# CHARACTER ANALOGUES OF COHEN-TYPE IDENTITIES AND RELATED VORONOI SUMMATION FORMULAS 

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#### Abstract

In [1], B. C. Berndt and A. Zaharescu introduced the twisted divisor sums associated with the Dirichlet character while studying the Ramanujan's type identity involving finite trigonometric sums and doubly infinite series of Bessel functions. Later, in a follow-up paper [2], S. Kim extended the definition of the twisted divisor sums to twisted sums of divisor functions. In this paper, we derive identities associated with the aforementioned weighted divisor functions and the modified $K$-Bessel function in light of recent results obtained by the first author and B. Maji [3]. Moreover, we provide a new expression for $L(1, \chi)$ from which we establish the positivity of $L(1, \chi)$ for any real primitive character $\chi$. In addition, we deduce Cohen-type identities and then exhibit the Voronoï-type summation formulas for them.


## 1. Introduction

We begin by reminiscing about a beautiful identity due to Ramanujan involving the $K$-Bessel function, which is recorded on page 253 of his lost notebook. If $\alpha$ and $\beta$ are any two positive numbers such that $\alpha \beta=\pi^{2}$ and $\nu$ is any complex number, then

$$
\begin{align*}
& \sqrt{\alpha} \sum_{n=1}^{\infty} \sigma_{-\nu}(n) n^{\nu / 2} K_{\nu / 2}(2 n \alpha)-\sqrt{\beta} \sum_{n=1}^{\infty} \sigma_{-\nu}(n) n^{\nu / 2} K_{\nu / 2}(2 n \beta) \\
& =\frac{1}{4} \Gamma\left(\frac{\nu}{2}\right) \zeta(\nu)\left\{\beta^{(1-\nu) / 2}-\alpha^{(1-\nu) / 2}\right\}+\frac{1}{4} \Gamma\left(-\frac{\nu}{2}\right) \zeta(-\nu)\left\{\beta^{(1+\nu) / 2}-\alpha^{(1+\nu) / 2}\right\}, \tag{1.1}
\end{align*}
$$

where $\sigma_{k}(n)=\sum_{d \mid n} d^{k}$ and $K_{\nu}(z)$ denotes the modified Bessel function of order $\nu$ [4, p. 78], which is defined as the following

$$
\begin{equation*}
K_{\nu}(z):=\frac{\pi}{2} \frac{I_{-\nu}(z)-I_{\nu}(z)}{\sin \pi \nu}, z \in \mathbb{C}, \nu \notin \mathbb{Z} \tag{1.2}
\end{equation*}
$$

with $I_{\nu}$ being the Bessel function of the imaginary argument [4, p. 77] given by

$$
\begin{equation*}
I_{\nu}(z):=\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} z\right)^{\nu+2 n}}{n!\Gamma(\nu+n+1)}, z \in \mathbb{C} . \tag{1.3}
\end{equation*}
$$

Later in 1955, Guinand [5] derived a formula almost similar to (1.1) by appealing to a formula due to Watson [6] involving the $K$-Bessel function. One can use Ramanujan's formula (1.1) to derive Koshliakov's formula (7], given by

$$
\begin{equation*}
\sqrt{\alpha}\left(\frac{1}{4} \gamma-\frac{1}{4} \log (4 \beta)+\sum_{n=1}^{\infty} d(n) K_{0}(2 n \alpha)\right)=\sqrt{\beta}\left(\frac{1}{4} \gamma-\frac{1}{4} \log (4 \alpha)+\sum_{n=1}^{\infty} d(n) K_{0}(2 n \beta)\right), \tag{1.4}
\end{equation*}
$$

where $\gamma$ denotes Euler's constant and $K_{0}(z)$ is defined by the limit

$$
\begin{equation*}
K_{0}(z):=\lim _{\nu \rightarrow 0} K_{\nu}(z) \tag{1.5}
\end{equation*}
$$

Koshliakov, in 1929, proved the formula (1.4) by employing the Voronoï summation formula [8], which reads as the following

$$
\begin{equation*}
\sum_{a \leq n \leq b} ' d(n) f(n)=\int_{a}^{b}(\log (x)+2 \gamma) f(x) d x+\sum_{n=1}^{\infty} d(n) \int_{a}^{b} f(x)\left(4 K_{0}(4 \pi \sqrt{n x})-2 \pi Y_{0}(4 \pi \sqrt{n x})\right) d x \tag{1.6}
\end{equation*}
$$

[^0]where the prime ' on the summation of the left-hand side implies that if $a$ or $b$ is an integer, only $f(a) / 2$ or $f(b) / 2$ is counted, respectively. Here $f(x)$ is a function of bounded variation in $(a, b)$ with $0<a<b$, and $K_{0}(z)$ is defined in (1.5), and $Y_{\nu}(z)$ denotes the Weber-Bessel function of order $\nu$ [4, p. 64] given by
\[

$$
\begin{align*}
& Y_{\nu}(z):=\frac{J_{\nu}(z) \cos \pi \nu-J_{-\nu}(z)}{\sin \pi \nu}, z \in \mathbb{C}, \nu \notin \mathbb{Z}  \tag{1.7}\\
& Y_{n}(z):=\lim _{\nu \rightarrow 0} Y_{\nu}(z), \quad z \in \mathbb{C}, n \in \mathbb{Z} \tag{1.8}
\end{align*}
$$
\]

and $J_{\nu}(z)$ denotes the Bessel function of the first kind of order $\nu$ [4, p. 40]

$$
\begin{equation*}
J_{\nu}(z):=\sum_{n=0}^{\infty} \frac{(-1)^{n}\left(\frac{1}{2} z\right)^{\nu+2 n}}{n!\Gamma(\nu+n+1)}, z \in \mathbb{C} \tag{1.9}
\end{equation*}
$$

After Voronoï's remarkable discovery of (1.6), many number theorists examined the formula (1.6) and provided proofs under different conditions on the function $f(x)$. A. L. Dixon and W. L. Ferrar [9] gave proof for a bounded second differential coefficient function $f(x)$ in $(a, b)$. Koshliakov proved (1.6) for the analytic function $f$ inside a closed contour strictly containing the interval $[a, b]$ with $0<a<b$. J. R. Wilton 10] proved (1.6) for the function $f$, which has compact support in the interval $[a, b]$ such that $\lim _{\varepsilon \rightarrow 0} V_{\alpha}^{\beta-\varepsilon} f(x)=V_{\alpha}^{\beta-0} f(x)$ where $V_{\alpha}^{\beta}$ denotes the total variation of $f(x)$ over $(\alpha, \beta)$. In 1987, M. Jutila [11] gave a Voronoii-type summation formula involving an exponential factor. One can refer to [12, 13, 14, 15] for details and developments on Voronoil's summation formulas. Apart from its connection to different fields of mathematics, Voronoï-type summation formulas also have some applications in physics, especially in quantum graph theory [16].

After Koshliakov, many mathematicians studied his formula (1.4). In 1936, Ferrar (17] reproved (1.4) by appealing to the functional equation of $\zeta(s)$. Later in 1966, K. Soni [18] showed that the functional equation of $\zeta^{2}(s)$ is equivalent to the Voronoï summation formula (1.6) and is equivalent to Koshliakov's formula (1.4). In 1972, Oberhettinger and Soni [19] established that the functional equation of $\zeta(s)$ and Koshliakov's formula are equivalent using the methods of Hamburger. In 2008, B. C. Berndt, Y. Lee, and J. Sohn [20] proved (1.1) by elaborating Guinand's method. They rediscovered Koshliakov's formula (1.4) by taking $\nu \rightarrow 0$ in (1.1). However, A. Dixit in [21] gave an extended version of Ramanujan's formula (1.1) by appealing to the Cauchy residue theorem and the theory of the Mellin transform. Further analysis of identities analogous to (1.1) and (1.4) have been done by B. C. Berndt, S. Kim and A. Zaharescu in [22]. They studied character analogues of Koshliakov's formula (1.4) for even characters. They replaced the classical divisor function $d(n)$ with the twisted divisor sums, namely,

$$
\begin{equation*}
d_{\chi}(n)=\sum_{d \mid n} \chi(d), \quad d_{\chi_{1}, \chi_{2}}(n)=\sum_{d \mid n} \chi_{1}(d) \chi_{2}(n / d) \tag{1.10}
\end{equation*}
$$

where $\chi, \chi_{1}$ and $\chi_{2}$ are the Dirichlet characters, and they proved the following beautiful identity

$$
\frac{q L(1, \chi)}{4 \tau(\chi)}+\sum_{n=1}^{\infty} d_{\chi}(n) K_{0}\left(\frac{2 \pi n z}{\sqrt{q}}\right)=\frac{\sqrt{q} L(1, \chi)}{4 z}+\frac{\tau(\chi)}{z \sqrt{q}} \sum_{n=1}^{\infty} d_{\bar{\chi}}(n) K_{0}\left(\frac{2 \pi n}{z \sqrt{q}}\right)
$$

where $\chi$ is a non-principal even primitive character $\bmod q, \Re(z)>0$, and $\tau(\chi)$ is the Gauss sum defined in (2.2), and $K_{0}(z)$ is defined in (1.5). In particular, for even real character $\chi$, they established the positivity of $L(1, \chi)$, which is instrumental in proving Dirichlet's theorem on primes in arithmetic progressions. The weighted divisor sums defined in (1.10) were introduced by B. C. Berndt and A. Zaharescu [1], where they showed that the twisted or weighted divisor sums could be studied in connection with identities associated with $r_{2}(n)$. However, S. Kim [2] extended the definition of twisted divisor sums to twisted sums of divisor functions, namely,

$$
\begin{equation*}
\sigma_{k, \chi}(n):=\sum_{d \mid n} d^{k} \chi(d), \quad \bar{\sigma}_{k, \chi}(n):=\sum_{d \mid n} d^{k} \chi(n / d), \quad \sigma_{k, \chi_{1}, \chi_{2}}(n):=\sum_{d \mid n} d^{k} \chi_{1}(d) \chi_{2}(n / d), \tag{1.11}
\end{equation*}
$$

and they studied Riesz sum-type identities associated with them. Recently A. Dixit and A. Kesarwani 23] studied a new generalization of the modified Bessel function of the second kind. They derived a formula analogous to (1.1) associated with the generalized Bessel function. They proved that their formula is equivalent to the functional equation of a non-holomorphic Eisenstein series on $S L(2, \mathbb{Z})$.

The study of the infinite series in (1.1) is of prime importance as it is intimately connected with the Fourier series expansion of non-holomorphic Eisenstein series on $S L(2, \mathbb{Z})$ or Maass wave forms [24, 25, 26, 27]. Motivated by this fact, Cohen, in 2010 [28], established the following result, similar to (1.1),

$$
\begin{align*}
4 x^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\sigma_{\nu}(n)}{n^{\nu / 2}} K_{\nu / 2}(2 \pi n x)+\Lambda(s)\left(x^{(1-\nu) / 2}-x^{(\nu-1) / 2}\right) & =4 x^{-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\sigma_{\nu}(n)}{n^{\nu / 2}} K_{\nu / 2}\left(\frac{2 \pi n}{x}\right) \\
& +\Lambda(-s)\left(x^{-(1+\nu) / 2}-x^{(1+\nu) / 2}\right) \tag{1.12}
\end{align*}
$$

where $\Lambda(s)=\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ and $K_{\nu}(z)$ is defined in (1.2). As an application, he obtained the following beautiful identity involving the divisor function $\sigma_{s}(n)$ and the modified $K$-Bessel function.
Proposition 1.1. [2d, p. 62, Theorem 3.4] For $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$ and any integer $N$ such that $N \geq\left\lfloor\frac{\Re(\nu)+1}{2}\right\rfloor$, then

$$
\begin{align*}
8 \pi x^{\frac{\nu}{2}} \sum_{n=1}^{\infty} \sigma_{-\nu}(n) n^{\nu / 2} K_{\nu}(4 \pi \sqrt{n x})= & -\frac{\Gamma(\nu) \zeta(\nu)}{(2 \pi)^{\nu-1}}+\frac{\Gamma(1+\nu) \zeta(1+\nu)}{\pi^{\nu+1} 2^{\nu} x}+\left\{\frac{\zeta(\nu) x^{\nu-1}}{\sin \left(\frac{\pi \nu}{2}\right)}+\frac{2}{\sin \left(\frac{\pi \nu}{2}\right)} \sum_{j=1}^{N} \zeta(2 j) \zeta(2 j-\nu) x^{2 j-1}\right. \\
& \left.-\pi \frac{\zeta(\nu+1) x^{\nu}}{\cos \left(\frac{\pi \nu}{2}\right)}+\frac{2}{\sin \left(\frac{\pi \nu}{2}\right)} \sum_{n=1}^{\infty} \sigma_{-\nu}(n) \frac{x^{2 N+1}}{\left(n^{2}-x^{2}\right)}\left(n^{\nu-2 N}-x^{\nu-2 N}\right)\right\} . \tag{1.13}
\end{align*}
$$

In addition to (1.13), he derived several interesting identities involving the divisor function $\sigma_{s}(n)$ and the modified $K$-Bessel function. Later, B. C. Berndt, A. Dixit, A. Roy, and A. Zaharescu [29], in their seminal work, showed that Cohen-type identity (1.13) can be used to derive the Voronoï-type summation formula for $\sigma_{s}(n)$.

Proposition 1.2. [29, p. 841, Theorem 6.1] Let $0<\alpha<\beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let $f$ denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that $-\frac{1}{2}<\Re(\nu)<\frac{1}{2}$. Then

$$
\begin{aligned}
& \sum_{\alpha<j<\beta} \sigma_{-\nu}(j) f(j)=\int_{\alpha}^{\beta} f(t)\left(\zeta(1-\nu, \chi) t^{-\nu}+\zeta(\nu+1)\right) d t \\
& +2 \pi \sum_{n=1}^{\infty} \sigma_{-\nu}(n) n^{\nu / 2} \int_{\alpha}^{\beta} f(t)(t)^{-\frac{\nu}{2}}\left\{\left(\frac{2}{\pi} K_{\nu}(4 \pi \sqrt{n t})-Y_{\nu}(4 \pi \sqrt{n t})\right) \cos \left(\frac{\pi \nu}{2}\right)-J_{\nu}(4 \pi \sqrt{n t}) \sin \left(\frac{\pi \nu}{2}\right)\right\} d t
\end{aligned}
$$

Inspired by Cohen's results [28], the first author and B. Maji [3] studied the infinite series involving the generalised divisor function and the modified $K$-Bessel functions. More precisely, they studied the following infinite series, for $r \in \mathbb{Z}, z \in \mathbb{C}$ and $a$ and $x$ be any two positive real numbers,

$$
\begin{equation*}
\sum_{n=1}^{\infty} \sigma_{z}^{(r)}(n) n^{\frac{\nu}{2}} K_{\nu}(a \sqrt{n x}) \tag{1.14}
\end{equation*}
$$

where $\sigma_{z}^{(r)}(n)=\sum_{d^{r} \mid n} d^{z}$ and $\nu$ is a complex number with $\Re(\nu) \geq 0$. It is important to note that $\sigma_{z}^{(1)}(n)=\sigma_{z}(n)$. Hence almost all the Cohen-type identities can be derived from their results. In this article, we are interested in the character analogues of (1.14). That is, we study the following infinite series

$$
\begin{equation*}
\sum_{n=1}^{\infty} \sigma_{z, \chi}(n) n^{\frac{\nu}{2}} K_{\nu}(a \sqrt{n x}), \quad \sum_{n=1}^{\infty} \bar{\sigma}_{z, \chi}(n) n^{\frac{\nu}{2}} K_{\nu}(a \sqrt{n x}), \quad \sum_{n=1}^{\infty} \sigma_{z, \chi_{1}, \chi_{2}}(n) n^{\frac{\nu}{2}} K_{\nu}(a \sqrt{n x}), \tag{1.15}
\end{equation*}
$$

where $\sigma_{z, \chi}(n), \bar{\sigma}_{z, \chi}(n)$ and $\sigma_{z, \chi_{1}, \chi_{2}}(n)$ are defined in (1.11), and $\nu$ is a complex number with $\Re(\nu) \geq 0$. We derive Cohen-type identities for twisted sums of divisor functions $\sigma_{z, \chi}(n), \bar{\sigma}_{z, \chi}(n)$ and $\sigma_{z, \chi_{1}, \chi_{2}}(n)$ and obtain the Voronoï-type summation formula for them. The paper is organized as follows: Section 2 states the results for the twisted sums of divisor functions when $z \in \mathbb{Z}$. Section 3 provides Cohen-type identities for them. Section 4 states the Voronoï-type summation formula for twisted sums of divisor functions. Section 5 reviews several significant results needed to derive our main results. Sections 6,7 and 8 are devoted to the proofs of identities stated in Sections 2, 3 and 4, respectively.

## 2. Main Results for the case $z \in \mathbb{Z}_{\geq 0}$

In this section, we consider $z$ a non-negative integer and denote it by $k$. Throughout the paper, we assume that $a$ and $x$ are two strictly positive real numbers, which differ from 0 . Before proceeding further, we will mention some definitions and notations which will be used later.

The Dirichlet $L$-function is defined by

$$
\begin{equation*}
L(s, \chi):=\sum_{n=1}^{\infty} \frac{\chi(n)}{n^{s}}, \quad \Re(s)>1 \tag{2.1}
\end{equation*}
$$

where $\chi$ is a Dirichlet character modulo $q$. It can be meromorphically continued to the entire complex plane. Furthermore, if $\chi$ is principal, the corresponding Dirichlet $L$-function has a simple pole at $s=1$. Otherwise, the $L$-function is entire.

The Gauss sum of a Dirichlet character modulo $q$ is

$$
\begin{equation*}
\tau(\chi):=\sum_{h=1}^{q} \chi(h) e^{2 \pi i h / q} \tag{2.2}
\end{equation*}
$$

Since our results involve the modified $K$-Bessel function, it is important to state some related results. The asymptotic estimate for the $K$-Bessel function defined in (1.2) is [4, p. 202]

$$
K_{\nu}(x)=\left(\frac{\pi}{2 x}\right)^{\frac{1}{2}} e^{-x}+O\left(\frac{e^{-x}}{x^{\frac{3}{2}}}\right) \text { as } x \rightarrow \infty
$$

The above expression ensures the absolute convergence of all the infinite series defined in (1.15). Throughout this paper, we will consider $\Re(\nu) \geq 0$ as $K_{-\nu}(x)=K_{\nu}(x)$. We recall that $K_{0}(x)$ is defined by (1.5). From the integral representation of $K_{0}(x)$ [4, p. 446]

$$
K_{0}(x)=\int_{0}^{\infty} e^{-x \cosh t} d t
$$

one can see that $K_{0}(x)$ is positive and monotonically decreasing on the interval $(0, \infty)$. We also note the series representation of $K_{0}(x)$ [4, p. 80]

$$
K_{0}(x)=-\log \left(\frac{x}{2}\right) I_{0}(x)+\sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2 m}}{(m!)^{2}} \frac{\Gamma^{\prime}(m+1)}{\Gamma(m+1)}
$$

where $I_{0}(x)$ is defined in (1.3). From its series representation mentioned above, one can infer that $K_{0}(x)$ tends to $+\infty$ as $x$ decreases to 0 .
2.1. Identities involving odd characters. In this subsection, we will consider $k$ to be an even, non-negative integer and $\chi$ an odd primitive character.

Theorem 2.1. Let $k$ be an even, non-negative integer and $\chi$ be an odd primitive Dirichlet character modulo $q$. Then, for any $\Re(\nu)>0$,

$$
\begin{aligned}
\sum_{n=1}^{\infty} \sigma_{k, \chi}(n) n^{\frac{\nu}{2}} K_{\nu}(a \sqrt{n x})= & \delta_{k} \frac{2^{\nu+1}}{a^{\nu+2}} \Gamma(1+\nu) L(1, \chi) x^{-\frac{\nu}{2}-1}+\frac{(-1)^{\frac{k}{2}} i q^{k}}{a^{\nu} 2^{k+2-\nu} \pi^{k+1}} \Gamma(\nu) \tau(\chi) \Gamma(k+1) L(k+1, \bar{\chi}) x^{-\frac{\nu}{2}} \\
& -\frac{(-1)^{\frac{k}{2}} i a^{\nu} q^{\nu+k} x^{\frac{\nu}{2}}}{2^{3 \nu+k+2} \pi^{2 \nu+k+1}} \Gamma(\nu+k+1) \tau(\chi) \sum_{n=1}^{\infty} \frac{\bar{\sigma}_{k, \bar{\chi}}(n)}{\left(n+\frac{a^{2} q x}{16 \pi^{2}}\right)^{\nu+k+1}}
\end{aligned}
$$

where $\delta_{k}$ is given by

$$
\delta_{k}= \begin{cases}1, & \text { if } k=0  \tag{2.3}\\ 0, & \text { if } k>0\end{cases}
$$

Our next result corresponds to $\nu=0$ is as follows

Theorem 2.2. Let $k$ be an even, non-negative integer and $\chi$ be an odd primitive Dirichlet character modulo q. Then

$$
\begin{align*}
\sum_{n=1}^{\infty} \sigma_{k, \chi}(n) K_{0}(a \sqrt{n x})= & \delta_{k} \frac{2}{a^{2} x} L(1, \chi)-\frac{L(-k, \chi)}{4}\left(\log \left(\frac{8 \pi}{a^{2}}\right)+\frac{L^{\prime}(-k, \chi)}{L(-k, \chi)}-2 \gamma\right)+\frac{L(-k, \chi)}{4} \log x \\
& +(-1)^{\frac{k}{2}} \frac{i k!q^{k}}{2(2 \pi)^{k+1}} \tau(\chi) \sum_{n=1}^{\infty} \bar{\sigma}_{k, \bar{\chi}}(n)\left(\frac{1}{n^{k+1}}-\frac{1}{\left(n+\frac{a^{2} q x}{16 \pi^{2}}\right)^{k+1}}\right) \tag{2.4}
\end{align*}
$$

where $\delta_{k}$ is defined in (2.3).
Remark 1. Let us assume that $\chi$ is a real odd primitive Dirichlet character modulo $q$. Now setting $k=0$ and then employing the functional equation (5.16) in (2.4), we obtain

$$
\begin{align*}
\sum_{n=1}^{\infty} d_{\chi}(n) K_{0}(a \sqrt{n x})= & \frac{L(1, \chi)}{x}\left(\frac{2}{a^{2}}-\frac{i \tau(\chi)}{4 \pi} x \log x\right)-\frac{L(0, \chi)}{4}\left(\log \left(\frac{8 \pi}{a^{2}}\right)+\frac{L^{\prime}(0, \chi)}{L(0, \chi)}-2 \gamma\right) \\
& +\frac{i a^{2} q x}{64 \pi^{3}} \tau(\chi) \sum_{n=1}^{\infty} \frac{d_{\bar{\chi}}(n)}{n\left(n+\frac{a^{2} q x}{16 \pi^{2}}\right)} \tag{2.5}
\end{align*}
$$

Now we can easily show that $d_{\chi}(n)$ is non-negative for each $n$ from the Euler product on the left-hand side of (5.10). More precisely, the factors in its Euler product are of the forms

$$
\left(1-\frac{1}{p^{s}}\right)^{-1},\left(1-\frac{1}{p^{s}}\right)^{-2}, \text { or }\left(1-\frac{1}{p^{2 s}}\right)^{-1}
$$

according as to whether $\chi(p)=0,1$ or -1 respectively. Therefore, by rewriting the Euler product as a Dirichlet series, one can easily notice that $d_{\chi}(n) \geq 0$ for all $n$. In addition, it is clear from (1.10) that $d_{\chi}(n) \geq 1$ whenever $n$ is a perfect square. We have already mentioned the fact that $K_{0}(x)$ tends to $+\infty$ as $x$ decreases to 0 at the beginning of this section. Therefore, the left-hand side of (2.5) approaches $+\infty$ as $x$ decreases to 0 . Let us examine the right-hand side of (2.5). Noting that i $\tau(\chi)$ is real for real odd primitive Dirichlet character [30, Theorem 9.9, p. 288], we can easily deduce that the infinite series on the right-hand side of (2.5) tends to 0 as $x$ decreases to 0 . Next noting that $i \tau(\chi)$ is real and $x \log x$ tends to 0 as $x$ decreases to 0 , we infer that $\frac{L(1, \chi)}{x}$ tends to $+\infty$ as $x$ decreases to 0 , which ensures the strict positivity of $L(1, \chi)$.

Theorem 2.3. Let $k \geq 2$ be an even integer and $\chi$ be an odd primitive Dirichlet character modulo $q$. Then, for any $\Re(\nu)>0$,

$$
\begin{aligned}
\sum_{n=1}^{\infty} \bar{\sigma}_{k, \chi}(n) n^{\frac{\nu}{2}} K_{\nu}(a \sqrt{n x})= & \frac{2^{\nu+2 k+1}}{a^{\nu+2 k+2}} \Gamma(k+1) \Gamma(\nu+k+1) L(1+k, \chi) x^{-\frac{\nu}{2}-k-1} \\
& -\frac{(-1)^{\frac{k}{2}} i(a q)^{\nu} x^{\frac{\nu}{2}}}{2^{3 \nu+k+2} \pi^{2 \nu+k+1}} \Gamma(\nu+k+1) \tau(\chi) \sum_{n=1}^{\infty} \frac{\sigma_{k, \bar{\chi}}(n)}{\left(n+\frac{a^{2} q x}{16 \pi^{2}}\right)^{\nu+k+1}}
\end{aligned}
$$

The result corresponding to $\nu=0$ is as follows
Theorem 2.4. Let $k \geq 2$ be an even integer and $\chi$ be an odd primitive Dirichlet character modulo $q$. Then

$$
\begin{aligned}
\sum_{n=1}^{\infty} \bar{\sigma}_{k, \chi}(n) K_{0}(a \sqrt{n x})= & \frac{2^{2 k+1}}{a^{2 k+2}} \Gamma^{2}(k+1) L(k+1, \chi) \frac{1}{x^{k+1}}+\frac{1}{2} \zeta^{\prime}(-k) L(0, \chi) \\
& +\frac{(-1)^{\frac{k}{2}} i k!\tau(\chi)}{2(2 \pi)^{k+1}} \sum_{n=1}^{\infty} \sigma_{k, \bar{\chi}}(n)\left(\frac{1}{n^{k+1}}-\frac{1}{\left(n+\frac{a^{2} q x}{16 \pi^{2}}\right)^{k+1}}\right)
\end{aligned}
$$

Remark 2. The case $k=0$ is excluded from Theorem 2.3 and Theorem 2.4 because of the fact that $\bar{\sigma}_{0, \chi}(n)=$ $\sigma_{0, \chi}(n)=d_{\chi}(n)$ and $\sum_{n=1}^{\infty} d_{\chi}(n) n^{\nu / 2} K_{\nu}(a \sqrt{n x})$ for $\Re(\nu)>0$ and $\sum_{n=1}^{\infty} d_{\chi}(n) K_{0}(a \sqrt{n x})$ are already considered in Theorems 2.1] and 2.2, respectively.
2.2. Identities involving even characters. In this subsection, we present similar results when $k$ is an odd positive integer and $\chi$ is a non-principal even primitive character.
Theorem 2.5. Let $k \geq 1$ be an odd integer and $\chi$ be a non-principal even primitive Dirichlet character modulo q. Then, for any $\Re(\nu)>0$,

$$
\begin{aligned}
\sum_{n=1}^{\infty} \sigma_{k, \chi}(n) n^{\frac{\nu}{2}} K_{\nu}(a \sqrt{n x})= & \frac{(-1)^{\frac{k-1}{2}} q^{k}}{a^{\nu} 2^{k+2-\nu} \pi^{k+1}} \Gamma(\nu) \tau(\chi) \Gamma(k+1) L(1+k, \bar{\chi}) x^{-\frac{\nu}{2}} \\
& +\frac{(-1)^{\frac{k+1}{2}} a^{\nu} q^{\nu+k} x^{\frac{\nu}{2}}}{2^{3 \nu+k+2} \pi^{2 \nu+k+1}} \Gamma(\nu+k+1) \tau(\chi) \sum_{n=1}^{\infty} \frac{\sigma_{k, \bar{\chi}}(n)}{\left(n+\frac{a^{2} q x}{16 \pi^{2}}\right)^{\nu+k+1}} .
\end{aligned}
$$

The result corresponding to $\nu=0$ is as follows
Theorem 2.6. Let $k \geq 1$ be an odd integer and $\chi$ be a non-principal even primitive Dirichlet character modulo q. Then

$$
\begin{aligned}
\sum_{n=1}^{\infty} \sigma_{k, \chi}(n) K_{0}(a \sqrt{n x})= & -\frac{L(-k, \chi)}{4}\left(\log \left(\frac{8 \pi}{a^{2}}\right)+\frac{L^{\prime}(-k, \chi)}{L(-k, \chi)}-2 \gamma\right)+\frac{L(-k, \chi)}{4} \log x \\
& +(-1)^{\frac{k-1}{2}} \frac{k!q^{k}}{2(2 \pi)^{k+1}} \tau(\chi) \sum_{n=1}^{\infty} \bar{\sigma}_{k, \bar{\chi}}(n)\left(\frac{1}{n^{k+1}}-\frac{1}{\left(n+\frac{a^{2} q x}{16 \pi^{2}}\right)^{k+1}}\right)
\end{aligned}
$$

Theorem 2.7. Let $k \geq 1$ be an odd integer and $\chi$ be a non-principal even primitive Dirichlet character modulo q. Then, for any $\Re(\nu)>0$,

$$
\begin{aligned}
\sum_{n=1}^{\infty} \bar{\sigma}_{k, \chi}(n) n^{\frac{\nu}{2}} K_{\nu}(a \sqrt{n x})= & \frac{2^{\nu+2 k+1}}{a^{\nu+2 k+2}} \Gamma(k+1) \Gamma(\nu+k+1) L(1+k, \chi) x^{-\frac{\nu}{2}-k-1} \\
& +\frac{(-1)^{\frac{k+1}{2}}(a q)^{\nu} x^{\frac{\nu}{2}}}{2^{3 \nu+k+2} \pi^{2 \nu+k+1}} \Gamma(\nu+k+1) \tau(\chi) \sum_{n=1}^{\infty} \sigma_{k, \bar{\chi}}(n) \frac{1}{\left(n+\frac{a^{2} q x}{16 \pi^{2}}\right)^{\nu+k+1}}
\end{aligned}
$$

The result corresponding to $\nu=0$ is as follows
Theorem 2.8. Let $k \geq 1$ be an odd integer and $\chi$ be a non-principal even primitive Dirichlet character modulo q. Then

$$
\begin{aligned}
\sum_{n=1}^{\infty} \bar{\sigma}_{k, \chi}(n) K_{0}(a \sqrt{n x})= & \frac{2^{2 k+1}}{a^{2 k+2}} \Gamma^{2}(k+1) L(k+1, \chi) \frac{1}{x^{k+1}}+\frac{1}{2} \zeta^{\prime}(-k) L(0, \chi) \\
& +\frac{(-1)^{\frac{k-1}{2}} k!}{2(2 \pi)^{k+1}} \tau(\chi) \sum_{n=1}^{\infty} \sigma_{k, \bar{\chi}}(n)\left(\frac{1}{n^{k+1}}-\frac{1}{\left(n+\frac{a^{2} q x}{16 \pi^{2}}\right)^{k+1}}\right)
\end{aligned}
$$

The next result corresponds to the case $\nu=0$ and $k=0$. We can also claim the positivity of $L(1, \chi)$ for even real character $\chi$ from the following identity.
Theorem 2.9. Let $\chi$ be a non-principal even primitive Dirichlet character modulo $q$. Then we have

$$
\begin{equation*}
\sum_{n=1}^{\infty} d_{\chi}(n) K_{0}(a \sqrt{n x})=\frac{2}{a^{2} x} L(1, \chi)-\frac{\tau(\chi)}{8} L(1, \bar{\chi})+\frac{a^{2} q x}{32 \pi^{4}} \tau(\chi) \sum_{n=1}^{\infty} d_{\bar{\chi}}(n) \frac{\log \left(\frac{16 \pi^{2} n}{a^{2} q x}\right)}{n^{2}-\left(\frac{a^{2} q x}{16 \pi^{2}}\right)^{2}} \tag{2.6}
\end{equation*}
$$

provided $\frac{a^{2} q x}{16 \pi^{2}} \notin \mathbb{Z}_{+}$.
Remark 3. When $\chi$ is any real even primitive Dirichlet character modulo $q$, we can show that $d_{\chi}(n)$ is nonnegative for each $n$ by similar arguments given in Remark 1. From (1.10), it can be easily seen that $d_{\chi}(n) \geq 1$ whenever $n$ is a perfect square. As $K_{0}(x)$ tends to $+\infty$ as $x$ decreases to 0 , the left-hand side of (2.6) approaches $+\infty$ as $x$ decreases to 0 . Now the infinite series in the right-hand side of (2.6) decreases rapidly as $x$ decreases to 0 . Therefore, we arrive at the conclusion that $\frac{2}{a^{2} x} L(1, \chi)$ tends to $+\infty$ as $x$ decreases to 0 which proves the strict positivity of $L(1, \chi)$.
2.3. Identities involving two characters. In this subsection, we provide the identities corresponding to $\sigma_{k, \chi_{1}, \chi_{2}}(n)=\sum_{d / n} d^{k} \chi_{1}(d) \chi_{2}(n / d)$, where $\chi_{1}$ and $\chi_{2}$ are Dirichlet characters modulo $p$ and $q$, respectively.
Theorem 2.10. Let $k \geq 1$ be an odd integer. Let $\chi_{1}$ and $\chi_{2}$ be primitive characters modulo $p$ and $q$, respectively, such that either both are non-principal even characters or both are odd characters. Then, for any $\Re(\nu)>0$,

$$
\sum_{n=1}^{\infty} \sigma_{k, \chi_{1}, \chi_{2}}(n) n^{\frac{\nu}{2}} K_{\nu}(a \sqrt{n x})=\frac{(-1)^{\frac{k+1}{2}}(a q)^{\nu} p^{\nu+k} x^{\frac{\nu}{2}}}{2^{3 \nu+k+2} \pi^{2 \nu+k+1}} \tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right) \Gamma(\nu+k+1) \sum_{n=1}^{\infty} \frac{\sigma_{k, \bar{\chi}_{2}, \chi_{1}}(n)}{\left(n+\frac{a^{2} p q x}{16 \pi^{2}}\right)^{\nu+k+1}}
$$

The result corresponding to $\nu=0$ is as follows
Theorem 2.11. Let $k \geq 1$ be an odd integer. Assume that $\chi_{1}$ and $\chi_{2}$ are primitive characters modulo $p$ and $q$, respectively, such that either both are non-principal even characters or both are odd characters, then

$$
\sum_{n=1}^{\infty} \sigma_{k, \chi_{1}, \chi_{2}}(n) K_{0}(a \sqrt{n x})=\frac{1}{2} c_{k, \chi_{1}, \chi_{2}}+\frac{(-1)^{\frac{k-1}{2}} k!p^{k}}{2(2 \pi)^{k+1}} \tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right) \sum_{n=1}^{\infty} \sigma_{k, \bar{\chi}_{2}, \bar{\chi}_{1}}(n)\left(\frac{1}{n^{k+1}}-\frac{1}{\left(n+\frac{a^{2} p q x}{16 \pi^{2}}\right)^{k+1}}\right)
$$

where $c_{k, \chi_{1}, \chi_{2}}$ is a constant defined as

$$
c_{k, \chi_{1}, \chi_{2}}= \begin{cases}L\left(-k, \chi_{1}\right) L^{\prime}\left(0, \chi_{2}\right), & \text { if both } \chi_{1} \text { and } \chi_{2} \text { are even },  \tag{2.7}\\ L^{\prime}\left(-k, \chi_{1}\right) L\left(0, \chi_{2}\right), & \text { if both } \chi_{1} \text { and } \chi_{2} \text { are odd. }\end{cases}
$$

Setting $\chi_{1}=\chi_{2}=\chi$ and observing $\sigma_{k, \chi, \chi}(n)=\chi(n) \sum_{d / n} d^{k}=\chi(n) \sigma_{k}(n)$ in Theorems 2.10 and 2.11, we obtain the following interesting identities.

Corollary 2.12. Let $k \geq 1$ be an odd integer and $\chi$ be a non-principal primitive character modulo $q$. Then, for any $\Re(\nu)>0$,

$$
\sum_{n=1}^{\infty} \sigma_{k}(n) \chi(n) n^{\frac{\nu}{2}} K_{\nu}(a \sqrt{n x})=\frac{(-1)^{\frac{k+1}{2}} a^{\nu} q^{2 \nu+k} x^{\frac{\nu}{2}}}{2^{3 \nu+k+2} \pi^{2 \nu+k+1}} \tau^{2}(\chi) \Gamma(\nu+k+1) \sum_{n=1}^{\infty} \frac{\sigma_{k}(n) \bar{\chi}(n)}{\left(n+\frac{a^{2} q^{2} x}{16 \pi^{2}}\right)^{\nu+k+1}}
$$

Corollary 2.13. Let $k \geq 1$ be an odd integer and $\chi$ be a non-principal primitive character modulo $q$. For $\nu=0$, we have

$$
\sum_{n=1}^{\infty} \sigma_{k}(n) \chi(n) K_{0}(a \sqrt{n x})=\frac{1}{2} c_{k, \chi, \chi}+\frac{(-1)^{\frac{k-1}{2}} k!q^{k}}{2(2 \pi)^{k+1}} \tau^{2}(\chi) \sum_{n=1}^{\infty} \sigma_{k}(n) \bar{\chi}(n)\left(\frac{1}{n^{k+1}}-\frac{1}{\left(n+\frac{a^{2} q^{2} x}{16 \pi^{2}}\right)^{k+1}}\right)
$$

where $c_{k, \chi, \chi}$ is defined in (2.7).
The results corresponding to $\nu=0$ and $k=0$ are as follows
Theorem 2.14. Let $\chi_{1}$ and $\chi_{2}$ be non-principal even primitive characters modulo $p$ and $q$, respectively. Then

$$
\sum_{n=1}^{\infty} d_{\chi_{1}, \chi_{2}}(n) K_{0}(a \sqrt{n x})=\frac{a^{2} p q x}{32 \pi^{4}} \tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right) \sum_{n=1}^{\infty} d_{\bar{\chi}_{1}, \bar{\chi}_{2}}(n) \frac{\log \left(\frac{16 \pi^{2} n}{a^{2} p q x}\right)}{n^{2}-\left(\frac{a^{2} p q x}{16 \pi^{2}}\right)^{2}}
$$

provided $\frac{a^{2} p q x}{16 \pi^{2}} \notin \mathbb{Z}_{+}$.
Theorem 2.15. Let $\chi_{1}$ and $\chi_{2}$ be odd primitive characters modulo $p$ and $q$, respectively. Then we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} d_{\chi_{1}, \chi_{2}}(n) K_{0}(a \sqrt{n x})= & \frac{1}{2} L\left(0, \chi_{1}\right) L\left(0, \chi_{2}\right)\left(-2 \gamma+\log \left(\frac{4}{a^{2} x}\right)+\frac{L^{\prime}\left(0, \chi_{1}\right)}{L\left(0, \chi_{1}\right)}+\frac{L^{\prime}\left(0, \chi_{2}\right)}{L\left(0, \chi_{2}\right)}\right) \\
& +\frac{a^{4} p^{2} q^{2}}{512 \pi^{4}} x^{2} \tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right) \sum_{n=1}^{\infty} \frac{d_{\bar{\chi}_{1}, \bar{\chi}_{2}}(n) \log \left(\frac{a^{2} p q x}{16 \pi^{2} n}\right)}{n\left(n^{2}-\left(\frac{a^{2} p q x}{16 \pi^{2}}\right)^{2}\right)},
\end{aligned}
$$

provided $\frac{a^{2} p q x}{16 \pi^{2}} \notin \mathbb{Z}_{+}$.

Theorem 2.16. Let $k$ be an even, non-negative integer. Assume that $\chi_{1}$ and $\chi_{2}$ are primitive characters modulo $p$ and $q$, respectively, such that one is a non-principal even character and the other is an odd character. Then, for any $\Re(\nu)>0$,

$$
\sum_{n=1}^{\infty} \sigma_{k, \chi_{1}, \chi_{2}}(n) n^{\frac{\nu}{2}} K_{\nu}(a \sqrt{n x})=\frac{(-1)^{\frac{k}{2}}(a q)^{\nu} p^{\nu+k} x^{\frac{\nu}{2}}}{i 2^{3 \nu+k+2} \pi^{2 \nu+k+1}} \tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right) \Gamma(\nu+k+1) \sum_{n=1}^{\infty} \frac{\sigma_{k, \chi_{2}, \overline{\chi_{1}}}(n)}{\left(n+\frac{a^{2} p q x}{16 \pi^{2}}\right)^{\nu+k+1}} .
$$

The result corresponding to $\nu=0$ is as follows
Theorem 2.17. Let $k$ be an even, non-negative integer. If $\chi_{1}$ and $\chi_{2}$ are primitive characters modulo $p$ and $q$, respectively, such that one is a non-principal even character and the other is an odd character, then

$$
\sum_{n=1}^{\infty} \sigma_{k, \chi_{1}, \chi_{2}}(n) K_{0}(a \sqrt{n x})=\frac{1}{2} e_{k, \chi_{1}, \chi_{2}}+(-1)^{\frac{k}{2}} \frac{i k!p^{k}}{2(2 \pi)^{k+1}} \tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right) \sum_{n=1}^{\infty} \sigma_{k, \bar{\chi}_{2}, \bar{\chi}_{1}}(n)\left(\frac{1}{n^{k+1}}-\frac{1}{\left(n+\frac{a^{2} p q x}{16 \pi^{2}}\right)^{k+1}}\right)
$$

where

$$
e_{k, \chi_{1}, \chi_{2}}= \begin{cases}L\left(-k, \chi_{1}\right) L^{\prime}\left(0, \chi_{2}\right), & \text { if } \chi_{1} \text { is odd and } \chi_{2} \text { is even },  \tag{2.8}\\ L^{\prime}\left(-k, \chi_{1}\right) L\left(0, \chi_{2}\right), & \text { if } \chi_{1} \text { is even and } \chi_{2} \text { is odd. }\end{cases}
$$

## 3. Cohen-Type Identities

This section deals with $z=-\nu$ with $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. We will assume that $x$ is a strictly positive real number.
3.1. Identities involving even characters and specializations. In this subsection, we present the identities associated with $\sigma_{-\nu, \bar{\chi}}(n)$ and $\bar{\sigma}_{-\nu, \bar{\chi}}(n)$ when $\chi$ is a non-principal even primitive character.
Theorem 3.1. Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Let $\chi$ be a non-principal even primitive character modulo $q$. If $N$ is any integer such that $N \geq\left\lfloor\frac{\Re(\nu)+1}{2}\right\rfloor$, then

$$
\begin{align*}
& 8 \pi x^{\nu / 2} \sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}}(n) n^{\nu / 2} K_{\nu}(4 \pi \sqrt{n x})=-\frac{\Gamma(\nu) L(\nu, \bar{\chi})}{(2 \pi)^{\nu-1}}+\frac{2 \Gamma(1+\nu) L(1+\nu, \bar{\chi})}{(2 \pi)^{\nu+1}} x^{-1} \\
& \quad+\frac{2 q^{1-\nu}}{\tau(\chi) \sin \left(\frac{\pi \nu}{2}\right)}\left\{\sum_{j=1}^{N} \zeta(2 j) L(2 j-\nu, \chi)(q x)^{2 j-1}+(q x)^{2 N+1} \sum_{n=1}^{\infty} \bar{\sigma}_{-\nu, \chi}(n)\left(\frac{n^{\nu-2 N}-(q x)^{\nu-2 N}}{n^{2}-(q x)^{2}}\right)\right\}, \tag{3.1}
\end{align*}
$$

provided $q x \notin \mathbb{Z}_{+}$.
The specialization of the above theorem to $\nu=1 / 2$ is as follows
Corollary 3.2. We have

$$
2 \pi \sum_{n=1}^{\infty} \sigma_{-\frac{1}{2}, \bar{\chi}}(n) e^{-4 \pi \sqrt{n x}}=-\pi L(1 / 2, \bar{\chi})+\frac{1}{4 \pi} L(3 / 2, \bar{\chi}) x^{-1}+\frac{2 q^{3 / 2}}{\tau(\chi)} x \sum_{n=1}^{\infty} \bar{\sigma}_{-\frac{1}{2}, \chi}(n) \frac{1}{(n+q x)(\sqrt{n}+\sqrt{q x})} .
$$

Theorem 3.3. Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Let $\chi$ be a non-principal even primitive character modulo $q$. If $N$ is any integer such that $N \geq\left\lfloor\frac{\Re(\nu)+1}{2}\right\rfloor$, then

$$
\begin{aligned}
& 8 \pi x^{\nu / 2} \sum_{n=1}^{\infty} \bar{\sigma}_{-\nu, \bar{\chi}}(n) n^{\nu / 2} K_{\nu}(4 \pi \sqrt{n x})=\frac{q}{\tau(\chi)}\left\{\frac{L(\nu, \chi)}{\sin \left(\frac{\pi \nu}{2}\right)}(q x)^{\nu-1}-\frac{\pi L(1+\nu, \chi)}{\cos \left(\frac{\pi \nu}{2}\right)}(q x)^{\nu}\right. \\
& \left.\quad+\frac{2}{\sin \left(\frac{\pi \nu}{2}\right)} \sum_{j=1}^{N} \zeta(2 j-\nu) L(2 j, \chi)(q x)^{2 j-1}+\frac{2}{\sin \left(\frac{\pi \nu}{2}\right)}(q x)^{2 N+1} \sum_{n=1}^{\infty} \sigma_{-\nu, \chi}(n)\left(\frac{n^{\nu-2 N}-(q x)^{\nu-2 N}}{n^{2}-(q x)^{2}}\right)\right\},
\end{aligned}
$$

provided $q x \notin \mathbb{Z}_{+}$.
The result corresponding to $\nu=1 / 2$ is as follows

Corollary 3.4. We have

$$
2 \pi \sum_{n=1}^{\infty} \bar{\sigma}_{-\frac{1}{2}, \bar{\chi}}(n) e^{-4 \pi \sqrt{n x}}=\frac{q^{1 / 2}}{\tau(\chi)} L(1 / 2, \chi) x^{-\frac{1}{2}}-\frac{\pi q^{3 / 2}}{\tau(\chi)} L(3 / 2, \chi) x^{\frac{1}{2}}+\frac{2 q^{2}}{\tau(\chi)} x \sum_{n=1}^{\infty} \frac{\sigma_{-\frac{1}{2}, \chi}(n)}{(n+q x)(\sqrt{n}+\sqrt{q x})} .
$$

3.2. Identities involving odd characters and specializations. In this subsection, we state the identities associated with $\sigma_{-\nu, \bar{\chi}}(n)$ and $\bar{\sigma}_{-\nu, \bar{\chi}}(n)$ when $\chi$ is an odd primitive character.

Theorem 3.5. Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Let $\chi$ be an odd primitive character modulo $q$. If $N$ is any integer such that $N \geq\left\lfloor\frac{\Re(\nu)+1}{2}\right\rfloor$, then

$$
\begin{aligned}
& 8 \pi x^{\nu / 2} \sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}}(n) n^{\nu / 2} K_{\nu}(4 \pi \sqrt{n x})=-\frac{\Gamma(\nu) L(\nu, \bar{\chi})}{(2 \pi)^{\nu-1}}+\frac{2 \Gamma(1+\nu) L(1+\nu, \bar{\chi})}{(2 \pi)^{\nu+1}} x^{-1}+\frac{2 i q^{1-\nu}}{\tau(\chi) \cos \left(\frac{\pi \nu}{2}\right)} \\
& \times\left\{\zeta(\nu+1) L(1, \chi)(q x)^{\nu}-\sum_{j=1}^{N} \zeta(2 j) L(2 j-\nu, \chi)(q x)^{2 j-1}-(q x)^{2 N+1} \sum_{n=1}^{\infty} \frac{\bar{\sigma}_{-\nu, \chi}(n)}{n}\left(\frac{n^{\nu+1-2 N}-(q x)^{\nu+1-2 N}}{n^{2}-(q x)^{2}}\right)\right\},
\end{aligned}
$$

provided $q x \notin \mathbb{Z}_{+}$.
Setting $\nu=1 / 2$ in the above theorem, we obtain the following
Corollary 3.6. We have

$$
\begin{aligned}
2 \pi \sum_{n=1}^{\infty} \sigma_{-\frac{1}{2}, \bar{\chi}}(n) e^{-4 \pi \sqrt{n x}}= & -\pi L(1 / 2, \bar{\chi})+\frac{1}{4 \pi} L(3 / 2, \bar{\chi}) x^{-1}+\frac{2 i q}{\tau(\chi)} \zeta(3 / 2) L(1, \chi) x^{1 / 2} \\
& -\frac{2 i q^{3 / 2}}{\tau(\chi)} x \sum_{n=1}^{\infty} \bar{\sigma}_{-\frac{1}{2}, \chi}(n) \frac{(n+\sqrt{n q x}+q x)}{n(n+q x)\left(n^{\frac{1}{2}}+(q x)^{\frac{1}{2}}\right)} .
\end{aligned}
$$

Theorem 3.7. Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Let $\chi$ be an odd primitive character modulo $q$. If $N$ is any integer such that $N \geq\left\lfloor\frac{\Re(\nu)+1}{2}\right\rfloor$, then

$$
\begin{aligned}
& 8 \pi x^{\nu / 2} \sum_{n=1}^{\infty} \bar{\sigma}_{-\nu, \bar{\chi}}(n) n^{\nu / 2} K_{\nu}(4 \pi \sqrt{n x})=\frac{2 \Gamma(\nu) \zeta(\nu) L(0, \bar{\chi})}{(2 \pi)^{\nu-1}}+\frac{i q}{\tau(\chi)}\left\{\frac{L(\nu, \chi)}{\cos \left(\frac{\pi \nu}{2}\right)}(q x)^{\nu-1}+\frac{\pi L(1+\nu, \chi)}{\sin \left(\frac{\pi \nu}{2}\right)}(q x)^{\nu}\right. \\
& \left.\quad+\frac{2}{\cos \left(\frac{\pi \nu}{2}\right)} \sum_{j=1}^{N-1} \zeta(2 j+1-\nu) L(2 j+1, \chi)(q x)^{2 j}+\frac{2}{\cos \left(\frac{\pi \nu}{2}\right)}(q x)^{2 N} \sum_{n=1}^{\infty} \sigma_{-\nu, \chi}(n)\left(\frac{n^{\nu+1-2 N}-(q x)^{\nu+1-2 N}}{n^{2}-(q x)^{2}}\right)\right\},
\end{aligned}
$$

provided $q x \notin \mathbb{Z}_{+}$.
The result corresponding to $\nu=1 / 2$ is as follows
Corollary 3.8. We have

$$
\begin{aligned}
2 \pi \sum_{n=1}^{\infty} \bar{\sigma}_{-\frac{1}{2}, \bar{\chi}}(n) e^{-4 \pi \sqrt{n x}}= & 2 \pi \zeta(1 / 2) L(0, \bar{\chi})+\frac{i q^{1 / 2}}{\tau(\chi)} L(1 / 2, \chi) x^{-\frac{1}{2}}+\frac{\pi i q^{3 / 2}}{\tau(\chi)} L(3 / 2, \chi) x^{\frac{1}{2}} \\
& +\frac{2 i q}{\tau(\chi)} \sum_{n=1}^{\infty} \sigma_{-\frac{1}{2}, \chi}(n) \frac{(n+\sqrt{n q x}+q x)}{(n+q x)(\sqrt{n}+\sqrt{q x})}
\end{aligned}
$$

3.3. Identities involving two characters and specializations. Here we state the identities corresponding to $\sigma_{-\nu, \chi_{1}, \chi_{2}}(n)=\sum_{d / n} d^{-\nu} \chi_{1}(d) \chi_{2}(n / d)$, where $\chi_{1}$ and $\chi_{2}$ are the Dirichlet characters modulo $p$ and $q$, respectively.
Theorem 3.9. Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Both $\chi_{1}$ and $\chi_{2}$ are non-principal even primitive characters modulo $p$ and $q$, respectively. If $N$ is any integer such that $N \geq\left\lfloor\frac{\Re(\nu)+1}{2}\right\rfloor$, then

$$
8 \pi x^{\frac{\nu}{2}} \sum_{n=1}^{\infty} \sigma_{-\nu, \chi_{1}, \bar{\chi}_{2}}(n) n^{\nu / 2} K_{\nu}(4 \pi \sqrt{n x})=\frac{2 p^{1-\nu} q}{\tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right) \sin \left(\frac{\pi \nu}{2}\right)}\left\{\sum_{j=1}^{N} L\left(2 j, \chi_{2}\right) L\left(2 j-\nu, \chi_{1}\right)(p q x)^{2 j-1}\right.
$$

$$
\left.+(p q x)^{2 N+1} \sum_{n=1}^{\infty} \sigma_{-\nu, \chi_{2}, \chi_{1}}(n)\left(\frac{n^{\nu-2 N}-(p q x)^{\nu-2 N}}{n^{2}-(p q x)^{2}}\right)\right\},
$$

provided pqx $\notin \mathbb{Z}_{+}$.
Setting $\chi_{1}=\chi_{2}=\chi$ in the above theorem, we get the following
Corollary 3.10. Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Let $\chi$ be a non-principal even primitive character modulo $q$. If $N$ is any integer such that $N \geq\left\lfloor\frac{\Re(\nu)+1}{2}\right\rfloor$, then

$$
\begin{aligned}
8 \pi x^{\frac{\nu}{2}} \sum_{n=1}^{\infty} \sigma_{-\nu}(n) \bar{\chi}(n) n^{\nu / 2} K_{\nu}(4 \pi \sqrt{n x})= & \frac{2 q^{2-\nu}}{\tau^{2}(\chi) \sin \left(\frac{\pi \nu}{2}\right)}\left\{\sum_{j=1}^{N} L(2 j, \chi) L(2 j-\nu, \chi)\left(q^{2} x\right)^{2 j-1}\right. \\
& \left.+\left(q^{2} x\right)^{2 N+1} \sum_{n=1}^{\infty} \sigma_{-\nu}(n) \chi(n)\left(\frac{n^{\nu-2 N}-\left(q^{2} x\right)^{\nu-2 N}}{n^{2}-\left(q^{2} x\right)^{2}}\right)\right\}
\end{aligned}
$$

provided $q^{2} x \notin \mathbb{Z}_{+}$.
Theorem 3.11. Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Both $\chi_{1}$ and $\chi_{2}$ are odd primitive characters modulo $p$ and $q$, respectively. If $N$ is any integer such that $N \geq\left\lfloor\frac{\Re(\nu)+1}{2}\right\rfloor$, then

$$
\begin{aligned}
& 8 \pi x^{\nu / 2} \sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}_{1}, \chi_{2}}(n) n^{\nu / 2} K_{\nu}(4 \pi \sqrt{n x})=\Gamma(\nu) L\left(\nu, \overline{\chi_{1}}\right) L\left(0, \overline{\chi_{2}}\right) \frac{2}{(2 \pi)^{\nu-1}} \\
& -\frac{2 p^{1-\nu} q}{\tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right) \sin \left(\frac{\pi \nu}{2}\right)}\left\{-L\left(\nu+1, \chi_{2}\right) L\left(1, \chi_{1}\right)(p q x)^{\nu}+\sum_{j=1}^{N-1} L\left(2 j+1, \chi_{2}\right) L\left(2 j+1-\nu, \chi_{1}\right)(p q x)^{2 j}\right. \\
& \left.+(p q x)^{2 N} \sum_{n=1}^{\infty} \frac{\sigma_{-\nu, \chi_{2}, \chi_{1}}(n)}{n}\left(\frac{n^{\nu-2 N+2}-(p q x)^{\nu-2 N+2}}{n^{2}-(p q x)^{2}}\right)\right\},
\end{aligned}
$$

provided $p q x \notin \mathbb{Z}_{+}$.
Taking $\chi_{1}=\chi_{2}=\chi$ in the above theorem, we get the following
Corollary 3.12. Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Let $\chi$ be an odd primitive character modulo $p$. If $N$ is any integer such that $N \geq\left\lfloor\frac{\Re(\nu)+1}{2}\right\rfloor$, then

$$
\begin{aligned}
& 8 \pi x^{\frac{\nu}{2}} \sum_{n=1}^{\infty} \sigma_{-\nu}(n) \bar{\chi}(n) n^{\nu / 2} K_{\nu}(4 \pi \sqrt{n x})=\Gamma(\nu) L(\nu, \bar{\chi}) L(0, \bar{\chi}) \frac{2}{(2 \pi)^{\nu-1}} \\
& -\frac{2 p^{2-\nu}}{\tau^{2}(\chi) \sin \left(\frac{\pi \nu}{2}\right)}\left\{-L\left(\nu+1, \chi_{2}\right) L\left(1, \chi_{1}\right)\left(p^{2} x\right)^{\nu}+\sum_{j=1}^{N-1} L(2 j+1, \chi) L(2 j+1-\nu, \chi)\left(p^{2} x\right)^{2 j}\right. \\
& \left.\quad+\left(p^{2} x\right)^{2 N} \sum_{n=1}^{\infty} \frac{\sigma_{-\nu}(n) \chi(n)}{n}\left(\frac{n^{\nu-2 N+2}-\left(p^{2} x\right)^{\nu-2 N+2}}{n^{2}-\left(p^{2} x\right)^{2}}\right)\right\},
\end{aligned}
$$

provided $p^{2} x \notin \mathbb{Z}_{+}$.
Theorem 3.13. Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Let $\chi_{1}$ be a non-principal even primitive character modulo $p$ and $\chi_{2}$ be an odd primitive character modulo $q$. If $N$ is any integer such that $N \geq\left\lfloor\frac{\Re(\nu)+1}{2}\right\rfloor$, then

$$
\begin{aligned}
& 8 \pi x^{\nu / 2} \sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}_{1}, \overline{\chi_{2}}}(n) n^{\nu / 2} K_{\nu}(4 \pi \sqrt{n x})=\frac{2}{(2 \pi)^{\nu-1}} \Gamma(\nu) L\left(\nu, \overline{\chi_{1}}\right) L\left(0, \overline{\chi_{2}}\right)+\frac{2 i p^{1-\nu} q}{\tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right) \cos \left(\frac{\pi \nu}{2}\right)} \\
& \times\left\{\sum_{j=1}^{N-1} L\left(2 j+1, \chi_{2}\right) L\left(2 j+1-\nu, \chi_{1}\right)(p q x)^{2 j}+(p q x)^{2 N} \sum_{n=1}^{\infty} \sigma_{-\nu, \chi_{2}, \chi_{1}}(n)\left(\frac{n^{\nu-2 N+1}-(p q x)^{\nu-2 N+1}}{n^{2}-(p q x)^{2}}\right)\right\},
\end{aligned}
$$

provided $p q x \notin \mathbb{Z}_{+}$.

Theorem 3.14. Let $\nu \notin \mathbb{Z}$ such that $\Re(\nu) \geq 0$. Let $\chi_{1}$ be an odd primitive character modulo $p$ and $\chi_{2}$ be $a$ non-principal even primitive character modulo $q$. If $N$ is any integer such that $N \geq\left\lfloor\frac{\Re(\nu)+1}{2}\right\rfloor$, then

$$
\begin{aligned}
& 8 \pi x^{\nu / 2} \sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}_{1}, \chi_{2}}(n) n^{\nu / 2} K_{\nu}(4 \pi \sqrt{n x})=\frac{2 i p^{1-\nu} q}{\tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right) \cos \left(\frac{\pi \nu}{2}\right)}\left\{L\left(\nu+1, \chi_{2}\right) L\left(1, \chi_{1}\right)(p q x)^{\nu}\right. \\
& \left.\quad-\sum_{j=1}^{N} L\left(2 j, \chi_{2}\right) L\left(2 j-\nu, \chi_{1}\right)(p q x)^{2 j-1}-(p q x)^{2 N+1} \sum_{n=1}^{\infty} \frac{\sigma_{-\nu, \chi_{2}, \chi_{1}}(n)}{n}\left(\frac{n^{\nu-2 N+1}-(p q x)^{\nu-2 N+1}}{n^{2}-(p q x)^{2}}\right)\right\},
\end{aligned}
$$

provided $p q x \notin \mathbb{Z}_{+}$.

## 4. Connection with Voronoï summation formula

In this section, we offer Voronoii-type summation formulas for $\sigma_{z, \chi}(n), \bar{\sigma}_{z, \chi}(n)$ and $\sigma_{z, \chi_{1}, \chi_{2}}(n)$ defined in (1.11).

### 4.1. Identities involving even characters.

Theorem 4.1. Let $0<\alpha<\beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let $f$ denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that $\chi$ is a non-principal even primitive character modulo $q$. For $0<\Re(\nu)<\frac{1}{2}$, we have

$$
\begin{aligned}
& \frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} \sum_{\alpha<j<\beta} \bar{\sigma}_{-\nu, \chi}(j) f(j)=\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} L(1-\nu, \chi) \int_{\alpha}^{\beta} \frac{f(t)}{t^{\nu}} \mathrm{d} t+2 \pi \sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}}(n) n^{\nu / 2} \int_{\alpha}^{\beta} f(t)(t)^{-\frac{\nu}{2}} \\
& \times\left\{\left(\frac{2}{\pi} K_{\nu}\left(4 \pi \sqrt{\frac{n t}{q}}\right)-Y_{\nu}\left(4 \pi \sqrt{\frac{n t}{q}}\right)\right) \cos \left(\frac{\pi \nu}{2}\right)-J_{\nu}\left(4 \pi \sqrt{\frac{n t}{q}}\right) \sin \left(\frac{\pi \nu}{2}\right)\right\} d t
\end{aligned}
$$

Theorem 4.2. Let $0<\alpha<\beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let $f$ denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that $\chi$ is a non-principal even primitive character modulo q. For $0<\Re(\nu)<\frac{1}{2}$, we have

$$
\begin{aligned}
& \frac{q^{1+\frac{\nu}{2}}}{\tau(\chi)} \sum_{\alpha<j<\beta} \sigma_{-\nu, \chi}(j) f(j)=\frac{q^{1+\frac{\nu}{2}}}{\tau(\chi)} L(1+\nu, \chi) \int_{\alpha}^{\beta} f(t) \mathrm{d} t+2 \pi \sum_{n=1}^{\infty} \bar{\sigma}_{-\nu, \bar{\chi}}(n) n^{\nu / 2} \int_{\alpha}^{\beta} f(t)(t)^{-\frac{\nu}{2}} \\
& \times\left\{\left(\frac{2}{\pi} K_{\nu}\left(4 \pi \sqrt{\frac{n t}{q}}\right)-Y_{\nu}\left(4 \pi \sqrt{\frac{n t}{q}}\right)\right) \cos \left(\frac{\pi \nu}{2}\right)-J_{\nu}\left(4 \pi \sqrt{\frac{n t}{q}}\right) \sin \left(\frac{\pi \nu}{2}\right)\right\} d t
\end{aligned}
$$

### 4.2. Identities involving odd characters.

Theorem 4.3. Let $0<\alpha<\beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let $f$ denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that $\chi$ is an odd primitive character modulo $q$. For $0<\Re(\nu)<\frac{1}{2}$, we have

$$
\begin{aligned}
& \frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} \sum_{\alpha<j<\beta} \frac{\bar{\sigma}_{-\nu, \chi}(j)}{j} f(j)=\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} L(1-\nu, \chi) \int_{\alpha}^{\beta} \frac{f(t)}{t^{\nu+1}} \mathrm{~d} t-2 \pi i \sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}}(n) n^{\nu / 2} \int_{\alpha}^{\beta} f(t)(t)^{-\frac{\nu}{2}-1} \\
& \times\left\{\left(\frac{2}{\pi} K_{\nu}\left(4 \pi \sqrt{\frac{n t}{q}}\right)-Y_{\nu}\left(4 \pi \sqrt{\frac{n t}{q}}\right)\right) \sin \left(\frac{\pi \nu}{2}\right)+J_{\nu}\left(4 \pi \sqrt{\frac{n t}{q}}\right) \cos \left(\frac{\pi \nu}{2}\right)\right\} d t .
\end{aligned}
$$

Theorem 4.4. Let $0<\alpha<\beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let $f$ denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that $\chi$ is an odd primitive character modulo $q$. For $0<\Re(\nu)<\frac{1}{2}$, we have

$$
\begin{aligned}
& \frac{q^{1+\frac{\nu}{2}}}{\tau(\chi)} \sum_{\alpha<j<\beta} \sigma_{-\nu, \chi}(j) f(j)=\frac{q^{1+\frac{\nu}{2}}}{\tau(\chi)} L(1+\nu, \chi) \int_{\alpha}^{\beta} f(t) \mathrm{d} t+2 \pi i \sum_{n=1}^{\infty} \bar{\sigma}_{-\nu, \bar{\chi}}(n) n^{\nu / 2} \int_{\alpha}^{\beta} f(t)(t)^{-\frac{\nu}{2}} \\
& \times\left\{\left(\frac{2}{\pi} K_{\nu}\left(4 \pi \sqrt{\frac{n t}{q}}\right)+Y_{\nu}\left(4 \pi \sqrt{\frac{n t}{q}}\right)\right) \sin \left(\frac{\pi \nu}{2}\right)-J_{\nu}\left(4 \pi \sqrt{\frac{n t}{q}}\right) \cos \left(\frac{\pi \nu}{2}\right)\right\} d t .
\end{aligned}
$$

4.3. Identities involving two characters. In this subsection, we state Voronoï-type summation formula associated with $\sigma_{-\nu, \chi_{1}, \chi_{2}}(n)=\sum_{d / n} d^{-\nu} \chi_{1}(d) \chi_{2}(n / d)$, where $\chi_{1}$ and $\chi_{2}$ are Dirichlet characters modulo $p$ and $q$, respectively.
Theorem 4.5. Let $0<\alpha<\beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let $f$ denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that $\chi_{1}$ and $\chi_{2}$ are non-principal even primitive characters modulo $p$ and $q$, respectively. For $0<\Re(\nu)<\frac{1}{2}$, we have

$$
\begin{aligned}
& \frac{p^{1-\frac{\nu}{2}} q^{1+\frac{\nu}{2}}}{\tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right)} \sum_{\alpha<j<\beta} \sigma_{-\nu, \chi_{2}, \chi_{1}}(j) f(j)=2 \pi \sum_{n=1}^{\infty} \sigma_{-\nu, \overline{\chi_{1}, \chi_{2}}}(n) n^{\nu / 2} \int_{\alpha}^{\beta} f(t)(t)^{-\frac{\nu}{2}} \\
& \times\left\{\left(\frac{2}{\pi} K_{\nu}\left(4 \pi \sqrt{\frac{n t}{p q}}\right)-Y_{\nu}\left(4 \pi \sqrt{\frac{n t}{p q}}\right)\right) \cos \left(\frac{\pi \nu}{2}\right)-J_{\nu}\left(4 \pi \sqrt{\frac{n t}{p q}}\right) \sin \left(\frac{\pi \nu}{2}\right)\right\} d t .
\end{aligned}
$$

Substituting $\chi_{1}=\chi_{2}=\chi$ in the above theorem, we get the following
Corollary 4.6. Let $0<\alpha<\beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let $f$ denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that $\chi$ is a non-principal even primitive character modulo $q$. For $0<\Re(\nu)<\frac{1}{2}$, we have

$$
\begin{aligned}
& \frac{q^{2}}{\tau^{2}(\chi)} \sum_{\alpha<j<\beta} \sigma_{-\nu}(j) \chi(j) f(j)=2 \pi \sum_{n=1}^{\infty} \sigma_{-\nu}(n) \bar{\chi}(j) n^{\nu / 2} \int_{\alpha}^{\beta} f(t)(t)^{-\frac{\nu}{2}} \\
& \times\left\{\left(\frac{2}{\pi} K_{\nu}\left(4 \pi \sqrt{\frac{n t}{q^{2}}}\right)-Y_{\nu}\left(4 \pi \sqrt{\frac{n t}{q^{2}}}\right)\right) \cos \left(\frac{\pi \nu}{2}\right)-J_{\nu}\left(4 \pi \sqrt{\frac{n t}{q^{2}}}\right) \sin \left(\frac{\pi \nu}{2}\right)\right\} d t
\end{aligned}
$$

Theorem 4.7. Let $0<\alpha<\beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let $f$ denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that $\chi_{1}$ and $\chi_{2}$ are odd primitive characters modulo $p$ and $q$, respectively. For $0<\Re(\nu)<\frac{1}{2}$, we have

$$
\begin{aligned}
& \frac{p^{1-\frac{\nu}{2}} q^{1+\frac{\nu}{2}}}{\tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right)} \sum_{\alpha<j<\beta} \frac{\sigma_{-\nu, \chi_{2}, \chi_{1}}(j) f(j)}{j}=-2 \pi \sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}_{1}, \chi_{2}}(n) n^{\nu / 2} \int_{\alpha}^{\beta} f(t)(t)^{-\frac{\nu}{2}-1} \\
& \times\left\{\left(\frac{2}{\pi} K_{\nu}\left(4 \pi \sqrt{\frac{n t}{p q}}\right)+Y_{\nu}\left(4 \pi \sqrt{\frac{n t}{p q}}\right)\right) \cos \left(\frac{\pi \nu}{2}\right)+J_{\nu}\left(4 \pi \sqrt{\frac{n t}{p q}}\right) \sin \left(\frac{\pi \nu}{2}\right)\right\} d t
\end{aligned}
$$

Substituting $\chi_{1}=\chi_{2}=\chi$ in the above theorem, we get the following
Corollary 4.8. Let $0<\alpha<\beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let $f$ denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that $\chi$ is an odd primitive character modulo $q$. For $0<\Re(\nu)<\frac{1}{2}$, we have

$$
\begin{aligned}
& \frac{q^{2}}{\tau^{2}(\chi)} \sum_{\alpha<j<\beta} \frac{\sigma_{-\nu}(j) \chi(j) f(j)}{j}=-2 \pi \sum_{n=1}^{\infty} \sigma_{-\nu}(n) \bar{\chi}(j) n^{\nu / 2} \int_{\alpha}^{\beta} f(t)(t)^{-\frac{\nu}{2}-1} \\
& \times\left\{\left(\frac{2}{\pi} K_{\nu}\left(4 \pi \sqrt{\frac{n t}{q^{2}}}\right)+Y_{\nu}\left(4 \pi \sqrt{\frac{n t}{q^{2}}}\right)\right) \cos \left(\frac{\pi \nu}{2}\right)+J_{\nu}\left(4 \pi \sqrt{\frac{n t}{q^{2}}}\right) \sin \left(\frac{\pi \nu}{2}\right)\right\} d t .
\end{aligned}
$$

Theorem 4.9. Let $0<\alpha<\beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let $f$ denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that $\chi_{1}$ is a non-principal even primitive character modulo $p$ and $\chi_{2}$ is an odd primitive character modulo $q$. For $0<\Re(\nu)<\frac{1}{2}$, we have

$$
\begin{aligned}
& \frac{p^{1-\frac{\nu}{2}} q^{1+\frac{\nu}{2}}}{\tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right)} \sum_{\alpha<j<\beta} \sigma_{-\nu, \chi_{2}, \chi_{1}}(j) f(j)=2 \pi i \sum_{n=1}^{\infty} \sigma_{-\nu, \overline{\chi_{1}, \chi_{2}}}(n) n^{\nu / 2} \int_{\alpha}^{\beta} f(t)(t)^{-\frac{\nu}{2}} \\
& \times\left\{\left(\frac{2}{\pi} K_{\nu}\left(4 \pi \sqrt{\frac{n t}{p q}}\right)+Y_{\nu}\left(4 \pi \sqrt{\frac{n t}{p q}}\right)\right) \sin \left(\frac{\pi \nu}{2}\right)-J_{\nu}\left(4 \pi \sqrt{\frac{n t}{p q}}\right) \cos \left(\frac{\pi \nu}{2}\right)\right\} d t .
\end{aligned}
$$

Theorem 4.10. Let $0<\alpha<\beta$ and $\alpha, \beta \notin \mathbb{Z}$. Let $f$ denote a function analytic inside a closed contour strictly containing $[\alpha, \beta]$. Assume that $\chi_{1}$ is an odd primitive character modulo $p$ and $\chi_{2}$ is a non-principal
even primitive character modulo $q$. For $0<\Re(\nu)<\frac{1}{2}$, we have

$$
\begin{aligned}
& \frac{p^{1-\frac{\nu}{2}} q^{1+\frac{\nu}{2}}}{\tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right)} \sum_{\alpha<j<\beta} \frac{\sigma_{-\nu, \chi_{2}, \chi_{1}}(j) f(j)}{j}=-2 \pi i \sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}_{1}, \chi_{2}}(n) n^{\nu / 2} \int_{\alpha}^{\beta} f(t) t^{-\frac{\nu}{2}-1} \\
& \times\left\{\left(\frac{2}{\pi} K_{\nu}\left(4 \pi \sqrt{\frac{n t}{p q}}\right)-Y_{\nu}\left(4 \pi \sqrt{\frac{n t}{p q}}\right)\right) \sin \left(\frac{\pi \nu}{2}\right)+J_{\nu}\left(4 \pi \sqrt{\frac{n t}{p q}}\right) \cos \left(\frac{\pi \nu}{2}\right)\right\} d t .
\end{aligned}
$$

## 5. Preliminaries

We begin this section by recalling and proving some important results which will be used throughout the paper.

The Mellin transform of a locally integrable function $f(x)$ on $(0, \infty)$ is defined by

$$
\begin{equation*}
\mathcal{M}[f ; s]=F(s)=\int_{0}^{\infty} f(t) t^{s-1} d t \tag{5.1}
\end{equation*}
$$

provided the integral converges. The basic properties of the Mellin transform follow immediately from those of the Laplace transform since these transforms are intimately connected. The integral in (5.1) defines the Mellin transform in a vertical strip in the $s$ plane whose boundaries are determined by the analytic structure of $f(x)$ as $x \rightarrow 0+$ and $x \rightarrow+\infty$. If we assume that $f(x)$ satisfies the following growth condition

$$
f(x)= \begin{cases}O\left(x^{-a-\varepsilon}\right) & \text { as } x \rightarrow 0+  \tag{5.2}\\ O\left(x^{-b+\varepsilon}\right) & \text { as } x \rightarrow+\infty\end{cases}
$$

where $\varepsilon>0$ and $a<b$, then the integral (5.1) converges absolutely in the strip $a<\Re(s)<b$ and defines an analytic function there in the strip. This strip is known as the strip of analyticity of $\mathcal{M}[f ; s]$. Furthermore, the inversion formula for (5.1) follows directly from the corresponding inversion formula for the bilateral Laplace transform. Thus,

$$
\begin{equation*}
f(x)=\frac{1}{2 \pi i} \int_{c-i \infty}^{c+i \infty} x^{-s} \mathcal{M}[f ; s] d s \quad(a<c<b), \tag{5.3}
\end{equation*}
$$

which is valid at all points $x \geq 0$ where $f(x)$ is continuous. For example, $\mathcal{M}\left[e^{x} ; s\right]=\Gamma(s)$ for $\Re(s)>0$, and we have the corresponding Mellin's inversion formula

$$
e^{-y}=\frac{1}{2 \pi i} \int_{(c)} \Gamma(s) y^{-s} \mathrm{~d} s
$$

valid for $\Re(y)>0$. The functional relations for $\Gamma(s)$ are given by 31, p. 73]

$$
\begin{gather*}
\Gamma(s+1)=s \Gamma(s), \quad \Gamma(s) \Gamma\left(s+\frac{1}{2}\right)=2^{1-2 s} \sqrt{\pi} \Gamma(2 s)  \tag{5.4}\\
\Gamma(s) \Gamma(1-s)=\frac{\pi}{\sin (\pi s)} \tag{5.5}
\end{gather*}
$$

The following lemma states the asymptotic behaviour of $\Gamma(s)$.
Lemma 5.1. [32, p. 38] In a vertical strip, for $s=\sigma+$ it with $a \leq \sigma \leq b$ and $|t| \geq 1$,

$$
|\Gamma(s)|=(2 \pi)^{\frac{1}{2}}|t|^{\sigma-\frac{1}{2}} \exp ^{-\frac{1}{2} \pi|t|}\left(1+O\left(\frac{1}{|t|}\right)\right)
$$

In our investigation, we shall require the following results related to the Mellin transform of derivatives of a function.

Lemma 5.2. Let $n \in \mathbb{N}$. Assume that $\phi$ is $n$-times differentiable function and

$$
\begin{equation*}
\mathcal{M}[\phi(t) ; s]=\int_{0}^{\infty} \phi(t) t^{s-1} d t=\Phi(s) \tag{5.6}
\end{equation*}
$$

If $\phi$ satisfies (5.2), then

$$
\begin{equation*}
\mathcal{M}\left[\phi^{(n)}(t) t^{n} ; s\right]=(-1)^{n} \frac{\Gamma(s+n)}{\Gamma(s)} \Phi(s), \tag{5.7}
\end{equation*}
$$

where $s \in\{w \in \mathbb{C} ; a<\Re(w)<b\}$, provided

$$
\begin{equation*}
\lim _{t \rightarrow 0, \infty} t^{s+n-j-1} \phi^{(n-j-1)}(t)=0 \quad j=0,1, \cdots, n-1 . \tag{5.8}
\end{equation*}
$$

Proof. The proof relies on mathematical induction. Using integration by parts, we have

$$
\mathcal{M}\left[x \phi^{\prime}(x) ; s\right]=\int_{0}^{\infty} \phi^{\prime}(t) t^{s} d t=\left[t^{s} \phi(t)\right]_{0}^{\infty}-s \int_{0}^{\infty} \phi(t) t^{s-1} d t
$$

Noting $\phi(t)$ satisfies (5.2), we can claim that

$$
\mathcal{M}\left[x \phi^{\prime}(x) ; s\right]=-s \Phi(s) \text { for } a<\Re(s)<b .
$$

Suppose the statement of the theorem is true for $n=N$ and $\phi$ is $N+1$-times differentiable function and satisfies (5.8). Then

$$
\begin{aligned}
\mathcal{M}\left[x^{N+1} \phi^{(N+1)}(x) ; s\right] & =\int_{0}^{\infty} t^{N+1} \phi^{(N+1)}(t) t^{s-1} d t \\
& =\left[t^{s+N} \phi^{(N)}(t)\right]_{0}^{\infty}-(s+N) \int_{0}^{\infty} t^{N} \phi^{(N)}(t) t^{s-1} d t .
\end{aligned}
$$

As $\phi$ satisfies (5.8), so we have

$$
\mathcal{M}\left[\phi^{(N+1)}(t) t^{N+1} ; s\right]=-(s+N) \int_{0}^{\infty} t^{N} \phi^{(N)}(t) t^{s-1} d t=(-1)^{N+1} \frac{\Gamma(s+N+1)}{\Gamma(s)} \Phi(s),
$$

and this completes the proof.
Lemma 5.3. [33, p. 91, Formula (3.3.9)] We have

$$
\mathcal{M}\left[(1+x)^{-a} ; s\right]=\frac{\Gamma(s) \Gamma(a-s)}{\Gamma(a)},
$$

for $0<\Re(s)<\Re(a)$.
As an immediate consequence of Lemma 5.3 we get,
Lemma 5.4. For any $n \in \mathbb{N}$,

$$
\mathcal{M}\left[\frac{a(a+1) \cdots(a+n-1) x^{n}}{(1+x)^{a+n}} ; s\right]=\frac{\Gamma(s+n) \Gamma(a-s)}{\Gamma(a)},
$$

whenever $0<\Re(s)<\Re(a)$.
Proof. By Lemma 5.3, we can write

$$
\mathcal{M}\left[(1+x)^{-a} ; s\right]=\frac{\Gamma(s) \Gamma(a-s)}{\Gamma(a)}
$$

for $0<\Re(s)<\Re(a)$. The function $\phi(t)=\frac{1}{(1+t)^{a}}$ for $t \geq 0$ is a continuous function and satisfies all the conditions of Lemma 5.2. Furthermore,

$$
\phi^{(n)}(t)=(-1)^{n} \frac{a(a+1) \cdots(a+n-1)}{(1+t)^{(a+n)}} .
$$

We have

$$
\Phi(s)=\frac{\Gamma(a-s) \Gamma(s)}{\Gamma(a)} \text { for } 0<\Re(s)<\Re(a) .
$$

Hence by Lemma 5.2,

$$
\mathcal{M}\left[\frac{a(a+1) \cdots(a+n-1) t^{n}}{(1+x)^{(a+1)}} ; s\right]=\frac{\Gamma(s+n)}{\Gamma(s)} \Phi(s)=\frac{\Gamma(s+n) \Gamma(a-s)}{\Gamma(a)}
$$

for $0<\Re(s)<\Re(a)$.
Lemma 5.5. Let $n \geq 0$ be any integer and $t>0$ be any real number. Then

$$
\frac{1}{2 \pi i} \int_{(c)} \Gamma(s+n) \Gamma(a-s) t^{-s} \mathrm{~d} s=\frac{\Gamma(a+n)}{(1+t)^{a+n}} t^{n},
$$

for $0<c<\Re(a)$.

Proof. We get our desired result by combining Lemmas 5.3 and 5.4 and applying Mellin's inversion formula.
Lemma 5.6. [34, p. 346, Formula (20)] We have

$$
\mathcal{M}\left[\frac{\log t}{t-1} ; s\right]=\frac{\pi^{2}}{\sin ^{2}(\pi s)},
$$

for $0<\Re(s)<1$. The integral is convergent in the sense of Cauchy's principal value.
Lemma 5.7. We have

$$
\begin{equation*}
\mathcal{M}\left[\frac{4 \log x}{x^{2}-1} ; s\right]=\frac{\pi^{2}}{\sin ^{2}\left(\frac{\pi s}{2}\right)}, \tag{5.9}
\end{equation*}
$$

for $0<\Re(s)<2$. The integral is convergent in the sense of Cauchy's principal value.
Proof. This is a direct consequence of Lemma 5.6.
Now, we record a few important results related to the modified $K$-Bessel function $K_{\nu}(x)$ defined by (1.2).
Lemma 5.8. [3, p. 10, Lemma 3.3] Let $\nu \in \mathbf{C}$. For any $c>\max \{0,-\Re(\nu)\}$, we have

$$
t^{\frac{\nu}{2}} K_{\nu}(a \sqrt{t x})=\frac{1}{2}\left(\frac{2}{a \sqrt{x}}\right)^{\nu} \frac{1}{2 \pi i} \int_{(c)} \Gamma(s) \Gamma(s+\nu)\left(\frac{4}{a^{2} x}\right)^{s} t^{-s} d s .
$$

We first observe that the generating functions for $\sigma_{z, \chi}(n)$ and $\bar{\sigma}_{z, \chi}(n)$ and $\sigma_{z, \chi_{1}, \chi_{2}}(n)$ defined in (1.11) are the following

$$
\begin{gather*}
\zeta(s) L(s-z, \chi)=\sum_{m=1}^{\infty} \frac{1}{m^{s}} \sum_{d=1}^{\infty} \frac{d^{z} \chi(d)}{d^{s}}=\sum_{n=1}^{\infty} \frac{\sigma_{z, \chi}(n)}{n^{s}},  \tag{5.10}\\
\zeta(s-z) L(s, \chi)=\sum_{m=1}^{\infty} \frac{1}{m^{s-z}} \sum_{d=1}^{\infty} \frac{\chi(d)}{d^{s}}=\sum_{n=1}^{\infty} \frac{\bar{\sigma}_{z, \chi}(n)}{n^{s}},  \tag{5.11}\\
L\left(s-z, \chi_{1}\right) L\left(s, \chi_{2}\right)=\sum_{d=1}^{\infty} \frac{d^{z} \chi_{1}(d)}{d^{s}} \sum_{m=1}^{\infty} \frac{\chi_{2}(m)}{m^{s}}=\sum_{n=1}^{\infty} \frac{\sigma_{z, \chi_{1}, \chi_{2}}(n)}{n^{s}}, \tag{5.12}
\end{gather*}
$$

for $\Re(s)>\max (\Re(z)+1,1)$, where $\zeta(s)$ denotes the the Riemann zeta function and $L(s, \chi)$ denotes the Dirichlet $L$-function defined by (2.1) for $\Re(s)>1$. We recall that the functional equation of $\zeta(s)$ [35, p. 234]

$$
\begin{equation*}
\zeta(s)=2^{s} \pi^{s-1} \sin \left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) . \tag{5.13}
\end{equation*}
$$

Replacing $s$ by $1-s$ in (5.13), we obtain

$$
\begin{equation*}
\Gamma(s) \zeta(s)=\frac{\pi^{s} \zeta(1-s)}{2^{1-s} \cos \left(\frac{\pi s}{2}\right)} \tag{5.14}
\end{equation*}
$$

Next, we write the functional equation for $L(s, \chi)$ 31, p. 71]

$$
\begin{equation*}
L(s, \chi)=\frac{\tau(\chi)}{i^{\kappa} \sqrt{q}}\left(\frac{\pi}{q}\right)^{s-1 / 2} \frac{\Gamma\left(\frac{1-s+\kappa}{2}\right)}{\Gamma\left(\frac{s+\kappa}{2}\right)} L(1-s, \bar{\chi}), \tag{5.15}
\end{equation*}
$$

where

$$
\kappa=\kappa(\chi)= \begin{cases}0, & \text { if } \quad \chi(-1)=1 \\ 1, & \text { if } \quad \chi(-1)=-1\end{cases}
$$

Employing (5.4) and (5.5) in (5.15), we obtain [30, Corrolary 10.9, p. 333]

$$
\begin{equation*}
L(s, \chi)=i^{-\kappa} \frac{\tau(\chi)}{\pi}\left(\frac{(2 \pi)}{q}\right)^{s} \Gamma(1-s) \sin \frac{\pi(s+\kappa)}{2} L(1-s, \bar{\chi}) . \tag{5.16}
\end{equation*}
$$

Now replacing $s$ by $s-z$ in (5.16), we get

$$
L(s-z, \chi)=i^{-\kappa} \frac{\tau(\chi)}{\pi}\left(\frac{(2 \pi)}{q}\right)^{s-z} \Gamma(1+z-s) \sin \frac{\pi(s+\kappa-z)}{2} L(1+z-s, \bar{\chi}) .
$$

So, we can rewrite the above equation as

$$
\begin{equation*}
\Gamma(1+z-s) L(1+z-s, \bar{\chi})=i^{\kappa} \frac{\pi}{\tau(\chi)}\left(\frac{q}{2 \pi}\right)^{s-z} \frac{L(s-z, \chi)}{\sin \pi\left(\frac{s+\kappa-z}{2}\right)} . \tag{5.17}
\end{equation*}
$$

We will also note that [31, p. 69, p. 71]

$$
\tau(\chi) \tau(\bar{\chi})= \begin{cases}-q, & \text { for odd primitive } \chi \bmod q  \tag{5.18}\\ q, & \text { for even non-principal primitive } \chi \bmod q\end{cases}
$$

6. Proof of results when $z \in \mathbb{Z}_{+}$

We will start this section by considering a more general setup. Let $\chi$ be any Dirichlet character modulo $q$ and $z \in \mathbb{C}$. Let $f_{z}(n)$ be one of the arithmetical functions $\sigma_{z, \chi}(n)$ or $\bar{\sigma}_{z, \chi}(n)$ or $\sigma_{z, \chi_{1}, \chi_{2}}(n)$ defined in (1.11). We denote

$$
\begin{equation*}
F_{z}(s):=\sum_{n=1}^{\infty} \frac{f_{z}(n)}{n^{s}}, \Re(s)>1 \tag{6.1}
\end{equation*}
$$

Hence $F_{z}(s)$ is one of the Dirichlet series given in (5.10) or (5.11) or (5.12). As mentioned in the previous section, we will consider $\Re(\nu)>0$ and $\nu=0$. Employing Lemma 5.8 with $t=n$ and subsequently interchanging the summation and integration, we get

$$
\begin{align*}
\sum_{n=1}^{\infty} f_{z}(n) n^{\nu / 2} K_{\nu}(a \sqrt{n x}) & =\frac{1}{2}\left(\frac{2}{a \sqrt{x}}\right)^{\nu} \frac{1}{2 \pi i} \int_{(c)} \Gamma(s) \Gamma(s+\nu)\left(\frac{4}{a^{2} x}\right)^{s} \sum_{n=1}^{\infty} f_{z}(n) n^{-s} \mathrm{~d} s \\
& =\frac{1}{2} X^{\nu / 2} \frac{1}{2 \pi i} \int_{(c)} \Gamma(s) \Gamma(s+\nu) F_{z}(s) X^{s} \mathrm{~d} s \tag{6.2}
\end{align*}
$$

where $c>\Re(z)+1$ and $X=\frac{4}{a^{2} x}$. Here the notation $(c)$ denotes the vertical line $[c-i \infty, c+i \infty]$. Next, we investigate the following integral

$$
\begin{equation*}
I_{z}^{(\nu)}(X):=\frac{1}{2 \pi i} \int_{(c)} \Gamma(s+\nu) \Gamma(s) F_{z}(s) X^{s} \mathrm{~d} s \tag{6.3}
\end{equation*}
$$

We shall use the Cauchy residue theorem to evaluate this line integral in (6.3). We consider the contour formed by the line segments $[c-i T, c+i T],[c+i T,-d+i T],[-d+i T,-d-i T],[-d-i T, c-i T]$, where the choice for $d$ is as follows: $0<d<\min \{1, \Re(\nu)\}$ whenever $\Re(\nu)>0$ and $0<d<1$ otherwise. Here, $T$ is taken to be a large positive number. The possible poles of the integrand function in (6.3) are at $s=0,1$ and $z+1$. Now letting $T \rightarrow \infty$ and invoking Lemma 5.1, one can show that the integrals along the horizontal segments $[c+i T,-d+i T]$ and $[-d-i T, c-i T]$ vanish and get

$$
\begin{equation*}
I_{z}^{(\nu)}(X)=R_{z+1}+R_{1}+R_{0}+\frac{1}{2 \pi i} \int_{(-d)} \Gamma(s+\nu) \Gamma(s) F_{z}(s) X^{s} \mathrm{~d} s \tag{6.4}
\end{equation*}
$$

where $R_{z+1}, R_{1}$ and $R_{0}$ are the residues at $s=z+1,1$ and $s=0$, respectively. It is easy to see that $R_{z+1}=0$ whenever $z=0$. Hence combining (6.2) and (6.3) together with (6.4), we obtain

$$
\begin{equation*}
\sum_{n=1}^{\infty} f_{z}(n) n^{\nu / 2} K_{\nu}(a \sqrt{n x})=\frac{1}{2} X^{\nu / 2}\left(R_{z+1}+R_{1}+R_{0}+J_{z}^{(\nu)}(X)\right) \tag{6.5}
\end{equation*}
$$

where $J_{z}^{(\nu)}(X)$ is defined by

$$
\begin{equation*}
J_{z}^{(\nu)}(X):=\frac{1}{2 \pi i} \int_{(-d)} \Gamma(s+\nu) \Gamma(s) F_{z}(s) X^{s} \mathrm{~d} s \tag{6.6}
\end{equation*}
$$

Next, we will offer the proofs of the theorems corresponding to $z=k$, where $k$ is a non-negative integer.
Proof of Theorem 2.1. Letting $f_{k}(n)=\sigma_{k, \chi}(n)$ where $\chi$ being an odd primitive character modulo $q$ and $k$ an even, non-negative integer in (6.5), we obtain

$$
\begin{equation*}
\sum_{n=1}^{\infty} \sigma_{k, \chi}(n) n^{\nu / 2} K_{\nu}(a \sqrt{n x})=\frac{1}{2} X^{\nu / 2}\left(R_{k+1}+R_{1}+R_{0}+J_{z}^{(\nu)}(X)\right) \tag{6.7}
\end{equation*}
$$

where $\Re(\nu)>0$ and $J_{k}^{(\nu)}(X)$ is defined in (6.6) with $F_{k}(s)=\zeta(s) L(s-k, \chi)$. It is easy to see that $R_{k+1}=0$ as the integrand function in (6.6) does not have any pole at $s=k+1$. Here, one can notice that $L(s-k, \chi)$ has a zero at $s=1$ when $k \geq 2$ is an even integer and $\chi$ is odd. Therefore, we will not get any contribution from the pole of $\zeta(s)$ at $s=1$. However, if $k=0$, the integrand in (6.6) will encounter a pole at $s=1$. Therefore, we can get

$$
R_{1}= \begin{cases}0, & \text { if } k>0  \tag{6.8}\\ \Gamma(1+\nu) L(1, \chi) X, & \text { if } k=0\end{cases}
$$

The integrand also has a pole at $s=0$ with residue $R_{0}$ given by

$$
\begin{equation*}
R_{0}=-\frac{\Gamma(\nu) L(-k, \chi)}{2}=\frac{(-1)^{\frac{k}{2}} i \tau(\chi)}{2 \pi}\left(\frac{(2 \pi)}{q}\right)^{-k} \Gamma(1+k) \Gamma(\nu) L(1+k, \bar{\chi}) \tag{6.9}
\end{equation*}
$$

where in the last step, we have applied functional equation (5.16). Collecting (6.8) and (6.9) and $R_{k+1}=0$ and then substituting them in (6.7), we get

$$
\begin{align*}
X^{-\frac{\nu}{2}} \sum_{n=1}^{\infty} \sigma_{k, \chi}(n) n^{\frac{\nu}{2}} K_{\nu}(a \sqrt{n x})= & \frac{(-1)^{\frac{k}{2}} i \tau(\chi)}{4 \pi}\left(\frac{(2 \pi)}{q}\right)^{-k} \Gamma(1+k) \Gamma(\nu) L(1+k, \bar{\chi}) \\
& +\delta_{k} \frac{\Gamma(1+\nu) L(1, \chi)}{2} X+\frac{1}{2} J_{k}^{(\nu)}(X) \tag{6.10}
\end{align*}
$$

where $\delta_{k}$ is defined in (2.3). To evaluate $J_{k}^{(\nu)}(X)$ defined in (6.6), we invoke the functional equations (5.14) and (5.16),

$$
\begin{aligned}
J_{k}^{(\nu)}(X) & =\frac{h_{k}}{2 \pi i} \int_{(-d)} \Gamma(s+\nu) \Gamma(1+k-s) \zeta(1-s) L(1-s+k, \bar{\chi}) Y^{s} \mathrm{~d} s \\
& =\frac{Y h_{k}}{2 \pi i} \int_{(1+d)} \Gamma(1-s+\nu) \Gamma(k+s) \zeta(s) L(s+k, \bar{\chi}) Y^{-s} \mathrm{~d} s \\
& =Y h_{k} \sum_{n=1}^{\infty} \sigma_{-k, \bar{\chi}}(n) \frac{1}{2 \pi i} \int_{(1+d)} \Gamma(1-s+\nu) \Gamma(k+s)(n Y)^{-s} \mathrm{~d} s
\end{aligned}
$$

where $h_{k}=\frac{(-1)^{1+\frac{k}{2}} i \tau(\chi)}{2 \pi}\left(\frac{q}{2 \pi}\right)^{k}$ and $Y=\frac{4 \pi^{2}}{q} X$ with $X=\frac{4}{a^{2} x}$. As $0<d<\Re(\nu)$, we can apply Lemma 5.5 with $n=k$ and $a=1+\nu$ to obtain

$$
\begin{align*}
J_{k}^{(\nu)}(X) & =Y^{k+1} \Gamma(1+\nu) h_{k} \sum_{n=1}^{\infty} \sigma_{-k, \bar{\chi}}(n) \frac{(\nu+1) \cdots(\nu+k) n^{k}}{(1+n Y)^{1+\nu+k}} \\
& =Y^{k+1} \Gamma(1+\nu+k) h_{k} \sum_{n=1}^{\infty} \frac{\bar{\sigma}_{k, \chi}(n)}{(1+n Y)^{1+\nu+k}} \tag{6.11}
\end{align*}
$$

where in the penultimate step we have used the fact $n^{k} \sigma_{-k, \chi}(n)=\bar{\sigma}_{k, \chi}(n)$. Therefore, remarking $Y=\frac{16 \pi^{2}}{a^{2} q x}$ and inserting (6.11) in (6.10) and simplifying, we can complete the proof.
Proof of Theorem 2.2. Let us begin the proof by taking $f_{k}(n)=\sigma_{k, \chi}(n)$ with $\chi$ being an odd primitive character modulo $q$ and $k \geq 0$ an even integer and $\nu=0$ in (6.5). The corresponding Dirichlet series, in this case, is $F_{k}(s)=\zeta(s) L(s-k, \chi)$. Therefore, we obtain

$$
\begin{equation*}
\sum_{n=1}^{\infty} \sigma_{k, \chi}(n) K_{0}(a \sqrt{n x})=\frac{1}{2}\left(R_{k+1}+R_{1}+R_{0}+J_{k}^{(0)}(X)\right) \tag{6.12}
\end{equation*}
$$

where $J_{k}^{(0)}(X)$ is defined in (6.6). It is clear that $R_{k+1}=0$ for $k \geq 0 . L(s-k, \chi)$ has a zero at $s=1$ in case $k \geq 2$ is an even integer, and $\chi$ is odd. So we will not get any contribution from the pole of $\zeta(s)$ at $s=1$. However, in the case of $k=0$, the integrand in (6.6) will encounter a pole at $s=1$. Hence, we can write

$$
R_{1}= \begin{cases}0, & \text { if } k>0  \tag{6.13}\\ L(1, \chi) X, & \text { if } k=0\end{cases}
$$

and the integrand in (6.6) encounters a double pole at $s=0$ with residue $R_{0}$ given by

$$
\begin{equation*}
R_{0}=-\frac{L(-k, \chi)}{2}\left(\log (2 \pi X)+\frac{L^{\prime}(-k, \chi)}{L(-k, \chi)}-2 \gamma\right) . \tag{6.14}
\end{equation*}
$$

Now using (6.13) and (6.14) and the fact $R_{k+1}=0$ in (6.12), we obtain

$$
\begin{equation*}
\sum_{n=1}^{\infty} \sigma_{k, \chi}(n) K_{0}(a \sqrt{n x})=\frac{\delta_{k}}{2} L(1, \chi) X-\frac{L(-k, \chi)}{4}\left(\log (2 \pi X)+\frac{L^{\prime}(-k, \chi)}{L(-k, \chi)}-2 \gamma\right)+\frac{1}{2} J_{k}^{(0)}(X) \tag{6.15}
\end{equation*}
$$

where $\delta_{k}$ is defined in (2.3). For $J_{k}^{(0)}(X)$, we employ the functional equations (5.14) and (5.16) to obtain

$$
\begin{align*}
J_{k}^{(0)}(X) & =\frac{h_{k}}{2 \pi i} \int_{(-d)} \Gamma(s) \Gamma(1+k-s) \zeta(1-s) L(1-s+k, \bar{\chi}) Y^{s} \mathrm{~d} s \\
& =\frac{Y h_{k}}{2 \pi i} \int_{(1+d)} \Gamma(1-s) \Gamma(k+s) \zeta(s) L(s+k, \bar{\chi}) Y^{-s} \mathrm{~d} s \\
& =Y h_{k} \sum_{n=1}^{\infty} \sigma_{-k, \bar{\chi}}(n) \frac{1}{2 \pi i} \int_{(1+d)} \Gamma(1-s) \Gamma(k+s)(n Y)^{-s} \mathrm{~d} s \\
& =\pi Y h_{k} \sum_{n=1}^{\infty} \sigma_{-k, \bar{\chi}}(n) \frac{1}{2 \pi i} \int_{(1+d)} \frac{\Gamma(k+s)}{\Gamma(s) \sin (\pi s)}(n Y)^{-s} \mathrm{~d} s \\
& =\pi Y h_{k}\left(\sum_{n \leq Y-1}+\sum_{n>Y^{-1}}\right) \sigma_{-k, \bar{\chi}}(n) \frac{1}{2 \pi i} \int_{(1+d)} \frac{\Gamma(k+s)}{\Gamma(s) \sin (\pi s)}(n Y)^{-s} \mathrm{~d} s \tag{6.16}
\end{align*}
$$

where $h_{k}=\frac{(-1)^{1+\frac{k}{2}} i \tau(\chi)}{2 \pi}\left(\frac{q}{2 \pi}\right)^{k}$ and $Y=\frac{4 \pi^{2}}{q} X$ with $X=\frac{4}{a^{2} x}$. In the second last step, we have used the reflection formula (5.5).

We will first investigate the infinite sum $\sum_{n>Y^{-1}}$. To evaluate this inner line integral in (6.16), we shall use the Cauchy residue theorem with the contour consisting of the line segments $[1+d-i T, 1+d+i T],[1+d+$ $\left.i T, M+\frac{1}{2}+i T\right],\left[M+\frac{1}{2}+i T, M+\frac{1}{2}-i T\right],\left[M+\frac{1}{2}-i T, 1+d-i T\right]$ where $M \in \mathbb{N}$ is a large number and $T$ is a large positive number. The poles of the integrand function in (6.16) are at $2,3, \cdots, M$, and they are simple. The residue at $s=m$ is given by

$$
\begin{equation*}
\mathcal{R}_{m}:=\frac{1}{\pi}(-1)^{m} m(m+1) \ldots(m+k-1)(n Y)^{-m} \tag{6.17}
\end{equation*}
$$

where $m=2,3, \cdots, M$. Employing Lemma 5.1, we can show that both the integrals along the horizontal lines $\left[1+d+i T, M+\frac{1}{2}+i T\right]$ and $\left[M+\frac{1}{2}-i T, 1+d-i T\right]$ vanish as $T \rightarrow \infty$. From (6.17), we arrive at

$$
\begin{aligned}
\frac{1}{2 \pi i} \int_{(1+d)} \frac{\Gamma(k+s)}{\Gamma(s) \sin (\pi s)}(n Y)^{-s} \mathrm{~d} s & =-\sum_{m=2}^{M} \mathcal{R}_{m}+\frac{1}{2 \pi i} \int_{\left(M+\frac{1}{2}\right)} \frac{\Gamma(k+s)}{\Gamma(s) \sin (\pi s)}(n Y)^{-s} \mathrm{~d} s \\
& =-\frac{1}{\pi} \sum_{m=2}^{M}(-1)^{m} m(m+1) \ldots(m+k-1)(n Y)^{-m}+O_{k}\left(\frac{M^{k}}{(n Y)^{M+1 / 2}}\right)
\end{aligned}
$$

where we have used $|\sin \pi(\sigma+i t)| \gg e^{\pi|t|}$ for $|t| \geq 1$ to bound the integral $\int_{\left(M+\frac{1}{2}\right)}$ and the implied constant depends on $k$. Next, allowing $M \rightarrow \infty$, the error term goes to 0 as $n>Y^{-1}$. Now simplifying, we readily obtain that

$$
\frac{1}{2 \pi i} \int_{(1+d)} \frac{\Gamma(k+s)}{\Gamma(s) \sin (\pi s)}(n Y)^{-s} \mathrm{~d} s=-\frac{k!}{\pi}\left(\frac{1}{n Y}\right)\left(1-\frac{n^{k+1}}{\left(Y^{-1}+n\right)^{k+1}}\right)
$$

and we easily deduce from the above expression that

$$
\begin{equation*}
\sum_{n>Y^{-1}} \sigma_{-k, \bar{\chi}}(n) \frac{1}{2 \pi i} \int_{(1+d)} \frac{\Gamma(k+s)}{\Gamma(s) \sin (\pi s)}(n Y)^{-s} \mathrm{~d} s=-\frac{k!}{\pi Y} \sum_{n>Y^{-1}} \frac{\sigma_{-k, \bar{\chi}}(n)}{n}\left(1-\frac{n^{k+1}}{\left(Y^{-1}+n\right)^{k+1}}\right) . \tag{6.18}
\end{equation*}
$$

Similarly, by shifting the line of integration to the left, we obtain

$$
\begin{equation*}
\sum_{n \leq Y^{-1}} \sigma_{-k, \bar{\chi}}(n) \frac{1}{2 \pi i} \int_{(1+d)} \frac{\Gamma(k+s)}{\Gamma(s) \sin (\pi s)}(n Y)^{-s} \mathrm{~d} s=-\frac{k!}{\pi Y} \sum_{n \leq Y^{-1}} \frac{\sigma_{-k, \bar{\chi}}(n)}{n}\left(1-\frac{n^{k+1}}{\left(Y^{-1}+n\right)^{k+1}}\right) . \tag{6.19}
\end{equation*}
$$

Inserting (6.18) and (6.19) in (6.16),

$$
\begin{equation*}
J_{k}^{(0)}(X)=-k!h_{k} \sum_{n=1}^{\infty} \frac{\sigma_{-k, \bar{\chi}}(n)}{n}\left(1-\frac{n^{k+1}}{\left(Y^{-1}+n\right)^{k+1}}\right) . \tag{6.20}
\end{equation*}
$$

We finish the proof by noting $Y=\frac{16 \pi^{2}}{a^{2} q x}$, substituting ( (6.20) in (6.15) and then simplifying.
Proof of Theorem 2.3. Here we will take $f_{k}(n)=\bar{\sigma}_{k, \chi}(n)$ and $\Re(\nu)>0$ in (6.5). Similar to the previous theorems, $\chi$ is odd and $k \geq 2$ is an even integer, and the corresponding Dirichlet series is $F_{k}(s)=\zeta(s-k) L(s, \chi)$. It is clear that $R_{1}=0$. When $k \geq 2$ is an even integer, $\zeta(s-k)$ has a zero at $s=0$. Therefore, we will not get any contribution from the pole of $\Gamma(s)$ at $s=0$. But the integrand in (6.6) will encounter a pole at $s=k+1$ with the residue $R_{k+1}$ given by

$$
R_{k+1}=\Gamma(k+1) \Gamma(\nu+k+1) L(k+1, \chi) X^{k+1}
$$

The calculation for $J_{k}^{(0)}(X)$ will be similar as given in the proof of Theorem 2.1. To avoid repetition, we skip the detail of the proof.
Proof of Theorem 2.4. Here we will consider $f_{k}(n)=\bar{\sigma}_{k, \chi}(n)$ with $\chi$ being an odd primitive character modulo $q$ and $k \geq 2$ an even integer and $\nu=0$ in (6.5). We skip the detail of the proof because of its similarity with the proof of Theorem [2.2.
Proofs of Theorems 2.5 and 2.6. Here, we will take $f_{k}(n)=\sigma_{k, \chi}(n)$ and $\chi$ being a non-principal even primitive Dirichlet character modulo $q$ and $k \geq 1$ an odd integer in (6.5). We can see that Theorem 2.5 deals with the case $\Re(\nu)>0$ while Theorem [2.6 concerns with $\nu=0$. Proceeding by almost identically the same argument as in the proof of Theorems 2.1 and 2.2, one can deduce Theorems 2.5 and 2.6, respectively. We leave the details of the proofs for the reader.
Proofs of Theorems 2.7 and 2.8. The proofs are similar to the corresponding proofs of Theorems 2.3 and 2.4 for the odd character.

Proof of Theorem 2.9. It deals with the special case $k=0$ and $\nu=0$ when $\chi$ is a non-principal even primitive character modulo $q$. Thus setting $f_{0}(n)=d_{\chi}(n)$ in (6.5), we obtain

$$
\begin{equation*}
\sum_{n=1}^{\infty} d_{\chi}(n) K_{0}(a \sqrt{n x})=\frac{1}{2}\left(R_{1}+R_{0}+J_{0}^{(0)}(X)\right) \tag{6.21}
\end{equation*}
$$

where the residues $R_{1}$ and $R_{0}$ are given by

$$
\begin{align*}
& R_{1}=L(1, \chi) X,  \tag{6.22}\\
& R_{0}=-\frac{1}{2} L^{\prime}(0, \chi)=-\frac{\tau(\chi)}{4} L(1, \bar{\chi}), \tag{6.23}
\end{align*}
$$

where in (6.23), we have used [36, p. 181, equation (3.2)]. Next, we evaluate $J_{0}^{(0)}(X)$ defined in (6.6) with $F_{0}(s)=\zeta(s) L(s, \chi)$. Utilizing the functional equations (5.14) and (5.16), one can get

$$
\begin{equation*}
J_{0}^{(0)}(X)=\frac{\tau(\chi)}{4} Y \sum_{n=1}^{\infty} d_{\bar{\chi}}(n) \frac{1}{2 \pi i} \int_{(1+d)} \frac{(n Y)^{-s}}{\sin ^{2}(\pi s / 2)} \mathrm{d} s \tag{6.24}
\end{equation*}
$$

where $Y=\frac{4 \pi^{2} X}{q}$. As $0<d<1$, applying inverse Mellin transform to (5.9) of Lemma 5.7 and then employing the formula in (6.24), we deduce that

$$
\begin{equation*}
J_{0}^{(0)}(X)=\frac{\tau(\chi)}{\pi^{2}} Y \sum_{n=1}^{\infty} d_{\bar{\chi}}(n) \frac{\log (n Y)}{(n Y)^{2}-1} . \tag{6.25}
\end{equation*}
$$

Inserting (6.22), (6.23) and (6.25) in (6.21) and noting $Y=\frac{16 \pi^{2}}{a^{2} q x}$, one can complete the proof.
Next, we are going to investigate the identities involving two characters.

Proof of Theorem 2.10, We will take $f_{k}(n)=\sigma_{k, \chi_{1}, \chi_{2}}(n)$ and $k \geq 1$ an odd integer and $\Re(\nu)>0$ in (6.5). By assumption, $\chi_{1}$ and $\chi_{2}$ are primitive characters modulo $p$ and $q$, respectively, such that either both are non-principal even characters or both are odd characters. In the notation of (6.1), $F_{k}(s)=L\left(s-k, \chi_{1}\right) L\left(s, \chi_{2}\right)$. We get

$$
\begin{equation*}
\sum_{n=1}^{\infty} \sigma_{k, \chi_{1}, \chi_{2}}(n) K_{\nu}(a \sqrt{n x})=\frac{1}{2} X^{\nu / 2}\left(R_{k+1}+R_{1}+R_{0}+J_{k}^{(\nu)}(X)\right) \tag{6.26}
\end{equation*}
$$

where $J_{k}^{(\nu)}(X)$ is defined in (6.6). It is easy to see that $R_{k+1}=0$ and $R_{1}=0$. When both $\chi_{1}$ and $\chi_{2}$ are nonprincipal even primitive characters, $L\left(s, \chi_{2}\right)$ has a zero at $s=0$. Hence we will not be getting any contribution from the pole of $\Gamma(s)$ at $s=0$. As a result, we will get $R_{0}=0$. If both $\chi_{1}$ and $\chi_{2}$ are odd primitive characters, $L\left(s-k, \chi_{1}\right)$ has a zero at $s=0$ since $k$ is an odd integer. Again, there will be no contribution of the pole from $\Gamma(s)$ at $s=0$. Therefore $R_{0}=0$. Now utilizing the facts $R_{k+1}=0, R_{1}=0$ and $R_{0}=0$ in (6.26), we obtain

$$
\begin{equation*}
\sum_{n=1}^{\infty} \sigma_{k, \chi_{1}, \chi_{2}}(n) K_{\nu}(a \sqrt{n x})=\frac{1}{2} X^{\nu / 2} J_{k}^{(\nu)}(X) \tag{6.27}
\end{equation*}
$$

To evaluate $J_{k}^{(\nu)}(X)$, we utilize the functional equations (5.16), (5.17) with (5.18)

$$
\begin{equation*}
J_{k}^{(\nu)}(X)=Y g_{k} \sum_{n=1}^{\infty} \sigma_{-k, \bar{\chi}_{1}, \overline{\chi_{2}}}(n) \frac{1}{2 \pi i} \int_{(1+d)} \Gamma(1-s+\nu) \Gamma(k+s)(n Y)^{-s} \mathrm{~d} s \tag{6.28}
\end{equation*}
$$

where $g_{k}=\frac{(-1)^{\frac{k+1}{2}} p^{k} \tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right)}{(2 \pi)^{k+1}}$ and $Y=\frac{4 \pi^{2}}{p q} X$. As $0<d<1$, appealing to Lemma 5.5 with $n=k$ and $a=1+\nu$, we deduce

$$
\begin{equation*}
J_{k}^{(\nu)}(X)=Y^{k+1} g_{k} \Gamma(1+\nu+k) \sum_{n=1}^{\infty} \frac{\sigma_{k, \bar{\chi}_{2}, \bar{\chi}_{1}}(n)}{(1+n Y)^{1+\nu+k}} \tag{6.29}
\end{equation*}
$$

where we have used the fact $\sigma_{-k, \bar{\chi}_{1}, \bar{\chi}_{2}}(n)=n^{-k} \sigma_{k, \bar{\chi}_{2}, \bar{\chi}_{1}}(n)$. We complete the proof by substituting (6.29) in (6.27) and remarking $Y=\frac{16 \pi^{2}}{a^{2} p q x}$.

Proof of Theorem 2.11. We leave the proof to the reader for its similarity with the proofs of Theorems 2.2 and 2.6.
Proofs of Theorems 2.14 and 2.15, We begin the proof by setting $k=0$ and $\nu=0$ and $f_{0}(n)=d_{\chi_{1}, \chi_{2}}(n)$ in (6.5). This will give

$$
\begin{equation*}
\sum_{n=1}^{\infty} d_{\chi_{1}, \chi_{2}}(n) K_{0}(a \sqrt{n x})=\frac{1}{2}\left(R_{0}+R_{1}+J_{0}^{(0)}(X)\right) \tag{6.30}
\end{equation*}
$$

where $\left.J_{0}^{(0)}(X)\right)$ is defined in (6.6) with $F_{0}(z)=L\left(s, \chi_{1}\right) L\left(s, \chi_{2}\right)$. Here we will have $R_{1}=0$. Now we will discuss the following two cases.
Case 1: When $\chi_{1}$ and $\chi_{2}$ are even non-principal primitive characters modulo $p$ and $q$, respectively. Both $L\left(s, \chi_{1}\right)$ and $L\left(s, \chi_{2}\right)$ have simple zero at $s=0$ which will get cancelled by the double pole of $\Gamma^{2}(s)$ at $s=0$. Hence we have $R_{0}=0$. Employing functional relation (5.16), we obtain

$$
\begin{equation*}
J_{0}^{(0)}(X)=\frac{\tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right)}{4} Y \sum_{n=1}^{\infty} d_{\bar{\chi}_{1}, \bar{\chi}_{2}}(n) \frac{1}{2 \pi i} \int_{(1+d)} \frac{(n Y)^{-s}}{\sin ^{2}\left(\frac{\pi s}{2}\right)} \mathrm{d} s \tag{6.31}
\end{equation*}
$$

where $Y=\frac{4 \pi^{2}}{p q} X$. Note that integral in (6.31) can be treated similarly as in the proof of Theorem [2.9, To avoid repetitions, we skip the detail.
Case 2: When $\chi_{1}$ and $\chi_{2}$ are odd primitive characters modulo $p$ and $q$, respectively. In this case, the integrand will encounter a double pole at $s=0$. Hence the residue $R_{0}$ is given by

$$
\begin{equation*}
R_{0}=L\left(0, \chi_{1}\right) L\left(0, \chi_{2}\right)\left(-2 \gamma+\log (X)+\frac{L^{\prime}\left(0, \chi_{1}\right)}{L\left(0, \chi_{1}\right)}+\frac{L^{\prime}\left(0, \chi_{2}\right)}{L\left(0, \chi_{2}\right)}\right) \tag{6.32}
\end{equation*}
$$

Employing (6.32) and $R_{1}=0$ in (6.30), we obtain

$$
\begin{equation*}
\sum_{n=1}^{\infty} d_{\chi_{1}, \chi_{2}}(n) K_{0}(a \sqrt{n x})=\frac{1}{2} L\left(0, \chi_{1}\right) L\left(0, \chi_{2}\right)\left(-2 \gamma+\log (X)+\frac{L^{\prime}\left(0, \chi_{1}\right)}{L\left(0, \chi_{1}\right)}+\frac{L^{\prime}\left(0, \chi_{2}\right)}{L\left(0, \chi_{2}\right)}\right)+\frac{1}{2} J_{0}^{(0)}(X) . \tag{6.33}
\end{equation*}
$$

Now appealing to functional relation (5.16), we will have

$$
\begin{align*}
J_{0}^{(0)}(X) & =-\frac{\tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right)}{4} Y \sum_{n=1}^{\infty} d_{\bar{\chi}_{1}, \bar{\chi}_{2}}(n) \frac{1}{2 \pi i} \int_{(1+d)} \frac{(n Y)^{-s}}{\cos ^{2}\left(\frac{\pi s}{2}\right)} \mathrm{d} s \\
& =-\frac{\tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right) Y}{4}\left(\sum_{n<Y^{-1}}+\sum_{n>Y^{-1}}\right) d_{\bar{\chi}_{1}, \bar{\chi}_{2}}(n) \frac{1}{2 \pi i} \int_{(1+d)} \frac{(n Y)^{-s}}{\cos ^{2}\left(\frac{\pi s}{2}\right)} \mathrm{d} s, \tag{6.34}
\end{align*}
$$

where $Y=\frac{16 \pi^{2}}{a^{2} p q x}$ and $Y^{-1} \notin \mathbb{Z}_{+}$.
We first evaluate the inner line integral on the sum $\sum_{n>Y^{-1}}$. We shall use the Cauchy residue theorem with the contour formed by the lines $[1+d-i T, 1+d+i T],\left[1+d+i T, M+\frac{1}{2}+i T\right],\left[M+\frac{1}{2}+i T, M+\frac{1}{2}-i T\right],[M+$ $\left.\frac{1}{2}-i T, 1+d-i T\right]$ where $M \in \mathbb{N}$ is any odd large number and $T$ is a large positive number. The poles of the integrand function in (6.34) are at $3,5, \cdots, M$, and they are double poles. The residue at $s=m$ is given by

$$
\begin{equation*}
\mathcal{R}_{m}:=-\frac{4}{\pi^{2}}(n Y)^{-m} \log (n Y), \tag{6.35}
\end{equation*}
$$

where $m=3,5, \cdots, M$. Employing Lemma 5.1, we can show both the integrals along the horizontal line segments $\left[1+d+i T, M+\frac{1}{2}+i T\right]$ and $\left[M+\frac{1}{2}-i T, 1+d-i T\right]$ vanish as $T \rightarrow \infty$. Utilising (6.35), we arrive at

$$
\begin{align*}
\frac{1}{2 \pi i} \int_{(1+d)} \frac{(n Y)^{-s}}{\cos ^{2}\left(\frac{\pi s}{2}\right)} \mathrm{d} s & =-\sum_{m=1}^{\frac{M-1}{2}} \mathcal{R}_{2 m+1}+\frac{1}{2 \pi i} \int_{\left(M+\frac{1}{2}\right)} \frac{(n Y)^{-s}}{\cos ^{2}\left(\frac{\pi s}{2}\right)} \mathrm{d} s \\
& =\frac{4}{\pi^{2}} \sum_{m=1}^{\frac{M-1}{2}} \frac{\log (n Y)}{(n Y)^{2 m+1}}+O\left(\frac{1}{(n Y)^{M+1 / 2}}\right) \tag{6.36}
\end{align*}
$$

Letting $M \rightarrow \infty$, the error term in (6.36) goes to 0 as $n>Y^{-1}$. Thus simplifying, we can readily deduce that

$$
\frac{1}{2 \pi i} \int_{(1+d)} \frac{(n Y)^{-s}}{\cos ^{2}\left(\frac{\pi s}{2}\right)} \mathrm{d} s=\frac{4}{\pi^{2}} \sum_{m=1}^{\infty} \frac{\log (n Y)}{(n Y)^{2 m+1}}=\frac{4}{\pi^{2} Y^{3}} \frac{1}{n\left(n^{2}-Y^{-2}\right)} \log (n Y)
$$

and subsequently, we get

$$
\begin{equation*}
\sum_{n>Y^{-1}} d_{\bar{\chi}_{1}, \bar{\chi}_{2}}(n) \frac{1}{2 \pi i} \int_{(1+d)} \frac{(n Y)^{-s}}{\cos ^{2}\left(\frac{\pi s}{2}\right)} \mathrm{d} s=\frac{4}{\pi^{2}} \sum_{n>Y^{-1}} d_{\bar{\chi}_{1}, \bar{\chi}_{2}}(n) \frac{Y^{-3}}{n\left(n^{2}-Y^{-2}\right)} \log (n Y) . \tag{6.37}
\end{equation*}
$$

Similarly, by shifting the integration line to the left, we get

$$
\begin{equation*}
\sum_{n<Y^{-1}} d_{\bar{\chi}_{1}, \bar{\chi}_{2}}(n) \frac{1}{2 \pi i} \int_{(1+d)} \frac{(n Y)^{-s}}{\cos ^{2}\left(\frac{\pi s}{2}\right)} \mathrm{d} s=\frac{4}{\pi^{2}} \sum_{n \leq Y^{-1}} d_{\bar{\chi}_{1}, \bar{\chi}_{2}}(n) \frac{Y^{-3}}{n\left(n^{2}-Y^{-2}\right)} \log (n Y) . \tag{6.38}
\end{equation*}
$$

Hence combining (6.37) and (6.38) with (6.34), we obtain

$$
\begin{equation*}
J_{0}^{(0)}(X)=-\frac{\tau\left(\chi_{1}\right) \tau\left(\chi_{2}\right)}{\pi^{2}} \sum_{n=1}^{\infty} d_{\bar{\chi}_{1}, \bar{\chi}_{2}}(n) \frac{Y^{-2}}{n\left(n^{2}-Y^{-2}\right)} \log (n Y) . \tag{6.39}
\end{equation*}
$$

Inserting (6.39) in (6.33) and remarking that $Y=\frac{16 \pi^{2}}{a^{2} p q x}$, we get the desired result.
Proofs of Theorems 2.16 and 2.17. The proofs are similar to the proofs of Theorems 2.10 and 2.11.

## 7. Proof of Cohen-Type Identities

This section is devoted to the proof of Cohen-type identities. Throughout this section, we will deal with $z=-\nu \notin \mathbb{Z}$ and $\Re(\nu) \geq 0$. In general set up, if we consider $\Re(\nu) \geq 0$ with $\nu \notin \mathbb{Z}$ and $a=4 \pi$, then (6.5) becomes

$$
\begin{equation*}
\sum_{n=1}^{\infty} f_{-\nu}(n) n^{\nu / 2} K_{\nu}(4 \pi \sqrt{n x})=\frac{1}{2} X^{\nu / 2}\left(R_{1-\nu}+R_{1}+R_{0}+R_{-\nu}+J_{-\nu}^{(\nu)}(X)\right) \tag{7.1}
\end{equation*}
$$

where $R_{-\nu}$ is the residue corresponding to the pole $s=-\nu$. It is easy to see that the pole at $s=-\nu$ appears if $\Re(\nu)=0$ and $\nu \notin \mathbb{Z}$. The expression $J_{-\nu}^{(\nu)}(X)$ defined in (6.6), can be rewritten as

$$
\begin{equation*}
J_{-\nu}^{(\nu)}(X):=\frac{1}{2 \pi i} \int_{(-d)} \Gamma(s+\nu) \Gamma(s) F_{-\nu}(s) X^{s} \mathrm{~d} s \tag{7.2}
\end{equation*}
$$

where $F_{-\nu}(s)$ defined in (6.1) is the Dirichlet series associated with the arithmetical function $f_{-\nu}(n)$. We will note that $X=\frac{1}{4 \pi^{2} x}$.
Proof of Theorem 3.1 Letting $f_{-\nu}(n)=\sigma_{-\nu, \bar{\chi}}(n)$ where $\chi$ being a non-principal even primitive character modulo $q$ in (7.1), we obtain

$$
\begin{equation*}
\sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}}(n) n^{\nu / 2} K_{\nu}(4 \pi \sqrt{n x})=\frac{1}{2} X^{\nu / 2}\left(R_{1-\nu}+R_{1}+R_{0}+R_{-\nu}+J_{-\nu}^{(\nu)}(X)\right) \tag{7.3}
\end{equation*}
$$

where $J_{-\nu}^{(\nu)}(X)$ is given in (7.2) and $F_{-\nu}(s)=\zeta(s) L(s+\nu, \bar{\chi})$. The integrand in (7.2) will encounter simple poles at $s=1$ and $s=0$ with residues $R_{1}$ and $R_{0}$ given by

$$
\begin{equation*}
R_{1}=\Gamma(1+\nu) L(1+\nu, \bar{\chi}) X, \quad \text { and } \quad R_{0}=-\frac{\Gamma(\nu) L(\nu, \bar{\chi})}{2} \tag{7.4}
\end{equation*}
$$

respectively. It is easy to see that $R_{1-\nu}=0$. As $\bar{\chi}$ is a non-principal even primitive character, $L(s+\nu, \bar{\chi})$ has a zero at $s=-\nu$. Therefore, we will not be getting any contribution from the pole of $\Gamma(s+\nu)$ at $s=-\nu$. Hence $R_{-\nu}=0$. Now employing (7.4) together with the facts $R_{1-\nu}=0$ and $R_{-\nu}=0$ in (7.3), we obtain

$$
\begin{equation*}
\sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}}(n) n^{\nu / 2} K_{\nu}(4 \pi \sqrt{n x})=\frac{1}{2} X^{\nu / 2}\left(\Gamma(1+\nu) L(1+\nu, \bar{\chi}) X-\frac{\Gamma(\nu) L(\nu, \bar{\chi})}{2}+J_{-\nu}^{(\nu)}(X)\right) \tag{7.5}
\end{equation*}
$$

where $X=\frac{1}{4 \pi^{2} x}$. Next, we evaluate the following integral $J_{-\nu}^{(\nu)}(X)$. Replacing $s$ by $1-s$ and then employing (5.14), (5.17), we obtain

$$
\begin{align*}
J_{-\nu}^{(\nu)}(X) & =\frac{1}{2 \pi i} \int_{(1+d)} \Gamma(1-s+\nu) \Gamma(1-s) \zeta(1-s) L(1-s+\nu, \bar{\chi}) X^{1-s} \mathrm{~d} s \\
& =\left(\frac{2 \pi}{q}\right)^{\nu} \frac{\pi^{2} X}{\tau(\chi)} \frac{1}{2 \pi i} \int_{(1+d)} \frac{\zeta(s) L(s-\nu, \chi)}{\left((2 \pi)^{2} q^{-1} X\right)^{s} \sin \left(\frac{\pi s}{2}\right) \sin \left(\frac{\pi(s-\nu)}{2}\right)} \mathrm{d} s \\
& =\frac{(2 \pi)^{\nu} q^{-\nu}}{4 x \tau(\chi)} \frac{1}{2 \pi i} \int_{(1+d)} \frac{\zeta(s) L(s-\nu, \chi)(q x)^{s}}{\sin \left(\frac{\pi s}{2}\right) \sin \left(\frac{\pi(s-\nu)}{2}\right)} \mathrm{d} s \tag{7.6}
\end{align*}
$$

To evaluate the integral in (7.6), we employ the Cauchy residue theorem with the rectangular contour formed by the lines $[1+d-i T, 1+d+i T],[1+d+i T, 2 N+\delta+i T],[2 N+\delta+i T, 2 N+\delta-i T],[2 N+\delta-i T, 1+d-i T]$ with $N \geq\left[\frac{\Re(\nu)+1}{2}\right]$ and $\{\Re(\nu)+1\}<\delta<1$ and $T$ is a large positive number. One can note that the simple poles of $\sin ^{-1}\left(\frac{\pi(s-\nu)}{2}\right)$ at $s=\nu, \nu-2, \cdots$ will get canceled by the simple zeroes of $L(s-\nu, \chi)$. Hence the poles of the integrand function in (7.6) are at $2,4, \cdots, 2 N$, and $\nu+2, \cdots, \nu+2 b_{N}$ where $b_{N}=\left\lfloor\frac{2 N+\delta-\nu}{2}\right\rfloor$, and they are simple. Utilising the fact $|\sin \pi(\sigma+i t)| \gg e^{\pi|t|}$ for $|t| \geq 1$, one can see that the integrals along the horizontal lines $[1+d+i T, 2 N+\delta+i T]$ and $[2 N+\delta-i T, 1+d-i T]$ vanish as $T \rightarrow \infty$. Hence we get

$$
\begin{equation*}
\frac{1}{2 \pi i} \int_{(1+d)} \frac{\zeta(s) L(s-\nu, \chi)(q x)^{s}}{\sin \left(\frac{\pi s}{2}\right) \sin \left(\frac{\pi(s-\nu)}{2}\right)} \mathrm{d} s=-\sum_{j=1}^{N} H_{2 j}-\sum_{r=1}^{b_{N}} \mathcal{H}_{2 r}+\frac{1}{2 \pi i} \int_{(2 N+\delta)} \frac{\zeta(s) L(s-\nu, \chi)(q x)^{s}}{\sin \left(\frac{\pi s}{2}\right) \sin \left(\frac{\pi(s-\nu)}{2}\right)} \mathrm{d} s \tag{7.7}
\end{equation*}
$$

where $H_{2 j}$ is the residue at $s=2 j$ given by

$$
H_{2 j}=-\frac{2 \zeta(2 j) L(2 j-\nu, \chi)(q x)^{2 j}}{\pi \sin \left(\frac{\pi \nu}{2}\right)}
$$

for $j=1,2, \cdots, N$ and $\mathcal{H}_{2 r}$ is the residue at $s=\nu+2 r$ given by

$$
\mathcal{H}_{2 r}=2 \zeta(\nu+2 r) L(2 r, \chi) \frac{(q x)^{\nu+2 r}}{\pi \sin \left(\frac{\pi \nu}{2}\right)}=\frac{2}{\pi \sin \left(\frac{\pi \nu}{2}\right)} \sum_{n=1}^{\infty} \sigma_{\nu, \chi}(n)\left(n^{-1} q x\right)^{\nu+2 r},
$$

for $r=1,2, \cdots, b_{N}$. In the above expression, we have applied the series representation of function $\zeta(\nu+$ $2 r) L(2 r, \chi)$ for $r \geq 1$. Now let us evaluate the integral in (7.7):

$$
\begin{align*}
& \frac{1}{2 \pi i} \int_{(2 N+\delta)} \frac{\zeta(s) L(s-\nu, \chi)(q x)^{s}}{\sin \left(\frac{\pi s}{2}\right) \sin \left(\frac{\pi(s-\nu)}{2}\right)} \mathrm{d} s \\
& =\sum_{n=1}^{\infty} \sigma_{\nu, \chi}(n) \frac{1}{2 \pi i} \int_{(2 N+\delta)} \frac{\left(n^{-1} q x\right)^{s}}{\sin \left(\frac{\pi s}{2}\right) \sin \left(\frac{\pi(s-\nu)}{2}\right)} \mathrm{d} s \\
& =\left(\sum_{n<q x}+\sum_{n>q x}\right) \sigma_{\nu, \chi}(n) \frac{1}{2 \pi i} \int_{(2 N+\delta)} \frac{\left(n^{-1} q x\right)^{s}}{\sin \left(\frac{\pi s}{2}\right) \sin \left(\frac{\pi(s-\nu)}{2}\right)} \mathrm{d} s, \tag{7.8}
\end{align*}
$$

noting that $q x \notin \mathbb{Z}_{+}$.
Next, we will first investigate the sum $\sum_{n>q x}$. To evaluate this inner line integral in (7.8), we shall use the Cauchy residue theorem with the contour consisting of the lines $[2 N+\delta-i T, 2 N+\delta+i T],[2 N+\delta+$ $\left.i T, 2 M+\frac{1}{2}+i T\right],\left[2 M+\frac{1}{2}+i T, 2 M+\frac{1}{2}-i T\right],\left[2 M+\frac{1}{2}-i T, 2 N+\delta-i T\right]$ where $M \in \mathbb{N}$ is a large number and $T$ is a large positive number. The poles of the integrand function in (7.8) are at $2 N+2,2 N+4, \cdots, 2 M$ and $\nu+2 b_{N}+2, \nu+2 b_{N}+4, \cdots, \nu+2 a_{M}$ where $a_{M}=\left\lfloor M+\frac{1}{4}-\frac{\nu}{2}\right\rfloor$, and they are simple. Now taking into account the fact that both the integrals along the horizontal lines $\left[2 M+\frac{1}{2}-i T, 2 N+\delta-i T\right]$ and $\left[2 N+\delta+i T, 2 M+\frac{1}{2}+i T\right]$ vanish as $T \rightarrow \infty$, we obtain

$$
\begin{aligned}
& \frac{1}{2 \pi i} \int_{(2 N+\delta)} \frac{\left(n^{-1} q x\right)^{s}}{\sin \left(\frac{\pi s}{2}\right) \sin \left(\frac{\pi(s-\nu)}{2}\right)} \mathrm{d} s \\
& =\frac{2}{\pi \sin \left(\frac{\pi \nu}{2}\right)}\left(\sum_{r=N+1}^{M}\left(n^{-1} q x\right)^{2 r}-\sum_{r=b_{N}+1}^{a_{M}}\left(n^{-1} q x\right)^{\nu+2 r}\right)+\frac{1}{2 \pi i} \int_{\left(2 M+\frac{1}{2}\right)} \frac{\left(n^{-1} q x\right)^{s}}{\sin \left(\frac{\pi s}{2}\right) \sin \left(\frac{\pi(s-\nu)}{2}\right)} \mathrm{d} s \\
& =\frac{2}{\pi \sin \left(\frac{\pi \nu}{2}\right)}\left(\sum_{r=N+1}^{M}\left(n^{-1} q x\right)^{2 r}-\sum_{r=b_{N}+1}^{a_{M}}\left(n^{-1} q x\right)^{\nu+2 r}\right)+O\left(\left(n^{-1} q x\right)^{2 M+\frac{1}{2}}\right) .
\end{aligned}
$$

Letting $M \rightarrow \infty$, the error term in the above expression goes to 0 since $n^{-1} q x<1$. Therefore, we get

$$
\frac{1}{2 \pi i} \int_{(2 N+\delta)} \frac{\left(n^{-1} q x\right)^{s}}{\sin \left(\frac{\pi s}{2}\right) \sin \left(\frac{\pi(s-\nu)}{2}\right)} \mathrm{d} s=\frac{2}{\pi \sin \left(\frac{\pi \nu}{2}\right)}\left(\sum_{r=N+1}^{\infty}\left(n^{-1} q x\right)^{2 r}-\sum_{r=b_{N}+1}^{\infty}\left(n^{-1} q x\right)^{\nu+2 r}\right)
$$

which in turn will give

$$
\begin{aligned}
& \sum_{n>q x} \sigma_{\nu, \chi}(n) \frac{1}{2 \pi i} \int_{(2 N+\delta)} \frac{\left(n^{-1} q x\right)^{s}}{\sin \left(\frac{\pi s}{2}\right) \sin \left(\frac{\pi(s-\nu)}{2}\right)} \mathrm{d} s \\
& =\frac{2}{\pi \sin \left(\frac{\pi \nu}{2}\right)} \sum_{n>q x} \sigma_{\nu, \chi}(n)\left(\sum_{r=N+1}^{\infty}\left(n^{-1} q x\right)^{2 r}-\sum_{r=b_{N}+1}^{\infty}\left(n^{-1} q x\right)^{\nu+2 r}\right) .
\end{aligned}
$$

From the above expression, one can deduce

$$
\begin{align*}
& \sum_{n>q x} \sigma_{\nu, \chi}(n) \frac{1}{2 \pi i} \int_{(2 N+\delta)} \frac{\left(n^{-1} q x\right)^{s}}{\sin \left(\frac{\pi s}{2}\right) \sin \left(\frac{\pi(s-\nu)}{2}\right)} \mathrm{d} s-\frac{2}{\pi \sin \left(\frac{\pi \nu}{2}\right)} \sum_{n>q x} \sigma_{\nu, \chi}(n) \sum_{r=1}^{b_{N}}\left(n^{-1} q x\right)^{\nu+2 r} \\
& =\frac{2}{\pi \sin \left(\frac{\pi \nu}{2}\right)} \sum_{n>q x} \sigma_{\nu, \chi}(n)\left(\sum_{r=N+1}^{\infty}\left(n^{-1} q x\right)^{2 r}-\sum_{r=1}^{\infty}\left(n^{-1} q x\right)^{\nu+2 r}\right) \\
& =\frac{2}{\pi \sin \left(\frac{\pi \nu}{2}\right)} \sum_{n>q x} \sigma_{\nu, \chi}(n) \frac{(q x)^{2 N+2}}{n^{\nu}} \frac{\left(n^{\nu-2 N}-(q x)^{\nu-2 N}\right)}{n^{2}-q^{2} x^{2}} \\
& =\frac{2(q x)^{2 N+2}}{\pi \sin \left(\frac{\pi \nu}{2}\right)} \sum_{n>q x} \bar{\sigma}_{-\nu, \chi}(n)\left(\frac{n^{\nu-2 N}-(q x)^{\nu-2 N}}{n^{2}-q^{2} x^{2}}\right) . \tag{7.9}
\end{align*}
$$

Similarly, by shifting the line of integration to the left, $\sum_{n \leq q x}$ can be evaluated as

$$
\begin{align*}
& \sum_{n<q x} \sigma_{\nu, \chi}(n) \frac{1}{2 \pi i} \int_{(2 N+\delta)} \frac{\left(n^{-1} q x\right)^{s}}{\sin \left(\frac{\pi s}{2}\right) \sin \left(\frac{\pi(s-\nu)}{2}\right)} \mathrm{d} s-\frac{2}{\pi \sin \left(\frac{\pi \nu}{2}\right)} \sum_{n<q x} \sigma_{\nu, \chi}(n) \sum_{r=1}^{b_{N}}\left(n^{-1} q x\right)^{\nu+2 r} \\
& =-\frac{2}{\pi \sin \left(\frac{\pi \nu}{2}\right)} \sum_{n<q x} \sigma_{\nu, \chi}(n)\left(\sum_{r=-N}^{\infty}\left(n(q x)^{-1}\right)^{2 r}-\sum_{r=0}^{\infty}\left(n(q x)^{-1}\right)^{-\nu+2 r}\right) \\
& =-\frac{2}{\pi \sin \left(\frac{\pi \nu}{2}\right)} \sum_{n<q x} \sigma_{\nu, \chi}(n)\left(\left(\frac{q x}{n}\right)^{2 N} \frac{(q x)^{2}}{(q x)^{2}-n^{2}}-\left(\frac{q x}{n}\right)^{\nu} \frac{(q x)^{2}}{(q x)^{2}-n^{2}}\right) \\
& =\frac{2(q x)^{2 N+2}}{\pi \sin \left(\frac{\pi \nu}{2}\right)} \sum_{n<q x} \bar{\sigma}_{-\nu, \chi}(n)\left(\frac{n^{\nu-2 N}-(q x)^{\nu-2 N}}{n^{2}-q^{2} x^{2}}\right) . \tag{7.10}
\end{align*}
$$

Now substituting (7.9) and (7.10) in (7.8),

$$
\begin{align*}
\frac{1}{2 \pi i} \int_{(2 N+\delta)} \frac{\zeta(s) L(s-\nu, \chi)(q x)^{s}}{\sin \left(\frac{\pi s}{2}\right) \sin \left(\frac{\pi(s-\nu)}{2}\right)} \mathrm{d} s & =\frac{2}{\pi \sin \left(\frac{\pi \nu}{2}\right)} \sum_{n=1}^{\infty} \sigma_{\nu, \chi}(n) \sum_{r=1}^{b_{N}}\left(n^{-1} q x\right)^{\nu+2 r} \\
& +\frac{2(q x)^{2 N+2}}{\pi \sin \left(\frac{\pi \nu}{2}\right)} \sum_{n=1}^{\infty} \bar{\sigma}_{-\nu, \chi}(n)\left(\frac{n^{\nu-2 N}-(q x)^{\nu-2 N}}{n^{2}-q^{2} x^{2}}\right) \tag{7.11}
\end{align*}
$$

Inserting (7.11) in (7.7) and then simplifying, we obtain

$$
\begin{align*}
\frac{1}{2 \pi i} \int_{(1+d)} \frac{\zeta(s) L(s-\nu, \chi)(q x)^{s}}{\sin \left(\frac{\pi s}{2}\right) \sin \left(\frac{\pi(s-\nu)}{2}\right)} \mathrm{d} s & =\frac{2}{\pi \sin \left(\frac{\pi \nu}{2}\right)} \sum_{j=1}^{N} \zeta(2 j) L(2 j-\nu, \chi)(q x)^{2 j} \\
& +\frac{2}{\pi \sin \left(\frac{\pi \nu}{2}\right)}(q x)^{2 N+2} \sum_{n=1}^{\infty} \bar{\sigma}_{-\nu, \chi}(n)\left(\frac{n^{\nu-2 N}-(q x)^{\nu-2 N}}{n^{2}-q^{2} x^{2}}\right) . \tag{7.12}
\end{align*}
$$

Combining (7.12) with (7.6), we deduce that

$$
\begin{align*}
& J_{-\nu}^{(\nu)}(X)=\frac{(2 \pi)^{\nu} q^{-\nu}}{2 \pi x \tau(\chi) \sin \left(\frac{\pi \nu}{2}\right)}\left\{\sum_{j=1}^{N} \zeta(2 j) L(2 j-\nu, \chi)(q x)^{2 j}\right. \\
&\left.+(q x)^{2 N+2} \sum_{n=1}^{\infty} \bar{\sigma}_{-\nu, \chi}(n)\left(\frac{n^{\nu-2 N}-(q x)^{\nu-2 N}}{n^{2}-q^{2} x^{2}}\right)\right\} . \tag{7.13}
\end{align*}
$$

Next, by substituting (7.13) in (7.5), one can finish the proof.
The proofs of other remaining theorems in Section 3 can be proved similarly. We leave the explanations for the readers.

## 8. Proof of Voronoï-type Summation formulas

In this section, we prove Theorem 4.1. The proofs of other theorems will be similar, so we will skip the proofs of other theorems. To prove Theorem 4.1, we will adapt the method introduced by B. C. Berndt, A. Dixit, A. Roy, and A. Zaharescu in [29].
Proof of Theorem 4.1. Let us recall the Theorem 3.1. One can see that identity (3.1) in Theorem 3.1 is valid not only for $x>0$ but also for $-\pi<\arg x<\pi$ by analytic continuation. If we set $N=1$ in (3.1), then the condition $\left\lfloor\frac{\Re(\nu)+1}{2}\right\rfloor \leq 1$ will imply that $0 \leq \Re(\nu)<3$. We consider $0<\Re(\nu)<\frac{1}{2}$. Replace $x$ by $i z / q$ in (3.1) for $-\pi<\arg z<\frac{\pi}{2}$ and then by $-i z / q$ for $-\frac{\pi}{2}<\arg z<\pi$. Now the common region of the resultant identities is $-\frac{\pi}{2}<\arg z<\frac{\pi}{2}$. So we add the resulting two identities and simplify, in the region $-\frac{\pi}{2}<\arg z<\frac{\pi}{2}$, to obtain

$$
\begin{equation*}
\Lambda(z, \nu)=\Psi_{1}(z, \nu) \tag{8.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\Lambda(z, \nu)=2 z^{-\frac{\nu}{2}} \sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}}(n) n^{\nu / 2}\left\{e^{\frac{i \pi \nu}{4}} K_{\nu}\left(4 \pi e^{\frac{i \pi}{4}} \sqrt{\frac{n z}{q}}\right)+e^{\frac{-i \pi \nu}{4}} K_{\nu}\left(4 \pi e^{\frac{-i \pi}{4}} \sqrt{\frac{n z}{q}}\right)\right\}, \tag{8.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\Psi_{1}(z, \nu)=-\frac{q^{\frac{\nu}{2}} \Gamma(\nu) L(\nu, \bar{\chi})}{(2 \pi)^{\nu}} z^{-\nu}+\frac{q^{1-\frac{\nu}{2}}}{\pi \tau(\chi)} \sum_{n=1}^{\infty} \frac{\bar{\sigma}_{-\nu, \chi}(n)}{n^{2}+z^{2}} z . \tag{8.3}
\end{equation*}
$$

Note that $\Psi_{1}(z, \nu)$ is an analytic function of $z$ in $\mathbb{C}$ except on negative real axis and at $z=i n$ where $n \in \mathbb{Z}$. Hence $\Psi_{1}(i z, \nu)$ is analytic in $\mathbb{C}$ except on the positive imaginary axis and at $z \in \mathbb{Z}$. Similarly, $\Psi_{1}(-i z, \nu)$ is analytic in $\mathbb{C}$ except on the negative imaginary axis and at $z \in \mathbb{Z}$. We deduce $\Psi_{1}(i z, \nu)+\Psi_{1}(-i z, \nu)$ is analytic in both the left and right half plane, except possibly when $z$ is an integer. Since

$$
\lim _{z \rightarrow \mp n}(z \pm n) \Psi_{1}(i z, \nu)=\frac{q^{1-\frac{\nu}{2}}}{2 \pi i \tau(\chi)} \bar{\sigma}_{-\nu, \chi}(n), \quad \lim _{z \rightarrow \mp n}(z \pm n) \Psi_{1}(-i z, \nu)=-\frac{q^{1-\frac{\nu}{2}}}{2 \pi i \tau(\chi)} \bar{\sigma}_{-\nu, \chi}(n)
$$

so we have

$$
\lim _{z \rightarrow \mp n}(z \pm n)\left(\Psi_{1}(i z, \nu)+\Psi_{1}(-i z, \nu)\right)=0 .
$$

Hence $\Psi_{1}(i z, \nu)+\Psi_{1}(-i z, \nu)$ is analytic in the entire right half plane. From (8.3), we observe that for $z$ lying inside an interval $(a, b)$ on the positive real line not containing an integer, we have

$$
\begin{equation*}
\Psi_{1}(i z, \nu)+\Psi_{1}(-i z, \nu)=-\frac{2 q^{\frac{\nu}{2}} \Gamma(\nu) L(\nu, \bar{\chi})}{(2 \pi)^{\nu}} \cos \left(\frac{\pi \nu}{2}\right) \frac{1}{z^{\nu}} \tag{8.4}
\end{equation*}
$$

Since both sides are analytic in the right half-complex plane as a function of $z$, by analytic continuation, the identity (8.4) holds for any $z$ in the right half-plane. Next employing functional equation for $L$-function (5.16) in (8.4) and simplifying, we obtain for $-\frac{\pi}{2}<\arg z<\frac{\pi}{2}$,

$$
\begin{equation*}
\Psi_{1}(i z, \nu)+\Psi_{1}(-i z, \nu)=-\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} L(1-\nu, \chi) \frac{1}{z^{\nu}} \tag{8.5}
\end{equation*}
$$

Next, Let $f$ be an analytic function of $z$ in a closed contour $\gamma^{\prime}$ intersecting the real axis in $\alpha$ and $\beta$ where $0<\alpha<\beta, m-1<\alpha<m, n-1<\beta<n$ and $m, n \in \mathbb{Z}$. Now $\gamma^{\prime}$ consists of two parts $\gamma_{1}$ and $\gamma_{2}$ where $\gamma_{1}$ is the portion of the contour in the upper half-plane, and $\gamma_{2}$ is the portion corresponding to lower half-plane. Now $\alpha \gamma_{1} \beta$ and $\alpha \gamma_{2} \beta$ denote the paths from $\alpha$ to $\beta$ in the upper and lower half planes, respectively. By the Cauchy residue theorem, we have

$$
\int_{\alpha \gamma_{2} \beta \gamma_{1} \alpha} f(z) \Psi_{1}(i z, \nu) d z=\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} \sum_{\alpha<j<\beta} \bar{\sigma}_{-\nu, \chi}(j) f(j),
$$

where $\frac{q^{1-\frac{\nu}{2}}}{2 \pi i \tau(\chi)} \bar{\sigma}_{-\nu, \chi}(j) f(j)$ is the residue of $f(z) \Psi_{1}(i z, \nu)$ at each integer $j$ where $\alpha<j<\beta$. Hence the above expression can be rewritten as

$$
\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} \sum_{\alpha<j<\beta} \bar{\sigma}_{-\nu, \chi}(j) f(j)=\int_{\alpha \gamma_{2} \beta} f(z) \Psi_{1}(i z, \nu) d z-\int_{\alpha \gamma_{1} \beta} f(z) \Psi_{1}(i z, \nu) d z
$$

$$
\begin{equation*}
=\int_{\alpha \gamma_{2} \beta} f(z) \Psi_{1}(i z, \nu) d z+\int_{\alpha \gamma_{1} \beta} f(z)\left\{\Psi_{1}(-i z, \nu)+\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} L(1-\nu, \chi) \frac{1}{z^{\nu}}\right\} d z, \tag{8.6}
\end{equation*}
$$

where in the last step, we used (8.5). Again we make use of the Cauchy residue theorem and obtain

$$
\begin{equation*}
\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} L(1-\nu, \chi) \int_{\alpha \gamma_{1} \beta} \frac{f(z)}{z^{\nu}} \mathrm{d} z=\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} L(1-\nu, \chi) \int_{\alpha}^{\beta} \frac{f(t)}{t^{\nu}} \mathrm{d} t . \tag{8.7}
\end{equation*}
$$

From (8.1), $\Lambda(z, \nu)=\Psi_{1}(z, \nu)$ for $-\frac{\pi}{2}<\arg z<\frac{\pi}{2}$. So it is easy to see that $\Lambda(i z, \nu)=\Psi_{1}(i z, \nu)$ holds for $-\pi<\arg z<0$, and $\Lambda(-i z, \nu)=\Psi_{1}(-i z, \nu)$ holds for $0<\arg z<\pi$. Thus

$$
\left\{\begin{array}{l}
\int_{\alpha \gamma_{2} \beta} f(z) \Psi_{1}(i z, \nu) d z=\int_{\alpha \gamma_{2} \beta} f(z) \Lambda(i z, \nu) d z,  \tag{8.8}\\
\int_{\alpha \gamma_{1} \beta} f(z) \Psi_{1}(-i z, \nu) d z=\int_{\alpha \gamma_{1} \beta} f(z) \Lambda(-i z, \nu) d z .
\end{array}\right.
$$

Here we notice that the series $\Lambda(i z, \nu)$ in (8.2) is uniformly convergent in compact subintervals of $-\pi<\arg z<0$, and series $\Lambda(-i z, \nu)$ is uniformly convergent in compact subintervals of $0<\arg z<\pi$. Thus, interchanging the order of summation and integration in (8.8) and inserting them in (8.6) together with (8.7), we get

$$
\begin{aligned}
& \frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} \sum_{\alpha<j<\beta} \bar{\sigma}_{-\nu, \chi}(j) f(j)=\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} L(1-\nu, \chi) \int_{\alpha}^{\beta} \frac{f(t)}{t^{\nu}} \mathrm{d} t+2 \sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}}(n) n^{\nu / 2} \\
& \quad \times \int_{\alpha \gamma_{2} \beta} f(z)(i z)^{-\frac{\nu}{2}}\left\{e^{\frac{i \pi \nu}{4}} K_{\nu}\left(4 \pi e^{\frac{i \pi}{4}} \sqrt{\frac{i n z}{q}}\right)+e^{\frac{-i \pi \nu}{4}} K_{\nu}\left(4 \pi e^{\frac{-i \pi}{4}} \sqrt{\frac{i n z}{q}}\right)\right\} \mathrm{d} z \\
& +2 \sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}}(n) n^{\nu / 2} \int_{\alpha \gamma_{1} \beta} f(z)(-i z)^{-\frac{\nu}{2}}\left\{e^{\frac{i \pi \nu}{4}} K_{\nu}\left(4 \pi e^{\frac{i \pi}{4}} \sqrt{\frac{-i n z}{q}}\right)+e^{\frac{-i \pi \nu}{4}} K_{\nu}\left(4 \pi e^{\frac{-i \pi}{4}} \sqrt{\frac{-i n z}{q}}\right)\right\} \mathrm{d} z .
\end{aligned}
$$

Simplifying we get

$$
\begin{aligned}
& \quad \frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} \sum_{\alpha<j<\beta} \bar{\sigma}_{-\nu, \chi}(j) f(j)=\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} L(1-\nu, \chi) \int_{\alpha}^{\beta} \frac{f(t)}{t^{\nu}} \mathrm{d} t \\
& +2 \sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}}(n) n^{\nu / 2} \int_{\alpha \gamma_{2} \beta} f(z) z^{-\frac{\nu}{2}}\left\{K_{\nu}\left(4 \pi i \sqrt{\frac{n z}{q}}\right)+e^{\frac{-i \pi \nu}{2}} K_{\nu}\left(4 \pi \sqrt{\frac{n z}{q}}\right)\right\} \mathrm{d} z \\
& \quad+2 \sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}}(n) n^{\nu / 2} \int_{\alpha \gamma_{1} \beta} f(z) z^{-\frac{\nu}{2}}\left\{e^{\frac{i \pi \nu}{2}} K_{\nu}\left(4 \pi \sqrt{\frac{n z}{q}}\right)+K_{\nu}\left(-4 \pi i \sqrt{\frac{n z}{q}}\right)\right\} \mathrm{d} z .
\end{aligned}
$$

Employing the residue theorem again, this time for each of the integrals inside the two sums, and simplifying, we obtain

$$
\begin{align*}
& \frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} \sum_{\alpha<j<\beta} \bar{\sigma}_{-\nu, \chi}(j) f(j)=\frac{q^{1-\frac{\nu}{2}}}{\tau(\chi)} L(1-\nu, \chi) \int_{\alpha}^{\beta} \frac{f(t)}{t^{\nu}} \mathrm{d} t+2 \sum_{n=1}^{\infty} \sigma_{-\nu, \bar{\chi}}(n) n^{\nu / 2} \\
& \times \int_{\alpha}^{\beta} f(t) t^{-\frac{\nu}{2}}\left\{K_{\nu}\left(4 \pi i \sqrt{\frac{n t}{q}}\right)+K_{\nu}\left(-4 \pi i \sqrt{\frac{n t}{q}}\right)+2 \cos \left(\frac{\pi \nu}{2}\right) K_{\nu}\left(4 \pi \sqrt{\frac{n t}{q}}\right)\right\} \mathrm{d} t \tag{8.9}
\end{align*}
$$

Here by [29, p. 848, equation (7.15)], we have

$$
\begin{equation*}
K_{\nu}(i x)+K_{\nu}(-i x)=-\pi\left(J_{\nu}(x) \sin \left(\frac{\pi \nu}{2}\right)+Y_{\nu}(x) \cos \left(\frac{\pi \nu}{2}\right)\right) \tag{8.10}
\end{equation*}
$$

where $J_{\nu}$ and $Y_{\nu}$ are the Bessel functions defined in (1.9) and (1.7), respectively. Now, we replace $x$ by $4 \pi \sqrt{n t / q}$ in (8.10) and substitute in (8.9), to get the desired result.

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