# Apéry-Like Sums and Colored Multiple Zeta Values

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#### Abstract

In this article we shall survey some recent progress on the study of Apéry-like sums which are multiple variable generalizations of the two sums Apéry used in his famous proof of the irrationality of  $\zeta(2)$  and  $\zeta(3)$ . We only allow the central binomial coefficients to appear in these infinite sums but they can appear either on the numerator or on the denominator. Special values of both types are closely related to the colored multiple zeta values and have played important roles in the calculations of the  $\varepsilon$ -expansion of multiloop Feynman diagrams. We will summarize several different approaches to computing these sums and prove a few conjectural identities of Z.-W. Sun as corollaries along the way.

**Keywords**: Apéry-like sum, central binomial coefficient, colored multiple zeta value, iterated integral, unital function, regularization.

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# 1 Multiple zeta values

### 1.1 Historical background.

The study of the Riemann zeta function  $\zeta(z)$  and its special values has been one of the major driving forces in contemporary and modern number theory. More than a century before Riemann defined his function for complex variable z, at the young age of 28, Euler already achieved his stardom status by finding the exact value of  $\zeta(2)$ , the renowned Basel problem at that time. In fact he gave three different proofs of  $\zeta(2) = \pi^2/6$  in [27] and presented his result in December of 1735 at St. Petersburg Academy of Sciences. Seven years later, Goldbach wrote to Euler from Moscow and referred him to some curious double sums which prompted Euler to begin his study of the double zeta values. Euler later returned a few more times to the same subject during his legendarily long math career.

In modern notation, the multiple zeta function of complex variables  $z_1, \ldots, z_d$  is defined by

$$\zeta(z_1, \dots, z_d) := \sum_{\substack{n_1 > \dots > n_d > 0}} \frac{1}{n_1^{z_1} \cdots n_d^{z_d}}.$$
(1.1)

It converges if  $\operatorname{Re}(z_1+\cdots+z_j)>j$  for all  $j=1,\ldots,d$  and similarly to  $\zeta(z)$ , it can be analytically continued to a memomorphic function to  $\mathbb{C}^d$  with points of indeterminacy explicitly described (see [4,56]). Its special value  $\zeta(k)$  at positive integer arguments  $k=(k_1,\ldots,k_d)$  is called a multiple zeta value (MZV). It converges if and only if  $k_1\geq 2$  in which case k is called admissible. Conventionally, d is called the depth and  $|k|=k_1+\cdots+k_d$  the weight of the MZV, respectively. A celebrated result of Euler states that if the weight w=a+b is odd then the double zeta value  $\zeta(a,b)$  can be expressed as a  $\mathbb{Q}$ -linear combination of  $\zeta(w)$  and products of Riemann zeta values  $\zeta(k)\zeta(w-k), 2\leq k\leq w-2$  (see [55, Prop. 7] for a modern constructive proof). The special case when w=3, often dubbed as Euler's identity, is particularly enticing:

$$\zeta(3) = \zeta(2,1). \tag{1.2}$$

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After Euler, the study of MZVs went into dormancy for more than 200 years. Two influential papers appeared during the last decade of 20th century. Around 1990, D. Zagier at Max Planck Institute for Mathematics in Germany started to consider the relations between MZVs, the mixed Tate motives over  $\mathbb{Z}$ , and the Vassiliev knot invariants and published his ideas in the seminal paper [54] (see [13] and [29] for more recent results). In particular, he pointed out the crucial representation of MZVs by Chen's iterated integrals first noticed by M. Kontsevitch.

Across the Atlantic, around almost the same time, M. Hoffman at the US Naval Academy began to study the rational relations among the MZVs in [30] and then discovered the algebraic structures hidden underneath them in his J. of Algebra paper [31] in 1997. The foundation laid down in this 1997 paper has since led to a series of important progresses in the study of not only MZVs themselves but also their various generalizations in recent years.

### 1.2 Chen's theory of iterated integrals.

For r > 1, we define inductively

$$\int_a^b f_1(t) dt \cdots f_r(t) dt = \int_a^b f_1(\tau) d\tau \cdots f_{r-1}(\tau) \left( \int_a^\tau f_r(t) dt \right) d\tau.$$

We set the integral to be 1 if r=0. More generally, let  $\phi_1, \ldots, \phi_r$  be some 1-forms (repetition allowed) on a manifold M and let  $\alpha:[0,1]\to M$  be a piecewise smooth path. Write  $\alpha^*\phi_i=f_i(t)\,dt$  and define the iterated integral

$$\int_{\Omega} \phi_1 \cdots \phi_r := \int_0^1 f_1(t) dt \cdots f_r(t) dt.$$

The following results are crucial in the application of Chen's theory of the iterated path integrals. For a sketch of the proof we refer the interested reader to [58, Lemma 2.1.2].

**Lemma 1.1.** Let  $\phi_i$   $(i \geq 1)$  be  $\mathbb{C}$ -valued 1-forms on a manifold M. Then

(i) If  $\alpha, \beta : [0,1] \longrightarrow M$  are composable paths (i.e.,  $\alpha(1) = \beta(0)$ ), then

$$\int_{\beta\alpha} \phi_1 \cdots \phi_r = \sum_{j=0}^r \int_{\beta} \phi_1 \cdots \phi_j \int_{\alpha} \phi_{j+1} \cdots \phi_r,$$

where we set  $\int_{\alpha} \phi_1 \cdots \phi_m = 1$  if m = 0 and  $\beta \alpha$  denotes the composition of  $\alpha$  and  $\beta$ .

(ii) For every path  $\alpha:[0,1] \longrightarrow M$ , let  $\alpha^{-1}(t) = \alpha(1-t)$ . Then

$$\int_{\alpha^{-1}} \phi_1 \cdots \phi_r = (-1)^r \int_{\alpha} \phi_r \cdots \phi_1.$$

(iii) For every path  $\alpha$ ,

$$\int_{\alpha} \phi_1 \cdots \phi_r \int_{\alpha} \phi_{r+1} \cdots \phi_{r+s} = \int_{\alpha} \phi_1 \cdots \phi_r \sqcup \phi_{r+1} \cdots \phi_{r+s},$$

where  $\coprod$  is the usual shuffle product of words<sup>1</sup>.

For any admissible composition  $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$  (i.e.,  $k_1 \geq 2$ ) we have

$$\zeta(\mathbf{k}) = \int_0^1 \left(\frac{dt}{t}\right)^{k_1 - 1} \frac{dt}{1 - t} \cdots \left(\frac{dt}{t}\right)^{k_d - 1} \frac{dt}{1 - t}.$$
 (1.3)

This can be proved easily if we replace the interval of integration by [0, x] on the right-hand side, and replace  $\zeta(\mathbf{k})$  by the single variable multiple polylogarithm

$$\operatorname{Li}_{k}(x) := \sum_{\substack{n_{1} > \dots > n_{d} > 0}} \frac{x^{n_{1}}}{n_{1}^{k_{1}} \cdots n_{d}^{k_{d}}}$$
(1.4)

on the left-hand side, and proceed by induction on the weight using differentiation by the Fundamental Theorem of Calculus. It follows immediately by the change of variables  $t_j \leftrightarrow 1 - t_j$  that the following duality relation holds. For  $l \in \mathbb{N}$ , we denote by  $1_l$  the sequence  $(1, \ldots, 1)$  with 1 repeating l times.

<sup>&</sup>lt;sup>1</sup>The shuffle product behaves exactly as the shuffling of two decks of cards.

**Theorem 1.2.** (Duality of MZVs) For all  $k_1, \ldots, k_d, l_1, \ldots, l_d \in \mathbb{N}$  we have

$$\zeta(k_1+1,1_{l_1-1},\ldots,k_d+1,1_{l_d-1})=\zeta(l_d+1,1_{k_d-1},\ldots,l_1+1,1_{k_1-1}).$$

#### 1.3 Double shuffle relations of MZVs.

The most fundamental question concerning MZVs is their relations over the rational number field  $\mathbb{Q}$ . Euler's proof of  $\zeta(2) = \pi^2/6$  implies that  $\zeta(2)$  is transcendental due to the transcendence of  $\pi$ . His idea can be easily extended to show that every Riemann zeta value at positive even integer  $\zeta(2n)$  is a rational multiple of  $\pi^{2n}$ . On the other hand, even though it is widely believed that  $\zeta(2n+1)$  is transcendental for every  $n \geq 1$  not a single case has been proved so far. In 1978, Apéry [6] shocked the math world by showing that  $\zeta(3)$  is irrational by a careful and clever manipulation of the following series

$$\zeta(3) = \frac{5}{2} \sum_{n \ge 1} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$
 (1.5)

It is therefore natural to ask, more generally, whether MZVs are all transcendental or not. As this is out of reach at the moment, we are led to the next problem: how can we find as many rational relations between MZVs as possible? The answer is given conjecturally by the so-called regularized double shuffle relations which, roughly speaking, are consequences of multiplying MZVs in two different ways.

When we multiply two MZVs, first we can apply their defining series and express the product as a  $\mathbb{Z}$ -linear combination of MZVs of the same weight. For example,

$$\begin{split} \zeta(m)\zeta(n) &= \sum_{k \geq 1} \frac{1}{k^m} \sum_{l \geq 1} \frac{1}{l^n} = \sum_{k > l \geq 1} \frac{1}{k^m l^n} + \sum_{k = l \geq 1} \frac{1}{k^m l^n} + \sum_{l > k \geq 1} \frac{1}{k^m l^n} \\ &= \zeta(m,n) + \zeta(m+n) + \zeta(n,m) \end{split}$$

for all positive integers  $m, n \geq 2$ . This clearly generalizes to MZVs of arbitrary depths. We call the terms such as the first and the third in the last line above "shuffle terms" and the others "stuffing terms" (i.e., obtained by some argument merges). This way of expanding the product is called the *stuffle product*, which was first coined in [11] although similar structures appeared implicitly already in Cartier's work [18] in early 1970s. For example, using stuffle product we see that

$$\zeta(2)^2 = 2\zeta(2,2) + \zeta(4). \tag{1.6}$$

The second way of multiplying the MZVs comes from their iterated integral expression (1.3). For this we need to apply Chen's Lemma 1.1(iii) using its shuffle product structure. For example, setting

$$\mathbf{x}_0 = \frac{dt}{t}, \qquad \mathbf{x}_1 = \frac{dt}{1-t} \tag{1.7}$$

we have

$$\zeta(2)^{2} = \int_{0}^{1} \mathbf{x}_{0} \mathbf{x}_{1} \sqcup \mathbf{x}_{0} \mathbf{x}_{1} = 2 \int_{0}^{1} \mathbf{x}_{0} \mathbf{x}_{1} \mathbf{x}_{0} \mathbf{x}_{1} + 4 \int_{0}^{1} \mathbf{x}_{0} \mathbf{x}_{0} \mathbf{x}_{1} \mathbf{x}_{1}$$
$$= 2\zeta(2, 2) + 4\zeta(3, 1). \tag{1.8}$$

Combining (1.8) with (1.6) we then derive a finite double shuffle relation (DBSF)

$$\zeta(4) = 4\zeta(3,1).$$

We see clearly that any finite DBSF must have weight at least four.

#### 1.4 Regularization and renormalization of MZVs.

A natural question now arises: is it possible to prove Euler's identity (1.2) by extending the above idea? This amounts to the question of how to define " $\zeta(1)$ " properly by generalizing both the series definition (1.1) and the integral expression (1.3) to include all non-admissible MZVs. This was done by K. Ihara, M. Kaneko and D. Zagier in their ground-breaking paper [33].

For any  $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$  one defines  $\mathbf{w}(\mathbf{k}) := \mathbf{x}_0^{k_1-1} \mathbf{x}_1 \cdots \mathbf{x}_0^{k_d-1} \mathbf{x}_1$  as its associated word. Using Hoffman's algebraic set-up laid out in [31], Ihara et al. showed that one can extend the shuffle product  $\square$  of words for admissible MZVs to that of non-admissible ones after suitable regularization. Further, one can define a stuffle product \* on words reflecting the stuffle product of admissible MZVs and then extend it to non-admissible ones.

To begin with, by abuse of notation, setting  $\zeta(\mathbf{w}(\mathbf{k})) = \zeta(\mathbf{k})$  and linearly extending  $\zeta$  over admissible words, we have, for admissible  $\mathbf{k}$  and  $\mathbf{l}$ ,

$$\zeta(\mathbf{k})\zeta(\mathbf{l}) = \zeta(\mathbf{w}(\mathbf{k}) \sqcup \mathbf{w}(\mathbf{l})) = \zeta(\mathbf{w}(\mathbf{k}) * \mathbf{w}(\mathbf{l})). \tag{1.9}$$

This is nothing but the finite DBSF. The key observation here is that for any  $\ell \geq 0$  and  $\mathbf{k} = (1_{\ell}, k_{\ell+1}, \dots, k_d) \in \mathbb{N}^d$   $(k_{\ell+1} \geq 2 \text{ if } \ell < d)$ , one can find the asymptotic expansions of the regularized iterated integral

$$\zeta^{(\varepsilon)}(\mathbf{k}) := \int_0^{1-\varepsilon} \mathbf{w}(\mathbf{k}) = P_{\ell}(\log \varepsilon) + O(\varepsilon \log^d \varepsilon) \text{ as } \varepsilon \to 0^+,$$

and the multiple harmonic sums

$$H_M(\mathbf{k}) := \sum_{M > n_1 > \dots > n_d > 0} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}} = Q_{\ell}(\log M + \gamma) + O(M^{-1} \log^d M)$$

as  $M \to \infty$ , where  $\gamma$  is Euler's constant,  $P_{\ell}(T)$  and  $Q_{\ell}(T)$  are polynomials in T of degree  $\ell$  whose coefficients of  $T^j$ , if nonzero, are  $\mathbb{Q}$ -linear combinations MZVs of weight  $|\mathbf{k}| - j$ . Moreover, if  $\ell = 0$  then  $P_{\ell}(T) = Q_{\ell}(T) = \zeta(\mathbf{k})$ .

**Definition 1.1.** The  $\sqcup$ - and \*-regularization of  $\zeta(k)$  are defined, respectively, by

$$\zeta_{\sqcup \sqcup}(\mathbf{k}) := P_{\ell}(T) \quad \text{and} \quad \zeta_{*}(\mathbf{k}) := Q_{\ell}(T).$$

$$\tag{1.10}$$

Furthermore, the value  $P_{\ell}(0)$  is called the *renormalization* or *renormalized value* of the divergent integral  $\int_{0}^{1} \mathbf{w}(\mathbf{k})$  representing the non-admissible MZV  $\zeta(\mathbf{k})$ , denoted by

$$\int_{0}^{1} \mathbf{w}(\mathbf{k}). \tag{1.11}$$

Ihara et al. showed that for any words  ${\bf u}$  and  ${\bf v}$  not ending with  ${\bf x}_0$  (representing either admissible or non-admissible MZVs)

$$\zeta_{\coprod}(\mathbf{u})\zeta_{\coprod}(\mathbf{v}) = \zeta_{\coprod}(\mathbf{u} \coprod \mathbf{v}) \quad \text{and} \quad \zeta_{*}(\mathbf{u})\zeta_{*}(\mathbf{v}) = \zeta_{*}(\mathbf{u} * \mathbf{v}).$$

Moreover, there is an  $\mathbb{R}$ -linear map  $\rho : \mathbb{R}[T] \to \mathbb{R}[T]$  defined by

$$\rho(\exp(Tu)) = \exp\left(\sum_{n>2} \frac{(-1)^n}{n} \zeta(n) u^n\right) \exp(Tu)$$

for a formal variable u, such that  $\rho \circ \zeta_* = \zeta_{\sqcup \sqcup}$ . This relation is called the *regularized* (or *extended*) DBSF.

As an application of the regularized DBSF we can now prove Euler's identity (1.2). Indeed, from the stuffle product of  $H_M(1)H_M(2)$  we have

$$\begin{split} \zeta_*(1)\zeta(2) &= \zeta_*(\mathtt{x}_0\mathtt{x}_1 * \mathtt{x}_1) \, = \zeta_*(\mathtt{x}_1\mathtt{x}_0\mathtt{x}_1 + \mathtt{x}_0\mathtt{x}_1\mathtt{x}_1 + \mathtt{x}_0^2\mathtt{x}_1) \\ &= \zeta_*(1,2) + \zeta(2,1) + \zeta(3). \end{split}$$

On the other hand, from the shuffle product of  $\zeta^{(\varepsilon)}(1)\zeta^{(\varepsilon)}(2)$ , we see that

$$\zeta_{\text{III}}(1)\zeta(2) = \zeta_{\text{III}}(\mathbf{x}_0\mathbf{x}_1 \coprod \mathbf{x}_1) = \zeta_{\text{III}}(\mathbf{x}_1\mathbf{x}_0\mathbf{x}_1 + 2\mathbf{x}_0\mathbf{x}_1\mathbf{x}_1) = \zeta_{\text{III}}(1,2) + 2\zeta(2,1).$$

Hence the regularized DBSF implies that

$$\zeta(2)T - 2\zeta(2,1) = \zeta_{11}(1,2) = \rho(\zeta_{*}(1,2)) = \zeta(2)T - \zeta(2,1) - \zeta(3),$$

since  $\rho(T) = T$ . Thus Euler's identity  $\zeta(2,1) = \zeta(3)$  follows immediately. Incidentally, we find that the renormalized value of " $\zeta(1,2)$ " is

$$\oint_0^1 \mathbf{x}_1 \mathbf{x}_0 \mathbf{x}_1 = \zeta_{\coprod}(1,2) \big|_{T=0} = -2\zeta(2,1).$$

# 2 Multiple polylogarithm and colored MZVs.

#### 2.1 Motivation.

Many mathematicians and physicists have been attracted to the study of MZVs because of the recent advancement in computer science which has enabled them to compute enormous integrals produced from very complicated Feynman diagrams. It turns out that what comes out of these computations are not only MZVs but also some special values of multiple polylogarithms at roots of unity which are defined as follows. Fix  $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ . For complex variables  $(z_1, \ldots, z_d) \in \mathbb{C}^d$  one defines the multiple polylogarithm of weight  $|\mathbf{k}| := k_1 + \cdots + k_d$  and depth d by

$$\operatorname{Li}_{k}(z_{1}, \dots, z_{d}) := \sum_{n_{1} > \dots > n_{d} > 0} \frac{z_{1}^{n_{1}} \cdots z_{d}^{n_{d}}}{n_{1}^{k_{1}} \cdots n_{d}^{k_{d}}}, \tag{2.1}$$

which converges if  $|z_1 \cdots z_j| < 1$  for all  $j = 1, \ldots, d$ . For convenience, if d = 0 we set  $k = \emptyset$  and  $\text{Li}_{\emptyset} = 1$ . As in the logarithm case, the multiple polylogarithm (2.1) can be analytically continued to a multi-valued memomorphic function on  $\mathbb{C}^d$  (see [58, Ch. 2]).

The name "Li", made standard in Lewin's book [39], comes from the French phrase "logarithmes itérés" due to the fact that (2.1) has an iterated integral representation. With an additional complex variable z, we have

$$\operatorname{Li}_{\mathbf{k}}(zz_1,\dots,z_d) = \int_0^z \left(\frac{dt}{t}\right)^{k_1-1} \frac{dt}{u_1-t} \cdots \left(\frac{dt}{t}\right)^{k_d-1} \frac{dt}{u_d-t},\tag{2.2}$$

where  $u_j = (z_1 \cdots z_j)^{-1}$ , if  $z_1 \cdots z_d \neq 0$ . This formula can be easily proved by induction on the weight  $|\mathbf{k}|$  by differentiating with respect to z. Note that  $\frac{dt}{t} = d \log t$  and  $\frac{dt}{u_1 - t} = -d \log(u_1 - t)$ . Formula (2.2) provides an easy way to analytically continue the multiple polylogarithm originally defined by (2.1).

Let  $N, d \in \mathbb{N}$  and  $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ . For any Nth roots of unity  $\eta_1, \dots, \eta_d$  one calls  $\text{Li}_k(\eta_1, \dots, \eta_d)$  defined by (2.1) a colored multiple zeta value (CMZV) of level N, which converges if and only if  $(k_1, \eta_1) \neq (1, 1)$  (see, e.g., [42, Lemma 3.3.16]). These values at lower levels appeared unexpectedly in the numerical computation of many Feynman integrals in the 1990s when the data from CERN and other labs around the world were analyzed by the mathematicians and theoretical physicists with the aid of high performance computers for the first time. We refer to the interested reader to [58, §14.7] for the fascinating history centered around this topic.

#### 2.2 Single variable multiple polylogarithm.

We have already seen that the single variable multiple polylogarithm defined by (1.4) is the key to the proof of the MZV duality relation. By setting  $z_1 = \cdots = z_d = 1$  in (2.2) we get

$$\operatorname{Li}_{k}(z) = \int_{0}^{z} \left(\frac{dt}{t}\right)^{k_{1}-1} \frac{dt}{1-t} \cdots \left(\frac{dt}{t}\right)^{k_{d}-1} \frac{dt}{1-t}.$$
 (2.3)

To express succinctly the fact that only 1-froms  $d \log t$  and  $d \log (1-t)$  are needed, we denote the right-hand side of (2.3) by  $\int_0^z [d \log t, d \log (1-t)]_w$  where  $w = |\mathbf{k}|$  is the number of such 1-forms. We may even regard this form as an analytic contination of  $\text{Li}_k(z)$ .

The following lemma is crucial for us to understand the behavior of  $\text{Li}_k(z)$  when  $z \to \infty$ . To state it more precisely, for s+1 compositions  $\mathbf{n}_j \in \mathbb{N}^{d_j}$   $(0 \le j \le s)$ , write  $\underline{\mathbf{n}} = (\mathbf{n}_0, \dots, \mathbf{n}_s)$ ,  $|\underline{\mathbf{n}}| = |\mathbf{n}_0| + \dots + |\mathbf{n}_s|$  and  $\text{dep}(\underline{\mathbf{n}}) = d_0 + \dots + d_s$ . Here we allow  $d_j = 0$  in which case we set  $\mathbf{n}_j = \emptyset$ ,  $\text{Li}_{\emptyset}(z) = \zeta(\emptyset) = 1$ .

**Lemma 2.1.** For any  $d \in \mathbb{N}$  and  $k \in \mathbb{N}^d$ , there are rational numbers  $q(a, b, \underline{n})$  such that

$$\operatorname{Li}_{\mathbf{k}}(1/z) = (-1)^{|\mathbf{k}|-d} \operatorname{Li}_{\mathbf{k}}(z) + \sum_{a,b,\underline{n}} q(a,b,\underline{n}) (2\pi i)^a \log^b(-z) \operatorname{Li}_{\mathbf{n}_0}(z) \prod_{j=1}^s \zeta(\mathbf{n}_j), \tag{2.4}$$

where a, b, s range over nonnegative integers and  $\underline{\mathbf{n}} = (\mathbf{n}_0, \dots, \mathbf{n}_s)$  ranges over s+1 compositions such that  $a+b+|\underline{\mathbf{n}}|=|\mathbf{k}|$  and  $\deg(\underline{\mathbf{n}})< d$ .

*Proof.* This follows from the proof of [41, Theorem 1.3].

Motivated by Lemma 2.1, we may define the regularized value  $\text{Li}_{k}^{\sqcup}(\infty; T)$  by replacing  $\log(-z)$  by T and setting  $\mathbf{n}_{0} = \emptyset$  in (2.4) since  $\text{Li}_{\mathbf{n}_{0}}(0) = 0$  for all  $\mathbf{n}_{0} \neq \emptyset$ . Moreover, we can now obtain its renormalized value by setting T = 0:

$$\operatorname{Li}_{k}^{\sqcup}(\infty) = \sum_{a, n_{1}, \ldots, n_{s}} q(a, 0, \emptyset, n_{1}, \ldots, n_{s}) (2\pi i)^{a} \prod_{j=1}^{s} \zeta(n_{j}).$$

In fact, by staying in the world of real numbers when taking the limit we see that a must be even so that we may use  $\zeta(2)$  instead of powers of  $2\pi i$ . Hence we may rewrite the above as

$$\operatorname{Li}_{k}^{\square}(\infty) := \oint_{0}^{\infty} \mathbf{w}(k) := \sum_{2|a, \mathbf{n}_{1}, \dots, \mathbf{n}_{s}} q'(a, 0, \emptyset, \mathbf{n}_{1}, \dots, \mathbf{n}_{s}) \zeta(2)^{a/2} \prod_{j=1}^{s} \zeta(\mathbf{n}_{j})$$

$$(2.5)$$

for some  $q' \in \mathbb{Q}$ . The example on the bottom of [41, p. 96] can be written as

$$\operatorname{Li}_{2,1}(1/z) = \operatorname{Li}_3(z) - \operatorname{Li}_{2,1}(z) + \zeta(3) - \log(-z)\operatorname{Li}_2(z) - \frac{1}{6}\log^3(-z).$$

Thus  $\operatorname{Li}_{2,1}^{\coprod}(\infty) = \zeta(3)$ .

We will see in a moment that  $\text{Li}_k(z)$  plays an important role in Au's recent paper [7] in which he proved the following result concerning some Apéry-like sums (see the next section). Let  $\mathsf{CMZV}_w^N$  be the  $\mathbb{Q}$ -span of all CMZVs of weight w and level N.

Theorem 2.2. The Apéry-like sums

$$\sum_{n=1}^{\infty}\frac{4^nH_{n-1}(\mathbf{k})}{n^l\binom{2n}{n}}\in\mathsf{CMZV}^2_{|\mathbf{k}|+l},\qquad \sum_{n=1}^{\infty}\frac{2^nH_{n-1}(\mathbf{k})}{n^l\binom{2n}{n}}\in\mathsf{CMZV}^4_{|\mathbf{k}|+l}.$$

The key idea in Au's proof is his observation that

$$\frac{1}{n\binom{2n}{n}} = \int_0^1 \frac{x^{n-1}}{(1+x)^{2n}} dx \tag{2.6}$$

and that for the rational function  $U_N(x) = 8x/(N(1+x)^2)$ , N=2 or 4, the integrals

$$\int_0^1 \frac{1}{x} \operatorname{Li}_{k}(U_N(x)) \, dx \in \mathsf{CMZV}_{|k|+1}^N.$$

To prove this we need the concept of N-unital functions (see §4).

# 3 Apéry-like sums

In his celebrated proof of irrationality of  $\zeta(2)$  and  $\zeta(3)$  in 1979, Apéry used crucially the following two identities

$$\zeta(2) = 3 \sum_{n>1} \frac{1}{n^2 \binom{2n}{n}} \quad \text{and} \quad \zeta(3) = \frac{5}{2} \sum_{n>1} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$
(3.1)

These were extended to formulas expressing other Riemann zeta values in terms of series similar to those on the right-hand side of (3.1) (see [5, 9, 37]).

Also motivated by Apéry's proof, Leshchiner [38] generalized these identities to higher weight (alternating) Riemann zeta values and their odd-indexed analogs. At arbitrary depth, one may define two types of generalizations with the binomial coefficients appearing on either the numerator or the denominator. One may further introduce other binomial coefficients such as  $\binom{3n}{n}$  and  $\binom{4n}{2n}$ , which we will not consider in this paper.

If the central binomial coefficients appear on the numerator (resp. denominator) then we call the sum a Apéry-type binomial series (resp. inverse binomial series). For example, for any  $d \in \mathbb{N}$ ,  $s = (s_1, \ldots, s_d) \in \mathbb{N}^d$ , and a complex variable z we can define a class of Apéry-type inverse binomial series by

$$\sigma(s;z) := \sum_{\substack{n_1 > n_2 > \dots > n_d > 0}} {2n_1 \choose n_1}^{-1} \frac{(2z)^{2n_1}}{(2n_1)^{s_1} \cdots (2n_d)^{s_d}}.$$
 (3.2)

Here we have renormalized the series so that its odd-indexed variation behaves similarly. By Stirling's formula

$$4^n \binom{2n}{n}^{-1} \sim \sqrt{\pi n}.$$

We will again call the sum  $|s| := s_1 + \cdots + s_d$  the weight of the series  $\sigma(s; z)$  and d the depth. We see that  $\sigma(s; z)$  converges for all |z| < 1 and, if s is admissible,  $\sigma(s; z)$  converges even for |z| = 1. In view of (3.1), one naturally wonders if such series are related to CMZVs for some special values of z. More specifically, we would like to answer the following question.

**Question.** Can we express  $\sigma(s;z)$  in terms of CMZVs for  $z=\sqrt{j}/2$  for j=1,2,4?

Another equally important motivation to study Apéry-like sums comes from theoretical physics. It turns out that such series, like CMZVs, play significant roles in the evaluation of Feynman integrals. Many experimental work emerged around the beginning of this century, e.g., see [12,25,26] for inverse binomial series and [34] for binomial series. In fact, odd variations of both types already appeared implicitly in [26, (1.1)] and [34, (A.25)], respectively.

In the theoretical framework of MZVs and their alternating version, considering their odd-indexed variations is also a very fruitful idea, see [32,36,51]. These are essentially some level two CMZVs.

Many different approaches have been devised to evaluate the Apéry-like sums and their odd variants. Ablinger [1] first converts many Apéry-type inverse binomial series into iterated integrals, which are then computed using so-called cyclotomic harmonic polylogarithms. Chu [21, 22] and his collaborators have applied hypergeometric series transformation to express many elegant  $\mathbb{Q}$ -linear combinations of these sums in terms of Riemann zeta values and proved quite a few conjectures of Z.-W. Sun. However, his method has been applied to Apéry-type inverse binomial series only. Campbell and his collaborators [15–17] use the idea of Fourier-Legendre series and the properties of the complete elliptic integral of the first kind to derive many compact formulas for Apéry-type binomial series but few inverse ones. Moreover, some multiple integrals associated with N-posets can also be used to prove some relations among the Apéry-like sums [53].

In the rest of this paper, we consider four more methods of evaluating these sums. The main tools are (i) single variable multiple polylogathms composed with unital functions, (ii) the Clausen functions, (iii) the trigonometric iterated integrals, and (iv) the contour integrals, respectively.

# 4 Apéry-like sums via N-unital functions.

### 4.1 N-unital functions.

Let N be a positive integer. We say a non-constant rational function  $U(x) \in \mathbb{C}(x)$  is N-unital if every zero and pole of both U(x) and 1-U(x) is either 0 or an Nth roots of unity. It follows from [40] or [14, Cor. 2.2] that there are only finitely many N-unital functions for any fixed  $N \in \mathbb{N}$ . Let  $U_N$  be the set of all such functions. For example, it is not hard to show that  $U_1 = \langle x \rangle_6$  where for each function  $f, \langle f \rangle_6$  is its  $S_3$ -orbit defined by

$$\langle f \rangle_6 := \left\{ f, 1 - f, \frac{1}{f}, \frac{f}{f - 1}, \frac{f - 1}{f}, \frac{-1}{f - 1} \right\}.$$

In fact, we are more interested in the set of all the possible values of U(0). For small levels this is given by the next result.

**Proposition 4.1.** ([57]) Let  $N \le 4$  and  $\mu_j = \exp(2j\pi i/N)$  for  $1 \le j < N$ . Define

$$C^{N} := \{0, 1, \infty\} \cup \begin{cases} \bigcup_{j=1}^{\frac{N}{2}} \langle \mu_{j} \rangle_{6}, & \text{if } N \text{ is even;} \\ \bigcup_{\varepsilon = \pm 1}^{\frac{N-1}{2}} \bigcup_{j=1}^{\infty} \langle \varepsilon \mu_{j} \rangle_{6}, & \text{if } N \text{ is odd.} \end{cases}$$

$$(4.1)$$

Then  $U(0) \in \mathcal{C}^N$  for all  $U(x) \in \mathsf{U}_N$ .

*Proof.* The proof is by an explicit computation of  $U_N$  via a tedious albeit elementary process. Precisely, there are 6 1-unital, 36 2-unital, 84 3-unital, and 252 4-unital functions.

**Remark 4.2.** When  $N \leq 4$  we verified the unital functions using the MAGMA program kindly shared with us by F. Brunault. The program shows that there are 336 5-unital functions. Unfortunately, even for N=5 the set  $\mathcal{C}^5$  in the proposition is too small to contain all values of 5-unital functions at 0. Concretely, setting  $\mu=\mu_5=\exp(2\pi i/5)$  we have found the following exceptional 5-unital functions:

$$f(x) = \frac{(\mu^3 + \mu^2 + 2)x^2 + (\mu^4 + \mu^2)x - \mu^2 + \mu - 1}{x^2 - (\mu^4 + \mu^2)x + \mu},$$
$$g(x) = \frac{(\mu^2 + \mu + 1)x - \mu^2 - \mu - 1}{x - \mu^2}.$$

One can easily check that

$$f(x) = \frac{(\mu^3 + \mu^2 + 2)(x - 1)(x - \mu)}{(x - \mu^2)(x - \mu^4)}, \quad g(x) = \frac{(\mu^2 + \mu + 1)(x - 1)}{x - \mu^2} \in \mathsf{U}_5$$

since

$$1-f(x)=-\frac{(\mu^3+\mu^2+1)(x-\mu^3)^2}{(x-\mu^2)(x-\mu^4)}, \quad 1-g(x)=\frac{-(\mu^2+\mu)(x-\mu^4)}{x-\mu^2}\in \mathsf{U}_5.$$

But none of the following values are in  $C^5$  defined by (4.1):

$$\begin{split} \langle f(0) \rangle_6 &= \Big\{ \frac{3-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}, \frac{3+\sqrt{5}}{2}, -\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{\sqrt{5}+1}{2} \Big\}, \\ \langle g(0) \rangle_6 &= \Big\{ \mu + \mu^2, 1-\mu-\mu^2, -\mu-\mu^2, 1+\mu+\mu^2, -\mu^3-\mu^4, 1+\mu^3+\mu^4 \Big\}. \end{split}$$

# 4.2 Iterated integral involving N-unital functions.

We will see that the following conjecture due to Au is crucial in the evaluation of Apéry-like sums via unital functions.

Conjecture 4.3. Let  $k \in \mathbb{N}^d$ . For all  $U(x) \in \mathsf{U}_N$ , the renormalized value

$$\int_0^1 \frac{1}{x} \operatorname{Li}_{\mathbf{k}}(U(x)) \, dx \in \mathsf{CMZV}^N_{|\mathbf{k}|+1}.$$

First, we should explain how the regularization is defined in this context. We start by considering

$$\int_{\eta}^{1-\varepsilon} \frac{1}{x} \operatorname{Li}_{\mathbf{k}}(U(x)) \, dx$$

for arbitrarily small numbers  $\eta, \varepsilon > 0$ . The main philosophy is that under ideal situations like ours, we may rewrite the integrand as a finite sum of "well-behaved" functions so that the integral of each part can be expressed as a polynomial of  $\log \eta$  and  $\log \varepsilon$  modulo  $O(\eta \log^J \eta)$  and  $O(\varepsilon \log^J \varepsilon)$  where  $J = \operatorname{dep}(\mathbf{k}) + 1$ . This process, which is completely similar to that for obtaining the regularized DBSF of MZVs, is again called *regularization*. Although each summand may diverge as  $\eta, \varepsilon \to 0$ , we can just consider the constant term of each summand since we know the total sum itself has finite limit. This constant term is then called the *renormalization* of the corresponding summand, completely similar to the renormalizations of

(1.11) and (2.5). We again denote this regularized and then renormalized value by  $\int_{0}^{1} \frac{1}{x} \operatorname{Li}_{k}(U(x)) dx$ .

Using the notation from the last section, if  $U(x) \in U_N$  then by (2.3)

$$\operatorname{Li}_{\mathbf{k}}(U(x)) = \int_{0}^{U(x)} \phi_{1} \cdots \phi_{w},$$

where  $w = |\mathbf{k}|$  is the weight and the 1-forms  $\phi_j$  are either  $d \log t$  or  $-d \log(1-t)$ . Then Lemma 1.1(i) immediately yields that

$$\oint_{0}^{1} \frac{1}{x} \operatorname{Li}_{k}(U(x)) dx = \sum_{j=1}^{w} \left( \oint_{0}^{1} \frac{1}{x} \int_{U(0)}^{U(x)} \phi_{1} \cdots \phi_{j} dx \right) \left( \oint_{0}^{U(0)} \phi_{j+1} \cdots \phi_{w} \right).$$
(4.2)

Here we have removed the j=0 term since it vanishes after regularization and renormalization.

It is clear to see that under the change of variables  $t \to U(t)$  the first factor in the products above is reduced to an iterated integral over [0, x] of 1-forms of the shape  $d \log U(x)$  and  $d \log (1 - U(x))$ . Then the renormalized value

$$\int_0^1 \frac{dx}{x} \int_{U(0)}^{U(x)} [d\log t, d\log(1-t)]_j = \int_0^1 \frac{1}{x} \int_0^x [d\log U(x), d\log(1-U(x))]_j$$

must lie in  $\mathsf{CMZV}_{i+1}^N$  by the definition of N-unitality.

However, it is much trickier to treat the second factor in (4.2), which amounts to finding the set  $U_N$  of all possible N-unital functions U(x) and then prove that the iterated integrals  $\int_0^{U(0)} \phi_{j+1} \cdots \phi_w$  lie in the correct space.

**Theorem 4.4.** Let  $C^N$  be defined by (4.1). For any composition of positive integers  $\mathbf{k}$  and  $U(x) \in \mathsf{U}_N$ , if  $U(0) \in C^N$  then

$$\iint_0^1 \frac{1}{x} \operatorname{Li}_{\pmb{k}}(U(x)) \, dx \in \mathsf{CMZV}^N_{|\pmb{k}|+1}.$$

*Proof.* By (4.2) it suffices to prove the claim that for all weight w and N-unital function U(x) with  $U(0) \in \mathcal{C}^N$ ,

$$\int_{0}^{U(0)} [d\log t, d\log(1-t)]_{w} \in \mathsf{CMZV}_{w}^{N}.$$

If U(0) = 0 or 1 then the claim is trivial. If  $U(0) = \infty$ , then the claim follows immediately from (2.5). This complete the proof of N = 1 case.

Assume  $N \geq 2$ . For  $U(0) \neq 0, 1, \infty$  since  $\mu_j^{-1}$  is still an Nth root of unity, for simplicity, we may just assume  $\mu = \mu_j$  for some j = 1, ..., N - 1.

- (1)  $U(0) = \mu$ . The claim follows quickly from the substitution  $t = \mu t$ .
- (2)  $U(0) = 1 \mu$ . Applying Lemma 1.1(i) again we get

$$\label{eq:sum_energy} \oint_0^{1-\mu} [\mathbf{x}_0,\mathbf{x}_1]_w = \sum_{j=0}^w \oint_1^{1-\mu} [\mathbf{x}_0,\mathbf{x}_1]_j \cdot \oint_0^1 [\mathbf{x}_0,\mathbf{x}_1]_{w-j}.$$

Then by change of variables  $t \to 1 - t$  (so that  $\mathbf{x}_0 \leftrightarrow \mathbf{x}_1$ ) in the first factor we reduce it to Case 1) by Lemma 1.1(ii) (notice the order of integration is reversed by this change of variables).

(3)  $U(0) = \frac{1}{1-u}$ . Applying Lemma 1.1(i) again we get

$$\oint_0^{\frac{1}{1-\mu}} [\mathbf{x}_0,\mathbf{x}_1]_w = \sum_{j=0}^w \oint_1^{\frac{1}{1-\mu}} [\mathbf{x}_0,\mathbf{x}_1]_j \cdot \oint_0^1 [\mathbf{x}_0,\mathbf{x}_1]_{w-j}.$$

Then by change of variable  $t \to 1/t$  (so that  $x_0 \to -x_0, x_1 \to x_1 - x_0$ ) in the first factor we get

$$\oint_{1}^{\frac{1}{1-\mu}} [\mathbf{x}_{0},\mathbf{x}_{1}]_{j} = \oint_{1}^{1-\mu} [-\mathbf{x}_{0},\mathbf{x}_{1}-\mathbf{x}_{0}]_{j} \in \mathsf{CMZV}_{j}^{N}$$

by Case (2) and Lemma 1.1(ii).

(4)  $U(0) = -\mu \neq -1$ . Put  $x_{\mu} = dt/(\mu - t)$ . Then

$$\label{eq:state_equation} \oint_0^{-\mu} [\mathbf{x}_0,\mathbf{x}_1]_w = \sum_{j=0}^w \oint_1^{-\mu} [\mathbf{x}_0,\mathbf{x}_1]_j \cdot \oint_0^1 [\mathbf{x}_0,\mathbf{x}_1]_{w-j}.$$

For the first factor, by substitution  $t = \frac{t-\mu}{1-\mu t}$  we see that  $\mathbf{x}_0 \to \mathbf{x}_\mu - \mathbf{x}_{\bar{\mu}}$  and  $\mathbf{x}_1 \to \mathbf{x}_1 - \mathbf{x}_{\bar{\mu}}$ . Thus by Lemma 1.1(ii)

$$\oint_{1}^{-\mu} [\mathbf{x}_{0}, \mathbf{x}_{1}]_{w} = \oint_{1}^{0} [\mathbf{x}_{\mu} - \mathbf{x}_{\bar{\mu}}, \mathbf{x}_{1} - \mathbf{x}_{\bar{\mu}}]_{w}$$

$$= (-1)^w \iint_0^1 \tau[\mathbf{x}_{\mu} - \mathbf{x}_{\bar{\mu}}, \mathbf{x}_1 - \mathbf{x}_{\bar{\mu}}]_w \in \mathsf{CMZV}_w^N,$$

where  $\tau$  means the order of the iterated integral is reversed.

(5)  $U(0) = 1 + \mu$ . By substitution  $t = \frac{(1+\mu)(1-t)}{1-\mu t}$  we see that  $\mathbf{x}_0 \to \mathbf{x}_1 - \mathbf{x}_{\bar{\mu}}$  and  $\mathbf{x}_1 \to \mathbf{x}_{\mu} - \mathbf{x}_{\bar{\mu}}$ . Thus by Lemma 1.1(ii)

$$\begin{split} \oint_0^{1+\mu} [\mathbf{x}_0,\mathbf{x}_1]_w &= \oint_1^0 [\mathbf{x}_1 - \mathbf{x}_{\bar{\mu}},\mathbf{x}_{\mu} - \mathbf{x}_{\bar{\mu}}]_w \\ &= (-1)^w \oint_0^1 \tau [\mathbf{x}_1 - \mathbf{x}_{\bar{\mu}},\mathbf{x}_{\mu} - \mathbf{x}_{\bar{\mu}}]_w \in \mathsf{CMZV}_w^N. \end{split}$$

(6)  $U(0) = \frac{1}{1+\mu}$ . Applying Lemma 1.1(i) again we get

$$\oint_0^{\frac{1}{1+\mu}} [\mathbf{x}_0,\mathbf{x}_1]_w = \sum_{j=0}^w \oint_1^{\frac{1}{1+\mu}} [\mathbf{x}_0,\mathbf{x}_1]_j \cdot \oint_0^1 [\mathbf{x}_0,\mathbf{x}_1]_{w-j}.$$

Then by change of variable  $t \to 1/t$  (so that  $x_0 \to -x_0, x_1 \to x_1 - x_0$ ), in the first factor we get

$$\iint_{1}^{\frac{1}{1+\mu}} [\mathbf{x}_0,\mathbf{x}_1]_j = \iint_{1}^{1+\mu} [-\mathbf{x}_0,\mathbf{x}_1-\mathbf{x}_0]_j \in \mathsf{CMZV}_j^N$$

by Case (5) and Lemma 1.1(ii).

This completes the proof of the theorem.

**Remark 4.5.** In Remark 4.2 we have found f(x),  $g(x) \in \mathsf{U}_5$  such that  $\langle f(0) \rangle_6 \cap \mathcal{C}^5 = \langle g(0) \rangle_6 \cap \mathcal{C}^5 = \emptyset$ . Au's computation in [8, Table 1] does contain the three values f(0), 1 - f(0), f(0)/(f(0) - 1) but not the others, nor any values in  $\langle g(0) \rangle_6$ . These are essentially the only exceptions one need to check to prove Theorem 4.4 for N = 5.

# 4.3 Evaluation of Apéry-like sums via N-unital functions.

As Apéry showed us that central binomial series such as (1.5) can play very important roles in number theory. Many works on these have appeared in recent years even though mathematicians studied them much earlier. In this section, using a key observation of Au [7] relating such series to CMZVs via unital functions we will present, for each level N > 2, a family of such series.

functions we will present, for each level  $N \geq 2$ , a family of such series. Fix  $N \geq 2$  and  $1 \leq j \leq N-1$ . Put  $\theta = \theta_j^{(N)} := 2j\pi/N$ ,  $\mu = \mu_j^{(N)} := e^{i\theta}$ , and  $a = \mu + \bar{\mu} = 2\cos\theta$ . Define

$$f_n = f_n^{(N,j)} := \int_0^1 \frac{1}{x} \left( \frac{(2-a)x}{x^2 - ax + 1} \right)^n dx,$$
$$g_n = g_n^{(N,j)} := \int_0^1 \left( \frac{(2-a)x}{x^2 - ax + 1} \right)^n dx.$$

By direct computation we see that

$$f_1 = \frac{(2-a)}{2\sin\theta}(\pi - \theta) = (\pi - \theta)\tan\frac{\theta}{2}, \quad f_2 = \frac{2-a}{4\sin^2\theta}(af_1 + 2 - a),$$
  
$$g_0 = 1, \quad g_1 = (2-a)\log\left(2\sin\frac{\theta}{2}\right) + \frac{a(2-a)}{4\sin\theta}(\pi - \theta), \quad g_2 = \frac{1}{a}(2f_2 - 2 + a).$$

The only exception is when N=2 and j=1 in which case we can compute directly that  $g_1=4\log 2-2$ . Assuming  $n\geq 1$ , by integration by parts we have

$$f_n = \frac{(2-a)^n x^n}{n(x^2 - ax + 1)^n} \bigg|_0^1 + \int_0^1 \frac{(2-a)^n x^n (2x - a)}{(x^2 - ax + 1)^{n+1}} dx$$

$$= \frac{1}{n} + \frac{2}{2-a}g_{n+1} - \frac{a}{2-a}f_{n+1}.$$

Similarly,

$$g_n = \frac{(2-a)^n x^{n+1}}{(n+1)(x^2 - ax + 1)^n} \Big|_0^1 + \frac{n}{n+1} \int_0^1 \frac{(2-a)^n x^{n+1} (2x - a)}{(x^2 - ax + 1)^{n+1}} dx$$

$$= \frac{1}{n+1} + \frac{n}{n+1} \int_0^1 \frac{(2-a)^n [2 \cdot (x^{n+2} - ax^{n+1} + x^n) + ax^{n+1} - 2x^n]}{(x^2 - ax + 1)^{n+1}} dx$$

$$= \frac{1}{n+1} + \frac{n}{n+1} \left( 2g_n + \frac{a}{2-a} g_{n+1} - \frac{2}{2-a} f_{n+1} \right).$$

Therefore we get the second order recurrence for both  $\{f_n\}$  and  $\{g_n\}$ : for all  $n \geq 2$ 

$$(a+2)nf_{n+1} - a(2n-1)f_n - (2-a)(n-1)f_{n-1} = 0,$$
  
$$(a+2)n(n-1)g_{n+1} - a(2n-1)(n-1)g_n - (2-a)n(n-2)g_{n-1} = 2-a.$$

**Theorem 4.6.** Let  $N \geq 2$ ,  $1 \leq j < N$  and  $\theta = 2j\pi/N$ . Set

$$F_{1} = \begin{cases} 1, & \text{if } N = 2; \\ (\pi - \theta) \csc \theta, & \text{if } N \ge 3, \end{cases} \quad F_{2} = \begin{cases} 4/3, & \text{if } N = 2; \\ (\cos \theta F_{1} + 1) \csc^{2} \theta, & \text{if } N \ge 3, \end{cases}$$

and

$$n\sin^2\theta F_{n+1} - (2n-1)\cos\theta F_n - (n-1)F_{n-1} = 0 \quad \forall n \ge 2.$$

Then we have

$$\sum_{n\geq 1} \frac{H_{n-1}(\mathbf{k})}{n^l} (1-\cos\theta)^n F_n \in \mathsf{CMZV}^N_{|\mathbf{k}|+l+1}.$$

In particular, when l = 1 and 2 we have

$$\begin{split} \sum_{n\geq 1} \frac{(1-\cos\theta)^n F_n}{n} &= 2\zeta(2) - \mathrm{Li}_2(\mu) - \mathrm{Li}_2(\bar{\mu}) \in \mathsf{CMZV}_2^N, \\ \sum_{n\geq 1} \frac{(1-\cos\theta)^n F_n}{n^2} &= \sum_{\eta=\mu,\bar{\mu}} \left( 2\,\mathrm{Li}_{2,1}(\eta,\bar{\eta}) - \mathrm{Li}_{2,1}(\eta,1) - \mathrm{Li}_{2,1}(\eta,\bar{\eta}^2) \right) \\ &+ 2\zeta(3) - \mathrm{Li}_3(\mu) - \mathrm{Li}_3(\bar{\mu}) \in \mathsf{CMZV}_3^N. \end{split}$$

*Proof.* It is easy to see that  $f_n = (1 - \cos \theta)^n F_n$ . Put

$$U(x) := \frac{(2-a)x}{x^2 - ax + 1} = \frac{(2-a)x}{(x-\mu)(x-\bar{\mu})}, \quad 1 - U(x) = \frac{(x-1)^2}{(x-\mu)(x-\bar{\mu})} \in \mathsf{U}_N.$$

We see that

$$\begin{split} \sum_{n \geq 1} \frac{H_{n-1}(\mathbf{k})}{n^l} f_n &= \int_0^1 \frac{1}{x} \sum_{n \geq 1} \frac{H_{n-1}(\mathbf{k})}{n^l} \big( U(x) \big)^n \, dx, \\ &= \int_0^1 \frac{1}{x} \operatorname{Li}_{l,\mathbf{k}} \big( U(x) \big) \, dx \\ &= \int_0^1 \frac{1}{x} \left( \int_0^{U(x)} [\mathbf{x}_0, \mathbf{x}_1]_{|\mathbf{k}| + l} \, dt \right) \, dx \\ &= \int_0^1 \mathbf{x}_0 [\mathbf{x}_0 + \mathbf{x}_\mu + \mathbf{x}_{\bar{\mu}}, 2\mathbf{x}_1 - \mathbf{x}_\mu - \mathbf{x}_{\bar{\mu}}]_{|\mathbf{k}| + l} \in \mathsf{CMZV}_{|\mathbf{k}| + l}^N \end{split}$$

since

$$d\log U(t) = d\log t - d\log(\mu - t) - d\log(\bar{\mu} - t),$$

$$d\log(1 - U(t)) = 2d\log(1 - t) - d\log(\mu - t) - d\log(\bar{\mu} - t).$$

In particular, when  $k = \emptyset$  we have

$$\sum_{n \geq 1} \frac{f_n}{n^l} = \int_0^1 \mathbf{x}_0 (\mathbf{x}_0 + \mathbf{x}_\mu + \mathbf{x}_{\bar{\mu}})^{l-1} (2\mathbf{x}_1 - \mathbf{x}_\mu - \mathbf{x}_{\bar{\mu}}).$$

This implies quickly the two special cases when l = 1 and l = 2.

Corollary 4.7. For all  $l \geq 2$  we have

$$\sum_{n \geq 1} \frac{4^n H_{n-1}(\mathbf{k})}{n^l \binom{2n}{n}} \in \mathsf{CMZV}^2_{|\mathbf{k}| + l + 1}, \quad \sum_{n \geq 1} \frac{\pi \binom{2n}{n}}{4^n \cdot (2n+1)^l} \in \mathsf{CMZV}^4_{l+1}.$$

Note that if  $k = \emptyset$  the first sum has been studied by many authors previously by a variety of methods. One can find its briefly history in [49, §3] and even a MAPLE package for its computation. One can also find the evaluation of many other similar Apéry-like sums in [7].

*Proof.* The first claim, as treated in [7], follows from (2.6) after taking N=2, j=1 in the theorem.

For the second sum, set N = 4, j = 1, u = i and a = 0. Using the Gamma integral we may compute  $f_n$  as follows:

$$f_n = \int_0^1 \frac{1}{x} \left( \frac{2x}{x^2 + 1} \right)^n dx = 2^n \int_0^{\pi/4} \frac{\tan^{n-1} t}{\sec^{2n-2} t} dt$$
$$= 2^{n-1} \int_0^{\pi/2} \sin^{n-1} t \cdot \cos^{n-1} t \, dt = 2^{n-2} \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{n}{2})}{\Gamma(n)}.$$

Hence

$$f_{2k} = 2^{2k-2} \frac{(k-1)!^2}{(2k-1)!} = \frac{4^k}{2k\binom{2k}{k}}, \quad f_{2k+1} = \frac{(2k-1)!!^2}{2(2k)!} \pi = \frac{\pi\binom{2k}{k}}{2\cdot 4^k}.$$

We see that

$$\sum_{n \geq 1} \frac{f_n}{n^l} = \sum_{k \geq 1} \frac{4^k}{(2k)^{l+1} \binom{2k}{k}} + \sum_{k \geq 0} \frac{\pi \binom{2k}{k}}{2(2k+1)^l \cdot 4^k} \in \mathsf{CMZV}_{l+1}^4.$$

Thus the second claim follows immediately from the first.

Remark 4.8. One can show that

$$\sum_{n\geq 1} \frac{\binom{2n}{n}}{4^n \cdot (2n+1)^l} \in i\mathsf{CMZV}^4_{|k|+l} \tag{4.3}$$

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by the following identity

$$\sum_{n\geq 1} \frac{\binom{2n}{n}}{4^n \cdot (2n+1)^l} = \sum_{n\geq 1} \frac{(-1)^n \binom{-1/2}{n}}{(2n+1)^l} = \int_0^1 \left(\frac{dt}{t}\right)^{l-1} \frac{dt}{\sqrt{1-t^2}}.$$

With the substitution  $t \to 2t/(1+t^2)$ , we get  $\mathbf{x}_0 \to \mathbf{x}_0 + \mathbf{x}_i + \mathbf{x}_{-i}$  and  $\frac{dt}{\sqrt{1-t^2}} \to \frac{2dt}{1+t^2} = i(\mathbf{x}_i - \mathbf{x}_{-i})$ . Thus (4.3) holds.

**Example 4.9.** By taking N = 2, j = 1 and using (2.6), as in [7], we can find

$$\sum_{k>1} \frac{4^k}{k^{l+1} \binom{2k}{k}} = \begin{cases} 3\zeta(2), & \text{if } l=1; \\ 6\zeta(2)\log 2 - \frac{7}{2}\zeta(3), & \text{if } l=2. \end{cases}$$
 (4.4)

This is consistent with [24, Example 8], [48, Theorem 3.1], and [49, Example 3.1].

**Example 4.10.** Take N=4. If l=1, by Theorem 4.6 and a straight-forward calculation we get

$$\sum_{n\geq 1} \frac{f_n}{n} = 2\zeta(2) - \text{Li}_2(i) - \text{Li}_2(-i) = \frac{9}{4}\zeta(2),$$

$$\sum_{n\geq 1} \frac{f_n}{n^2} = 2\zeta(3) - \text{Li}_3(i) - \text{Li}_3(-i) + \sum_{\eta=\pm i} \left(2 \text{Li}_{2,1}(\eta, \bar{\eta}) - \text{Li}_{2,1}(\eta, 1) - \text{Li}_{2,1}(\eta, \bar{\eta}^2)\right)$$

$$= \frac{21}{16}\zeta(3) + \frac{3}{4}\zeta(2)\log 2 + M(\bar{2}, \bar{1})$$
(4.5)

by manipulating the summation indices suitably, where

$$M(\bar{2},\bar{1}) = \sum_{m>n>0} \frac{4(-1)^{m+n}}{(2m-1)^2(2n-1)} = \frac{3}{2}\zeta(2)\log 2 - \frac{7}{4}\zeta(3)$$

is a double mixed value studied in [51]. To show the second equality in (4.5) we have

$$\operatorname{Li}_{2}(i) + \operatorname{Li}_{2}(-i) = \sum_{m \geq 1} \frac{i^{m} + (-i)^{m}}{m^{2}} = \sum_{m \geq 1} \frac{2(-1)^{m}}{(2m)^{2}} = \frac{1}{2}\zeta(\bar{2}) = -\frac{1}{4}\zeta(2),$$

where the evaluation of the alternating double zeta value  $\zeta(\bar{2}) := \sum_{m \geq 1} \frac{(-1)^m}{m^2}$  can be found easily by the well-known relation  $\zeta(2) + \zeta(\bar{2}) = \frac{1}{2}\zeta(2)$ . By (4.4) this implies

$$\sum_{k>0} \frac{\binom{2k}{k}}{(2k+1)\cdot 4^k} = \frac{\pi}{2}, \quad \sum_{k>0} \frac{\pi\binom{2k}{k}}{2(2k+1)^2\cdot 4^k} = \frac{7}{4}\zeta(3) + M(\bar{2},\bar{1}).$$

Hence

$$\sum_{k>0} \frac{\binom{2k}{k}}{(2k+1)^2 \cdot 4^k} = \frac{3}{\pi} \zeta(2) \log 2 = \frac{\pi \log 2}{2}.$$

# 5 Apéry-like sums via Clausen function.

#### 5.1 Some conjectures of Z.-W. Sun.

Since late 1980s Z.-W. Sun has discovered many beautiful and highly nontrivial identities of infinite series, often involving interesting objects such as Fibonacci numbers, Bernoulli numbers, Euler numbers, etc. In [45–47] he proposed a few families of conjectures of such identities by generalizing the Lucas numbers to Lucas sequences. We strongly recommend his recent book [46] even though it is in Chinese because the readers can still understand most of the 820 conjectures even if they do not know a single Chinese character.

Let A and B be integers. Sun defines the more general Lucas sequence  $\{v_n(A, B)\}_{n\geq 0}$  as follows (see [44]):

$$v_0(A, B) := 2, \quad v_1(A, B) := A,$$
  
 $v_{n+1}(A, B) = Av_n(A, B) - Bv_{n-1}(A, B) \quad \forall n \ge 1.$ 

The characteristic equation  $x^2 - Ax + B = 0$  of the sequence  $\{v_n\}_{n \geq 0}$  has two roots

$$\alpha = \frac{A + \sqrt{A^2 - 4B}}{2}$$
 and  $\beta = \frac{A - \sqrt{A^2 - 4B}}{2}$ .

It is well known that for any integer  $n \geq 0$  we have

$$v_n(A, B) = \alpha^n + \beta^n.$$

For examples, setting  $\phi = \frac{\sqrt{5}+1}{2}$  we have

$$v_n(3,1) = L_{2n} = \phi^{2n} + \phi^{-2n},$$

$$v_n(4,1) = (2 - \sqrt{3})^n + (2 + \sqrt{3})^n,$$
  

$$v_n(4,2) = (2 - \sqrt{2})^n + (2 + \sqrt{2})^n,$$
  

$$v_n(5,5) = (\sqrt{5}\phi)^n + (\sqrt{5}\phi^{-1})^n.$$

Define  $H_0 := 0$  and the classical harmonic numbers

$$H_n := H_n(1) = \sum_{k=1}^n \frac{1}{k} \quad \forall n \ge 1.$$

In this section, we will prove the following results.

Theorem 5.1. We have

$$\sum_{n=1}^{\infty} \frac{L_{2n}}{n^2 \binom{2n}{n}} \left( H_{2n} - H_{n-1} \right) = \frac{41\zeta(3) + 4\pi^2 \log(\phi)}{25},\tag{5.1}$$

$$\sum_{n=1}^{\infty} \frac{v_n(5,5)}{n^2 \binom{2n}{n}} (H_{2n} - H_{n-1}) = \frac{124\zeta(3) + \pi^2 \log(5^5 \phi^6)}{50},$$
(5.2)

$$\sum_{n=1}^{\infty} \frac{v_n(4,1)}{n^2 \binom{2n}{n}} (H_{2n} - H_{n-1}) = \frac{23\zeta(3) + 2\pi^2 \log(2 + \sqrt{3})}{12},\tag{5.3}$$

$$\sum_{n=1}^{\infty} \frac{v_n(4,2)}{n^2 \binom{2n}{n}} (H_{2n} - H_{n-1}) = \frac{259\zeta(3) + 2\pi^2 (5\log(2) + 8\log(1+\sqrt{2}))}{128}.$$
 (5.4)

Motivated by his study of the congruences in [44], Sun proposed identities (5.1) and (5.2) as [46, Conjectures 10.62 and 10.63] (see also [45, Eqs. (3.11) and (3.12)]). He further conjectured (5.3) and (5.4) in [47].

### 5.2 Proof of Theorem 5.1 by Clausen function.

Recall that for any positive integer m the Clausen function  $Cl_m(\theta)$  is defined by

$$\operatorname{Cl}_{2m-1}(\theta) := \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^{2m-1}}$$
 and  $\operatorname{Cl}_{2m}(\theta) := \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^{2m}}$   $\forall \theta \in [0, \pi].$ 

Many special values of the Clausen function are closely related to the Riemann zeta values. For example,  $\text{Cl}_{2m-1}(0) = \zeta(2m-1)$ ,

$$\operatorname{Cl}_{2m-1}(\pi) = (2^{2-2m} - 1)\zeta(2m - 1), \text{ and } \operatorname{Cl}_{2m-1}\left(\frac{\pi}{2}\right) = \frac{\operatorname{Cl}_{2m-1}(\pi)}{2^{2m-1}}.$$
 (5.5)

The following relations among special values of the Clausen function are well known (see, e.g., [39, Eq. (6.51)]): for any positive integer r we have

$$\operatorname{Cl}_3\left(\frac{\pi}{r}\right) + \operatorname{Cl}_3\left(\frac{3\pi}{r}\right) + \dots + \operatorname{Cl}_3\left(\frac{(2r-1)\pi}{r}\right) = -\frac{3}{4r^2}\zeta(3),\tag{5.6}$$

$$\operatorname{Cl}_3\left(\frac{2\pi}{r}\right) + \operatorname{Cl}_3\left(\frac{4\pi}{r}\right) + \dots + \operatorname{Cl}_3\left(\frac{(2r-2)\pi}{r}\right) = -\left(1 - \frac{1}{r^2}\right)\zeta(3).$$
 (5.7)

Setting r = 4, 5, 6 in (5.6) and r = 5 in (5.7) one obtains

$$\operatorname{Cl}_{3}\left(\frac{\pi}{4}\right) + \operatorname{Cl}_{3}\left(\frac{3\pi}{4}\right) = -\frac{3}{128}\zeta(3), \qquad \operatorname{Cl}_{3}\left(\frac{\pi}{5}\right) + \operatorname{Cl}_{3}\left(\frac{3\pi}{5}\right) = \frac{9}{25}\zeta(3), \tag{5.8}$$

$$\operatorname{Cl}_3\left(\frac{\pi}{6}\right) + \operatorname{Cl}_3\left(\frac{5\pi}{6}\right) = \frac{1}{12}\zeta(3), \qquad \operatorname{Cl}_3\left(\frac{2\pi}{5}\right) + \operatorname{Cl}_3\left(\frac{4\pi}{5}\right) = -\frac{12}{25}\zeta(3),$$
 (5.9)

since  $Cl_3(\theta) = Cl_3(2\pi \pm \theta)$ ,  $Cl_3(\pi) = -\frac{3}{4}\zeta(3)$  and  $Cl_3(\frac{\pi}{2}) = -\frac{3}{32}\zeta(3)$  by (5.5).

On the other hand, Davydychev and Kalmykov have shown that (see [26, (2.36), (2.37) and (2.67)]) the Apéry-type inverse binomial series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}} u^n = 2 \operatorname{Cl}_3(\theta) + 2\theta \operatorname{Cl}_2(\theta) - 2\zeta(3) + \theta^2 \log\left(2\sin\frac{\theta}{2}\right),$$

$$\sum_{n=1}^{\infty} \frac{H_{n-1}}{n^2 \binom{2n}{n}} u^n = 4 \operatorname{Cl}_3(\pi - \theta) - 2\theta \operatorname{Cl}_2(\pi - \theta) + 3\zeta(3),$$

$$\sum_{n=1}^{\infty} \frac{H_{2n-1}}{n^2 \binom{2n}{n}} u^n = -2 \operatorname{Cl}_3(\theta) + 4 \operatorname{Cl}_3(\pi - \theta) - 2\theta \operatorname{Cl}_2(\pi - \theta) - \theta \operatorname{Cl}_2(\theta) + 5\zeta(3),$$

where  $u = 4\sin^2\frac{\theta}{2}$  for any  $\theta \in [0, \pi]$ . From the above three equations we get

$$\sum_{n=1}^{\infty} \frac{u^n}{n^2 \binom{2n}{n}} (H_{2n} - H_{n-1}) = \sum_{n=1}^{\infty} \frac{u^n}{n^2 \binom{2n}{n}} \left( H_{2n-1} - H_{n-1} + \frac{1}{2n} \right)$$
$$= \zeta(3) - \text{Cl}_3(\theta) + \frac{\theta^2}{2} \log \left( 2 \sin \frac{\theta}{2} \right). \tag{5.10}$$

Taking special values of u we obtain

$$u = \left(\frac{\sqrt{5} \pm 1}{2}\right)^2 = 4\sin^2\frac{\theta}{2} \implies \sin\frac{\theta}{2} = \frac{\sqrt{5} \pm 1}{4} \text{ and } \theta = \frac{2\pi}{5} \pm \frac{\pi}{5},$$

$$u = \frac{5 \pm \sqrt{5}}{2} = 4\sin^2\frac{\theta}{2} \implies \sin\frac{\theta}{2} = \sqrt{\frac{5 \pm \sqrt{5}}{8}} \text{ and } \theta = \frac{3\pi}{5} \pm \frac{\pi}{5},$$

$$u = 2 \pm \sqrt{3} = 4\sin^2\frac{\theta}{2} \implies \sin\frac{\theta}{2} = \frac{\sqrt{2 \pm \sqrt{3}}}{2} \text{ and } \theta = \frac{\pi}{2} \pm \frac{\pi}{3},$$

$$u = 2 \pm \sqrt{2} = 4\sin^2\frac{\theta}{2} \implies \sin\frac{\theta}{2} = \frac{\sqrt{2 \pm \sqrt{2}}}{2} \text{ and } \theta = \frac{\pi}{2} \pm \frac{\pi}{4}.$$

Therefore, (5.8)–(5.10) yield

$$\sum_{n=1}^{\infty} \frac{L_{2n}}{n^2 \binom{2n}{n}} (H_{2n} - H_{n-1}) = 2\zeta(3) - \text{Cl}_3\left(\frac{3\pi}{5}\right) - \text{Cl}_3\left(\frac{\pi}{5}\right) + \frac{4\pi^2}{25} \log(\phi)$$

$$= \frac{41\zeta(3) + 4\pi^2 \log(\phi)}{25},$$

$$\sum_{n=1}^{\infty} \frac{v_n(5,5)}{n^2 \binom{2n}{n}} (H_{2n} - H_{n-1}) = 2\zeta(3) - \text{Cl}_3\left(\frac{2\pi}{5}\right) - \text{Cl}_3\left(\frac{4\pi}{5}\right) + \frac{\pi^2 \log\left(5^5\phi^6\right)}{50}$$

$$= \frac{124\zeta(3) + \pi^2 \log\left(5^5\phi^6\right)}{50},$$

$$\sum_{n=1}^{\infty} \frac{v_n(4,1)}{n^2 \binom{2n}{n}} (H_{2n} - H_{n-1}) = 2\zeta(3) - \text{Cl}_3\left(\frac{\pi}{6}\right) - \text{Cl}_3\left(\frac{5\pi}{6}\right) + \frac{\pi^2 \log(2 + \sqrt{3})}{6}$$

$$= \frac{23\zeta(3) + 2\pi^2 \log(2 + \sqrt{3})}{12},$$

$$\sum_{n=1}^{\infty} \frac{v_n(4,2)}{n^2 \binom{2n}{n}} (H_{2n} - H_{n-1}) = 2\zeta(3) - \text{Cl}_3\left(\frac{\pi}{4}\right) - \text{Cl}_3\left(\frac{3\pi}{4}\right)$$

$$+ \frac{\pi^2(5\log(2) + 8\log(1 + \sqrt{2}))}{64}$$

$$= \frac{259\zeta(3) + 2\pi^2(5\log(2) + 8\log(1 + \sqrt{2}))}{129}.$$

This concludes the proof of Theorem 5.1.

# 6 Apéry-like sums via trigonometric integrals

# 6.1 Akhilesh's *n*-tails of Apéry-type inverse binomial series.

In the two sequential papers [2,3] Akhilesh discovered some very intriguing and surprising connections between MZVs and the Apéry-type inverse binomial series (3.2), with a slightly different normalization.

His ingenious idea is to study the *n*-tails (and more generally, double tails) of such series. After obtaining the explicit formula for the depth one case one can then iterate this process to arbitrary depth. To make it more transparent we reformulate one of his key results as follows. Set  $\binom{0}{0} = 1$ ,

$$b_n(z) = \frac{4^n z^{2n}}{\binom{2n}{n}}$$
 and  $b_n = b_n(1) = \frac{4^n}{\binom{2n}{n}}$   $\forall n \ge 0$ .

Define

$$g_s(t) = \begin{cases} \tan t \, dt, & \text{if } s = 1; \\ dt \circ (\cot t \, dt)^{s-2} \circ dt, & \text{if } s \ge 2, \end{cases}$$
 (6.1)

and their non-trigonometric counterpart

$$G_s(t) = \begin{cases} \omega_2 & \text{if } s = 1; \\ \omega_1 \omega_0^{s-2} \omega_1 & \text{if } s \ge 2, \end{cases}$$

$$(6.2)$$

where

$$\omega_0 := \frac{dt}{t}, \quad \omega_1 := \frac{dt}{\sqrt{1 - t^2}}, \quad \omega_2 := \frac{t \, dt}{1 - t^2}.$$
 (6.3)

**Theorem 6.1.** (cf. [3, Theorem 4]) Let  $n \in \mathbb{N}_0$ ,  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ ,  $y \in (-\pi/2, \pi/2)$  if  $s_1 = 1$  and  $y \in [-\pi/2, \pi/2]$  if  $s_1 \geq 2$ . Then the n-tail of  $\sigma(\mathbf{s}; \sin y)$ 

$$\sigma(\mathbf{s}; \sin y)_n := \sum_{n_1 > \dots > n_d > n} \frac{b_{n_1}(\sin y)}{(2n_1)^{s_1} \cdots (2n_d)^{s_d}}$$
$$= \frac{d}{dy} \int_0^y g_{s_1} \circ \dots \circ g_{s_d} \circ b_n(\sin t) dt.$$

Using non-trigonometric 1-forms, for all  $z \in (-1,1)$  we have

$$\sigma(\mathbf{s}; z)_n := \sum_{n_1 > \dots > n_d > n} \frac{b_{n_1}(z)}{(2n_1)^{s_1} \cdots (2n_d)^{s_d}}$$
$$= \sqrt{1 - z^2} \frac{d}{dz} \int_0^z G_{s_1} \circ \cdots \circ G_{s_d} \circ b_n(t) \omega_1.$$

We now provide a new proof of the following result conjectured by Sun [45, (3.10)] first proved by Ablinger [1, (5-56)].

Theorem 6.2. We have

$$\sum_{n\geq 1} \frac{3^n}{n^2 \binom{2n}{n}} \left( H_n + \frac{1}{2n} \right) = \frac{\pi^2}{3} \log 3. \tag{6.4}$$

*Proof.* First, since  $H_n = H_{n-1} + 1/n$  we can rewrite (6.4) as

$$8\sigma(2,1;\sqrt{3}/2) + 12\sigma(3;\sqrt{3}/2) = \frac{\pi^2}{3}\log 3.$$

By Akhilesh's Theorem 6.1 it suffices to show

$$8A + 12B = \frac{\pi^2}{3}\log 3,\tag{6.5}$$

where

$$A = \int_0^{\pi/3} dt (\tan t \, dt) \, dt, \quad B = \int_0^{\pi/3} \frac{dt}{\tan t} \, dt \, dt = \int_0^{\pi/3} (t^2 \cot t) dt.$$

By integrating by parts, we get

$$A = \int_0^{\pi/3} \left( \int_0^t (x \tan x) dx \right) dt = t \int_0^t (x \tan x) dx \Big|_0^{\pi/3} - \int_0^{\pi/3} (t^2 \tan t) dt$$

$$= \frac{\pi}{3} \int_0^{\pi/3} (x \tan x) dx - \int_0^{\pi/3} (t^2 \tan t) dt.$$

We can verify by differentiation that (see also [3, Lemma 1])

$$\int_{0}^{z} (t \tan t) dt = \frac{i}{2} \left( z^{2} + \text{Li}_{2}(-e^{2iz}) + \frac{1}{2} \zeta(2) \right) - z \log(1 + e^{2iz}),$$

$$\int_{0}^{z} (t^{2} \tan t) dt = \frac{i}{3} z^{3} - z^{2} \log(1 + e^{2iz}) + iz \text{Li}_{2}(-e^{2iz}) - \frac{1}{2} \text{Li}_{3}(-e^{2iz}) - \frac{3}{8} \zeta(3),$$

$$\int_{0}^{z} (t^{2} \cot t) dt = -\frac{i}{3} z^{3} + z^{2} \log(1 + e^{iz}) + z^{2} \log(1 - e^{iz}) - 2iz \text{Li}_{2}(e^{iz})$$

$$-2iz \text{Li}_{2}(-e^{iz}) + 2 \text{Li}_{3}(e^{iz}) + 2 \text{Li}_{3}(-e^{iz}) - \frac{1}{2} \zeta(3),$$

where we have used the fact that  $\text{Li}_2(-1) = \zeta(\bar{2}) = -\frac{1}{2}\zeta(2) = -\frac{\pi^2}{12}$  and  $\text{Li}_3(-1) = \zeta(\bar{3}) = -\frac{3}{4}\zeta(3)$ . Setting  $\eta = e^{2\pi i/3}$ , after a lot of simplification and cancelations we obtain

$$8A + 12B = \frac{\pi^2 \log 3}{3} - \frac{2\pi i}{3} \left( \frac{\pi^2}{9} + 2\operatorname{Li}_2(-\eta) + 3\operatorname{Li}_2(\eta) \right). \tag{6.6}$$

Noting that by the series definition

$$\operatorname{Li}_{2}(-\eta) + \operatorname{Li}_{2}(\eta) = \sum_{k=1}^{\infty} \frac{(-\eta)^{k} + \eta^{k}}{k^{2}} = \frac{1}{2}\operatorname{Li}_{2}(\eta^{2}),$$
 (6.7)

we know the quantity in the parenthesis in (6.6)

$$\frac{\pi^2}{9} + 2\operatorname{Li}_2(-\eta) + 3\operatorname{Li}_2(\eta) = \frac{\pi^2}{9} + \operatorname{Li}_2(\eta^2) + \operatorname{Li}_2(\eta)$$
(6.8)

is real (invariant under conjugation) so that it must be zero as 8A + 12B is real. Thus (6.4) follows. We can also prove directly (6.8) vanishes as follows. By (6.7) we have

$$2 \operatorname{Li}_{2}(-\eta) + 3 \operatorname{Li}_{2}(\eta) + \operatorname{Li}_{2}(1) = \operatorname{Li}_{2}(\eta^{2}) + \operatorname{Li}_{2}(\eta) + \operatorname{Li}_{2}(1)$$
$$= \sum_{k=1}^{\infty} \frac{\eta^{2k} + \eta^{k} + 1}{k^{2}} = \frac{1}{3} \operatorname{Li}_{2}(1)$$

since  $\eta^{2k} + \eta^k + 1 = 0$  if  $3 \nmid k$  and  $\eta^{2k} + \eta^k + 1 = 3$  if  $3 \mid k$ . This proves (6.8) since  $\text{Li}_2(1) = \zeta(2) = \pi^2/6$ . We have completed the proof of the theorem.

### 6.2 Odd-indexed variations of Apéry-type inverse binomial series.

By extending Akhilesh's idea it is not too difficult to discover the following recursive formulas. Define

$$f_1(t) := 1, \quad f_2(t) := \frac{t}{\sqrt{1 - t^2}}, \quad f_3(t) = \frac{1}{t}, \quad f_{20}(t) := \frac{1}{t(1 \mp t^2)}, \quad f_5(t) = t.$$

Then by applying  $t \to \sin^{-1} t$  in Theorem 6.1 and its odd-indexed variations we find the following iterative formulas.

**Theorem 6.3.** For any positive integer n and |z| < 1 we have

$$\sum_{m>n} \frac{b_m(z)}{2m} = f_2(z) \int_0^z b_n(t)\omega_1, \tag{6.9}$$

$$\sum_{m>n} \frac{b_m(z)}{(2m)^s} = f_1(z) \int_0^z \omega_0^{s-2} \omega_1 \, b_n(t) \omega_1 \quad \forall s \ge 2, \tag{6.10}$$

$$\sum_{m \ge n} \frac{b_m(z)}{2m+1} = f_{20}(z) \int_0^z b_n(t)\omega_1, \tag{6.11}$$

$$\sum_{m \ge n} \frac{b_m(z)}{(2m+1)^s} = f_3(z) \int_0^z \omega_0^{s-2} \omega_3 \, b_n(t) \omega_1 \quad \forall s \ge 2, \tag{6.12}$$

$$\sum_{m>n} \frac{b_m(z)}{2m-1} = f_5(z) \int_0^z \omega_3 \, b_n(t) \omega_1 + f_2(z) \int_0^z b_n(t) \omega_1, \tag{6.13}$$

$$\sum_{m>n} \frac{b_m(z)}{(2m-1)^s} = f_5(z) \int_0^z (\omega_0 + 1)\omega_0^{s-2} \omega_3 \, b_n(t)\omega_1 \quad \forall s \ge 2, \tag{6.14}$$

where  $\omega_j$   $(0 \le j \le 2)$  are defined by (6.3),

$$\omega_3 = \frac{dt}{t\sqrt{1-t^2}}$$
 and  $\omega_{20} = \omega_2 + \omega_0 = \frac{dt}{t(1-t^2)}$ 

such that  $f_j(t)\omega_1 = \omega_j$  for all  $j \geq 1$ .

*Proof.* We will only sketch the idea of the proof of (6.9)–(6.14). Let  $u_n(z)$  (resp.  $v_n(z)$ ) be the left-hand (resp. right-hand) side of any one of the equations above. Then one can show by differentiation that  $u_n(z) - u_{n+1}(z) = v_n(z) - v_{n+1}(z)$ . Since  $\lim_{n\to\infty} u_n(z) = \lim_{n\to\infty} v_n(z) = 0$ , the equation follows from telescoping. We leave the details to the interested read since the verification is a routine application of calculus.

**Example 6.4.** As an easy example of Theorem 6.3 we apply (6.10) followed by (6.11) to get

$$\sum_{n_1 > n_2 > 0} \frac{b_{n_1}}{n_1^2 (2n_2 + 1)} = 4 \int_0^1 \omega_1 \circ \omega_{20} \circ \omega_1.$$

By the change of variables  $t \to (1-t^2)/(1+t^2)$  we have

$$\sum_{n_1 > n_2 > 0} \frac{b_{n_1}}{n_1^2 (2n_2 + 1)} = -4 \int_0^1 (\mathbf{x}_{-i} - \mathbf{x}_i) \circ (\mathbf{x}_0 + \mathbf{x}_{-1} + \mathbf{x}_1) \circ (\mathbf{x}_{-i} - \mathbf{x}_i) = 7\zeta(3),$$

where we have used Au's Mathematica package [7] at the last step.

# 6.3 Apéry-type binomial series.

It turns out Akhilesh's method of using iteration to express Apéry-type inverse binomial series can be generalized to binomial series and even to the case where the summation indices can be either even or odd. To do this, we need to extend the definition of iterated integrals. For any 1-forms  $f_1(t) dt$ ,  $f_2(t) dt$  and functions F(t) and G(t), we define

$$(F(t) + f_1(t) dt \circ G(t)) \circ f_2(t) dt := F(t) f_2(t) dt + f_1(t) dt \circ G(t) f_2(t) dt.$$

Put 
$$a_0(x) = 1$$
,  $a_n(x) = \binom{2n}{n} x^{2n} / 4^n$  and  $a_n = a_n(1)$  for all  $n \ge 0$ . Define 
$$p_s(t) = \tan t \, dt \, (\cot t \, dt)^{s-1} (1 - \csc t \, dt \circ \sec t).$$

**Theorem 6.5.** For all  $n \in \mathbb{N}_0$ ,  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$  and  $y \in [-\pi/2, \pi/2]$  we have

$$\alpha(\mathbf{s}; \sin y) := \sum_{n_1 > \dots > n_d > n} \frac{a_{n_1}(\sin y)}{(2n_1)^{s_1} \cdots (2n_d)^{s_d}}$$

$$= \cot y \frac{d}{dy} \int_0^y p_{s_1} \circ \cdots \circ p_{s_d} \circ a_n(\sin t) \tan t \, dt. \tag{6.16}$$

(6.15)

Here if  $y = \pm \pi/2$  then the last expression is defined to be the limit as  $y \to \pm \pi/2$ .

When computing sums with both even and odd summation indices we may use the following iterative formulas. Let  $y \in (-\pi/2, \pi/2)$  and  $F_s = (\cot t \, dt)^{s-1}$  for  $s \in \mathbb{N}$ . Then

$$\sum_{m>n} \frac{a_m(\sin y)}{(2m)^s} = \int_0^y F_s(1 - \csc t \, dt \circ \sec t) a_n(\sin t) \tan t \, dt, \tag{6.17}$$

$$\sum_{m>n} \frac{a_m(\sin y)}{(2m+1)^s} = \csc y \int_0^y F_s(1 - dt \circ \csc t \sec t) a_n(\sin t) \sin t \tan t \, dt. \tag{6.18}$$

$$\sum_{m>n} \frac{a_m(\sin y)}{2m-1} = \cos y \int_0^y a_n(\sin t) \tan t \sec t \, dt, \tag{6.19}$$

$$\sum_{m>n} \frac{a_m(\sin y)}{(2m-1)^s} = \sin y \int_0^y F_{s-1} \frac{dt}{\tan^2 t} a_n(\sin t) \tan t \sec t \, dt \quad (s \ge 2).$$
 (6.20)

All the formulas (6.17)–(6.20) as well as Theorem 6.5 can be proved using exactly the same idea as that of Theorem 6.3. We thus omit their proofs here.

**Example 6.6.** We now compute the following Apéry-like sum as an illuminating example using the above formulas:

$$\sum_{n>m>0} \frac{a_n^2}{(2n-1)m^2} = \sum_{n>m>0} {2n \choose n}^2 \frac{1}{16^n (2n-1)m^2}.$$

First, by (6.19) and then (6.17) we get

$$\sum_{n>m>0} \frac{a_n \sin^{2n} y}{(2n-1)(2m)^2} = \cos y \int_0^y \sum_{m>0} \frac{a_m(\sin t)}{(2m)^2} \tan t \sec t \, dt$$

$$= \cos y \int_0^y \tan t \sec t \, dt \cot t \, dt \, (1 - \csc t \, dt \circ \sec t) \tan t \, dt$$

$$= \cos y \int_0^y \tan t \sec t \, dt \cot t \, dt \, (\tan t \, dt - \csc t \, dt \circ \sec t \tan t \, dt)$$

$$= \cos y \int_0^y \tan t \sec t \, dt \cot t \, dt \, (\csc t - \cot t) \, dt. \tag{6.21}$$

Note that by the beta integral

$$\int_0^{\pi/2} \sin^{2n} t \, dt = \frac{1}{2} B\left(n + \frac{1}{2}, \frac{1}{2}\right) = \frac{\pi}{2} a_n. \tag{6.22}$$

Thus integrating (6.21) over  $(0, \pi/2)$  and dividing by  $\pi/2$  we get

$$\sum_{n>m>0} \frac{a_n^2}{(2n-1)m^2} = \frac{8}{\pi} \int_0^{\pi/2} \cos t \, dt \, d(\sec t) \cot t \, dt \Big( \csc t - \cot t \Big) \, dt$$
$$= \frac{8}{\pi} \int_0^{\pi/2} dt \, \cot t \, dt \Big( \csc t - \cot t \Big) \, dt - (1-\sin t) \csc t \, dt \Big( \csc t - \cot t \Big) \, dt.$$

For any Nth roots of unity  $\xi$  and  $\xi'$  we set  $\mathbf{x}_{\xi} = dt/(\xi - t)$  and  $\mathbf{d}_{\xi,\xi'} = \mathbf{x}_{\xi} - \mathbf{x}_{\xi'}$ . Then under the change of variables  $t \to \sin^{-1}[(1-t^2)/(1+t^2)]$  we have

$$dt \to i \mathbf{d}_{-i,i}, \cot t \, dt \to \mathbf{y}, \csc t \, dt \to \mathbf{d}_{-1,1},$$
 (6.23)

where  $y = x_{-i} + x_i - x_{-1} - x_1$  and  $z = -a - x_{-i} - x_i$ . Hence

$$\sum_{n>m>0} \frac{a_n^2}{(2n-1)m^2} = \frac{8i}{\pi} \int_0^1 \left\{ \frac{(2\mathbf{x}_{-1} - \mathbf{x}_{-i} - \mathbf{x}_i)(1-\mathbf{y})\mathbf{d}_{-i,i}}{-(2\mathbf{x}_{-1} - \mathbf{x}_{-i} - \mathbf{x}_i)\mathbf{d}_{-1,1}} \right\}.$$

Now we may convert all the iterated integral terms into CMZVs of level 4 and then use Au's Mathmatica package [7] to reduce the above to

$$\sum_{n>m>0} \frac{a_n^2}{(2n-1)m^2} = \frac{3\pi^2}{2} - \frac{\pi}{3} - 6\log^2 2 + 8\log 2 - \frac{64}{\pi} \operatorname{Im} \operatorname{Li}_3\left(\frac{1+i}{2}\right) - \frac{16G}{\pi},$$

where Im z is the imaginary part of z and  $G = \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)^2}$  is Catalan's constant. One can check that numerically the sum is approximately 0.14117237.

Using the iterations (6.17)–(6.20) and by the same change of variables  $t \to \sin^{-1}[(1-t^2)/(1+t^2)]$  one can show that all the Apéry-like sums  $\sigma(s;1)$  defined by (3.2),  $\alpha(s;1)$  defined by (6.16) and their odd-indexed variations can be expressed as  $\mathbb{Q}$ -linear combinations of the real or the imaginary part of CMZVs of level 4. Furthermore, by applying the beta integral (6.22), similar results hold when one replaces  $a_n$  and  $b_n$  by  $a_n^2$  and  $b_n^2$ , respectively, with the only difference being that sometimes an extra factor of  $1/\pi$  is needed as shown by Example 6.6 above. For more details we refer the readers to [52].

### 6.4 Alternating Apéry-like sums.

One such sum in (1.5) already appeared in Apéry's original work. Generally, for any  $n \in \mathbb{N}$ ,  $\eta = (\eta_1, \ldots, \eta_d) \in \{\pm 1\}^d$ ,  $\mathbf{s} = (s_1, \ldots, s_d) \in \mathbb{N}^d$  and a complex variable z, we define

$$\alpha(\mathbf{s}; \boldsymbol{\eta}; z)_n := \sum_{\substack{n_1 > \dots > n_d > n}} \frac{a_{n_1}(z)\eta_1^{n_1} \cdots \eta_d^{n_d}}{(2n_1)^{s_1} \cdots (2n_d)^{s_d}},$$

$$\sigma(\mathbf{s}; \boldsymbol{\eta}; z)_n := \sum_{\substack{n_1 > \dots > n_d > n}} \frac{b_{n_1}(z)\eta_1^{n_1} \cdots \eta_d^{n_d}}{(2n_1)^{s_1} \cdots (2n_d)^{s_d}},$$

which are called the n-tails of the alternating Apéry-type series

$$\alpha(s; \eta; z) := \alpha(s; \eta; z)_0$$
 and  $\sigma(s; \eta; z) := \sigma(s; \eta; z)_0$ 

respectively. To save space we will put a bar on top of  $s_j$  if the corresponding  $\eta_j = -1$ . For example,

$$\sigma(\bar{3},4,\bar{1};z) = \sum_{n_1 > n_2 > n_3 > 0} \frac{b_{n_1}(z)(-1)^{n_1 + n_3}}{(2n_1)^3 (2n_2)^4 (2n_3)}.$$

By modifying the results in the previous sections we can derive the corresponding formulas for alternating sums. The key is to replace the trigonometric 1-forms by hyperbolic ones. To fix notation, we put

$$sh z = -i \sin(iz) = \frac{e^z - e^{-z}}{2}, \quad ch z = \cos(iz) = \frac{e^z + e^{-z}}{2}, 
th z = -i \tan(iz) = \frac{e^z - e^{-z}}{e^z + e^{-z}}, \quad cth z = i \cot(iz) = \frac{e^z + e^{-z}}{e^z - e^{-z}}.$$

Define the hyperbolic counterpart of  $p_s(t)$  and  $g_s(t)$  by

$$\tilde{p}_s(t) = \operatorname{th} t \, dt \, (\operatorname{cth} t \, dt)^{s-1} (1 - \operatorname{csch} t \, dt \circ \operatorname{sech} t)$$

$$\tilde{g}_s(t) = \begin{cases} \operatorname{th} t \, dt, & \text{if } s = 1; \\ dt \circ (\operatorname{cth} t \, dt)^{s-2} \circ dt, & \text{if } s \ge 2. \end{cases}$$

**Theorem 6.7.** Set  $\psi = \operatorname{sh}^{-1} 1 = \log(1 + \sqrt{2})$ . For all  $n \in \mathbb{N}_0$ ,  $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$  and  $y \in [-\psi, \psi]$  we have

$$\alpha(\mathbf{s}; i \operatorname{sh} y)_n = (-1)^d \operatorname{cth} y \frac{d}{dy} \int_0^y \tilde{p}_{s_1} \circ \cdots \circ \tilde{p}_{s_d} \circ a_n(i \operatorname{sh} t) \operatorname{th} t \, dt,$$
  
$$\sigma(\mathbf{s}; i \operatorname{sh} y)_n = (-1)^d \frac{d}{dy} \int_0^y \tilde{g}_{s_1} \circ \cdots \circ \tilde{g}_{s_d} \circ b_n(i \operatorname{sh} t) \, dt.$$

*Proof.* By the substitution  $y \to iy$  in Theorem 6.1 we obtain immediately

$$\sigma(\mathbf{s}; i \operatorname{sh} y)_n := \sum_{n_1 > \dots > n_d > n} \frac{b_{n_1}(i \operatorname{sh} y)}{(2n_1)^{s_1} \cdots (2n_d)^{s_d}}$$
$$= -i \frac{d}{dy} \int_0^y g_{s_1}(it) \circ \cdots \circ g_{s_d}(it) \circ b_n(i \operatorname{sh} t) d(it),$$

where we have used the fact that  $\sin it = i \sinh t$  and  $\cos it = \cot t$ . Note that

$$g_s(it) = \begin{cases} \tan(it) d(it), & \text{if } s = 1; \\ d(it) \circ (\cot(it) d(it))^{s-2} \circ d(it), & \text{if } s \ge 2 \end{cases}$$

$$= \begin{cases} -\operatorname{th} t \, dt, & \text{if } s = 1; \\ -dt \circ (\operatorname{cth} t \, dt)^{s-2} \circ dt, & \text{if } s \geq 2. \end{cases}$$

The claim for  $\sigma(s; i \operatorname{sh} y)_n$  follows immediately. The proof of the claim for  $\alpha(s; i \operatorname{sh} y)_n$  is completely similar so we omit it here.

By combining Theorem 6.1 and Theorem 6.7 we can express all possible alternating Apéry-type inverse binomial series by iterated integrals. Moreover, if  $(s_1, \eta_1) \neq 1$  then  $\sigma(s; \eta; 1)$  can be expressed as a  $\mathbb{Q}[i, \sqrt{2}]$ -linear combinations of CMZVs of level 8. We are interested in extending this result to all odd-indexed sums so we first consider the alternating form of (6.9)–(6.14). Write

$$b_n^+(z) := \frac{4^n z^{2n}}{\binom{2n}{n}}, \quad b_n^-(z) = (-1)^n b_n(z), \text{ and } b_n^- = b_n^-(1) \quad \forall n \ge 0.$$

Define

$$f_{\pm 1}(t) := 1, \ f_{\pm 2}(t) := \frac{t}{\sqrt{1 \mp t^2}}, \ f_{\pm 3}(t) = \frac{1}{t}, \ f_{\pm 20}(t) := \frac{1}{t(1 \mp t^2)}, \ f_{\pm 5}(t) = t,$$
$$\omega_{\pm 1} := \frac{dt}{\sqrt{1 \mp t^2}}, \qquad \omega_{\pm j} := f_{\pm j}(t) \cdot \omega_{\pm 1} \quad \forall j \ge 2.$$

Then one can show that  $\omega_{\pm 20} = \omega_0 \mp \omega_{\pm 2}$  and by (6.9)–(6.14)

$$\sum_{n_1 > n} \frac{b_{n_1}^{\pm}(z)}{2n_1} = \pm f_{\pm 2}(z) \int_0^z b_n^{\pm}(t)\omega_{\pm 1}, \tag{6.24}$$

$$\sum_{n_1 > n} \frac{b_{n_1}^{\pm}(z)}{(2n_1)^s} = \pm f_1(z) \int_0^z \omega_0^{s-2} \omega_{\pm 1} \, b_n^{\pm}(t) \omega_{\pm 1} \quad \forall s \ge 2, \tag{6.25}$$

$$\sum_{n \ge n} \frac{b_{n_1}^{\pm}(z)}{2n_1 + 1} = f_{\pm 20}(z) \int_0^z b_n^{\pm}(t)\omega_{\pm 1}, \tag{6.26}$$

$$\sum_{n \ge n} \frac{b_{n_1}^{\pm}(z)}{(2n_1 + 1)^s} = f_3(z) \int_0^z \omega_0^{s-2} \omega_{\pm 3} \, b_n^{\pm}(t) \omega_{\pm 1} \quad \forall s \ge 2, \tag{6.27}$$

$$\sum_{n_1 \ge n} \frac{b_{n_1}^{\pm}(z)}{2n_1 - 1} = \pm f_5(z) \int_0^z \omega_{\pm 3} \, b_n^{\pm}(t) \omega_{\pm 1} \pm f_{\pm 2}(z) \int_0^z b_n^{\pm}(t) \omega_{\pm 1}, \tag{6.28}$$

$$\sum_{n_1 > n} \frac{b_{n_1}^{\pm}(z)}{(2n_1 - 1)^s} = \pm f_5(z) \int_0^z (\omega_0 + 1)\omega_0^{s - 2} \omega_{\pm 3} b_n^{\pm}(t)\omega_{\pm 1} \quad \forall s \ge 2.$$
 (6.29)

We illustrate this with an example. For the proof of the above formulas in general we refer the interested reader to [52].

**Example 6.8.** In depth two, using (6.24) and (6.26) successively we get

$$S := \sum_{n > m > 0} \frac{(-1)^n b_n}{(2n)(2m+1)} = \frac{-1}{\sqrt{2}} \int_0^1 \sum_{m > 0} \frac{b_m^-(t)}{2m+1} \omega_{-1} = \frac{-1}{\sqrt{2}} \int_0^1 \omega_{-20} \omega_{-1}.$$

By the change of variables  $t \to \sqrt{2}t/\sqrt{1+t^4}$  we see that

$$\begin{split} \omega_0 := & \frac{dt}{t} & \rightarrow & \frac{(1-t^4)\,dt}{t(1+t^4)} &= \mathtt{a} + \frac{1}{2} \sum_{j=1}^4 \mathtt{x}_{\mu_j}, \\ \omega_{-1} := & \frac{dt}{\sqrt{1+t^2}} & \rightarrow & \frac{\sqrt{2}(1-t^2)dt}{1+t^4} &= \frac{\sqrt{2}}{4} \sum_{j=1}^4 (\mu_j - \mu_j^3) \mathtt{x}_{\mu_j}, \\ \omega_{-2} := & \frac{t\,dt}{1+t^2} & \rightarrow & \frac{2(1-t^2)dt}{(1+t^2)(1+t^4)} = \frac{1}{2} \sum_{j=1}^4 \mathtt{x}_{\mu_j} - \mathtt{x}_i - \mathtt{x}_{-i}, \end{split}$$

where  $\mu_j = \exp((2j-1)\pi i/4)$   $(1 \le j \le 4)$  are the four 8th roots of unity satisfying  $x^4 + 1 = 0$ . Thus

$$S = -\frac{1}{4} \int_{0}^{1} (\mathbf{a} + \mathbf{x}_{i} + \mathbf{x}_{-i}) \left( \sum_{j=1}^{4} (\mu_{j} - \mu_{j}^{3}) \mathbf{x}_{\mu_{j}} \right)$$

$$= -\frac{1}{4} \sum_{j=1}^{4} (\mu_{j} - \mu_{j}^{3}) \operatorname{Li}_{2}(\mu_{j}^{-1}) - \frac{1}{4} \sum_{j=1}^{4} \sum_{\varepsilon = \pm i} (\mu_{j} - \mu_{j}^{3}) \operatorname{Li}_{1,1}(\varepsilon^{-1}, \varepsilon/\mu_{j})$$

$$= \frac{\sqrt{2}}{4} \left( \log 2 \log \nu - \log^{2} \nu - 2 \operatorname{Li}_{2}(\nu^{-1}) \right) - \frac{\sqrt{2}\pi^{2}}{96} \approx -0.53464318757$$

by Au's Mathematica package [7].

Now we consider the alternating form of (6.17)–(6.20). By the change of variables  $y \to \sinh^{-1}(iz)$  we can easily get

$$\sum_{m>n} \frac{a_m^-(z)}{(2m)^s} = \int_0^z \omega_0^{s-1} \left(\omega_{-3} \circ \frac{1}{\sqrt{1+t^2}} - 1\right) a_n^-(t) \omega_{-2},\tag{6.30}$$

$$\sum_{m \ge n} \frac{a_m^-(z)}{(2m+1)^s} = \frac{1}{z} \int_0^z \omega_0^{s-1} \left(\frac{1}{t} + \omega_{-1} \circ \frac{1}{\sqrt{1+t^2}}\right) a_n^-(t) \omega_{-2},\tag{6.31}$$

$$\sum_{m>n} \frac{a_m^-(z)}{2m-1} = -\sqrt{1+z^2} \int_0^z a_n^-(t) \frac{t \, dt}{(1+t^2)^{3/2}},\tag{6.32}$$

$$\sum_{m>n} \frac{a_m^-(z)}{(2m-1)^s} = z \int_0^z \omega_0^{s-2} \frac{\sqrt{1+t^2} dt}{t^2} a_n^-(t) \frac{t dt}{(1+t^2)^{3/2}} \quad (s \ge 2).$$
 (6.33)

**Example 6.9.** Using (6.31) and (6.30), we get

$$\sum_{n\geq m>0} \frac{(-1)^n a_n}{(2n+1)(2m)} = \int_0^1 \left(\frac{1}{t} - \omega_{-1} \circ \frac{1}{\sqrt{1+t^2}}\right) \sum_{m>0} \frac{a_m^-(t)}{2m} \omega_{-2}$$

$$= \int_0^1 \left(\frac{1}{t} - \omega_{-1} \circ \frac{1}{\sqrt{1+t^2}}\right) \omega_{-2} \left(\omega_{-3} \circ \frac{1}{\sqrt{1+t^2}} - 1\right) \omega_{-2}$$

$$= \int_0^1 \left(\frac{dt}{1+t^2} - \omega_{-1} \circ \frac{t \, dt}{(1+t^2)^{3/2}}\right) \left(\omega_{-3} - \omega_{-20} - \omega_{-2}\right)$$

$$= \frac{1}{\sqrt{2}} \int_0^1 \omega_{-1} \left(\omega_{-3} - \omega_{-20} - \omega_{-2}\right).$$

Set  $\mu = \exp(2\pi i/8)$ . By the change of variables  $t \to i(1-t^2)/(1+t^2)$  we get

$$\begin{split} \sum_{n \geq m > 0} \frac{(-1)^n a_n}{(2n+1)(2m)} &= \frac{1}{\sqrt{2}} \int_1^\mu \mathsf{d}_{i,-i} \big( 2 \mathsf{x}_{-1} - \mathsf{x}_{-i} - \mathsf{x}_i \big) \\ &= \frac{1}{\sqrt{2}} \left( \int_1^0 \mathsf{d}_{i,-i} \big( 2 \mathsf{x}_{-1} - \mathsf{x}_{-i} - \mathsf{x}_i \big) + \int_0^\mu \mathsf{d}_{i,-i} \int_1^0 \big( 2 \mathsf{x}_{-1} - \mathsf{x}_{-i} - \mathsf{x}_i \big) \right. \\ &\qquad \qquad + \int_0^\mu \mathsf{d}_{i,-i} \big( 2 \mathsf{x}_{-1} - \mathsf{x}_{-i} - \mathsf{x}_i \big) \bigg) \\ &= \frac{1}{\sqrt{2}} \left( \int_0^1 \big( 2 \mathsf{x}_{-1} - \mathsf{x}_{-i} - \mathsf{x}_i \big) \mathsf{d}_{i,-i} - \int_0^1 \mathsf{d}_{i/\mu,-i/\mu} \int_0^1 \big( 2 \mathsf{x}_{-1} - \mathsf{x}_{-i} - \mathsf{x}_i \big) \right. \\ &\qquad \qquad + \int_0^1 \mathsf{d}_{i/\mu,-i/\mu} \big( 2 \mathsf{x}_{-1/\mu} - \mathsf{x}_{-i/\mu} - \mathsf{x}_{i/\mu} \big) \bigg) \\ &= \frac{\pi^2}{12} + 2 \log 2 \log (1 + \sqrt{2}) - \frac{3}{2} \log^2 (1 + \sqrt{2}) - 2 \operatorname{Li}_2(\sqrt{2} - 1) \approx -0.05598682674 \end{split}$$

by using Au's Mathematica package [7].

We will conclude this section by extending our question on page 7:

**Question.** Let  $d \in \mathbb{N}$ ,  $s = (s_1, \ldots, s_d) \in \mathbb{N}^d$  and  $\eta = (\eta_1, \ldots, \eta_d) \in \{\pm 1\}^d$ . Let  $l_j(n) = 2n$  or  $l_j(n) = 2n \pm 1$  for all  $1 \le j \le d$ . Suppose  $z = \sqrt{j}/2$  for j = 1, 2, 4. Is there a level N such that

$$\sum_{n_1 \succ n_2 \succ \cdots \succ n_d \succ 0} \frac{a_{n_1} z^{2n_1} \eta_1^{n_1} \cdots \eta_d^{n_d}}{l_1(n_1)^{s_1} \cdots l_d(n_d)^{s_d}} \in \mathsf{CMZV}^N \otimes \mathbb{Q}[\exp(2\pi i/N)],$$

$$\sum_{\substack{n_1 \succ n_2 \succ \cdots \succ n_d \succ 0}} \frac{b_{n_1} z^{2n_1} \eta_1^{n_1} \cdots \eta_d^{n_d}}{l_1(n_1)^{s_1} \cdots l_d(n_d)^{s_d}} \in \mathsf{CMZV}^N \otimes \mathbb{Q}[\exp(2\pi i/N)]$$

if the sums converge? Here " $\succ$ " can be either " $\geq$ " or ">". One may also ask the same question when  $a_n$  and  $b_n$  are replaced by their squares while adding an extra factor of  $1/\pi$  on the right-hand side for  $a_n^2$  version. Available evidence seems to suggest that N is at least 120.

# 7 Apéry-like sums via contour integrals

Let  $H_n^{(p)}$  be the generalized nth-harmonic number of order p defined by

$$H_n^{(p)} := \sum_{k=1}^n \frac{1}{k^p}, \quad H_n \equiv H_n^{(1)} \quad \text{and} \quad H_0^{(p)} := 0.$$

Motivated by Flajolet and Salvy's paper [28]. Flajolet and Salvy [28] introduced and used the method of contour integrals and residue computations to study the following kind of special Dirichlet series that involve harmonic numbers

$$S_{\mathbf{p},q} := \sum_{n=1}^{\infty} \frac{H_n^{(p_1)} H_n^{(p_2)} \cdots H_n^{(p_r)}}{n^q}, \tag{7.1}$$

we call them generalized (non-alternating) Euler sums. Moreover, if r > 1 in (7.1), they were called nonlinear Euler sums. Here  $\mathbf{p} := (p_1, p_2, \dots, p_r)$   $(r, p_i \in \mathbb{N}, i = 1, 2, \dots, r)$  with  $p_1 \leq p_2 \leq \dots \leq p_r$  and  $q \geq 2$ . The quantity  $w := p_1 + \dots + p_r + q$  is called the weight and the quantity r is called the degree (order). By the well-known "harmonic product" (also called "quasi-shuffle product") [31], we see that the generalized Euler sums  $S_{\mathbf{p},q}$  can be expressed as a rational linear combinations of multiple zeta values. See [50] for the explicit formula of generalized Euler sums via multiple zeta values.

Flajolet and Salvy [28] defined a kernel function  $\xi(s)$  by the two requirements: 1. it is meromorphic on the whole complex plane. 2. it satisfies  $\xi(s) = o(s)$  over an infinite collection of circles  $|s| = \rho_k$  with  $\rho_k \to \infty$ . Applying these two conditions of kernel function  $\xi(s)$ , Flajolet and Salvy discovered the following residue lemma.

**Lemma 7.1.** (cf. [28]) Let  $\xi(s)$  be a kernel function and let r(s) be a rational function which is  $O(s^{-2})$  at infinity. Then

$$\sum_{\alpha \in O} \operatorname{Res}(r(s)\xi(s), \alpha) + \sum_{\beta \in S} \operatorname{Res}(r(s)\xi(s), \beta) = 0,$$
(7.2)

where S is the set of poles of r(s) and O is the set of poles of  $\xi(s)$  that are not poles r(s). Here  $\operatorname{Res}[r(s),s=\alpha]$  denotes the residue of r(s) at  $s=\alpha$ .

Flajolet and Salvy considered the contour integration involving classical digamma function and used the residue computations to establish more explicit reductions of generalized Euler sums to Euler sums with lower degree. In particular, they proved the famous theorem that a nonlinear Euler sum  $S_{p_1p_2\cdots p_r,q}$  reduces to a combination of sums of lower orders whenever the weight  $p_1 + p_2 + \cdots + p_r + q$  and the order r are of the same parity by considering the kernels

$$\xi_r(s) := \pi \cot(\pi s) \frac{\psi^{(p_1 - 1)}(-s) \cdots \psi^{(p_r - 1)}(-s)}{(p_1 - 1)! \cdots (p_r - 1)!}$$

and the base function  $r(s) = s^{-q}$ .

#### 7.1 Main Methods.

By extending Flajolet and Salvy's idea it is not too difficult to discover the following fact that any polynomial form in

$$\pi \cot \pi s$$
,  $\frac{\pi}{\sin \pi s}$ ,  $\psi^{(j)}(\pm s)$ ,  $\frac{\Gamma^2(s+1)}{\Gamma(2s+1)}4^s$ 

is itself a kernel function with poles at a subset of the integers. Because by the Legendre duplication formula

$$\Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = \sqrt{\pi} \cdot 2^{1-2s}\Gamma(2s)$$

and the asymptotic expansion for the ratio of two gamma functions

$$\frac{\Gamma(s+a)}{\Gamma(s+b)} = s^{a-b} \left( 1 + O\left(\frac{1}{s}\right) \right), \quad \text{for } |\arg(s)| \le \pi - \varepsilon, \ \varepsilon > 0, \ |s| \to \infty$$

(see [35, Sections 2.3 and 2.11]), we have

$$\frac{\Gamma^2(s+1)}{\Gamma(2s+1)} 4^s = \sqrt{\pi s} \left( 1 + O\left(\frac{1}{s}\right) \right) = o(s), \quad |s| \to \infty.$$
 (7.3)

The purpose of this section is precisely to investigate the power of such kernels in connection with summation formulas and Apéry-like sums. We need the following lemmas.

**Lemma 7.2.** (cf. [49]) For |s| < 1, the following identities hold:

$$\Gamma(s+1) e^{\gamma s} = \sum_{n=0}^{\infty} C_n \frac{s^n}{n!}, \quad \{\Gamma(s+1) e^{\gamma s}\}^{-1} = \sum_{n=0}^{\infty} D_n \frac{s^n}{n!},$$

where  $\gamma := \lim_{n \to \infty} (H_n - \log n)$  is the Euler-Mascheroni constant, and

$$C_n := Y_n(0, 1!\zeta(2), -2!\zeta(3), \dots, (-1)^n(n-1)!\zeta(n)),$$
  

$$D_n := Y_n(0, -1!\zeta(2), 2!\zeta(3), \dots, (-1)^{n-1}(n-1)!\zeta(n)).$$

Here  $Y_n$  denote exponential complete Bell polynomials (see [23, Section 3.3] and [43, Section 2.8]) defined by

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k!}\right) = \sum_{n=0}^{\infty} Y_n(x_1, x_2, \dots, x_n) \frac{t^n}{n!},$$
(7.4)

and satisfy the recurrence

$$Y_n(x_1, x_2, \dots, x_n) = \sum_{j=0}^{n-1} {n-1 \choose j} x_{n-j} Y_j(x_1, x_2, \dots, x_j), \quad n \ge 1.$$
 (7.5)

According to Lemma 7.2, it is clear that  $C_k$  and  $D_k$  are rational linear combinations of products of zeta values. In particular, by (7.5), we have  $(C_k)_{k\in\mathbb{N}_0}=(1,0,\zeta(2),-2\zeta(3),\frac{27}{2}\zeta(4),\ldots)$  and  $(D_k)_{k\in\mathbb{N}_0}=(1,0,-\zeta(2),2\zeta(3),\frac{3}{2}\zeta(4),\ldots)$ . Furthermore, by the relations

$$\begin{split} \frac{\Gamma(s-n)}{\Gamma(s+1)} &= \prod_{k=0}^n \frac{1}{s-k} = \frac{(-1)^n}{n!} \frac{1}{s} \sum_{m=0}^\infty \zeta_n^{\star}(\{1\}_m) s^m \,, \\ \frac{\Gamma(s+1)}{\Gamma(s-n)} &= \prod_{k=0}^n (s-k) = (-1)^n n! s \sum_{m=0}^\infty (-1)^m \zeta_n(\{1\}_m) s^m \,, \end{split}$$

for  $n \geq 0$ , the next lemma can be established.

**Lemma 7.3.** For nonnegative integer n, when  $s \to -n$ , we have

$$\Gamma(s) e^{\gamma(z-1)} = \frac{(-1)^n}{n!} e^{-\gamma(n+1)} \sum_{k=0}^{\infty} A_k(n) (s+n)^{k-1},$$

$$\frac{1}{\Gamma(s) e^{\gamma(s-1)}} = (-1)^n n! e^{\gamma(n+1)} \sum_{k=0}^{\infty} B_k(n) (s+n)^{k+1},$$

where

$$A_k(n) := \sum_{\substack{k_1 + k_2 = k \\ k_1, k_2 \ge 0}} \zeta_n^{\star}(\{1\}_{k_1}) \frac{C_{k_2}}{k_2!} \,, \quad B_k(n) := \sum_{\substack{k_1 + k_2 = k \\ k_1, k_2 \ge 0}} (-1)^{k_1} \zeta_n(\{1\}_{k_1}) \frac{D_{k_2}}{k_2!} \,.$$

Here for positive integers  $k_1, k_2, \ldots, k_r$ , the multiple harmonic sums (MHSs)  $\zeta_n(k_1, k_2, \ldots, k_r)$  are defined by

$$\zeta_n(\mathbf{k}) := \sum_{n > n_1 > n_2 > \dots > n_r > 1} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}}.$$
 (7.6)

Additionally, similarly to MHSs, denote the multiple harmonic star sums  $\zeta_n^{\star}(k_1, k_2, \dots, k_r)$  by

$$\zeta_n^{\star}(\mathbf{k}) := \sum_{n \ge n_1 \ge n_2 \ge \dots \ge n_r \ge 1} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}}.$$
 (7.7)

Here when n < r, then  $\zeta_n(\mathbf{k}) := 0$ , and  $\zeta_n(\emptyset) = \zeta_n^*(\emptyset) := 1$ . When taking the limit  $n \to \infty$  in (7.6) and (7.7) with  $k_1 > 1$ , we get the so-called the multiple zeta values and the multiple zeta star values, respectively.

Now, we use the Lemmas 7.1 and 7.3 to establish some explicit relations of Apéry-like sums. For more details we refer the readers to our paper [49].

# 7.2 Evaluations of some Apéry-like sums via contour integrals.

Firstly, denote

$$G(\vec{k}) := G(k_1, k_2, k_3, k_4) = \frac{2^{k_3 + k_4} \log^{k_4}(2) C_{k_1} C_{k_2} D_{k_3}}{k_1! k_2! k_2! k_4!}. \tag{7.8}$$

Then we have a new closed form expression.

**Theorem 7.4.** For integer  $q \geq 2$ , we have

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n n^{q-1}} = (-1)^q \sum_{\substack{k_1 + k_2 + k_3 + k_4 = q-1, \\ k_1, \dots, k_4 > 0}} G(k_1, k_2, k_3, k_4).$$

Proof. Consider the function

$$F_0(s) := \frac{\Gamma^2(s+1)}{\Gamma(2s+1)} \frac{4^s}{s^q}.$$

Applying (7.3) yields

$$F_0(s) = \frac{\sqrt{\pi}}{s^{q-1/2}} \left( 1 + O\left(\frac{1}{s}\right) \right), \quad |s| \to \infty,$$

which implies that the integral  $\oint_{(\infty)} F_0(s) ds = 0$ , where  $\oint_{(\infty)}$  denotes integration along large circles, that is, the limit of integrals  $\oint_{|s|=\rho_k}$ . Hence, we can use the Lemma 7.1 to obtain an identity of infinite series. On the other hand, the function  $F_0(s)$  has poles only at all non-positive integers. For a negative integer -n, by Lemma 7.3, if  $s \to -n$ , we have

$$F_0(s) = -\frac{4^s}{s^q} \frac{(2n-1)!}{(n-1)!^2} \sum_{k_1, k_2, k_3 \ge 0} 2^{k_3+1} A_{k_1}(n-1) A_{k_2}(n-1) B_{k_3}(2n-1) (s+n)^{k_1+k_2+k_3-1}.$$

Hence, the residue is

Res
$$(F_0(s), -n) = (-1)^{q+1} \frac{\binom{2n}{n}}{4^n n^{q-1}}, \text{ for } n = 1, 2, \dots$$

Similarly, if  $s \to 0$ , we have

$$F_0(s) = \frac{4^s}{s^q} \sum_{k_1, k_2, k_3 > 0} \frac{2^{k_3} C_{k_1} C_{k_2} D_{k_3}}{k_1! k_2! k_3!} s^{k_1 + k_2 + k_3} ,$$

and the residue of the pole of order q at 0 is

Res
$$(F_0(s), 0) = \sum_{|k|_4 = q-1} G(k_1, k_2, k_3, k_4)$$
.

Using Lemma 7.1 and Summing these two contributions gives the statement of the theorem.  $\Box$ 

**Theorem 7.5.** For positive integer q > 1,

$$\sum_{n=1}^{\infty} \frac{n}{(n-1/2)^q} \cdot \frac{\binom{2n}{n}}{4^n} = \frac{2}{\pi} \sum_{\substack{k_1 + \dots + k_4 + 2k_5 = q-2, \\ k_1, \dots, k_5 > 0}} G(k_1, k_2, k_3, k_4) \widetilde{t}(2k_5 + 2), \tag{7.9}$$

where  $\tilde{t}(k) := 2^k \frac{1}{(2n-1)^k} = (2^k - 1)\zeta(k)$  for k > 1.

*Proof.* Similarly to Theorem 7.4, we consider the contour integral

$$\oint_{(\infty)} F_1(s)ds := \oint_{(\infty)} \pi \tan(\pi s) F_0(s) ds = 0.$$

The function  $F_1(s)$  has poles only at 0 and n-1/2 (n=1,2,3,...). By straightforward calculations, we obtain

Res
$$(F_1(s), n - 1/2) = -\pi \frac{n}{(n - 1/2)^q} \frac{\binom{2n}{n}}{4^n}$$

and

$$\operatorname{Res}(F_1(s), 0) = 2 \sum_{\substack{k_1 + \dots + k_4 + 2k_5 = q - 2, \\ k_1 + \dots + k_5 = q - 2, \\ k_2 + \dots + k_5 = q - 2,}} \frac{C_{k_1} C_{k_2} D_{k_3} 2^{k_3 + k_4} (\log(2))^{k_4} \widetilde{t}(2k_5 + 2)}{k_1! k_2! k_3! k_4!}.$$

Hence, we complete the proof of Theorem 7.5.

It is possible that some other identities of Apéry-like sums can be proved using techniques of the present section. For example, considering the function  $F_2(s) := (\psi(-s) + \gamma)F_0(s)$ , which has poles only at all integers. By residue computations, we obtain

$$\sum_{n=1}^{\infty} \frac{4^n}{n^q \binom{2n}{n}} = (-1)^q \sum_{n=1}^{\infty} \frac{H_{n-1} \binom{2n}{n}}{4^n n^{q-1}} + \sum_{\substack{k_1 + k_2 + \dots + k_5 = q-2, \\ k_1, \dots, k_5 > 0}} G(k_1, k_2, k_3, k_4) \zeta(k_5 + 2).$$
 (7.10)

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