, c_\infty)$ have been introduced as the elements of a parametric GKP triangle, how they can be efficiently computed will be explained. For general parameter values, it is difficult to extract a useful closed-form expression from (4.16), their bivariate EGF. They can be computed alternatively by (4.18a) from the Hsu–Shiue generalized Stirling numbers, which in turn are given by the rank-1 formula (4.7a). But the resulting formula for $E_{n,k}(a, b; c_0, c_\infty)$ is of rank 2: it involves a double summation.

It may be possible to simplify this, but a formula without a multi-sum can be worked out by another technique, which is of independent interest. Besides yielding the following rank-1 formula, the technique provides a Worpitzky-like interpretation of the $E_{n,k}(a, b; c_0, c_\infty)$ as connection coefficients.

Theorem 4.8. (i) *When $b \neq 0$, $E_{n,k}(a, b; c_0, c_\infty)$ is given by the rank-1 formula*

$$E_{n,k}(a, b; c_0, c_\infty) = \frac{1}{b^k k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (bn + c_0 + c_\infty)^{\overline{k-j, b}} (c_0 + c_\infty)^{\bar{j}, b} (bj + c_0)^{\underline{n, a}}. \quad (4.22)$$

(ii) In general the numbers $E_{n,k}(a, b; c_0, c_\infty)$, $0 \leq k \leq n$, satisfy

$$(c_0 + c_\infty)^{\overline{n},b}(x)^{\underline{2},a} = \sum_{k=0}^n E_{n,k}(a, b; c_0, c_\infty) (x - c_0)^{\underline{k},b} (x + c_\infty)^{\overline{n-k},b}, \quad (4.23)$$

which if $(c_0 + c_\infty)^{\overline{n},b} \neq 0$, defines them as coefficients that express the factorial polynomial $(x)^{\underline{2},a}$ with respect to a bifactorial basis of the $(n+1)$ -dimensional space of polynomials of degree $\leq n$.

Proof. Let $G(t, z)$ denote the EGF of the GKP triangle defining $E_{n,k}(a, b; c_0, c_\infty)$, and let $G_n(t)$, $n \geq 0$, be its row polynomials. That is,

$$G(t, z) = \left[\begin{array}{c|c} -a, & b \\ a + b, & -b \end{array} \middle| \begin{array}{c} c_0 \\ c_\infty \end{array} \right] (t, z), \quad G_n(t) = \left[\begin{array}{c|c} -a, & b \\ a + b, & -b \end{array} \middle| \begin{array}{c} c_0 \\ c_\infty \end{array} \right]_n (t), \quad (4.24)$$

and the PDE satisfied by $G(t, z)$ and the differential recurrence satisfied by $G_n(t)$ are given in Theorem 2.6. The lifting transformations previously considered were of the form $G^*(t^*, z^*) = G(t, z)$ where $(t, z) = (R(t^*), S(t^*)z^*)$. Consider the more general transformation

$$G^*(t^*, z^*) = Q(t^*) G(R(t^*), S(t^*)z^*) \quad (4.25)$$

specified by (Q, R, S) , where Q satisfies $Q(0) = 1$. It may be possible to choose (Q, R, S) so that $G^*(t^*, z^*)$ is the EGF of an infinite array $\left| \begin{smallmatrix} n \\ k \end{smallmatrix} \right|^*$ that satisfies a triangular recurrence of GKP type, though the array will not be lower-triangular if $Q \neq 1$.

Consider in particular the case when $(R(t^*), S(t^*)) = (t^*, (1 - t^*)^{-1})$ and the prefactor $Q(t^*)$ equals $(1 - t^*)^{-(c_0 + c_\infty)/b}$, so that $t = t^*$ and

$$G_n^*(t) = (1 - t)^{-n - (c_0 + c_\infty)/b} G_n(t). \quad (4.26)$$

The unlifted row polynomials $G_n(t)$, $n \geq 0$ satisfy the recurrence

$$G_{n+1} = \{[-a + (a + b)t]n + (c_0 + c_\infty t)\} G_n + b(1 - t)t G_n', \quad (4.27)$$

with $G_0(t) \equiv 1$. By direct computation the lifted $G_n^*(t)$, $n \geq 0$, which by (4.26) are not polynomials, satisfy the recurrence

$$G_{n+1}^* = (-an + c_0)G_n^* + bt(G_n^*)', \quad (4.28)$$

with $G_0^*(t) = (1 - t)^{-(c_0 + c_\infty)/b}$. Substituting $G_n^*(t) = \sum_{k=0}^{\infty} \left| \begin{smallmatrix} n \\ k \end{smallmatrix} \right|^* t^k$ into (4.28) reveals that the lifted array coefficients $\left| \begin{smallmatrix} n \\ k \end{smallmatrix} \right|^*$ satisfy a degenerate recurrence of the GKP type, namely

$$\left| \begin{smallmatrix} n + 1 \\ k \end{smallmatrix} \right|^* = (-an + bk + c_0) \left| \begin{smallmatrix} n \\ k \end{smallmatrix} \right|^*, \quad (n, k) \in \mathbb{N}^2, \quad (4.29)$$

with the non-GKP initial condition

$$\left| \begin{smallmatrix} 0 \\ k \end{smallmatrix} \right|^* = \binom{(c_0 + c_\infty)/b - 1 + k}{k}, \quad k \in \mathbb{N}. \quad (4.30)$$

The explicit formula

$$\left| \begin{matrix} n \\ k \end{matrix} \right|^* = \binom{(c_0 + c_\infty)/b - 1 + k}{k} (bk + c_0)^{\underline{n}, a}, \quad (n, k) \in \mathbb{N}^2, \quad (4.31)$$

follows by inspection.

Now consider (4.26) above. Expanding the prefactor and likewise its reciprocal in geometric series, and equating like powers of t , yields the pair

$$\left| \begin{matrix} n \\ k \end{matrix} \right|^* = \sum_{j=0}^k \binom{n + (c_0 + c_\infty)/b - 1 + k - j}{k - j} \left| \begin{matrix} n \\ j \end{matrix} \right|, \quad (4.32a)$$

$$\left| \begin{matrix} n \\ k \end{matrix} \right| = \sum_{j=0}^k (-1)^{k-j} \binom{n + (c_0 + c_\infty)/b}{k - j} \left| \begin{matrix} n \\ j \end{matrix} \right|^*, \quad (4.32b)$$

holding for all $(n, k) \in \mathbb{N}^2$. Here, $\left| \begin{matrix} n \\ k \end{matrix} \right|$ signifies $E_{n,k}(a, b; c_0, c_\infty)$. Taking (4.31) into account, one sees that (4.32b) is equivalent to the claimed formula (4.22) for $E_{n,k}(a, b; c_0, c_\infty)$.

Continuing, let $\delta := (c_0 + c_\infty)/b - 1$, so that (4.32a) says that

$$\binom{\delta + k}{k} (bk + c_0)^{\underline{n}, a} = \sum_{j=0}^k \binom{n + \delta + k - j}{k - j} E_{n,j}(a, b; c_0, c_\infty). \quad (4.33)$$

If $k \leq n$ the summation $\sum_{j=0}^k$ can obviously be replaced by $\sum_{j=0}^n$, and multiplying both sides by $k! (\delta + n)^{\overline{n-k}}$ then yields

$$(\delta + 1)^{\overline{n}} (bk + c_0)^{\underline{n}, a} = \sum_{j=0}^n (k)^{\underline{j}} (k + \delta + 1)^{\overline{n-j}} E_{n,j}(a, b; c_0, c_\infty). \quad (4.34)$$

By the formal substitution $k = (x - c_0)/b$ this becomes a statement that the equality

$$(\delta + 1)^{\overline{n}} (x)^{\underline{n}, a} = \sum_{j=0}^n ((x - c_0)/b)^{\underline{j}} ((x - c_0)/b + \delta + 1)^{\overline{n-j}} E_{n,j}(a, b; c_0, c_\infty) \quad (4.35)$$

holds when x takes on any of the $n + 1$ distinct values $bk + c_0$, $k = 0, 1, \dots, n$. But both sides are degree- n polynomials in x , so the equality must hold for all x . This is equivalent to the claim (4.23), the $b = 0$ case following by taking the $b \rightarrow 0$ limit. \square

An immediate corollary of the connection formula (4.23) is the involutive identity

$$E_{n, n-k}(a, b; c_0, c_\infty) = E_{n, k}(-a, b; c_\infty, c_0). \quad (4.36)$$

This also follows from the RT formula (2.9c), if one uses the definition (4.10) of $E_{n,k}$ as a GKP triangle. An instance of (4.36), coming from the choice

$(a, b; c_0, c_\infty) = (0, 1; 1, 1)$, or equivalently $(\alpha, \beta, \gamma; \alpha', \beta', \gamma') = (0, 1, 1; 1, -1, 1)$ as in (2.1c), is the classical Eulerian identity $\langle \binom{n+1}{n-k} \rangle = \langle \binom{n+1}{k} \rangle$, $0 \leq k \leq n$. In the traditional indexing this is $A_{n+1, n-k+1} = A_{n+1, k+1}$, $0 \leq k \leq n$.

Theorem 4.8(i) and its inverse, which comes from (4.32a) rather than (4.32b), can be rephrased in the following way.

Theorem 4.9. *For all $n \geq 0$ and choices of parameters $(a, b; c_0, c_\infty)$, one has the binomial transform pair*

$$b^k k! E_{n,k}(a, b; c_0, c_\infty) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (bn + c_0 + c_\infty)^{\overline{k-j, b}} (c_0 + c_\infty)^{\overline{j, b}} (bj + c_0)^{\underline{n, a}}, \quad (4.37a)$$

$$(c_0 + c_\infty)^{\overline{k, b}} (bk + c_0)^{\underline{n, a}} = \sum_{j=0}^k \binom{k}{j} (bn + c_0 + c_\infty)^{\overline{k-j, b}} b^j j! E_{n,j}(a, b; c_0, c_\infty). \quad (4.37b)$$

When $(a, b; c_0, c_\infty) = (0, 1; 1, 0)$, the formula (4.37a) reduces to the classical formula (1.15) for $\langle \binom{n}{k} \rangle$, which is well known [22, 33]. Slightly modified versions of this formula can be obtained by setting $(a, b; c_0, c_\infty) = (0, 1; 0, 1)$ and $(0, 1; 1, 1)$. It must be mentioned that He and Shiue [38, eqs. (15), (17)] recently obtained the $(a, b; c_0, c_\infty) = (\theta, 1; 1, 0)$ specialization not only of (4.37a), but of the inverse identity (4.37b) as well.

Theorem 4.9 should be compared with Theorem 4.7, which revealed that for all $n \geq 0$, the row sequences $E_{n,k}(a, b; c_0, c_\infty)$, $k = 0, \dots, n$ and $(c_0 + c_\infty)^{\overline{n-k, b}} S_{n, n-k}(-a, b; c_\infty)$, $k = 0, \dots, n$, form a UBT pair; and also with the following.

Theorem 4.10. *For all $n \geq 0$ and choices of parameters $(a, b; r)$, one has the LBT (lower binomial transform) pair*

$$b^k k! S_{n,k}(a, b; r) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (bj + r)^{\underline{n, a}}, \quad (4.38a)$$

$$(bk + r)^{\underline{n, a}} = \sum_{j=0}^k \binom{k}{j} b^j j! S_{n,j}(a, b; r). \quad (4.38b)$$

Proof. The first identity is from Theorem 4.5, and the LBT is inverted in the second. For the theory of LBT's, see Boyadzhiev [9]. \square

For any $n \geq 0$, the UBT pair in Theorem 4.7 and the LBT pair in Theorem 4.10 are of the classical forms

$$v_k = \sum_{j=k}^n (-1)^{j-k} \binom{j}{k} u_j \iff u_k = \sum_{j=k}^n \binom{j}{k} v_j \quad (4.39)$$

and

$$v_k = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} u_j \iff u_k = \sum_{j=0}^k \binom{k}{j} v_j, \quad (4.40)$$

respectively. In each, two sequences $(u_k)_{k=0}^n, (v_k)_{k=0}^n$ are related by a non-involutive binomial transform, or equivalently by its inverse. But the LBT pair in Theorem 4.9 is of a more general form than (4.40), namely

$$v_k = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (A)^{\overline{k-j}, b} u_j \iff u_k = \sum_{j=0}^k \binom{k}{j} (A)^{\overline{k-j}, b} v_j, \quad (4.41)$$

where $A \neq 0$ and b are constants. (Actually, in the theorem A depends affinely on the row index n , but because these transformations act row-wise, that is not a major matter.) When $A = 1$ and $b = 0$, (4.41) reduces to (4.40). Unusual LBT pairs of the type (4.41), which have not appeared widely in the literature, are briefly discussed at the end of the next subsection.

4.3. Generalized Stirling formulas

The generalized Eulerian numbers $E_{n,k}(a, b; c_0, c_\infty)$ can be computed by Theorem 4.7 from the Hsu–Shiue Stirling numbers $S_{n,k}$, as well as by the rank-1 summation formula in Theorem 4.8 (when $b \neq 0$). Because of the existence of the former method, some identities satisfied by the $S_{n,k}$ in general, and explicit formulas for certain $S_{n,k}$, will now be given. The formulas will incidentally lead to some sequence transformations that generalize the classical Stirling transform (for which see [9]). By exploiting the previously mentioned homogeneity property

$$S_{n,k}(\lambda a, \lambda b; \lambda r) = \lambda^{n-k} S_{n,k}(a, b; r), \quad (4.42)$$

additional explicit formulas can be generated.

Identities (i) and (ii) in the following theorem could be called ‘contiguous function relations’ in the spirit of Gauss.

Theorem 4.11. *For all $n \geq 0$, the numbers $S_{n,k}$ satisfy*

- (i) $S_{n,k}(a, b; r + a) = S_{n,k}(a, b; r) + an S_{n-1,k}(a, b; r)$,
- (ii) $S_{n,k}(a, b; r + b) = S_{n,k}(a, b; r) + b(k + 1) S_{n,k+1}(a, b; r)$,
- (iii) $S_{n,k}(-a, b; r) = S_{n,k}(a, b; r + a(n - 1))$,
- (iv) $S_{n,k}(a, -b; r) = S_{n,k}(a, b; r - bk)$,
- (v) $S_{n,k}(a, b; b - a) = S_{n+1,k+1}(a, b; 0)$,

when $0 \leq k \leq n$, with $k = -1$ also allowed in (ii) and (v). In (i), (ii), and (v), the convention that $S_{n,k} = 0$ if $k < 0$, $k > n$, or $n < 0$, is adhered to.

Proof. Each of (i),(ii),(iii) follows by elementary manipulation of the bivariate EGF (4.4), the cases $a = 0$ and $b = 0$ holding by continuity; though (iv) follows more easily from the finite-difference representation (4.7b). Identity (v) is an example of the left-trimming of a GKP triangle, as in Theorem 2.1(i). \square

Example 4.12. The r -Stirling subset numbers $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r = S_{n,k}(0, 1; r)$ and cycle numbers $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_r = S_{n,k}(-1, 0; r)$ (where usually $r \in \mathbb{N}$, see [11]), generalize $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$

and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$. For an $(n+r)$ -set, they count restricted partitions with $k+r$ blocks, resp. permutations with $k+r$ cycles, the restriction being that r distinguished elements of the set must be placed in distinct blocks, resp. cycles.⁴ The r -Stirling subset numbers can be computed by the rank-1 formula of Theorem 4.5. Additionally, setting $(a, b) = (0, 1)$ in part (ii) and $(-1, 0)$ in part (i) of Theorem 4.11 yields respectively that when $0 \leq k \leq n$,

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{r+1} - (k+1) \left\{ \begin{matrix} n \\ k+1 \end{matrix} \right\}_r, \quad (4.43a)$$

and

$$\left[\begin{matrix} n+1 \\ k \end{matrix} \right]_r = \left[\begin{matrix} n+1 \\ k \end{matrix} \right]_{r+1} - (n+1) \left[\begin{matrix} n \\ k \end{matrix} \right]_{r+1}. \quad (4.43b)$$

These ‘cross’ recurrences are equivalent to known ones with combinatorial interpretations [11, §3].

The following theorem says that in a sense, the parametric family of Hsu–Shiue Stirling number triangles is closed under the taking of (row-wise) upper binomial transforms.

Theorem 4.13. *For all $0 \leq k \leq n$ and δ , the $S_{n,k}$ satisfy*

$$\delta^{k,b} S_{n,k}(a, -b; r + \delta) = \sum_{j=k}^n \binom{j}{k} \delta^{j,b} S_{n,j}(a, b; r).$$

Proof. This follows from the UBT formula (2.10c), if one views $\delta^{k,b} S_{n,k}(a, b; r)$ as the GKP triangle $\left[\begin{smallmatrix} -a & b \\ 0 & -b \end{smallmatrix} \middle| \begin{smallmatrix} r \\ \delta \end{smallmatrix} \right]_{n,k}$. (See (1.9).) \square

To prove the next theorem, recall from (4.6) that the matrix $\mathfrak{S}(a, b; r) = (S_{n,k}(a, b; r))$ is an exponential Riordan array:

$$\mathfrak{S}(a, b; r) = \left[(1+az)^{r/a}, \frac{(1+az)^{b/a} - 1}{b} \right], \quad (4.44)$$

meaning that

$$S_{n,k}(a, b; r) = \frac{n!}{k!} [z^n] (1+az)^{r/a} \left[\frac{(1+az)^{b/a} - 1}{b} \right]^k. \quad (4.45)$$

(Taking the $a \rightarrow 0$ limit, if desired, is straightforward.) It is a fundamental fact [6] that (exponential) Riordan arrays form a group under matrix multiplication: if d_i, h_i are formal power series in z of respective orders 0, 1, for $i = 1, 2$, then $[d_1, h_1][d_2, h_2] = [(d_2 \circ h_1)d_1, h_2 \circ h_1]$ and $[d_1, h_1]^{-1} = [1/(d_1 \circ \bar{h}_1), \bar{h}_1]$, where \bar{h}_1 is the compositional inverse of h_1 .

Of the following three identities, (i) is presumably well known and (ii) appears in [40]; (iii) extends a pair of identities of Can and Değil [12, eqs. (30)–(31)], and appears to be new.

⁴In present notation, the r -Stirling numbers of [11] would be written as $\left\{ \begin{smallmatrix} n-r \\ k-r \end{smallmatrix} \right\}_r$ and $\left[\begin{smallmatrix} n-r \\ k-r \end{smallmatrix} \right]_r$.

Theorem 4.14. (i) *The parametric, infinite lower-triangular matrix $\mathcal{S}(a, b; r)$, i.e., $(S_{n,k}(a, b; r))$, satisfies the product formula*

$$\mathcal{S}(a, c; r_1 + r_2) = \mathcal{S}(a, b; r_1)\mathcal{S}(b, c; r_2);$$

and $\mathcal{S}(a, b; r)^{-1} = \mathcal{S}(b, a; -r)$, as $\mathcal{S}(a, a; 0) = \mathcal{I}$, the identity matrix, for any a .

(ii) *For any non-negative n and k, k_1, k_2 satisfying $k = k_1 + k_2$, one has the convolution formula*

$$\frac{k!}{k_1!k_2!}S_{n,k}(a, b; r_1 + r_2) = \sum \frac{n!}{n_1!n_2!}S_{n_1,k_1}(a, b; r_1)S_{n_2,k_2}(a, b; r_2),$$

the sum being over non-negative pairs n_1, n_2 satisfying $n = n_1 + n_2$, with $n_1 \geq k_1$ and $n_2 \geq k_2$.

(iii) *For any non-negative k and n_1, n, k_2 satisfying $n_1 = n + k_2$, one has the asymmetric convolution formula*

$$\frac{n_1!}{n!k_2!}S_{n,k}(a, b; r_1 + r_2) = \sum \frac{k_1!}{k!n_2!}S_{n_1,k_1}(a, b; r_1)S_{n_2,k_2}(b, a; r_2),$$

the sum being over non-negative pairs k_1, n_2 satisfying $k_1 = k + n_2$, with $n_1 \geq k_1$ and $n_2 \geq k_2$. (Note the interchange of a, b in the summand.)

Proof. (i) These facts are immediate corollaries of the connection formula (4.8). They also have a Riordan-array interpretation: as is easily verified, they come from the just-stated formulas that express $[d_1, h_1][d_2, h_2]$ and $[d_1, h_1]^{-1}$ as Riordan arrays, when d_i, h_i depend on z as shown in (4.44).

(ii) To prove this, substitute $(n, k) = (n_i, k_i)$ in the vertical EGF formula (4.6), and take the product of two copies of it: one with $i = 1$ and one with $i = 2$. Then, equate the coefficients of like powers of z on the left and right sides.

(iii) Consider the infinite matrix $B(a, b; r) = (B_{n,w}(a, b; r))$ defined by

$$B_{n,w}(a, b; r) = n![z^n](1 + az)^{r/a} \left[\frac{bz}{(1 + az)^{b/a} - 1} \right]^w. \quad (4.46)$$

This is an example of an improper Riordan array [3]. It is not a lower-triangular matrix indexed by $n, k \geq 0$. Rather, it is indexed by $n \geq 0, w \in \mathbb{Z}$. As a power series in z , the quantity raised to the w 'th power here is of order 0, not 1.

If $r = r_1 + r_2$, for any $k \geq 0$ one has by elementary manipulations

$$\begin{aligned} & (1 + az)^{r/a} \left[\frac{bz}{(1 + az)^{b/a} - 1} \right]^w \frac{1}{k!} \left[\frac{(1 + az)^{b/a} - 1}{b} \right]^k \\ &= \sum_{n=k}^{\infty} \left[\sum_{j=k}^n \frac{1}{j!(n-j)!} B_{n-j,w}(a, b; r_1) S_{j,k}(a, b; r_2) \right] z^n. \end{aligned} \quad (4.47)$$

If moreover $k \geq w$, this alternatively equals

$$\begin{aligned} \frac{z^w}{k!} (1+az)^{r/a} \left[\frac{(1+az)^{b/a} - 1}{b} \right]^{k-w} &= \frac{z^w (k-w)!}{k!} \sum_{n=k-w}^{\infty} S_{n,k-w}(a, b; r) \frac{z^n}{n!} \\ &= \sum_{n=k}^{\infty} \left[S_{n-w,k-w}(a, b; r) \frac{(k-w)!}{k!(n-w)!} \right] z^n. \end{aligned} \quad (4.48)$$

Equating the coefficients of like powers of z yields the identity

$$\frac{\binom{n}{k}}{\binom{n-w}{k-w}} S_{n-w,k-w}(a, b; r) = \sum_{j=k}^n \binom{n}{j} B_{n-j,w}(a, b; r_1) S_{j,k}(a, b; r_2), \quad (4.49)$$

which holds when $n, k \geq \max(w, 0)$.

In this, only the w 'th column of $B = (B_{n,w})$ appears. Define a parametric, lower-diagonal Toeplitz matrix $\mathcal{B}^{(w)}(a, b; r_1) = \left(\mathcal{B}_{n,j}^{(w)}(a, b; r_1) \right)$, indexed by $n, j \geq 0$, which depends only on this column vector:

$$\mathcal{B}_{n,j}^{(w)}(a, b; r_1) = \begin{cases} \binom{n}{j} B_{n-j,w}(a, b; r_1), & n \geq j, \\ 0, & \text{otherwise.} \end{cases} \quad (4.50)$$

The identity (4.49) can be rewritten as

$$\frac{\binom{n}{k}}{\binom{n-w}{k-w}} S_{n-w,k-w}(a, b; r) = \sum_{j=k}^n \mathcal{B}_{n,j}^{(w)}(a, b; r_1) S_{j,k}(a, b; r_2), \quad (4.51)$$

in which each side is an element of a matrix indexed by $n, k \geq \max(w, 0)$, and the right-hand side computes the product of two such matrices.

It is known that if $g(z) = c_0 + g_1 z + \dots$ is a formal power series and $M = M(g)$ is the lower-triangular Toeplitz matrix defined by $M_{n,j} = g_{n-j}$, the map $g \mapsto M(g)$ is an algebra isomorphism. It follows by examining (4.46) that the inverse of $\mathcal{B}^{(w)}(a, b; r_1)$ is $\mathcal{B}^{(-w)}(a, b; -r_1)$, because negating w, r in (4.46) replaces the power series in z to which $[z^n]$ is applied by its reciprocal.

Taking this into account and computing the matrix inverse of both sides of (4.51) yields the inverted identity

$$\frac{\binom{n}{k}}{\binom{n-w}{k-w}} S_{n-w,k-w}(b, a; -r) = \sum_{j=k}^n S_{n,j}(b, a; -r_2) \mathcal{B}_{j,k}^{(-w)}(a, b; -r_1), \quad (4.52)$$

which again holds when $n, k \geq \max(w, 0)$. The case when $w \geq 0$ is of interest here. Comparing (4.45) and (4.46) reveals that if $w \geq 0$,

$$B_{n,-w}(a, b; r) = \binom{n+w}{w}^{-1} S_{n+w,w}(a, b; r), \quad (4.53)$$

so that the inverted identity can be rewritten as

$$\binom{n}{w} S_{n-w, k-w}(b, a; -r) = \sum_{j=k}^n S_{n,j}(b, a; -r_2) \binom{j}{k-w} S_{j-k+w, w}(a, b; -r_1). \quad (4.54)$$

By negating r, r_1, r_2 and interchanging a, b , one sees that this is equivalent to the claimed asymmetric convolution formula. \square

The following theorem lists formulas for certain $S_{n,k}(a, b; r)$ Hsu–Shiue triangles, parametrized by r . They may be known but seem not to have not been assembled before in a single place. The notation used for a hypergeometric term in parts (iv),(v), and in the sequel, is similar to that often used in the series for ${}_2F_1$:

$${}_2F_1 \left(\begin{matrix} A, B \\ C \end{matrix} \middle| w \right) = \sum_{k=0}^{\infty} \frac{A^{\bar{k}} B^{\bar{k}}}{1^{\bar{k}} C^{\bar{k}}} w^k =: \sum_{k=0}^{\infty} \left[\begin{matrix} A, B \\ 1, C \end{matrix} \right]^{\bar{k}} w^k. \quad (4.55)$$

That is,

$$\left[\begin{matrix} A_1, \dots, A_p \\ C_1, \dots, C_q \end{matrix} \right]^{\bar{k}} := \frac{(A_1)^{\bar{k}} \cdots (A_p)^{\bar{k}}}{(C_1)^{\bar{k}} \cdots (C_q)^{\bar{k}}}. \quad (4.56)$$

By exploiting homogeneity and the identities of Theorem 4.14, one can derive formulas for additional triangles $S_{n,k}(a, b; r)$.

Theorem 4.15. *For all n, k with $n \geq k \geq 0$, and for all r ,*

$$(i) \quad S_{n,k}(0, 0; r) = \binom{n}{k} r^{n-k},$$

$$(ii) \quad S_{n,k}(1, 1; r) = \binom{n}{k} r^{n-k},$$

$$(iii) \quad S_{n,k}(-1, 1; r) = \binom{n}{k} (n+r-1)^{n-k},$$

$$(iv) \quad \begin{aligned} S_{n,k}(1, 2; r) &= \binom{n}{k} \frac{k!}{(2k-n)!} 2^{-(n-k)} {}_2F_1 \left(\begin{matrix} -r, -n+k \\ -n+2k+1 \end{matrix} \middle| 2 \right) \\ &= 2^{n-k} \left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2} \\ 1 \end{matrix} \right]^{\overline{n-k}} {}_2F_1 \left(\begin{matrix} -r, -n+k \\ -n+2k+1 \end{matrix} \middle| 2 \right), \\ S_{n,k}(-2, -1; r) &= \binom{n}{k} \frac{(2n-k)!}{n!} 2^{-(n-k)} {}_2F_1 \left(\begin{matrix} r-1, -n+k \\ -2n+k \end{matrix} \middle| 2 \right) \\ (v) \quad &= \left(-\frac{1}{2} \right)^{n-k} \left[\begin{matrix} -n, n+1 \\ 1 \end{matrix} \right]^{\overline{n-k}} {}_2F_1 \left(\begin{matrix} r-1, -n+k \\ -2n+k \end{matrix} \middle| 2 \right). \end{aligned}$$

Proof. A unified version of (i),(ii), namely $S_{n,k}(a, a; r) = \binom{n}{k} r^{n-k, a}$, appears as [46, eq. (3.6)]. Also, (i),(ii),(iii) are valid because by examination, each satisfies the appropriate GKP recurrence, given in Definition 4.4. Formulas equivalent to (iv),(v) were derived by Cheon, Jung, and Shapiro [21, eqs. (11),(14)], and each is restated here as a product of a (reversed) hypergeometric term and a terminating hypergeometric series. \square

The ${}_2F_1(2)$ series in parts (iv) and (v) must be interpreted with care. In (iv), the lower hypergeometric parameter $-n + 2k + 1$ may be non-positive, causing a division by zero in the terms of the series, but the division by $(2k - n)!$ compensates for this.⁵ It turns out that $S_{n,k}(1, 2; r)$ (when $r \in \mathbb{N}$) is nonzero if and only if $0 \leq n - k \leq \lfloor \frac{n+r}{2} \rfloor$. When $(n, k) = (0, 0)$, the ${}_2F_1(2)$ series in (v) is troublesome also, and must be interpreted as signifying unity. It should be noted that the $r = 0$ versions of the triangles $S_{n,k}(1, 2; r)$ and $S_{n,k}(-2, -1; r)$ can be left-trimmed, yielding the $r = 1$ versions.

For all $r \in \mathbb{N}$, the triangle $S_{n,k}(1, 1; r)$ in part (ii) has a combinatorial interpretation: its elements are rook numbers, which count the number of ways of placing $n - k$ non-attacking rooks on an $r \times n$ chessboard. Thus $S_{n,k}(1, 1; r)$ is nonzero only if $0 \leq n - k \leq r$. The $r = 0$ cases of the triangles $S_{n,k}(a, b; r)$ in parts (iii),(iv),(v) are classical also, and can be identified by examining the corresponding GKP recurrences. The elements $S_{n,k}(-1, 1; 0)$ are the (unsigned) Lah numbers $L_{n,k}$ (see [52, A271703]), and the $S_{n,k}(1, 2; 0)$ and $S_{n,k}(-2, -1; 0)$ are the second-kind Bessel numbers $B_{n,k}$ (see [52, A122848]), resp. the (unsigned) first-kind ones $\hat{b}_{n,k}$ (see [52, A132062]). The formulas for all three are well known [22, p. 158]. They agree with the $r = 0$ cases of the formulas in (iii),(iv),(v).

In fact for all $r \in \mathbb{N}$, the generalized Stirling triangles in (iii),(iv),(v) have combinatorial interpretations. The elements $S_{n,k}(-1, 1; 2r) =: L_{n,k}^{(2r)}$ are the r -Lah numbers, which count restricted partitions of an $(n + r)$ -set into $k + r$ lists, the restriction being that r distinguished elements of the set must be placed in distinct lists [51]. The elements $S_{n,k}(1, 2; r) =: B_{n,k}^{(r)}$ and $S_{n,k}(-2, -1; r) =: \hat{b}_{n,k}^{(r)}$ are the second-kind r -Bessel numbers, resp. the (unsigned) first-kind ones. For an $(n + r)$ -set, $B_{n,k}^{(r)}$ counts partitions with each block having size 1 or 2, subject to the restriction that r distinguished elements must be placed in distinct blocks [21]. This interpretation provides a proof that $S_{n,k}(1, 2; r)$ (when $r \in \mathbb{N}$) is nonzero if and only if $0 \leq n - k \leq \lfloor \frac{n+r}{2} \rfloor$.

A notable feature of the hypergeometric series in parts (iv),(v) for $S_{n,k}(1, 2; r)$ and $S_{n,k}(-2, -1; r)$ is that when $r \in \mathbb{Z}$ with $r \geq 0$, resp. $r \leq 1$, they terminate after $1 + \min(r, n - k)$ terms, resp. $1 + \min(1 - r, n - k)$ terms. That is, the number of terms does not grow with n . For any such r , this yields a ‘rank-0’ formula involving no summation at all, as in parts (i),(ii),(iii).

For example, $S_{n,k}(1, 2; r)$ equals

$$B_{n,k}^{(r)} = \binom{n}{k} \frac{k!}{(2k - n + r)!} 2^{-(n-k)} \times \begin{cases} 1, & r = 0, \\ n + 1, & r = 1, \\ n(n + 1) + 2(k + 1), & r = 2, \end{cases} \quad (4.57)$$

which holds when $2k - n + r \geq 0$, i.e., $0 \leq n - k \leq \lfloor \frac{n+r}{2} \rfloor$. Similarly,

⁵The useful notation introduced by Olver [53, Chapter 15] could be employed here. If C is the lower parameter of a ${}_2F_1$ function, ${}_2\mathbf{F}_1$ signifies $\Gamma(C)^{-1}{}_2F_1$. Unlike ${}_2F_1$, ${}_2\mathbf{F}_1$ is defined when C is any non-positive integer, by taking a limit.

$S_{n,k}(-2, -1; r)$ equals

$$\hat{b}_{n,k}^{(r)} = \binom{n}{k} \frac{(2n-k)!}{n!(2n-k)^{\underline{1-r}}} 2^{-(n-k)} \times \begin{cases} k(k+1) - 2n, & r = -1, \\ k, & r = 0, \\ 1, & r = 1, \end{cases} \quad (4.58)$$

which holds when $0 \leq k \leq n$, though if $2n - k + r \leq 0$ it must be interpreted in a limiting sense: for instance, $\hat{b}_{0,0}^{(r)}$ always equals unity. The curious identity

$$\hat{b}_{n+1,k+1}^{(r)} = B_{2n-k,n}^{(r)}, \quad 0 \leq k \leq n, \quad (4.59)$$

holds not only when $r = 0$, as has been noted [34], but also when $r = 1$.

By applying the coefficient extraction operator $[z^n]$ to the vertical univariate EGF (4.6), one can derive the alternative general formula

$$\begin{aligned} B_{n,k}^{(r)} &= \frac{2^{-(n-k)}}{(n-k)!} \binom{n+r}{r}^{-1} (n+r)^{\underline{2n-2k}} \sum_{\ell=0}^r \binom{k+\ell}{\ell} \binom{n-k}{r-\ell} \\ &= \frac{2^{-(n-k)}}{(n-k)!} \binom{n+r}{r}^{-1} (n+r)^{\underline{2n-2k}} \sum_{\ell=0}^r \binom{k+\lfloor \ell/2 \rfloor}{\lfloor \ell/2 \rfloor} \binom{n}{r-\ell}, \end{aligned} \quad (4.60)$$

which holds for all $r \in \mathbb{N}$ when $0 \leq k \leq n$, and subsumes (4.57). Another approach to deriving formulas for $B_{n,k}^{(r)}$ and $\hat{b}_{n,k}^{(r)}$ would be to apply the identities of Theorem 4.11, which allow the parameter r in $S_{n,k}(a, b; r)$ to be incremented repeatedly by a or by b .

For any r and (a, b) , such as the various choices in Theorem 4.15, the Hsu–Shiue array $S_{n,k}(a, b; r)$ can be used to perform sequence transformations. By Theorem 4.14(i), the infinite lower-triangular matrix $\mathcal{S}(a, b; r)$ has $\mathcal{S}(b, a; -r)$ as its inverse. Hence one has a ‘lower’ transform pair

$$v_k = \sum_{j=0}^k S_{k,j}(b, a; -r) u_j \iff u_k = \sum_{j=0}^k S_{k,j}(a, b; r) v_j, \quad (4.61)$$

and an ‘upper’ transform pair

$$v_k = \sum_{j=k}^{\infty} S_{j,k}(b, a; -r) u_j \iff u_k = \sum_{j=k}^{\infty} S_{j,k}(a, b; r) v_j. \quad (4.62)$$

One may need to require convergence in the latter, unless the sequences terminate.

Such Hsu–Shiue–Stirling transforms include binomial transforms (for which $(a, b; r)$ is $(0, 0; 1)$) and Stirling and r -Stirling transforms (for which it is $(0, 1; 0)$ or $(0, 1; r)$). Other choices of $(a, b; r)$ seem not to have been much investigated, such as the ones in parts (ii)–(v) of Theorem 4.15. The choice $(a, b; r) = (1, 1; r)$

in part (ii) is the most relevant here. When $r \neq 0$, it yields the pair of ‘lower rook number transforms’

$$v_k = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} r^{\overline{k-j}, -1} u_j \iff u_k = \sum_{j=0}^k \binom{k}{j} r^{\overline{k-j}, -1} v_j. \quad (4.63)$$

which up to normalization is identical to the unusual LBT (4.41), an example of which appeared in Theorem 4.9. The interplay between Hsu–Shiue–Stirling transforms and the generalized Eulerian numbers remains to be fully explored.

4.4. Generalized Eulerian formulas

In several cases, it is possible to derive a formula for the generalized Eulerian numbers $E_{n,k}(a, b; c_0, c_\infty)$ not based on a summation, or at least, not based on one in which the number of terms grows with n or k . Several such cases will now be explored. By exploiting the previously mentioned reflection and homogeneity properties

$$E_{n,k}(a, b; c_0, c_\infty) = E_{n,n-k}(-a, b; c_\infty, c_0), \quad (4.64)$$

$$E_{n,k}(\lambda a, \lambda b; \lambda c_0, \lambda c_\infty) = \lambda^n E_{n,k}(a, b; c_0, c_\infty), \quad (4.65)$$

additional ones can be generated.

Theorem 4.16. *For all c_0, c_∞ , one has*

- (i) $E_{n,k}(0, 0; c_0, c_\infty) = \binom{n}{k} c_0^{n-k} c_\infty^k$,
- (ii) $E_{n,k}(-1, 1; c_0, c_\infty) = \binom{n}{k} (c_0 + k)^{\overline{n-k}} (c_\infty)^{\underline{k}}$, and
- (iii) $E_{n,k}(a, b; c_0, -c_0) = (-1)^k \binom{n}{k} (c_0)^{\underline{a}}$, irrespective of b .

Proof. By examination, each of (i),(ii),(iii) satisfies the corresponding GKP recurrence, given in Definition 4.6. Note that formulas (i),(ii) can be unified: $E_{n,k}(-a, a; c_0, c_\infty)$ equals $\binom{n}{k} (c_0 + ka)^{\overline{n-k}, a} (c_\infty)^{\underline{k}, a}$. Another approach to (ii) and (iii) is to note that when the parameter vector $(a, b; c_0, c_\infty)$ of $E_{n,k}$ equals $(-1, 1; c_0, c_\infty)$, resp. $(a, b; c_0, -c_0)$, the bivariate EGF (4.16) reduces to

$$(1 - z)^{-c_0 - c_\infty} (1 - z + tz)^{c_\infty}, \quad \text{resp.} \quad (1 + az - atz)^{c_0/a},$$

from which (ii) and (iii) follow by repeated differentiation. A short approach to (ii) is to note that $E_{n,k}(-1, 1; c_0, c_\infty)$ and $(c_\infty)^{\underline{k}} S_{n,k}(-1, 1; c_0)$ are identical, as they are both solutions of the GKP recurrence with parameter array $\left[\begin{smallmatrix} \alpha, \beta \\ \alpha', \beta' \end{smallmatrix} \middle| \begin{smallmatrix} \gamma \\ \gamma' \end{smallmatrix} \right] = \left[\begin{smallmatrix} 1, & 1 \\ 0, & -1 \end{smallmatrix} \middle| \begin{smallmatrix} c_0 \\ c_\infty \end{smallmatrix} \right]$. Then, one refers to Theorem 4.15(iii). \square

A more sophisticated case when the $E_{n,k}(a, b; c_0, c_\infty)$ may be relatively easy to compute is the important ‘single progression’ case when $c_0 + c_\infty = b$, the first results on which were apparently obtained by Carlitz [14, §8], whose ‘degenerate’ Eulerian numbers are of the innocuously normalized form $E_{n,k}(\lambda, 1;$

$c_0, 1-c_0$). (When $c_0 = 1$, these numbers have recently been placed in a combinatorial setting [39].) Additional results are due to Charalambides [16]⁶, and Hsu and Shiue [41]. As mentioned in the introduction, the single-progression case includes the usual Eulerian triangle $E_{n,k}(0, 1; 1, 0) = \langle \binom{n}{k} \rangle$, the type- B Eulerian triangle $E_{n,k}(0, 2; 1, 1) = \langle \binom{n}{k} \rangle_B$, and others.

One can write

$$E_{n,k}(a, b; c_0, b - c_0) = \left[\begin{array}{c|c} -a, & b \\ a + b, & -b \end{array} \middle| \begin{array}{c} c_0 \\ b - c_0 \end{array} \right]_{n,k}, \quad (4.66)$$

and if $b \neq 0$,

$$E_{n,k}(a, b; c_0, b - c_0) = b^n \left[\begin{array}{c|c} 0, & 1, & \infty \\ a/b, & 1, & -a/b \\ c_0/b, & -1, & 1 - c_0/b \end{array} \right]_{n,k}, \quad (4.67)$$

in the new parametrization of Section 3. (Recall eqs. (3.3),(3.4).) One sees from the parameter-pair in the second column of this tableau how the single-progression case is special: not only is $r_1 = 1$, which is the sign of the generalized Eulerian case (A III), but also $g_1 = -1$.

When $c_0 + c_\infty = b$, the bivariate EGF (4.16) reduces to

$$\sum_{n=0}^{\infty} \sum_{k=0}^n E_{n,k}(a, b; c_0, b - c_0) t^k \frac{z^n}{n!} = \frac{(1-t)(1+az-atz)^{c_0/a}}{1-t(1+az-atz)^{b/a}} \quad (4.68)$$

(if $a \neq 0$; taking the $a \rightarrow 0$ limit is straightforward). Moreover, Theorems 4.7(i) and 4.8(i) reduce to

$$E_{n,k}(a, b; c_0, b - c_0) = \sum_{j=k}^n (-1)^{j-k} \binom{j}{k} b^{n-j} (n-j)! S_{n,n-j}(-a, b; b - c_0) \quad (4.69)$$

and the rank-1 formula

$$E_{n,k}(a, b; c_0, b - c_0) = \sum_{j=0}^k (-1)^{k-j} \binom{n+1}{k-j} (bj + c_0)^{\underline{n},a} \quad (4.70a)$$

$$= \nabla_x^{n+1} \left[(bx + c_0)^{\underline{n},a} \mathbf{1}_{0 \leq x \leq k} \right] \Big|_{x=k}, \quad (4.70b)$$

where ∇_x is the backward first difference operator with respect to x , defined by $\nabla_x f(x) = f(x) - f(x-1)$, and $\mathbf{1}$ signifies an indicator function. The expression (4.70b), a version of which was derived in [16], can be viewed as an unexpanded version of the preceding one.

⁶In present notation, the composition number $A(m, k, r, s)$ in [16] equals $E_{m,k}(1, s; r, s-r)$ or $E_{m,m-k}(-1, s; s-r, r)$, divided by $m!$. Some misprints in eqs. (2.26),(2.27) of [16] are corrected in what follows.

When $(a, b; c_0) = (0, 1; 1)$, resp. $(0, 1; 0)$, (4.70a) and (4.70b) become known formulas for the Eulerian numbers $\langle \binom{n}{k} \rangle$, resp. the traditionally indexed numbers $\langle \binom{n}{k-1} \rangle := A_{n,k}$. Similarly, when $(a, b; c_0) = (0, 2; 1)$, they become formulas for the type-*B* Eulerian numbers. Each of the formulas (4.69), (4.70a), (4.70b) is useful, but certain restrictions on parameters may lead to alternative summation representations in which the number of terms does not grow with n or k , as will now be seen.

In the following theorem, parts (i) and (ii) are analogues for the $E_{n,k}$ of the contiguity relations of Theorem 4.11, which applied to the $S_{n,k}$. (Restricted versions of (i) and (ii) were derived by Carlitz [14, §8].) Note that though part (ii) relates a pair of single-progression $E_{n,k}$'s, (i) holds more generally; and both can be iterated. Parts (iii), (iv), and (v) also relate parametric triangles $E_{n,k}$ which are not of the single-progression type. They indicate how generalized Eulerian triangles can be left-trimmed (if $c_0 = 0$), right-trimmed (if $c_\infty = 0$), and mid-trimmed (if $c_0 + c_\infty = 0$).

Theorem 4.17. *For all $n \geq 0$, the numbers $E_{n,k}$ satisfy*

- (i) $E_{n,k}(a, b; c_0 + a, c_\infty - a) = E_{n,k} + an(E_{n-1,k} - E_{n-1,k-1})$, the parameters of each number on the right-hand side being $(a, b; c_0, c_\infty)$, when $0 \leq k \leq n$;
- (ii) $E_{n,k}(a, b; c_0 + b, -c_0) = E_{n,k+1}(a, b; c_0, b - c_0) + (-1)^k (c_0)^{\underline{2}, a} \binom{n+1}{k+1}$, when $-1 \leq k \leq n$;
- (iii) $c_\infty E_{n,k}(a, b; b - a, c_\infty + a) = E_{n+1,k+1}(a, b; 0, c_\infty)$, when $-1 \leq k \leq n$;
- (iv) $c_0 E_{n,k}(a, b; c_0 - a, a + b) = E_{n+1,k}(a, b; c_0, 0)$, when $0 \leq k \leq n$;
- (v) $c[E_{n,k+1} - E_{n,k}](a, b; c - a, a - c) = E_{n+1,k+1}(a, b; c, -c)$, when $-1 \leq k \leq n$.

In these, the convention that $E_{n,k} = 0$ if $k < 0$, $k > n$, or $n < 0$, is adhered to.

Proof. Identity (i) follows readily from the bivariate EGF (4.16), and (ii) from its restriction (4.68) to the single-progression subcase, with the fact that the EGF of the term $(-1)^k (c_0)^{\underline{2}, a} \binom{n+1}{k+1}$ equals $[(t-1)/t](1+az-atz)^{c_0/a}$ taken into account. Identities (iii) and (iv) are specializations of Theorem 2.1, and (v) follows from a fact indicated in Remark 2.2: the row polynomials $G_{n+1}(t)$, $n \geq 0$, of any GKP triangle with $\gamma + \gamma' = 0$ and $\beta + \beta' = 0$ (with $\beta\beta' \neq 0$) have $1-t$ as a factor. This factor can be divided by, yielding (up to a constant factor, here c) the row polynomials $G_n^*(t)$ of a new, ‘mid-trimmed’ GKP triangle. \square

The following is an interesting formula for a triangle $E_{n,k}$ of single-progression type, having what amounts to a single discrete parameter: $1 - \zeta + 2p \in \mathbb{N}$.

Theorem 4.18. *For all $p \in \mathbb{N}$ and $\zeta \in \{0, 1\}$,*

$$E_{n,k}(-1, 2; 2 - \zeta + 2p, \zeta - 2p) = n! \binom{n+1}{2k+2p+1-\zeta} - (-1)^{k+p} \sum_{\ell=0}^{p-1} (-1)^\ell (2 - \zeta + 2\ell)^{\overline{p}} \binom{n+1}{k+p-\ell}.$$

Proof. The $\zeta = 0, 1$ versions are proved independently, by induction on p . The inductive step uses Theorem 4.17(ii). The base ($p = 0$) cases are respectively the triangles

$$E_{n,k}(-1, 2; 2 - \zeta, \zeta) = n! \binom{n+1}{2k+1-\zeta}, \quad \zeta = 0, 1, \quad (4.71)$$

which by examination satisfy the GKP recurrence of Definition 4.6. (These two number triangles, normalized by division by $n!$, appear as A034839 and A034867 in the OEIS [52].) \square

The number of terms in this summation formula does not grow with n or k . Besides applying Theorem 4.17(ii) to increment c_0 and decrement c_∞ repeatedly (by $b = 2$), as was done in the proof, one could also apply Theorem 4.17(i) to decrement c_0 and increment c_∞ repeatedly (by 1). The number of terms in the resulting formula for any desired $E_{n,k}(-1, 2; c_0, 2 - c_0)$, $c_0 \in \mathbb{Z}$, also will not grow with n or k .

Another notable feature of the single-progression case $c_0 + c_\infty = b$ is that in this case, the connection formula of Theorem 4.8(ii) reduces to the identity

$$b^n(x)^{\underline{n},a} = \frac{1}{n!} \sum_{k=0}^n E_{n,k}(a, b; c_0, b - c_0) [x - c_0 + b(n - k)]^{\underline{n},b}, \quad (4.72)$$

or equivalently

$$n! b^n(x)^{\underline{n},a} = \sum_{k=0}^n E_{n,k}(a, b; b - c_\infty, c_\infty) (x + c_\infty - bk)^{\overline{n},b}. \quad (4.73)$$

This is a generalized Worpitzky identity, as the $a = 0$ subcase makes clear. When $(a, b; c_0, c_\infty) = (0, 1; 1, 0)$, it reduces to the classical Worpitzky identity (1.13), and when $(a, b; c_0, c_\infty) = (0, 1; 0, 1)$, to a slightly modified version. Also, when $(a, b; c_0, c_\infty) = (0, 2; 1, 1)$, it reduces to the Worpitzky identity of type B , which displays the type- B Eulerian numbers $\langle \binom{n}{k} \rangle_B = E_{n,k}(0, 2; 1, 1)$ as connection coefficients [2].

The complementary subcase when $a \neq 0$ has its own logic. Setting $a = 1$ in (4.72) (which by homogeneity can be done without loss of generality), redefining the indeterminate x , and applying the reflection property (4.64), produces the rather symmetric identity

$$(bx + c_\infty)^{\underline{n}} = \sum_{k=0}^n \left[\frac{1}{n!} E_{n,k}(-1, b; b - c_\infty, c_\infty) \right] (x + k)^{\underline{n}}, \quad (4.74)$$

which holds for arbitrary b and c_∞ .

For all $n \geq 0$ and b , let a matrix $\mathcal{A}^{(n,b)} = (A_{k,j}^{(n,b)})$, $0 \leq k, j \leq n$, which is not triangular, be defined by

$$A_{k,j}^{(n,b)} = \frac{1}{n!} E_{n,j}(-1, b; b - k, k), \quad (4.75)$$

Then (4.74), restricted to the case when $c_\infty \in \{0, \dots, n\}$, says that

$$(bx + k)^n = \sum_{j=0}^n A_{k,j}^{(n,b)} (x + j)^n, \quad 0 \leq k \leq n, \quad (4.76)$$

which extends to

$$\Delta_x^r [(bx + k)^n] = \sum_{j=0}^n A_{k,j}^{(n,b)} n^r (x + j)^{n-r}, \quad 0 \leq k \leq n, \quad r \in \mathbb{N}. \quad (4.77)$$

Note that $\mathcal{A}^{(n,1)}$ equals J_{n+1} , the $(n+1)$ -by- $(n+1)$ identity matrix, as follows from Theorem 4.16(ii).

Thus for all $n \geq 0$ and $b \neq 0$, the coefficients that connect the two factorial bases $[(bx + k)^n]_{k=0}^n$ and $[(x + j)^n]_{j=0}^n$ of the $(n+1)$ -dimensional space of polynomials of degree $\leq n$ can be viewed as numbers in the n 'th rows of certain generalized Eulerian triangles of single-progression type, divided by $n!$. Moreover, it follows from (4.76) that for all $n \geq 0$, the map $b \mapsto \mathcal{A}^{(n,b)}$ is a homomorphism from \mathbb{C}^* , the multiplicative group of nonzero complex numbers, to $GL(n+1, \mathbb{C})$. This map and its image deserve further study; empirically, one finds that the eigenvalues of $\mathcal{A}^{(n,b)}$ are $\{1, b, b^2, \dots, b^n\}$.

When b is a positive integer, the preceding results make contact with known identities, and $b = 2$ is illustrative. It follows from Theorem 4.18 that

$$A_{k,j}^{(n,2)} = \binom{n+1}{2j-k+1}, \quad 0 \leq k, j \leq n. \quad (4.78)$$

(It is assumed that $0 \leq 2j - k + 1 \leq n + 1$, otherwise $A_{k,j}^{(n,2)}$ vanishes.) Hence (4.76) reduces when $b = 2$ to

$$(2x + k)^n = \sum_{j=0}^n \binom{n+1}{2j-k+1} (x + j)^n, \quad 0 \leq k \leq n. \quad (4.79)$$

(The summation includes only j for which $0 \leq 2j - k + 1 \leq n + 1$, i.e., terms with $\lfloor \frac{k}{2} \rfloor \leq j \leq \lfloor \frac{n+k}{2} \rfloor$.) This identity exhibits the coefficients of connection as binomial coefficients, and could be proved directly. When $b = 3$ the coefficients of connection become trinomial coefficients, and so forth.

Because of the group homomorphism, for all $b \neq 0$ one has

$$\mathcal{A}^{(n,b)} \mathcal{A}^{(n, \frac{1}{b})} = J_{n+1}. \quad (4.80)$$

The interesting pair of mutually inverse sequence transformations

$$v_k = \frac{1}{n!} \sum_{j=0}^n E_{n,j}(-1, \frac{1}{b}; \frac{1}{b} - k, k) u_j \iff u_k = \frac{1}{n!} \sum_{j=0}^n E_{n,j}(-1, b; b - k, k) v_j \quad (4.81)$$

follows from (4.80). Such pairs were first derived in [16]. However, deriving satisfactory n -dependent expressions for the elements of the matrix $\mathcal{A}^{(n,b)}$ when $b \notin \mathbb{Z}$ is difficult.

Even outside the single-progression case, i.e., even when $c_0 + c_\infty \neq b$, it may be possible to find a formula for $E_{n,k}(a, b; c_0, c_\infty)$ that is not based on a sum, or at most involves one in which the number of terms does not grow with n or k . Consider for instance the parametric triangle $E_{n,k}(-1, 2; c_0, 0)$, for which the bivariate EGF (see (4.16)) is a manageable function of its arguments. Versions of this triangle arise in several combinatorial contexts, and formulas for its elements when $c_0 = 3$ and $c_0 = 1$ were given by Ma, Ma, and Yeh [44].

$E_{n,k}(-1, 2; 3, 0)$ is the number of leaf-labeled rooted binary trees with $n + 2$ leaves and $k + 1$ cherries, i.e., interior vertices with exactly two descendant leaves. (See [56, Table 6] and [52, A306364].) Also, the normalized version $4^k E_{n,k}(-1, 2; c_0, 0)/(c_0)^{\overline{n}}$, when $c_0 = 3$, resp. 1, has as its n 'th row the γ -vector of a simplicial complex dual to the associahedron of type A_n , resp. B_n . (See [30] and [52, A055151, A105868].) The element $4^{n-k} E_{n,n-k}(-1, 2; 3, 0)/(3)^{\overline{n}}$ of the reflected triangle is the number of Motzkin paths of semi-length n with k steps, of types $U = (1, 1)$ and $H = (1, 0)$. (See [52, A107131].)

Theorem 4.19. *The triangles $E_{n,k}(-1, 2; c_0, 0)$ and $E_{n,k}(-1, 2; c_0 + 1, 1)$ have the hypergeometric term representations*

$$(c_0)^{\overline{n}} \left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2} \\ 1, \frac{c_0}{2} + \frac{1}{2} \end{matrix} \right]^{\overline{k}}, \quad \text{resp.} \quad (c_0 + 1)^{\overline{n}} \left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2} - \frac{1}{2} \\ 1, \frac{c_0}{2} + \frac{1}{2} \end{matrix} \right]^{\overline{k}},$$

which are nonzero only if $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, resp. $0 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$. Equivalently, their n 'th row polynomials have the hypergeometric representations

$$(c_0)^{\overline{n}} {}_2F_1 \left(\begin{matrix} -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2} \\ \frac{c_0}{2} + \frac{1}{2} \end{matrix} \middle| t \right), \quad \text{resp.} \quad (c_0 + 1)^{\overline{n}} {}_2F_1 \left(\begin{matrix} -\frac{n}{2}, -\frac{n}{2} - \frac{1}{2} \\ \frac{c_0}{2} + \frac{1}{2} \end{matrix} \middle| t \right).$$

Proof. The expressions for $E_{n,k}(-1, 2; c_0, 0)$ and $E_{n,k}(-1, 2; c_0 + 1, 1)$ can be obtained by repeatedly differentiating the bivariate EGF (4.16), or verified by by confirming that they satisfy the GKP recurrence of Definition 4.6. Also, the two are equivalent: the second comes from the first by a single application of Theorem 4.17(iv), the right-trimming identity. \square

Beginning with $E_{n,k}(-1, 2; c_0, 0)$, one can apply Theorem 4.17(i) repeatedly, so as to decrement c_0 and increment c_∞ by any desired positive integer. In this way, for all $c_\infty \in \mathbb{N}$ one can obtain a formula for $E_{n,k}(-1, 2; c_0, c_\infty)$ in which the number of terms increases with c_∞ , but not with n or k .

Experimentation along this line resulted in the following unusual conjecture, which involves the Bessel numbers $B_{\nu,\kappa} = S_{\nu,\kappa}(1, 2; 0)$ and the r -Bessel numbers $B_{\nu,\kappa}^{(r)} = S_{\nu,\kappa}(1, 2; r)$, rank-0 formulas for which appeared in Theorem 4.15(iv) and eq. (4.60). A proof and a combinatorial interpretation are currently lacking.

Conjecture 4.20. For all $p \in \mathbb{N}$ and $\zeta \in \{0, 1\}$, and all c ,

$$E_{n,k}(-1, 2; c + 2p + \zeta, 2p + \zeta) = \frac{(c + 2p + \zeta)^{\overline{n}}}{(c + 2p + 2\zeta)^{\overline{p}, 2}} \sum_{\ell=0}^p \frac{(c + 1)^{\overline{p}, 2}}{(c + 1)^{k+\ell, 2}} B_{2p+\zeta, 2p+\zeta-\ell} B_{n, n-k}^{(2p+\zeta-2\ell)}.$$

In addition to $E_{n,k}(-1, 2; c_0, c_\infty)$, the parametric triangle $E_{n,k}(-2, 1; c_0, c_\infty)$ may be worth investigating, though not many combinatorial applications of it seem to be known. It is not difficult to derive the striking pair of identities

$$E_{n+1,k}(-2, 1; c_0, 0) = c_0 n! [t^k] P_n^{(c_0+n+1, -c_0-n-1)}(-t), \quad (4.82a)$$

$$E_{n,k}(-2, 1; c_0 + 1, -1) = n! [t^k] P_n^{(c_0+n, -c_0-n)}(-t), \quad (4.82b)$$

where $P_n^{(A,B)}(t)$ is the degree- n Jacobi polynomial. The two are equivalent, as the second is a right-trimmed version of the first (with c_0 decremented by 1). The second is proved by verifying that the GKP row polynomial coming from its left-hand side,

$$G_n(t) = \left[\begin{array}{cc|c} 2, & 1 & c_0 + 1 \\ -1, & -1 & -1 \end{array} \right]_n (t), \quad (4.83)$$

satisfies the same differential recurrence on n (see Theorem 2.6(ii)) as the known recurrence satisfied by $n! P_n^{(c_0+n, -c_0-n)}(-t)$. Alternatively a generating function proof could be used, as the ordinary generating function of the sequence $P_n^{(c_0+n, -c_0-n)}(t)$, $n \geq 0$, is known [60].

It follows from the Jacobi-polynomial representation (4.82b) that if $c_0 = \frac{1}{2}$, the generalized Eulerian row polynomial $G_n(t)$ of (4.83) can be identified with $n! P_n(-t)$, where P_n is the n 'th Boros–Moll polynomial, originally introduced in the study of a quartic integral. (See [8] and [52, A223549].) But from a computational point of view, (4.82a) and (4.82b) are weaker results than the row-polynomial formulas of Theorem 4.19: the coefficient of t^k in $P_n^{(c_0+n, -c_0-n)}(t)$ can only be expressed as a sum, the number of terms in which grows with k , as n increases.

5. Generalized Narayana triangles

In Section 3, several cases when GKP recurrences can be solved in closed form were introduced. Case (A) was the generalized Stirling–Eulerian case, which gave rise in Section 4 to the generalized Stirling and Eulerian numbers, and many related identities. Case (B) will now be examined. As will be seen, the GKP triangles in case (B) include many triangles with combinatorial interpretations, including two now-standard triangles of Narayana numbers [54]. This is the reason for calling (B) the generalized Narayana case.

In case (BI), when $(r_0, r_1, r_\infty) = (-\frac{1}{2}, -\frac{1}{2}, 2)$, Theorem 3.6 supplies an expression for the bivariate EGF $G(t, z)$, based on a quadratic irrationality. Much as in case (A), cases (BII) and (BIII) are obtained from (BI) by applying the

appropriate elements of the S_3 -group: the row-wise sequence transformations RT and $\text{RT} \circ \text{UBT} \circ \text{RT}$. The resulting vectors (r_0, r_1, r_∞) are the permutations $(2, -\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, 2, -\frac{1}{2})$. Due to the analogy with the subcases of case (A), cases (BI), (BII), and (BIII) will be called the Stirling, reversed Stirling, and Eulerian subcases of the generalized Narayana triangle.

It will be recalled that the new-style parameters (r_0, r_1, r_∞) and (g_0, g_1, g_∞) , where the sums $r_0 + r_1 + r_\infty$ and $g_0 + g_1 + g_\infty$ are constrained to equal 1 and 0 respectively, can be converted to the traditional GKP parameters $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$ and vice versa, by the formulas in (3.4) and (3.6). In the following, 3-parameter generalized Narayana triangles $N_{n,k}^X$ (where $X = \text{S}, \text{rS}, \text{E}$, referring to (BI), (BII), (BIII)) are defined, in both the traditional and new parametrizations.

Definition 5.1.

(BI) The generalized Narayana triangle $N_{n,k}^{\text{S}}(b; c_0, c_\infty)$, of Stirling type, is defined by

$$N_{n,k}^{\text{S}} = N_{n,k}^{\text{S}}(b; c_0, c_\infty) := \left[\begin{array}{cc|c} b/2, & b & c_0 \\ -b, & -b & c_\infty \end{array} \right]_{n,k},$$

which if $b \neq 0$ equals

$$b^n \left[\begin{array}{ccc} 0, & 1, & \infty \\ \hline -\frac{1}{2}, & -\frac{1}{2}, & 2 \\ c_0/b, & -(c_0 + c_\infty)/b, & c_\infty/b \end{array} \right]_{n,k}.$$

(BII) The generalized Narayana triangle $N_{n,k}^{\text{rS}}(b; c_0, c_\infty)$, of reversed Stirling type, is defined by

$$N_{n,k}^{\text{rS}} = N_{n,k}^{\text{rS}}(b; c_0, c_\infty) := \left[\begin{array}{cc|c} -2b, & b & c_0 \\ 3b/2, & -b & c_\infty \end{array} \right]_{n,k},$$

which if $b \neq 0$ equals

$$b^n \left[\begin{array}{ccc} 0, & 1, & \infty \\ \hline 2, & -\frac{1}{2}, & -\frac{1}{2} \\ c_0/b, & -(c_0 + c_\infty)/b, & c_\infty/b \end{array} \right]_{n,k}.$$

(BIII) The generalized Narayana triangle $N_{n,k}^{\text{E}}(b; c_0, c_\infty)$, of Eulerian type, is defined by

$$N_{n,k}^{\text{E}} = N_{n,k}^{\text{E}}(b; c_0, c_\infty) := \left[\begin{array}{cc|c} b/2, & b & c_0 \\ 3b/2, & -b & c_\infty \end{array} \right]_{n,k},$$

which if $b \neq 0$ equals

$$b^n \left[\begin{array}{ccc} 0, & 1, & \infty \\ \hline -\frac{1}{2}, & 2, & -\frac{1}{2} \\ c_0/b, & -(c_0 + c_\infty)/b, & c_\infty/b \end{array} \right]_{n,k}.$$

The GKP recurrences satisfied by these triangles are manifest in their definition. Their common homogeneity properties include

$$N_{n,k}^X(\lambda b; \lambda c_0, \lambda c_\infty) = \lambda^n N_{n,k}^X(b; c_0, c_\infty), \quad X = S, rS, E. \quad (5.1)$$

On account of this homogeneity, one can restrict without loss of generality to a single nonzero value of b . To facilitate comparison with the standard Narayana triangles, the choice $b = 2$ will be convenient. In what follows, the bivariate EGF's $G(t, z)$ of the three types will be denoted by $N^X(b; c_0, c_\infty; t, z)$, and the corresponding row polynomials $G_n(t)$ by $N_n^X(b; c_0, c_\infty; t)$.

Theorem 5.2. *The following EGF formulas hold in a neighborhood of $(0, 0)$. (BI), i.e., $(r_0, r_1, r_\infty) = (-\frac{1}{2}, -\frac{1}{2}, 2)$, with $b = 2$:*

$$N^S(2; c_0, c_\infty; t, z) = \left[\begin{array}{cc|c} 1, & 2 & c_0 \\ -2, & -2 & c_\infty \end{array} \right] (t, z) = \left(\frac{s_+}{t_+} \right)^{c_0/2} \left(\frac{s_-}{t_-} \right)^{-(c_0+c_\infty)/2},$$

$$s_\pm = \frac{1}{2} \pm \frac{(2t-1)+z}{2\sqrt{1+2(2t-1)z+z^2}}, \quad t_\pm = \frac{1}{2} \pm \left(t - \frac{1}{2} \right).$$

(BII), i.e., $(r_0, r_1, r_\infty) = (2, -\frac{1}{2}, -\frac{1}{2})$, with $b = 2$:

$$N^{rS}(2; c_0, c_\infty; t, z) = \left[\begin{array}{cc|c} -4, & 2 & c_0 \\ 3, & -2 & c_\infty \end{array} \right] (t, z) = \left(\frac{s_+}{t_+} \right)^{c_\infty/2} \left(\frac{s_-}{t_-} \right)^{-(c_0+c_\infty)/2},$$

$$s_\pm = \frac{1}{2} \pm \frac{(2-t)+t^2z}{2t\sqrt{1+2(2-t)z+t^2z^2}}, \quad t_\pm = \frac{1}{2} \pm \frac{2-t}{2t}.$$

(BIII), i.e., $(r_0, r_1, r_\infty) = (-\frac{1}{2}, 2, -\frac{1}{2})$, with $b = 2$:

$$N^E(2; c_0, c_\infty; t, z) = \left[\begin{array}{cc|c} 1, & 2 & c_0 \\ 3, & -2 & c_\infty \end{array} \right] (t, z) = \left(\frac{s_+}{t_+} \right)^{c_0/2} \left(\frac{s_-}{t_-} \right)^{c_\infty/2},$$

$$s_\pm = \frac{1}{2} \pm \frac{(1+t)-(1-t)^2z}{2(t-1)\sqrt{1-2(1+t)z+(1-t)^2z^2}}, \quad t_\pm = \frac{1}{2} \pm \frac{t+1}{2(t-1)}.$$

Proof. The formula for $N^S(2; c_0, c_\infty; t, z)$ is that of Theorem 3.6, scaled by the factor $b = 2$. The $X = rS, E$ formulas come by replacing (t, z) by $(\frac{1}{t}, tz)$ and $(\frac{-t}{1-t}, (1-t)z)$ respectively, as stated in Theorem 3.8. \square

For general c_0, c_∞ , it is difficult to extract formulas for the generalized Narayana numbers $N_{n,k}^X(2; c_0, c_\infty)$ from the explicit EGF's in this theorem. However, each of the three EGF's satisfies a 'contiguity relation.' For instance, it follows by elementary algebra that

$$\begin{aligned} [(2t-1)+z] N^S(2; c_0, c_\infty; t, z) = \\ 2t N^S(2; c_0+1, c_\infty; t, z) - N^S(2; c_0-1, c_\infty+2; t, z). \end{aligned} \quad (5.2)$$

This relates the parametric triangle $\begin{bmatrix} n \\ k \end{bmatrix} = N_{n,k}^S(2; c_0, c_\infty)$ at three contiguous values of the parameter-pair (c_0, c_∞) . The space of such relations and the possibility of iterating them remain to be explored.⁷

By examination, there are three parametric restrictions (i.e., constraints on the pair (c_0, c_∞)) which simplify each EGF $N^X(2; c_0, c_\infty; t, z)$ sufficiently that a closed-form expression for the triangle entries $N_{n,k}^X(2; c_0, c_\infty)$ can be obtained. They will be denoted by (a),(b),(c). The resulting specialized EGF's are given in Theorem 5.3 below.

Recall that cases (B I),(B II),(B III) are really the same, up to permutation of the points $0, 1, \infty$ of the projective line: they constitute the case (B), when the unordered set of parameters $\{r_0, r_1, r_\infty\}$ is $\{-\frac{1}{2}, -\frac{1}{2}, 2\}$. The restrictions (a),(b),(c) can be viewed as constraining the unordered set of parameter-pairs $\{(r_0, g_0), (r_1, g_1), (r_\infty, g_\infty)\}$. For (a), it must be of the form $\{(-\frac{1}{2}, g), (-\frac{1}{2}, g), (2, -2g)\}$; for (b), of the form $\{(-\frac{1}{2}, \frac{1}{2} + h), (-\frac{1}{2}, \frac{1}{2} - h), (2, -1)\}$; and for (c), of the form $\{(-\frac{1}{2}, h), (-\frac{1}{2}, -h), (2, 0)\}$.

Theorem 5.3. *The following parametrically restricted EGF formulas hold in a neighborhood of $(0, 0)$.*

(B I), i.e., $(r_0, r_1, r_\infty) = (-\frac{1}{2}, -\frac{1}{2}, 2)$, with $b = 2$:

$$\begin{aligned} \text{(a)} \quad g_0 = g_1: \quad N^S(2; c, -2c; t, z) &= S^{-c}, \\ \text{(b)} \quad g_\infty = -1: \quad N^S(2; c, -2; t, z) &= S^{-1} \left[\frac{-1+2t+z+S}{2t} \right]^{c-1}, \\ \text{(c)} \quad g_\infty = 0: \quad N^S(2; c-1, 0; t, z) &= \left[\left(\frac{t-1}{t} \right) \frac{1-2t-z-S}{1-2t-z+S} \right]^{(c-1)/2}, \end{aligned}$$

where $S := \sqrt{1 + 2(2t-1)z + z^2}$.

(B II), i.e., $(r_0, r_1, r_\infty) = (2, -\frac{1}{2}, -\frac{1}{2})$, with $b = 2$:

$$\begin{aligned} \text{(a)} \quad g_1 = g_\infty: \quad N^{rS}(2; -2c, c; t, z) &= S^{-c}, \\ \text{(b)} \quad g_0 = -1: \quad N^{rS}(2; -2, c; t, z) &= S^{-1} \left[\frac{2-t+t^2z+tS}{2} \right]^{c-1}, \\ \text{(c)} \quad g_0 = 0: \quad N^{rS}(2; 0, c-1; t, z) &= \left[(1-t) \frac{2-t+t^2z+tS}{2-t+t^2z-tS} \right]^{(c-1)/2}, \end{aligned}$$

where $S := \sqrt{1 + 2(2-t)z + t^2z^2}$.

(B III), i.e., $(r_0, r_1, r_\infty) = (-\frac{1}{2}, 2, -\frac{1}{2})$, with $b = 2$:

$$\begin{aligned} \text{(a)} \quad g_0 = g_\infty: \quad N^E(2; c, c; t, z) &= S^{-c}, \\ \text{(b)} \quad g_1 = -1: \quad N^E(2; c, 2-c; t, z) &= S^{-1} \left[\frac{1+t-(1-t)^2z-(1-t)S}{2t} \right]^{c-1}, \\ \text{(c)} \quad g_1 = 0: \quad N^E(2; c-1, 1-c; t, z) &= \left[\left(\frac{1}{t} \right) \frac{1+t-(1-t)^2z-(1-t)S}{1+t-(1-t)^2z+(1-t)S} \right]^{(c-1)/2}, \end{aligned}$$

⁷Three-term relations resembling eq. (5.2) can be derived far more generally: in fact, from any GKP recurrence in which one of the normalized parameters r_0, r_1, r_∞ equals 1 or 2. An explicit EGF is not needed. The derivation employs the PDE satisfied by the EGF, eq. (2.2).

where $S := \sqrt{1 - 2(1+t)z + (1-t)^2z^2}$.

The EGF formulas of this theorem are more manageable than the general ones of Theorem 5.2. With some effort, one can extract from them the explicit formulas for 1-parameter generalized Narayana numbers shown in Table 2.

Surprisingly, in each case the triangle entry $N_{n,k}^X$ can be represented as a hypergeometric term, parametrized by n, k . Because of this, for any of the parametric triangles of Table 2, each row polynomial is a Gauss-hypergeometric polynomial, and could optionally be rewritten in terms of a Jacobi polynomial. It is straightforward to confirm each formula in the table, by verifying that it satisfies the appropriate GKP recurrence.

A few representations are omitted from the table because they are not ‘pure,’ in that they comprise more than a single hypergeometric term. For instance, one can prove by induction that

$$N_{n,k}^E(2; c-1, 1-c) = (c-1)^{\overline{n}} \left\{ \left[\begin{matrix} -n+1, -n+c \\ 1, c \end{matrix} \right]^{\overline{k}} - \left[\begin{matrix} -n+1, -n+c \\ 1, c \end{matrix} \right]^{\overline{k-1}} \right\}, \quad (5.3)$$

when $1 \leq k \leq n$. This formula illustrates the operation of mid-trimming. The n 'th row polynomial of the parametric triangle on the left-hand side, for all $n \geq 1$, is divisible by $1-t$; whence the two terms on the right-hand side.

Many more trimming relationships could be mentioned. In each of the three sections of the table, triangle (b) is a trimmed version of triangle (c); so the formulas given for the latter could be viewed as redundant. The trimming is respectively a right-, a left-, and a mid-trimming, and (5.3) is the identity resulting from the last.

Moreover, in each of the three sections the $c = 0$ and $c = 2$ cases of the (b) triangle can be trimmed into the $c = 3$ case of the (a) triangle. The trimmings are respectively left- and mid-trimming; right- and mid-trimming; and left- and right-trimming. Also worth noting is that in each section, the $c = 1$ cases of the (a) and (b) triangles coincide.

A final observation regarding Table 2 is that the triangles in sections (BI), (BII), and (BIII) are respectively invariant under the involutive S_3 -group operations UBT, $\text{RT} \circ \text{UBT} \circ \text{RT}$, and RT, each of which acts row-wise. The last invariance is manifest: the substitution $k \leftarrow n-k$ leaves (BIII)(a) invariant and merely replaces c by $2-c$ in (BIII)(b). Up to parametrization, RT acts in cycle notation as the permutation (I,II)(III), UBT as (II,III)(I), and $\text{RT} \circ \text{UBT} \circ \text{RT}$ as (I,III)(II). Although these operations permute the three sections of the table, they do not affect the specialization letters (a),(b),(c).

Many versions of the generalized Narayana triangles of Table 2 appear in the OEIS [52], in normalized forms appropriate for combinatorial applications. A list is given in Table 3, showing the value taken in each relevant OEIS entry by the single parameter c of the corresponding Narayana triangle. As indicated, in each OEIS entry the triangle elements $\binom{n}{k}$ are divided by a certain rising factorial, to reduce the triangle to lowest terms. For the (a) specializations, this factor is $(1)^{\overline{n}}, (1)^{\overline{n}}, (3)^{\overline{n}}$ when $c = 1, 2, 3$, and for the (b) ones, it is $(2)^{\overline{n}}, (1)^{\overline{n}}, (2)^{\overline{n}}$ when

$c = 0, 1, 2$. Also, when $X = S, rS, E$, the triangle elements $N_{n,k}^X$ have respective signs $(-)^k, (-)^{n-k}, (+)$, and in the OEIS the negative signs are omitted.

For each generalized Narayana triangle in the OEIS, a compact hypergeometric representation of its n 'th row polynomial is shown in the table. (It comes from the hypergeometric term representation of $\left| \begin{smallmatrix} n \\ k \end{smallmatrix} \right|$ given in Table 2, altered to agree with the normalization used in the OEIS.) The formulas listed for the $c = 0$ cases of B I(b) and B II(b) require comment. In both, $-2n$ is the lower parameter of the ${}_2F_1$, which would seem to cause a division by zero if $n = 0$; but $-n$ being an upper parameter, each ${}_2F_1$ is interpreted as unity if $n = 0$.

From a combinatorial point of view, the most important triangles in Table 3 may be the $c = 1$ and $c = 3$ cases of B I(a) and B III(a). Explicitly, they are

$$(-1)^k [(1)^{\overline{n}}]^{-1} N_{n,k}^S(2; 1, -2) = \binom{n}{k} \binom{n+k}{k}, \quad (5.4)$$

$$(-1)^k [(3)^{\overline{n}}]^{-1} N_{n,k}^S(2; 3, -6) = \binom{n}{k} \binom{n+k+2}{k} / (k+1), \quad (5.5)$$

and

$$[(1)^{\overline{n}}]^{-1} N_{n,k}^E(2; 1, 1) = \binom{n}{k}^2, \quad (5.6)$$

$$[(3)^{\overline{n}}]^{-1} N_{n,k}^E(2; 3, 3) = \binom{n}{k} \binom{n+1}{k} / (k+1). \quad (5.7)$$

For each n , the n 'th rows of these four triangles are respectively the f -vectors of simplicial complexes dual to the associahedra of types B_n and A_n , and the corresponding h -vectors [30]. As with the permutohedra briefly encountered in §4.1, the f -vectors are mapped to the h -vectors by what is essentially $\text{UBT} \circ \text{RT}$, or equivalently $\text{RT} \circ \text{UBT} \circ \text{RT}$, the h -vectors of these simplicial polytopes being reflection-invariant (the Dehn–Sommerville symmetry). The invariance of (5.6) and (5.7) under $k \leftarrow n - k$ is evident.

The triangles (5.6) and (5.7) are known outside the combinatorics of polytopes: they are the now-standard Narayana number triangles of types B and A . In the normalized triangle (5.7), the element $\left| \begin{smallmatrix} n \\ k \end{smallmatrix} \right|$ counts the non-crossing partitions of an ordered $(n+1)$ -set into $k+1$ blocks; in the normalized triangle (5.6), it counts ‘signed’ or type- B non-crossing partitions [20, 30].

	$N_{n,k}^X(2; c_0, c_\infty)$	${}_2F_1$ term representation	reversed representation
(B I)			
(a)	$N_{n,k}^S(2; c, -2c)$	$(c)^{\overline{n}} \left[\begin{matrix} -n, & n+c \\ 1, & \frac{1}{2} + \frac{c}{2} \end{matrix} \right]^{\overline{k}}$	$(-2c)^{\underline{n},4} \left[\begin{matrix} -n, & -n + \frac{1}{2} - \frac{c}{2} \\ 1, & -2n + 1 - c \end{matrix} \right]^{\overline{n-k}}$
(b)	$N_{n,k}^S(2; c, -2)$	$(c)^{\overline{n}} \left[\begin{matrix} -n, & n+1 \\ 1, & c \end{matrix} \right]^{\overline{k}}$	$(-2)^{\underline{n},4} \left[\begin{matrix} -n, & -n+1-c \\ 1, & -2n \end{matrix} \right]^{\overline{n-k}}$
(c)	$N_{n,k}^S(2; c-1, 0)$	$(c-1)^{\overline{n}} \left[\begin{matrix} -n+1, & n \\ 1, & c \end{matrix} \right]^{\overline{k}}$	—
(B II)			
(a)	$N_{n,k}^{rS}(2; -2c, c)$	$(-2c)^{\underline{n},4} \left[\begin{matrix} -n, & -n + \frac{1}{2} - \frac{c}{2} \\ 1, & -2n + 1 - c \end{matrix} \right]^{\overline{k}}$	$(c)^{\overline{n}} \left[\begin{matrix} -n, & n+c \\ 1, & \frac{1}{2} + \frac{c}{2} \end{matrix} \right]^{\overline{n-k}}$
(b)	$N_{n,k}^{rS}(2; -2, c)$	$(-2)^{\underline{n},4} \left[\begin{matrix} -n, & -n+1-c \\ 1, & -2n \end{matrix} \right]^{\overline{k}}$	$(c)^{\overline{n}} \left[\begin{matrix} -n, & n+1 \\ 1, & c \end{matrix} \right]^{\overline{n-k}}$
(c)	$N_{n,k}^{rS}(2; 0, c-1)$	—	$(c-1)^{\overline{n}} \left[\begin{matrix} -n+1, & n \\ 1, & c \end{matrix} \right]^{\overline{n-k}}$
(B III)			
(a)	$N_{n,k}^E(2; c, c)$	$(c)^{\overline{n}} \left[\begin{matrix} -n, & -n + \frac{1}{2} - \frac{c}{2} \\ 1, & \frac{1}{2} + \frac{c}{2} \end{matrix} \right]^{\overline{k}}$	$(c)^{\overline{n}} \left[\begin{matrix} -n, & -n + \frac{1}{2} - \frac{c}{2} \\ 1, & \frac{1}{2} + \frac{c}{2} \end{matrix} \right]^{\overline{n-k}}$
(b)	$N_{n,k}^E(2; c, 2-c)$	$(c)^{\overline{n}} \left[\begin{matrix} -n, & -n-1+c \\ 1, & c \end{matrix} \right]^{\overline{k}}$	$(2-c)^{\overline{n}} \left[\begin{matrix} -n, & -n+1-c \\ 1, & 2-c \end{matrix} \right]^{\overline{n-k}}$
(c)	$N_{n,k}^E(2; c-1, 1-c)$	—	—

Table 2: Certain 1-parameter generalized Narayana triangles with hypergeometric-term representations. There are three each of the Stirling, reversed Stirling, and Eulerian kinds, i.e., three parametrically restricted versions of GKP cases (B I), (B II), and (B III).

BI(a), $c = 1$	$(-1)^k [(1)^{\overline{n}}]^{-1} N_{n,k}^S(2; 1, -2)$	${}_2F_1(-n, n+1; 1; -t)$	A063007
BI(a), $c = 2$	$(-1)^k [(1)^{\overline{n}}]^{-1} N_{n,k}^S(2; 2, -4)$	$(n+1) {}_2F_1(-n, n+2; \frac{3}{2}; -t)$	A053124
BI(a), $c = 3$	$(-1)^k [(3)^{\overline{n}}]^{-1} N_{n,k}^S(2; 3, -6)$	${}_2F_1(-n, n+3; 2; -t)$	A033282
BI(b), $c = 0$	$(-1)^k [(2)^{\overline{n}}]^{-1} N_{n,k}^S(2; 0, -2)$	$[(2)^{\overline{n},4}/(2)^{\overline{n}}] t^n {}_2F_1(-n, -n+1; -2n; -\frac{1}{t})$	A086810
BI(b), $c = 1$	$(-1)^k [(1)^{\overline{n}}]^{-1} N_{n,k}^S(2; 1, -2)$	${}_2F_1(-n, n+1; 1; -t)$	A063007
BI(b), $c = 2$	$(-1)^k [(2)^{\overline{n}}]^{-1} N_{n,k}^S(2; 2, -2)$	${}_2F_1(-n, n+1; 2; -t)$	A088617
BII(a), $c = 1$	$(-1)^{n-k} [(1)^{\overline{n}}]^{-1} N_{n,k}^{rS}(2; -2, 1)$	$t^n {}_2F_1(-n, n+1; 1; -\frac{1}{t})$	A104684
BII(a), $c = 2$	$(-1)^{n-k} [(1)^{\overline{n}}]^{-1} N_{n,k}^{rS}(2; -4, 2)$	$(n+1) t^n {}_2F_1(-n, n+2; \frac{3}{2}; -\frac{1}{t})$	A053125
BII(a), $c = 3$	$(-1)^{n-k} [(3)^{\overline{n}}]^{-1} N_{n,k}^{rS}(2; -6, 3)$	$t^n {}_2F_1(-n, n+3; 2; -\frac{1}{t})$	A126216
BII(b), $c = 0$	$(-1)^{n-k} [(2)^{\overline{n}}]^{-1} N_{n,k}^{rS}(2; -2, 0)$	$[(2)^{\overline{n},4}/(2)^{\overline{n}}] {}_2F_1(-n, -n+1; -2n; t)$	A133336
BII(b), $c = 1$	$(-1)^{n-k} [(1)^{\overline{n}}]^{-1} N_{n,k}^{rS}(2; -2, 1)$	$t^n {}_2F_1(-n, n+1; 1; -\frac{1}{t})$	A104684
BII(b), $c = 2$	$(-1)^{n-k} [(2)^{\overline{n}}]^{-1} N_{n,k}^{rS}(2; -2, 2)$	$t^n {}_2F_1(-n, n+1; 2; -\frac{1}{t})$	A060693
BIII(a), $c = 1$	$[(1)^{\overline{n}}]^{-1} N_{n,k}^E(2; 1, 1)$	${}_2F_1(-n, -n; 1; t)$	A008459
BIII(a), $c = 2$	$[(1)^{\overline{n}}]^{-1} N_{n,k}^E(2; 2, 2)$	$(n+1) {}_2F_1(-n, -n - \frac{1}{2}; \frac{3}{2}; t)$	A091044
BIII(a), $c = 3$	$[(3)^{\overline{n}}]^{-1} N_{n,k}^E(2; 3, 3)$	${}_2F_1(-n, -n-1; 2; t)$	A001263
BIII(b), $c = 0$	$[(2)^{\overline{n}}]^{-1} N_{n,k}^E(2; 0, 2)$	$t^n {}_2F_1(-n, -n+1; 2, \frac{1}{t})$	A090181
BIII(b), $c = 1$	$[(1)^{\overline{n}}]^{-1} N_{n,k}^E(2; 1, 1)$	${}_2F_1(-n, -n; 1; t)$	A008459
BIII(b), $c = 2$	$[(2)^{\overline{n}}]^{-1} N_{n,k}^E(2; 2, 0)$	${}_2F_1(-n, -n+1; 2; t)$	A131198

Table 3: Generalized Narayana triangles in the OEIS [52], with hypergeometric-polynomial representations of their row polynomials.
(N.B.: Triangles A053124, A053125 in the OEIS are signed rather than signless, disagreeing with the convention adhered to here.)
(N.B.: When $n = 0$, each ${}_2F_1$ equals unity, by convention if necessary.)

6. Generalized secant–tangent triangles

In addition to the generalized Stirling–Eulerian case (A) and the generalized Narayana case (B), there is a third case when a GKP recurrence can be solved in closed form, or at least the bivariate EGF $G(t, z)$ can be computed explicitly by the method of characteristics. As noted in Section 3, this is the generalized secant–tangent case, case (C).

The GKP triangles in case (C) include important ones with combinatorial interpretations, but they are relatively few. The following treatment will briefly relate their EGF’s to previous work. The term ‘generalized secant–tangent’ will be justified. It comes from an alternative method of generating the row polynomials $G_n(t)$, $n \geq 0$, of any GKP triangle. (See Theorem 6.4.) This could be called the iterated derivation method, and is a specific application of the context-free grammar approach taken by Chen [19] and Dumont [26] to exponential structures in combinatorics. As another application, some final identities involving the generalized Eulerian numbers $E_{n,k}(a, b; c_0, c_\infty)$ will be derived.

In subcase (CI), when $(r_0, r_1, r_\infty) = (\frac{1}{2}, \frac{1}{2}, 0)$, Theorem 3.7 supplies a transcendental expression for the bivariate EGF $G(t, z)$. Much as with cases (A) and (B), subcases (CII) and (CIII) are obtained from (CI) by applying the appropriate elements of the S_3 -group: the row-wise sequence transformations RT and $\text{RT} \circ \text{UBT} \circ \text{RT}$. The resulting vectors (r_0, r_1, r_∞) are the permutations $(0, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2})$. Due to the analogy with the subcases of case (A), cases (CI), (CII), and (CIII) will be called the Stirling, reversed Stirling, and Eulerian subcases of the generalized secant–tangent triangle.

As always, the parameters (r_0, r_1, r_∞) and (g_0, g_1, g_∞) , where the sums $r_0 + r_1 + r_\infty$ and $g_0 + g_1 + g_\infty$ are constrained to equal 1 and 0 respectively, can be converted to the traditional GKP parameters $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$ and vice versa, by the formulas in (3.4) and (3.6). In the following, 3-parameter generalized secant–tangent triangles $W_{n,k}^X$ (where $X = \text{S}, \text{rS}, \text{E}$, referring to (CI), (CII), (CIII)) are defined, in both the traditional and new parametrizations. The bivariate EGF’s $G(t, z)$ of the three types will be denoted by $W^X(b; c_0, c_\infty; t, z)$,

Definition 6.1.

(CI) The generalized secant–tangent triangle $W_{n,k}^{\text{S}}(b; c_0, c_\infty)$, of Stirling type, is defined by

$$W_{n,k}^{\text{S}} = W_{n,k}^{\text{S}}(b; c_0, c_\infty) := \left[\begin{array}{cc|c} -b/2, & b & c_0 \\ b, & -b & c_\infty \end{array} \right]_{n,k},$$

which if $b \neq 0$ equals

$$b^n \left[\begin{array}{ccc} 0, & 1, & \infty \\ \frac{1}{2}, & \frac{1}{2}, & 0 \\ c_0/b, & -(c_0 + c_\infty)/b, & c_\infty/b \end{array} \right]_{n,k}.$$

(C II) The generalized secant–tangent triangle $W_{n,k}^{\text{rS}}(b; c_0, c_\infty)$, of reversed Stirling type, is defined by

$$W_{n,k}^{\text{rS}} = W_{n,k}^{\text{rS}}(b; c_0, c_\infty) := \left[\begin{array}{cc|c} 0, & b & c_0 \\ b/2, & -b & c_\infty \end{array} \right]_{n,k},$$

which if $b \neq 0$ equals

$$b^n \left[\begin{array}{ccc} 0, & 1, & \infty \\ 0, & \frac{1}{2}, & \frac{1}{2} \\ c_0/b, & -(c_0 + c_\infty)/b, & c_\infty/b \end{array} \right]_{n,k}.$$

(C III) The generalized secant–tangent triangle $W_{n,k}^{\text{E}}(b; c_0, c_\infty)$, of Eulerian type, is defined by

$$W_{n,k}^{\text{E}} = W_{n,k}^{\text{E}}(b; c_0, c_\infty) := \left[\begin{array}{cc|c} -b/2, & b & c_0 \\ b/2, & -b & c_\infty \end{array} \right]_{n,k},$$

which if $b \neq 0$ equals

$$b^n \left[\begin{array}{ccc} 0, & 1, & \infty \\ \frac{1}{2}, & 0, & \frac{1}{2} \\ c_0/b, & -(c_0 + c_\infty)/b, & c_\infty/b \end{array} \right]_{n,k}.$$

These parametric triangles have the common homogeneity property

$$W_{n,k}^{\text{X}}(\lambda b; \lambda c_0, \lambda c_\infty) = \lambda^n W_{n,k}^{\text{X}}(b; c_0, c_\infty), \quad \text{X} = \text{S, rS, E}. \quad (6.1)$$

To facilitate comparison with previous work, the choice $b = 2$ will now be made, without loss of generality.

Theorem 6.2. *The following EGF formulas hold in a neighborhood of $(0, 0)$.*

(C I), *i.e.*, $(r_0, r_1, r_\infty) = (\frac{1}{2}, \frac{1}{2}, 0)$, *with* $b = 2$:

$$W^{\text{S}}(2; c_0, c_\infty; t, z) = \left[\begin{array}{cc|c} -1, & 2 & c_0 \\ 2, & -2 & c_\infty \end{array} \right] (t, z) = \left(\frac{s_+}{t_+} \right)^{c_0/2} \left(\frac{s_-}{t_-} \right)^{-(c_0+c_\infty)/2},$$

$$s_\pm = \left[\sqrt{t_\pm} \cos(z\sqrt{t_+t_-}) \pm \sqrt{t_\mp} \sin(z\sqrt{t_+t_-}) \right]^2, \quad t_\pm = \frac{1}{2} \pm \left(t - \frac{1}{2} \right).$$

(C II), *i.e.*, $(r_0, r_1, r_\infty) = (0, \frac{1}{2}, \frac{1}{2})$, *with* $b = 2$:

$$W^{\text{rS}}(2; c_0, c_\infty; t, z) = \left[\begin{array}{cc|c} 0, & 2 & c_0 \\ 1, & -2 & c_\infty \end{array} \right] (t, z) = \left(\frac{s_+}{t_+} \right)^{c_\infty/2} \left(\frac{s_-}{t_-} \right)^{-(c_0+c_\infty)/2},$$

$$s_\pm = \left[\sqrt{t_\pm} \cos(z\sqrt{t_-/t_+}) \pm \sqrt{t_\mp} \sin(z\sqrt{t_-/t_+}) \right]^2, \quad t_\pm = \frac{1}{2} \pm \frac{2-t}{2t}.$$

(C III), i.e., $(r_0, r_1, r_\infty) = (\frac{1}{2}, 0, \frac{1}{2})$, with $b = 2$:

$$W^E(2; c_0, c_\infty; t, z) = \left[\begin{array}{cc|c} -1, & 2 & c_0 \\ 1, & -2 & c_\infty \end{array} \right] (t, z) = \left(\frac{s_+}{t_+} \right)^{c_0/2} \left(\frac{s_-}{t_-} \right)^{c_\infty/2},$$

$$s_\pm = \left[\sqrt{t_\pm} \cos(z\sqrt{t_+/t_-}) \pm \sqrt{t_\mp} \sin(z\sqrt{t_+/t_-}) \right]^2, \quad t_\pm = \frac{1}{2} \pm \frac{t+1}{2(t-1)}.$$

It should be noted that in each case, $s_+ + s_-$ and $t_+ + t_-$ equal unity.

Proof. The formula for $W^S(2; c_0, c_\infty; t, z)$ is that of Theorem 3.7, scaled by the factor $b = 2$. The X = rS, E formulas come by replacing (t, z) by $(\frac{1}{t}, tz)$ and $(\frac{-t}{1-t}, (1-t)z)$ respectively, as stated in Theorem 3.8. \square

It has long been known that case-(C) GKP recurrences, in particular case-(C II) ones, appear in the enumerative combinatorics of peaks and valleys of permutations. (See the papers of Ma [43] and Zhuang [67], with earlier work extending from André in the 1880's through Entringer in the 1960's and Gessel in the 1970's [32] remaining relevant.) $W_{n,k}^{rS}(2; 1, 0)$ counts the elements of the group S_n that have k left (or right) peaks [52, A008971], and $W_{n,k}^{rS}(2; 2, 0)$ the elements of S_{n+1} that have k peaks [52, A008303]. (In [43], these are denoted by $W_{n,k}^l$ and $W_{n+1,k}$.) Also, the triangle $W_{n,k}^{rS}(2; 0, 1)$ counts the elements of S_n that have k left-right peaks [67], but left-trimming this triangle reduces it to $W_{n,k}^{rS}(2; 2, 0)$. In a separate combinatorial application, the normalized triangle $2^{-n}4^k W_{n,k}^{rS}(2; 2, 0)$, resp. $4^k W_{n,k}^{rS}(2; 1, 0)$, has as its n 'th row the γ -vector of a simplicial complex dual to the permutohedron of type A_n , resp. B_n . (See [30] and [52, A101280].)

Proposition 6.3. *The following EGF formulas hold in a neighborhood of $(0, 0)$.*

$$W_{n,k}^{rS}(2; 1, 0; t, z) = \left[\begin{array}{cc|c} 0, & 2 & 1 \\ 1, & -2 & 0 \end{array} \right] (t, z)$$

$$= \frac{\sqrt{1-t}}{\sqrt{1-t} \cosh(z\sqrt{1-t}) - \sinh(z\sqrt{1-t})}, \quad (6.2)$$

and

$$W_{n,k}^{rS}(2; 2, 0; t, z) = \left[\begin{array}{cc|c} 0, & 2 & 2 \\ 1, & -2 & 0 \end{array} \right] (t, z)$$

$$= \left[\frac{\sqrt{1-t}}{\sqrt{1-t} \cosh(z\sqrt{1-t}) - \sinh(z\sqrt{1-t})} \right]^2 \quad (6.3)$$

$$= \frac{d}{dz} \left\{ [\sqrt{1-t} \coth(z\sqrt{1-t}) - 1]^{-1} \right\}.$$

These EGF formulas follow by some trigonometric manipulation from the one for $W^{rS}(2; c_0, c_\infty; t, z)$ in Theorem 6.2. They agree with those previously known [32, 43, 67], but the present derivation places them in analytic context as individual EGF's that belong to a family of EGF's that can be computed

by the method of characteristics. It should be noted that for case-(C) GKP recurrences, it is the bivariate triangle EGF's and not the triangle elements $\binom{n}{k}$ for which explicit, closed-form expressions are currently available.

The iterated derivation method of solving a GKP recurrence with parameters $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$, which is an instance of a more general grammar-based method of combinatorial enumeration, will now be summarized. Its applicability is not restricted to case-(C) recurrences.

Let D be a formal derivation satisfying Leibniz's rule, which acts on any reasonable function of the variables or indeterminates x, y ; such as a polynomial or a formal power series, though non-integral powers are allowed. The following is a known fact (cf. [35, Theorem 2.1] and [44, Lemma 8]), restated in present notation. It is equivalent to the differential recurrence of Theorem 2.6(ii) and the iterated operator formula of Theorem 2.6(iii).

Theorem 6.4. *If the variables x, y have the respective derivations*

$$D(x, y) = (x, y) * \left(x^\alpha y^{\alpha'}, x^{\alpha+\beta} y^{\alpha'+\beta'} \right), \quad (6.4)$$

where $*$ signifies the elementwise product, then for all $n \geq 0$,

$$D^n(x^\gamma y^{\gamma'}) = (x^\gamma y^{\gamma'}) (x^\alpha y^{\alpha'})^n \sum_{k=0}^n \left[\begin{array}{c} \alpha, \beta \\ \alpha', \beta' \end{array} \middle| \begin{array}{c} \gamma \\ \gamma' \end{array} \right]_{n,k} (x^\beta y^{\beta'})^k, \quad (6.5)$$

or equivalently

$$D^n(x^\gamma y^{\gamma'}) = (x^\gamma y^{\gamma'}) (x^\alpha y^{\alpha'})^n G_n(x^\beta y^{\beta'}),$$

where $G_n = G_n(t)$ is the n 'th row polynomial of the GKP triangle with parameter array $\left[\begin{array}{c} \alpha, \beta \\ \alpha', \beta' \end{array} \middle| \begin{array}{c} \gamma \\ \gamma' \end{array} \right]$.

It may be possible to realize x, y as functions of an independent variable w , and D as the derivative $D_w := d/dw$. If so, the following is immediate: it comes by exponentiating D_w .

Corollary 6.5. *For all δ , or equivalently as an equality between formal power series in δ ,*

$$\frac{(x^\gamma y^{\gamma'})(w + \delta)}{(x^\gamma y^{\gamma'})(w)} = G \left((x^\beta y^{\beta'})(w), \delta (x^\alpha y^{\alpha'})(w) \right),$$

where $G = G(t, z)$ is the bivariate EGF of the GKP triangle with parameter array $\left[\begin{array}{c} \alpha, \beta \\ \alpha', \beta' \end{array} \middle| \begin{array}{c} \gamma \\ \gamma' \end{array} \right]$.

There is also an easily checked homogeneity property: if the pair $x = x(w)$, $y = y(w)$ satisfy the differential equation (6.4), then for any λ , so do the pair $\lambda^p x(\lambda w)$, $\lambda^q y(\lambda w)$, where

$$p = \frac{\beta'}{\alpha\beta' - \alpha'\beta}, \quad q = \frac{-\beta}{\alpha\beta' - \alpha'\beta}, \quad (6.6)$$

it being assumed that $\alpha\beta' - \alpha'\beta \neq 0$. If $\beta' = -\beta$, these reduce to $\lambda^{1/(\alpha+\alpha')}x(\lambda w)$, $\lambda^{1/(\alpha+\alpha')}y(\lambda w)$. Moreover, w can be shifted without affecting (6.4), as it is an autonomous equation. Thus (6.4) has a 2-parameter space of solutions.

For the three case-(C) GKP triangles $W_{n,k}^X(b; c_0, c_\infty)$ of Definition 6.1, solutions x, y of (6.4) can be found by inspection. When $b = 2$, canonical ones are the following.

	$\begin{matrix} n \\ k \end{matrix}$	$(\alpha, \beta; \alpha', \beta')$	x, y
(CI):	$W_{n,k}^S, b = 2$	$(-1, 2; 2, -2)$	$x = \tan w, y = \sec w$
(CII):	$W_{n,k}^{rS}, b = 2$	$(0, 2; 1, -2)$	$x = \sec w, y = \tan w$
(CIII):	$W_{n,k}^E, b = 2$	$(-1, 2; 1, -2)$	$x = \cosh w, y = \sinh w$

In each case the pair γ, γ' equals c_0, c_∞ , and Theorem 6.4 yields the three elegant identities

$$\begin{aligned} & (\tan^{c_0} w \sec^{c_\infty} w)^{-1} D_w^n (\tan^{c_0} w \sec^{c_\infty} w) \\ &= (\sec^n w) \sum_{k=0}^n W_{n,k}^S(2; c_0, c_\infty) \csc^{n-2k} w, \end{aligned} \quad (6.7a)$$

$$\begin{aligned} & (\sec^{c_0} w \tan^{c_\infty} w)^{-1} D_w^n (\sec^{c_0} w \tan^{c_\infty} w) \\ &= (\tan^n w) \sum_{k=0}^n W_{n,k}^{rS}(2; c_0, c_\infty) \csc^{2k} w, \end{aligned} \quad (6.7b)$$

and

$$\begin{aligned} & (\cosh^{c_0} w \sinh^{c_\infty} w)^{-1} D_w^n (\cosh^{c_0} w \sinh^{c_\infty} w) \\ &= (\tanh^n w) \sum_{k=0}^n W_{n,k}^E(2; c_0, c_\infty) \coth^{2k} w. \end{aligned} \quad (6.7c)$$

Under the reflection operation (RT), the parametric triangles $W_{n,k}^S, W_{n,k}^{rS}$ are interchanged, with $c_0 \leftrightarrow c_\infty$. Also triangle $W_{n,k}^E$ is invariant, except that again, $c_0 \leftrightarrow c_\infty$. It should be noted that in (6.7c), \sinh, \cosh (and likewise \tanh, \coth) could optionally be interchanged, because $x(w) = \cosh w, y(w) = \sinh w$ are converted to $x(w) = \sinh w, y(w) = \cosh w$ by a complex shift of w .

As mentioned, the triangles $W_{n,k}^{rS}(2; 1, 0)$ and $W_{n,k}^{rS}(2; 2, 0)$ are of combinatorial significance. By (6.7b), their elements are the coefficients of trigonometric polynomials obtained by repeated differentiation of $\sec w$ and $\sec^2 w$, respectively. This fact is known [43], but one now sees that by a straightforward generalization one can generate any desired case-(C) GKP triangle, with or without a combinatorial interpretation. This explains the term ‘generalized secant–tangent triangle.’

Case-(A) and case-(B) triangles can also be generated by the iterated derivation method. It could be applied as follows in the generalized Stirling–Eulerian case (A). To treat subcase (AI) similarly to (CI), one should generate not the

Hsu–Shiue triangle $S_{n,k}(a, b; r)$ but rather $(s)^{\overline{k}}S_{n,k}(a, b; r)$ for arbitrary s , because it is the latter which satisfies a GKP recurrence with $\beta\beta' \neq 0$. (Recall eq. (1.9).) The matching (A II) triangle is the reflection $(r)^{\overline{n-k}}S_{n,n-k}(a, b; s)$, and the (A III) triangle is the generalized Eulerian triangle $E_{n,k}(a, b; c_0, c_\infty)$. The GKP parameters of these triangles, and canonical solutions x, y of (6.4) when $a \neq 0$ and $b \neq 0$ that can be found by inspection, are as follows.

	$\begin{matrix} n \\ k \end{matrix}$	$(\alpha, \beta; \alpha', \beta'), (\gamma, \gamma')$	x, y
(A I):	$s^{\overline{k}}S_{n,k}(a, b; r)$	$(-a, b; 0, 1)$ (r, s)	$x = (1 + aw)^{1/a}$ $y = b[1 - (1 + aw)^{b/a}]^{-1}$
(A II):	$r^{\overline{n-k}}S_{n,n-k}(a, b; s)$	$(1, -1; -a + b, -b)$ (r, s)	$x = b[1 - (1 + aw)^{b/a}]^{-1}$ $y = (1 + aw)^{1/a}$
(A III):	$E_{n,k}(a, b; c_0, c_\infty)$	$(-a, b; a + b, -b)$ (c_0, c_∞)	$x = [1 - (1 - aw)^{-b/a}]^{-1/b}$ $y = [(1 - aw)^{b/a} - 1]^{-1/b}$

As in case (C), the parametric (A I) and (A II) triangles are interchanged by the reflection operation, with $r \leftrightarrow s$, and so are the solutions x, y . The (A III) triangle is invariant, with $c_0 \leftrightarrow c_\infty$; and the solutions x, y could optionally be interchanged, as in case (C III).

The three corresponding identities come at once from Theorem 6.4. The (A II) identity is merely a reflected version of the (A I) one and is left to the reader. The (A I) and (A III) identities are respectively

$$\begin{aligned} & \left\{ (1 + aw)^{r/a} [1 - (1 + aw)^{b/a}]^{-s} \right\}^{-1} D_w^n \left\{ (1 + aw)^{r/a} [1 - (1 + aw)^{b/a}]^{-s} \right\} \\ &= (1 + aw)^{-n} \sum_{k=0}^n b^k s^{\overline{k}} S_{n,k}(a, b; r) \left\{ (1 + aw)^{b/a} [1 - (1 + aw)^{b/a}]^{-1} \right\}^k \end{aligned} \quad (6.8a)$$

and

$$\begin{aligned} & \left\{ (1 - aw)^{c_0/a} [(1 - aw)^{b/a} - 1]^{-(c_0 + c_\infty)/b} \right\}^{-1} \\ & \quad \times D_w^n \left\{ (1 - aw)^{c_0/a} [(1 - aw)^{b/a} - 1]^{-(c_0 + c_\infty)/b} \right\} \\ &= (1 - aw)^{-n} [(1 - aw)^{b/a} - 1]^{-n} \sum_{k=0}^n E_{n,k}(a, b; c_0, c_\infty) (1 - aw)^{kb/a}. \end{aligned} \quad (6.8b)$$

In the $a \rightarrow 0$ limit (the $b \rightarrow 0$ limit is not considered here), these become

$$\begin{aligned} & [e^{rw} (1 - e^{bw})^{-s}]^{-1} D_w^n [e^{rw} (1 - e^{bw})^{-s}] \\ &= \sum_{k=0}^n b^k s^{\overline{k}} S_{n,k}(0, b; r) [e^{bw} (1 - e^{bw})^{-1}]^k \end{aligned} \quad (6.9a)$$

and

$$\begin{aligned} & [e^{-c_0 w} (e^{-bw} - 1)^{-(c_0 + c_\infty)/b}]^{-1} D_w^n [e^{-c_0 w} (e^{-bw} - 1)^{-(c_0 + c_\infty)/b}] \\ &= (e^{-bw} - 1)^{-n} \sum_{k=0}^n E_{n,k}(0, b; c_0, c_\infty) e^{-kbw}. \end{aligned} \quad (6.9b)$$

By applying (6.8a) and (6.9a), the parameter s being arbitrary, one can compute Hsu–Shiue numbers $S_{n,k}(a, b; r)$ by repeated differentiation. For instance, the De Morgan numbers $\text{Surj}(n, k) = k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = (1)^{\overline{k}} S_{n,k}(0, 1; 0)$, which count the number of maps from an n -set onto a k -set (see Example 2.5), satisfy

$$(1 - e^w) D_w^n [(1 - e^w)^{-1}] = \sum_{k=0}^n \text{Surj}(n, k) [e^w (1 - e^w)^{-1}]^k. \quad (6.10)$$

The generalized Eulerian numbers $E_{n,k}(a, b; c_0, c_\infty)$ can be computed likewise from (6.8b) and (6.9b), with the latter applying in the Carlitz–Scoville $a = 0$ case, examples of which were mentioned in the introduction.

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