# ACTION OF HECKE ALGEBRA ON THE DOUBLE FLAG VARIETY OF TYPE AIII

#### LUCAS FRESSE AND KYO NISHIYAMA

ABSTRACT. Consider a connected reductive algebraic group G and a symmetric subgroup K. Let  $\mathfrak{X} = K/B_K \times G/P$  be a double flag variety of finite type, where  $B_K$  is a Borel subgroup of K, and P a parabolic subgroup of G. A general argument shows that the orbit space  $\mathbb{C}\mathfrak{X}/K$  inherits a natural action of the Hecke algebra  $\mathscr{H} = \mathscr{H}(K, B_K)$  of double cosets via convolutions. However, it is a quite different problem to find out the explicit structure of the Hecke module.

In this paper, for the double flag variety of type AIII, we determine the explicit action of  $\mathscr{H}$  on  $\mathbb{C}\mathfrak{X}/K$  in a combinatorial way using graphs. As a by-product, we also get the description of the representation of the Weyl group on  $\mathbb{C}\mathfrak{X}/K$  as a direct sum of induced representations.

#### Contents

1.	Double flag varieties and Hecke algebra actions	1
2.	Double flag variety of type AIII	3
3.	Description of $K$ -orbits on $\mathfrak{X}$	4
4.	Setting over finite fields	7
5.	Hecke algebra of double cosets	8
6.	Double cosets multiplications	10
7.	Explicit action of Hecke algebra on the double flag variety	12
8.	Representation of the Weyl group	15
9.	Appendix: Calculation of the stabilizer	15
References		18

#### 1. Double flag varieties and Hecke algebra actions

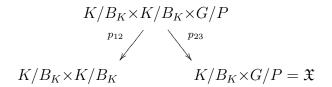
Let G be a connected reductive algebraic group with an involutive automorphism  $\theta$ . We denote by  $K = G^{\theta}$  the subgroup of fixed points of  $\theta$  in G. We assume K is connected for simplicity. Note that this assumption holds if G is simply connected.

<sup>2020</sup> Mathematics Subject Classification. Primary 20C08; Secondary 14M17, 14M15, 20G05.

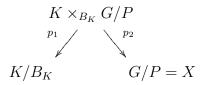
K. N. is supported by JSPS KAKENHI Grant Number #21K03184.

Let us consider a double flag variety  $\mathfrak{X} = K/B_K \times G/P$ , where  $B_K$  is a Borel subgroup of K and P is a parabolic subgroup of G. We assume  $\mathfrak{X}$  is of finite type, i.e., there are finitely many orbits with respect to the diagonal K action on  $\mathfrak{X}$  (see [12] and [7]). Since  $\mathfrak{X}/K \simeq B_K \backslash G/P$ , this is equivalent to saying that there are finitely many  $B_K$ -orbits on the partial flag variety G/P, or in other words, the natural action of K on G/P is spherical.

Let us denote the Hecke algebra of  $(K, B_K)$  by  $\mathscr{H} = \mathscr{H}(K, B_K)$ . Then there exists a general recipe to define an action of  $\mathscr{H}$  on the space of K-orbits  $\mathbb{C}\mathfrak{X}/K$  in the double flag variety  $\mathfrak{X}$  by using the convolution product and the following double fibration maps (see [2], for example).



In this diagram, K acts diagonally, and all the maps respect the K action. However, in practice we prefer a simpler picture with the left  $B_K$  action:



More generally, if X is a spherical K-variety, Hecke algebra actions are considered by Mars-Springer [11] and Knop [10].

Thus there exists an action of the Hecke algebra on the orbit space of  $\mathfrak{X}/K \simeq B_K \backslash G/P$  so that the orbit space  $\mathbb{C}\mathfrak{X}/K$  is a Hecke module. However, there is no definite way to determine this module structure, and it seems difficult to describe the module structure even for a given explicit double flag variety.

Here in this paper, we will describe the explicit and concrete module structure of the Hecke algebra  $\mathcal{H}$  for the case of the double flag variety of type AIII. The action is very explicit in terms of certain graphs, which represent K-orbits. See Theorem 7.5, which is the main theorem of this paper. From this theorem, we can also deduce the precise module structure of  $\mathbb{C} \mathfrak{X}/K$  as a representation of the Weyl group  $W_K$  of K, which is isomorphic to  $\mathfrak{S}_p \times \mathfrak{S}_q$  in our situation. The representation is described in terms of a sum of induced representations. See Theorem 8.1.

To state the results in detail, let us first explain what is our double flag variety, and the structure of the orbit space.

#### 2. Double flag variety of type AIII

In this section, the base field will be any field of characteristic other than 2. Later, we will consider the double flag varieties over finite fields.

From now on, we concentrate on the case of the symmetric space of type AIII.

- $G = GL_n$  denotes the general linear group of order n.
- $K = GL_p \times GL_q$  is a symmetric subgroup diagonally embedded into G, where p + q = n.
- $P = P_{(r,n-r)}$  denotes a standard maximal parabolic subgroup in G consisting of blockwise upper-triangular matrices with 2 diagonal blocks of size r and n-r.
- $B_K = B_p \times B_q$  is a Borel subgroup in K, where  $B_p$  denotes the subgroup of  $GL_p$  consisting of upper-triangular matrices.

Thus we have

$$\mathfrak{X} = K/B_K \times G/P = \left( \operatorname{GL}_p/B_p \times \operatorname{GL}_q/B_q \right) \times \operatorname{GL}_n/P_{(r,n-r)}$$
$$\simeq \left( \mathscr{F}\ell(V^+) \times \mathscr{F}\ell(V^-) \right) \times \operatorname{Gr}_r(V),$$

where

- V is an n-dimensional vector space with a polar decomposition  $V = V^+ \oplus V^-$  and  $\dim V^+ = p$ ,  $\dim V^- = q$ .
- $Gr_r(V)$  is the *Grassmannian* of r-dimensional subspaces of V, and
- $\mathscr{F}\ell(V^{\pm})$  denote the complete flag varieties of  $V^{\pm}$ .

It is not difficult to see

Lemma 2.1.  $\#\mathfrak{X}/K < \infty$ , i.e.,  $\mathfrak{X}$  is of finite type.

For general double flag varieties of finite type, we refer the readers to [7].

Write  $X = \operatorname{Gr}_r(V) \simeq G/P_{(r,n-r)}$ , then K acts on X spherically, i.e., the action  $B_K \cap X$  has finitely many orbits.

Lemma 2.2. There is a natural bijection

$$\begin{array}{ccc} \mathfrak{X}/K & \xrightarrow{\simeq} & X/B_K \\ \ni & & \ni \end{array}$$

$$K \cdot ([\tau], \mathcal{F}_0^+, \mathcal{F}_0^-) \longmapsto B_K \cdot [\tau]$$

where  $[\tau] \in Gr_r(V)$ , and  $\mathcal{F}_0^{\pm}$  denote the standard flags of  $V^{\pm}$  stabilized by  $B_p$  and  $B_q$ , respectively.

In the following, we will often identify  $\mathfrak{X}/K$  and  $X/B_K$  via the above explicit bijection.

#### 4

## 3. Description of K-orbits on $\mathfrak{X}$

Here we summarize the structure of the double flag variety and K-orbits on it from our previous works. For details we refer the readers to [3, 4, 5, 6].

3.1. Partial permutations. A partial permutation of size  $p \times r$  is a matrix  $\tau_1 \in M_{p,r}$ with entries in  $\{0,1\}$ , in which the number of 1's is less than or equal to 1 for any row and any column. (If p = r, we recover the set of partial permutation matrices considered in [4].) Let us denote by  $\mathfrak{T}_{p,r}$  the set of all the partial permutations in  $M_{p,r}$ . Put

$$\mathfrak{T}=\mathfrak{T}_{(p,q),r}:=\left\{\tau=\begin{pmatrix}\tau_1\\\tau_2\end{pmatrix}\in\mathfrak{T}_{p,r}\times\mathfrak{T}_{q,r}\;\middle|\;\mathrm{rank}\,\tau=r\right\}\subset\mathrm{M}_{p+q,r},$$

which is the set of pairs of partial permutations arranged vertically which are of full rank. Note that the symmetric group  $\mathfrak{S}_r$  of order r acts on this set from the right:  $\mathfrak{T} \cap \mathfrak{S}_r$ , and we denote by  $\overline{\mathfrak{T}} = \mathfrak{T}/\mathfrak{S}_r$  the quotient by the symmetric group action.

Let  $[\tau] := \operatorname{Im} \tau \in \operatorname{Gr}_r(V)$  denote the r-dimensional subspace generated by the column vectors of  $\tau$ .

**Theorem 3.1** ([5, Theorem 2.2]). The map  $\mathfrak{T} \ni \tau \mapsto [\tau] \in Gr_r(V)$  factors through to a bijection

$$\overline{\mathfrak{T}} = \mathfrak{T}/\mathfrak{S}_r \xrightarrow{\simeq} X/B_K \simeq \mathfrak{X}/K$$

so that we get the parametrization of the K-orbits in the double flag variety  $\mathfrak{X}/K \simeq \overline{\mathfrak{T}}$ .

If there is no confusion, we will identify a matrix  $\tau \in \mathfrak{T}$  with its representative in  $\overline{\mathfrak{T}}$ . Thus  $\tau$  often represents a K-orbit in  $\mathfrak{X}$ . Note also that the Weyl group  $W_K = \mathfrak{S}_p \times \mathfrak{S}_q$ acts on  $\overline{\mathfrak{T}}$  on the left in a natural way.

3.2. Graphs. There exists a convenient presentation of  $\tau \in \overline{\mathfrak{T}}$  by using graphs. Let us explain it.

For  $\tau \in \overline{\mathfrak{T}}$ , we consider a graph  $\Gamma(\tau)$  determined by the following rule.

- It has two kinds of *vertices*: "positive" vertices  $\mathcal{V}_p^+ = \{1^+, \dots, p^+\}$  and "negative" vertices  $\mathcal{V}_q^- = \{1^-, \dots, q^-\}$ , both being displayed along two horizontal lines. Draw *edges* between  $i^+ \in \mathcal{V}_p^+$  and  $j^- \in \mathcal{V}_q^-$  if  $\tau$  contains *two* 1's in the same
- column, at rows  $i^+$  and  $j^-$ .
- There are marked vertices: mark the vertex  $i^+$  (or  $j^-$ ) if  $\tau$  contains only one 1 at row  $i^+$  (or  $j^-$ ) in a column.
- As a result we get #(edges) + #(marked vertices) = r.

**Example 3.2.** To understand the graphs, let us give an example. When (p,q)=(5,3)and r = 4, we get

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \mapsto \qquad \Gamma(\tau) = \begin{pmatrix} 1^{+} & 2^{+} & 3^{+} & 4^{+} & 5^{+} \\ \bullet & \bullet & \bullet & \bullet \\ 1^{-} & 2^{-} & 3^{-} \end{pmatrix}$$

$$(3.1)$$

The set of graphs of this type will be denoted by  $\mathcal{G}((p,q),r) = \{\Gamma(\tau) \mid \tau \in \overline{\mathfrak{T}}\}$ . The graphs are characterized by the properties listed above. Note that every vertex is incident with at most one edge or mark, and that there is no edge joining two distinct vertices of the same sign.

We summarize the description of orbits using graphs into the following lemma.

**Lemma 3.3.** The graphs classify K-orbits in  $\mathfrak{X}$ .

$$\mathfrak{X}/K \simeq X/B_K \stackrel{\simeq}{\longleftarrow} \overline{\mathfrak{T}} \stackrel{\simeq}{\longrightarrow} \mathcal{G}((p,q),r)$$

$$\ni B_K \cdot [\tau] \longleftarrow \tau \longmapsto \Gamma(\tau)$$

- 3.3. Orbital invariants:  $a^{\pm}(\tau)$ ,  $b(\tau)$ ,  $c(\tau)$  and  $R(\tau) = (r_{i,j}(\tau))$ . For the graph  $\Gamma(\tau)$  we define:
  - - We set the degree of vertices as deg  $i^{\pm} := 0, 1, 2$ , depending on whether it is not incident with an edge nor marked, the end point of an edge, or marked,
    - $a^{\pm}(\tau) := \#\{(i^{\pm}, j^{\pm}) \mid i < j \text{ and } \deg(i^{\pm}) < \deg(j^{\pm})\}$
    - $b(\tau) := \#\{\text{edges}\}\$
    - $c(\tau) := \#\{crossings \text{ of edges}\}, \text{ i.e., the number of pairs of edges } (i^+, j^-) \text{ and }$  $(k^+, \ell^-)$  such that i < k and  $j > \ell$ .
    - $r_{i,j}(\tau) := \#(\text{edges}) + \#(\text{marked vertices}) \text{ with vertices among } \{1^+, \dots, i^+\} \times$
    - $R(\tau) := (r_{i,j}(\tau))_{0 \le i \le p, \ 0 \le j \le q} \in \mathcal{M}_{p+1,q+1}$ : the "rank matrix".

We need  $a^{\pm}(\tau)$ ,  $b(\tau)$ ,  $c(\tau)$  to give a dimension formula for the K-orbits in  $\mathfrak{X}$  below, while the matrices  $R(\tau)$  are to be used to describe the closure relations of orbits.

We also define a decomposition

$$\mathcal{V}_p^+ = \{1, \dots, p\} = I \sqcup L \sqcup L',$$

where I (resp. L, resp. L') denotes the set of elements  $i \in \{1, ..., p\}$  such that  $i^+$  is a vertex of  $\Gamma(\tau)$  of degree 1 (resp. 2, resp. 0).

A decomposition

$$\mathcal{V}_q^- = \{1, \dots, q\} = J \sqcup M \sqcup M'$$

is defined similarly. Namely, J (resp. M, resp. M') consists of the elements j such that  $j^-$  has degree 1 (resp. 2, resp. 0)

Let  $\sigma: J \to I$  be the *bijection* defined by  $\sigma(j) = i$  if  $(i^+, j^-)$  is an edge in  $\Gamma(\tau)$ .

Note that  $\tau$  is characterized by the subsets I, L, L', J, M, M' and the bijection  $\sigma: J \to I$ . Also note that we have  $b(\tau) = \#I = \#J$ , and  $c(\tau)$  is the number of inversions of  $\sigma$ .

Let  $e_i^{\pm}$  be a standard basis vector of  $V^{\pm}$ . For  $\tau$  as in (3.1), the associated Example 3.4. graph is given as

$$\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} e_2^+ & e_4^+ & e_5^+ & 0 \\ \hline e_3^- & e_1^- & 0 & e_2^- \end{pmatrix} \implies \Gamma(\tau) = \begin{pmatrix} 1^+ & 2^+ & 3^+ & 4^+ & 5^+ \\ \hline e_3^- & e_1^- & 0 & e_2^- \end{pmatrix} \implies \Gamma(\tau) = \begin{pmatrix} 1^+ & 2^+ & 3^+ & 4^+ & 5^+ \\ \hline e_3^- & e_1^- & 0 & e_2^- \end{pmatrix} \implies \Gamma(\tau) = \begin{pmatrix} 1^+ & 2^+ & 3^+ & 4^+ & 5^+ \\ \hline e_3^- & e_1^- & 0 & e_2^- \end{pmatrix} \implies \Gamma(\tau) = \begin{pmatrix} 1^+ & 2^+ & 3^+ & 4^+ & 5^+ \\ \hline e_3^- & e_1^- & 0 & e_2^- \end{pmatrix} \implies \Gamma(\tau) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$I = \{2, 4\}, \quad L = \{5\}, \quad L' = \{1, 3\},$$

$$J = \{1, 3\}, \quad M = \{2\}, \quad M' = \emptyset, \quad \sigma = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \in \text{Bij}(J, I)$$

$$[\tau] = \langle e_2^+ + e_2^-, e_4^+ + e_1^-, e_5^+, e_2^- \rangle.$$

3.4. Dimensions and closure relations of orbits. Recall the base point  $([\tau], \mathcal{F}_0^+, \mathcal{F}_0^-)$ in  $\mathfrak{X} = \operatorname{Gr}_r(V) \times \mathscr{F}\ell(V^+) \times \mathscr{F}\ell(V^-)$ .

**Theorem 3.5** ([5, Theorem 2.2]). Denote a K-orbit in  $\mathfrak{X}$  by  $\mathbb{O}_{\tau} := K \cdot ([\tau], \mathcal{F}_0^+, \mathcal{F}_0^-)$ .

(1) 
$$\dim \mathbb{O}_{\tau} = \frac{p(p-1)}{2} + \frac{q(q-1)}{2} + a^{+}(\tau) + a^{-}(\tau) + \frac{b(\tau)(b(\tau)+1)}{2} + c(\tau).$$
  
(2)  $\mathbb{O}_{\tau} = \{(W, \mathcal{F}^{+}, \mathcal{F}^{-}) \mid \dim W \cap (\mathcal{F}_{i}^{+} + \mathcal{F}_{j}^{-}) = r_{i,j}(\tau)$ 

(2) 
$$\mathbb{O}_{\tau} = \{(W, \mathcal{F}^{+}, \mathcal{F}^{-}) \mid \dim W \cap (\mathcal{F}_{i}^{+} + \mathcal{F}_{i}^{-}) = r_{i,i}(\tau) \}$$

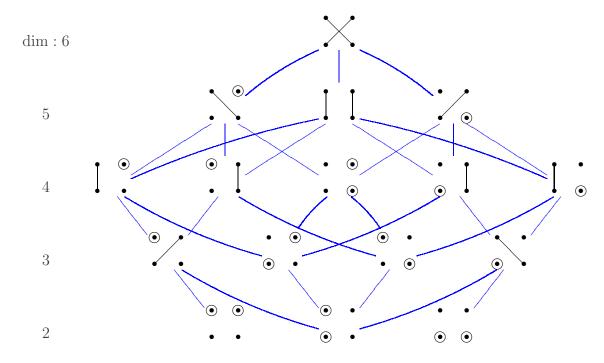
for any 
$$(i, j) \in \{0, \dots, p\} \times \{0, \dots, q\}\}.$$

(3) 
$$\overline{\mathbb{O}_{\tau}} \subset \overline{\mathbb{O}_{\tau'}} \iff r_{i,j}(\tau) \geq r_{i,j}(\tau') \quad \text{for any } (i,j) \in \{0,\ldots,p\} \times \{0,\ldots,q\}.$$

We can describe the cover relation of the closure of orbits, which is not presented here (see [5, Theorem 2.3]). Taking this for granted, we have

Corollary 3.6. If  $\mathbb{O}_{\tau'}$  covers  $\mathbb{O}_{\tau}$  then  $\dim \mathbb{O}_{\tau'} = \dim \mathbb{O}_{\tau} + 1$  holds.

Figure 1. Closure relations of K-orbits for p = q = r = 2



3.5. The number of orbits. Let (k, s, t) be nonnegative integers which satisfy

$$p > k + s$$
,  $q > k + t$ ,  $r = k + s + t$ .

Put s' = p - k - s and t' = q - k - t. Consider the subgroup  $H_{k,s,t} \subset \mathfrak{S}_p \times \mathfrak{S}_q$  defined by  $H_{k,s,t} = \{(a_1, a_2, a_3; a_1, b_2, b_3) \in (\mathfrak{S}_k \times \mathfrak{S}_s \times \mathfrak{S}_{s'}) \times (\mathfrak{S}_k \times \mathfrak{S}_t \times \mathfrak{S}_{t'})\}$   $\cong \Delta \mathfrak{S}_k \times \mathfrak{S}_s \times \mathfrak{S}_{s'} \times \mathfrak{S}_t \times \mathfrak{S}_{t'},$ 

where  $\Delta \mathfrak{S}_k \subset \mathfrak{S}_k^2$  stands for the diagonal subgroup.

**Theorem 3.7** ([5, Corollary 2.13]). The total number of K-orbits in  $\mathfrak{X}$  is given by

$$\#\mathfrak{X}/K = \sum_{(k,s,t)} \dim \operatorname{Ind}_{H_{k,s,t}}^{\mathfrak{S}_p \times \mathfrak{S}_q} \mathbf{1} = \sum_{(k,s,t)} \binom{p}{k,s,s'} \binom{q}{k,t,t'} k!,$$

where the sums are running over triples (k, s, t) as above.

#### 4. Setting over finite fields

Based on the classification of orbits, we will calculate the Hecke algebra action on the orbit space. For this, we follow the classical recipe of Iwahori [8], and we will consider everything over the finite field  $\mathbb{F} = \mathbb{F}_q$  of q-elements from now on. (The letter q is already used to denote the size of the second block for K. But the number of elements of a finite

field is customary denoted also by the letter "q". To distinguish them, we will use q instead of q for the finite field  $\mathbb{F}_{q}$ .)

Summary of the notation over the finite fields:

In addition to this, we also use the following notation.

- $s_i = (i, i+1)$ : simple reflection (a transposition in  $W_K$ ), and  $T_i = T_{s_i}$  is the corresponding generator in the Hecke algebra  $\mathscr{H} = \mathscr{H}(K, B_K)$ .
- Recall pairs of partial permutations  $\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \in \overline{\mathfrak{T}} = \mathfrak{T}/\mathfrak{S}_r$  of full rank r. The matrix  $\tau$  is identified with its image  $[\tau] \in X = \operatorname{Gr}_r(\mathbb{F}^n)$  (thus we often omit  $[\ ]$  below).
- Let  $\mathbb{O}_{\tau} = B_K \cdot \tau$  be a  $B_K$ -orbit in the Grassmannian X. Then  $\xi_{\tau}$  denotes the characteristic function of the orbit  $\mathbb{O}_{\tau}$ .

We are interested in the action of  $T_i$ ,  $T_i * \xi_{\tau}$  for  $\tau \in \mathfrak{T}$ . To calculate it, we recall some basic facts on the action of Hecke algebras.

#### 5. Hecke algebra of double cosets

In this section, we consider a general finite group and review some general properties of a Hecke algebra of double cosets. For that reason, we will denote by K a general finite group. This notation is effective only in this section, but there is no harm to consider it as the already defined K (over a finite field) above.

Let us take a subgroup  $B \subset K$  (again B does not necessarily mean a Borel subgroup) and consider the convolution algebra of B-spherical functions on K. Note that these functions are  $\mathbb{C}$ -valued functions. This algebra is called the Hecke algebra of double cosets and we denote it as  $\mathcal{H} = \mathcal{H}(K, B)$ . Namely,

$$\mathscr{H}(K,B) = \{ f : K \to \mathbb{C} \mid f(h_1kh_2) = f(k) \text{ for } h_1, h_2 \in B \text{ and } k \in K \}$$

and the convolution product is defined by

$$a * b(x) = \int_K a(k)b(k^{-1}x)dk = \frac{1}{\#K} \sum_{k \in K} a(k)b(k^{-1}x)$$

where the integral  $\int_K dk$  is taken with respect to the normalized Haar measure of the finite group K. As written above, the integral is just the pointwise sum divided by the whole volume #K, but we prefer the notation using integral  $\int_K$ .

Put  $W = B \setminus K/B = \{w\}$ , identified with the set of representatives in K, which we pick and fix once and for all. Let us consider characteristic functions on the double cosets  $BwB \subset K$  so that they form a basis of  $\mathcal{H}$ . However, since we would like to get an identity element for the double coset B = BeB, we will normalize the characteristic functions by #K/#B. Thus we put

$$T_w = \frac{\#K}{\#B} \cdot \mathbf{1}_{BwB} \qquad (w \in W).$$

Then  $\{T_w\}_{w\in W}$  forms a basis of  $\mathscr{H}$  over  $\mathbb{C}$ .

Let X be a finite set and assume K acts on X from the left. We consider the space of functions  $\operatorname{Fun}^B(X)$  on X which are B-invariant. The Hecke algebra  $\mathscr{H}$  acts on  $\operatorname{Fun}^B(X)$  via the convolution again:

$$f * \xi(x) := \int_K f(k)\xi(k^{-1}x)dk \qquad (f \in \mathcal{H}, \ \xi \in \operatorname{Fun}^B(X)).$$

We denote by  $\mathfrak{T} = X/B$  identified with the set of representatives in X.

We denote by  $\xi_{\tau}$  the characteristic function on a *B*-orbit  $B\tau \subset X$  so that  $\{\xi_{\tau}\}_{{\tau}\in\mathfrak{T}}$  is a basis of Fun<sup>B</sup>(X).

Let us calculate the convolution:

$$T_{w} * \xi_{\tau}(x) = \int_{K} T_{w}(g)\xi_{\tau}(g^{-1}x)dg$$

$$= \frac{1}{\#K} \sum_{g \in K} T_{w}(g)\xi_{\tau}(g^{-1}x)$$

$$= \frac{1}{\#K} \sum_{g \in BwB} T_{w}(g)\xi_{\tau}(g^{-1}x). \tag{5.1}$$

For this sum, only  $g \in BwB \cap aK_{\tau}B$  contributes, where  $a \in K$  is chosen as  $x = a\tau$  (we assume that  $x \in K\tau$ , otherwise the sum is zero), and  $K_{\tau} = \operatorname{Stab}_{K}(\tau)$  denotes the stabilizer of  $\tau \in X$ . In fact,  $\xi_{\tau}(g^{-1}x) \neq 0$  iff  $g^{-1}x \in B\tau$ . Since  $x = a\tau$ , we get

$$g^{-1}x = b\tau \iff g^{-1}a\tau = b\tau \iff \tau = a^{-1}gb\tau$$

which means  $a^{-1}gb \in K_{\tau} = \operatorname{Stab}_{K}(\tau)$ . Thus we get  $g \in aK_{\tau}B$ .

Since  $T_w * \xi_\tau \in \operatorname{Fun}^B(X)$ , it is a linear combination of various  $\xi_{\tau_i}$ 's for  $\tau_i \in \mathfrak{T}$ . From the last expression (5.1), it is easy to see that if  $g \in BwB$  contributes to the sum nontrivially then

$$g^{-1}x \in B\tau \iff x \in gB\tau \iff Bx \subset BwB\tau.$$

Let us decompose

$$BwB\tau = \bigsqcup_{i=1}^{N} B\tau_i \tag{5.2}$$

Thus we only have to consider the cases  $x = \tau_i$  ( $1 \le i \le N$ ). If we choose  $a_i$ 's which satisfy  $\tau_i = a_i \tau$ , then the above consideration tells us

**Theorem 5.1.** The Hecke operator  $T_w$   $(w \in W)$  acts on  $\operatorname{Fun}^B(X)$  by

$$T_w * \xi_\tau = \sum_{i=1}^N \frac{\#(a_i K_\tau B \cap BwB)}{\#B} \xi_{\tau_i}.$$

*Proof.* This theorem follows from the discussion above. Note that  $T_w$  is normalized by the constant #K/#B.

We shall apply this formula to our situation. (It's still interesting to consider vector bundle case in general. We postpone it as future study.)

#### 6. Double cosets multiplications

Let us return to the setting of §4, however we make the assumption that the ground field  $\mathbb{F}$  is algebraically closed, of characteristic  $\neq 2$ , which will take place only in this section.

Let  $s_i = (i, i + 1)$  be a simple reflection (a transposition in  $W_K$ ), and put  $T_i = T_{s_i}$  be the corresponding element in the Hecke algebra. We are interested in  $T_i * \xi_{\tau}$  for  $\tau \in \mathfrak{T}$ . As in §3,  $\tau \in \mathfrak{T}$  is often identified with a graph with two subsets of vertices  $\mathcal{V}_p^+$  and  $\mathcal{V}_q^-$  (of p and q elements respectively) which are equipped with several edges and marked vertices. Recall that  $W_K \simeq \mathfrak{S}_p \times \mathfrak{S}_q$  acts on  $\mathfrak{T}$  by the matrix multiplication from the left, which descends to the action on  $\overline{\mathfrak{T}}$ , the set of parameters of orbits. This action can be identified with the natural action of  $W_K$  on the graphs, induced by that on the vertices.

The following key lemma corresponds to Equation (5.2) in the present situation.

**Lemma 6.1.** A double coset  $B_K s_i B_K$  generates at most two  $B_K$ -orbits on the Grassmannian  $X = \operatorname{Gr}_r(\mathbb{F}^n)$ . Namely we have

$$B_K s_i B_K \cdot \tau = \begin{cases} B_K s_i \tau = B_K \tau & \text{if } s_i \tau = \tau \\ B_K s_i \tau \cup B_K \tau & \text{if } s_i \tau \neq \tau \text{ and } \tau \text{ is among (*)} \\ B_K s_i \tau & \text{if } s_i \tau \neq \tau \text{ and } \tau \text{ is among (**)} \end{cases}$$
 case (II)

where (\*) denotes the case of (1), (3), (6), (8) in Table 9 in Appendix §9, and (\*\*) denotes the case of (2), (4), (5), (7) (ibid.).

*Proof.* Let  $B_K \cdot \tau$  be a  $B_K$ -orbit of the Grassmannian  $X = \operatorname{Gr}_r(\mathbb{F}^n)$ . Let  $i \in \{1, \dots, p-1\}$  and let  $P_i = B_K \sqcup B_K s_i B_K$  be the corresponding minimal parabolic subgroup  $(s_i)$  is the corresponding simple reflection).

In Appendix §9, we compute the isotropy subgroup  $P_i^{\tau} := \{g \in P_i : g \cdot \tau = \tau\} \subset P_i$ . More precisely, let  $U_i$  be the unipotent radical of  $P_i$  and let  $L_i$  be the standard Levi subgroup of  $P_i$ . The quotient  $L_i/Z(L_i)$  is isomorphic to  $\operatorname{PGL}_2(\mathbb{F})$ . By considering the Levi decomposition  $P_i = L_i \ltimes U_i$ , we get a morphism of groups

$$\pi_i: P_i \to L_i \to L_i/Z(L_i) \cong \mathrm{PGL}_2(\mathbb{F})$$

and a morphism of Lie algebras

$$d\pi_i: \mathfrak{p}_i = \mathrm{Lie}(P_i) \to \mathfrak{sl}_2(\mathbb{F}).$$

In concrete terms, any element in  $P_i$  (resp.  $\mathfrak{p}_i$ ) is a blockwise upper triangular matrix with one block X of size 2 and the other blocks of size 1, and the map  $\pi_i$  (resp.  $d\pi_i$ ) is obtained by considering the projection of X to  $\mathrm{PGL}_2(\mathbb{F})$  (resp.  $\mathfrak{sl}_2(\mathbb{F})$ ). In Appendix §9, we have calculated the image of  $P_i^{\tau}$  (in fact, of  $\mathfrak{p}_i^{\tau} = \mathrm{Lie}(P_i^{\tau})$ ) by  $\pi_i$  (in fact,  $d\pi_i$ ). The calculations show the following alternative:

- (A)  $s_i \tau = \tau$ , in which case  $d\pi_i(\mathfrak{p}_i^{\tau}) = \mathfrak{sl}_2(\mathbb{F})$ ;
- (B)  $s_i \tau \neq \tau$ , in which case  $d\pi_i(\mathfrak{p}_i^{\tau})$  is a Borel subalgebra of  $\mathfrak{sl}_2(\mathbb{F})$ .

One can be more precise. There are in fact three cases. Here we refer to i, i+1 as vertices in the graphic representation of  $\tau$ .

- (I) If i, i + 1 are both of degree 0 or both of degree 2, then we are in case (A).
- (II) If  $\deg_{\tau}(i) < \deg_{\tau}(i+1)$  or i, i+1 are end points of two edges which have a crossing, then we are in case (B) and, moreover,  $d\pi_i(\mathfrak{p}_i^{\tau})$  is the subalgebra of *lower* triangular matrices in  $\mathfrak{sl}_2(\mathbb{F})$ ;
- (III) If  $\deg_{\tau}(i) > \deg_{\tau}(i+1)$  or i, i+1 are end points of two edges which do not have a crossing, then we are in case (B) and, moreover,  $d\pi_i(\mathfrak{p}_i^{\tau})$  is the subalgebra of upper triangular matrices in  $\mathfrak{sl}_2(\mathbb{F})$ .

In the language of Knop's paper [9, §3]:

- In case (A),  $\Phi(P_i)$  is of type  $G_0$ ;
- In case (B),  $\Phi(P_i)$  is of type  $S \cdot U_0$ .

Types  $T_0$  and  $N_0$  of [9, §3] do not appear in our situation. We can check this if we consider the type of the stabilizer and consider the claims just after [9, Lemma 3.1].

In particular, the information on isotropy subgroups/subalgebras can be used in combination with [9, table on p. 295] in order to determine  $B_K s_i B_K \cdot \tau$ . First, we note that  $P_i \cdot \tau$  always contains the orbits  $B_K \cdot \tau$  and  $B_K \cdot (s_i \tau)$ , which can be the same. In case (B), where  $\Phi(P_i)$  is of type  $S \cdot U_0$ , we also know from [9] that  $P_i \cdot \tau$  contains exactly two orbits, namely

$$P_i \cdot \tau = B_K \cdot \tau \cup B_K \cdot s_i \tau.$$

In this case,  $B_K s_i B_K \cdot \tau$  contains at most two orbits, hence we have either  $B_K s_i B_K \cdot \tau = B_K \cdot s_i \tau$  or  $B_K s_i B_K \cdot \tau = B_K \cdot \tau \cup B_K \cdot s_i \tau$ . It remains to determine in which case we have indeed two orbits.

Let  $B_K = TU$  where T is the standard maximal torus and  $U \subset B_K$  is the unipotent radical. Let  $X_i^{\pm} := \{u_i^{\pm}(t)\}_{t \in \mathbb{F}}$  be the one parameter subgroup of unipotent matrices attached to the root  $\pm \alpha_i$ . Thus

$$U = U_i X_i^+$$
 and  $s_i X_i^+ s_i^{-1} = X_i^-$ .

Hence

$$B_K s_i B_K = B_K s_i U_i X_i^+ = B_K s_i X_i^+ = B_K X_i^- s_i.$$

Whence

$$B_K s_i B_K \cdot \tau = B_K X_i^- \cdot s_i \tau.$$

- In case (I), we have  $s_i \tau = \tau$  and  $X_i^- \in P_i^{\tau}$ , hence  $B_K s_i B_K \cdot \tau = B_K \cdot \tau$  in this case.
- In case (II), we have that the projection  $\pi_i(P_i^{s_i\tau})$  of the isotropy group of  $s_i\tau$  consists of upper triangular matrices (since by applying  $s_i$  to  $\tau$ , we switch the configuration of the vertices i, i + 1). This means  $X_i^- \not\subset P_i^{s_i\tau}$ , and hence there exists  $g \in X_i^-$  such that  $g \cdot s_i\tau \neq s_i\tau$ . We claim that  $g \cdot s_i\tau \notin B_K \cdot s_i\tau$ , so that  $g \cdot s_i\tau \in B_K \cdot \tau$  and we must have  $B_K s_i B_K \cdot \tau = B_K \cdot \tau \cup B_K \cdot s_i\tau$  (as asserted in the lemma) in this case. Arguing by contradiction assume that  $g \cdot s_i\tau = b \cdot s_i\tau$  with some  $b \in B_K$ . Then  $g^{-1}b \in P_i^{s_i\tau}$ , which implies that  $\pi_i(g^{-1}b)$  must be upper triangular. But this is not the case, hence the claim is verified.
- In case (III), the projection  $\pi_i(P_i^{s_i\tau})$  of the isotropy group of  $s_i\tau$  consists of lower triangular matrices. Hence  $X_i^- \subset P_i^{s_i\tau}$ . Whence  $B_K s_i B_K \cdot \tau = B_K X_i^- \cdot s_i \tau = B_K \cdot s_i \tau$  in this case (as asserted in the lemma).

The proof of Lemma 6.1 is complete for i associated to  $\mathfrak{S}_p$ , i.e., 0 < i < p. The case for  $s_i \in \mathfrak{S}_q$  can be argued similarly.

# 7. Explicit action of Hecke algebra on the double flag variety

In this section,  $\mathbb{F}$  is a finite field of characteristic  $\neq 2$  again. As before we denote  $q = \#\mathbb{F}$ . Note that Lemma 6.1 is still valid in this context (by considering fixed points of the Frobenius map).

Recall that the Hecke algebra  $\mathscr{H} = \mathscr{H}(K, B_K)$  acts on the space of K-orbits  $\mathbb{C}\mathfrak{X}/K$  and the action is given by the general theory of spherical functions discussed in §5.

According to the theory, by Theorem 5.1 and Lemma 6.1, we get

$$T_i * \xi_\tau = \alpha \xi_\tau + \beta \xi_{s_i \tau}$$

for some coefficients  $\alpha, \beta \in \mathbb{Q}$  (one of which might be zero). Let us determine them.

7.1. Calculation of  $\alpha$ . Note that  $\alpha \neq 0$  only if we are in Cases (I) or (II) in Lemma 6.1. To compute it, we use the formula in Theorem 5.1 with  $a_i = e$  (identity). The numerator becomes (before counting the number)

$$K_{\tau}B_K \cap B_K s_i B_K$$
, where  $K_{\tau} = K \cap P_{[\tau]}$ ,

and  $P_{[\tau]} = \operatorname{Stab}_G([\tau])$  is the stabilizer of the r-dimensional space  $[\tau] \in \operatorname{Gr}_r(\mathbb{F}^n)$  generated by the columns of  $\tau$ . From a general argument,

$$B_K s_i B_K = X_i^+ s_i B_K$$
 with  $X_i^+ = U_{\alpha_i} \simeq \mathbb{F}$ 

where  $U_{\alpha_i} \subset B_K$  denotes the one parameter subgroup generated by a root vector  $x_{\alpha_i}$  corresponding to  $s_i = s_{\alpha_i}$ .

#### Lemma 7.1.

$$K_{\tau}B_K \cap B_K s_i B_K = \{u s_i b \in X_i^+ s_i B_K \mid s_i u^{-1} \tau \in B_K \tau\}.$$

The expression  $us_ib$  is unique.

*Proof.* Write  $us_ib \in B_K s_i B_K = X_i^+ s_i B_K$  for  $u \in X_i^+$  and  $b \in B_K$ .

$$us_ib \in K_\tau B_K \iff us_i \in K_\tau B_K \iff (us_i)^{-1} \in B_K K_\tau \iff s_i u^{-1} \tau \in B_K \tau.$$

**Lemma 7.2.** Assume we are in Case (I) so that  $s_i \tau = \tau$ . Then

$$K_{\tau}B_K \cap B_K s_i B_K = X_i^+ s_i B_K \simeq \mathbb{F} \times B_K.$$

This means  $\alpha = \#\mathbb{F} = \mathbf{q}$ .

*Proof.* We will apply Lemma 7.1. Since  $s_i\tau = \tau$ , if we denote by  $v = s_i u s_i$ , a generator of the one parameter subgroup corresponding to the negative root  $-\alpha_i$ , we get

$$s_i u^{-1} \tau = (s_i u s_i)^{-1} s_i \tau = v^{-1} \tau$$

and  $v^{-1}\tau \in B_K\tau$  holds for any v (according to Lemma 9.1(1)). Thus  $u \in X_i^+$  is arbitrary.

**Lemma 7.3.** Assume we are in Case (II) so that  $s_i \tau \neq \tau$ . Then

$$K_{\tau}B_K \cap B_K s_i B_K = (X_i^+ \setminus \{e\}) s_i B_K \simeq \mathbb{F}^{\times} \times B_K.$$

This means  $\alpha = \#\mathbb{F} - 1 = \mathbf{q} - 1$ .

*Proof.* As in the proof of Lemma 7.2, we denote  $v = s_i u s_i$ . We get

$$s_i u^{-1} \tau = (s_i u s_i)^{-1} s_i \tau = v^{-1} s_i \tau.$$

Since  $s_i \tau \neq \tau$ , this is in  $B_K \tau$  iff  $v^{-1} s_i \tau \notin B_K s_i \tau$ , iff  $v \neq e$  (this follows from a similar arguments as in the end of the proof of Lemma 6.1. See Lemma 9.1(2) also). This proves the lemma.

7.2. Calculation of  $\beta$ . The case  $\beta \neq 0$  only occurs for the cases (II) and (III) in Lemma 6.1. Thus we can assume  $s_i \tau \neq \tau$ .

To compute  $\beta$ , as in the case of  $\alpha$ , we use the formula in Theorem 5.1 with  $a_i = s_i$ . The numerator becomes (before counting the number)

$$s_i K_\tau B_K \cap B_K s_i B_K = s_i K_\tau B_K \cap X_i^+ s_i B_K.$$

Let us denote  $X_i^- = s_i X_i^+ s_i$ . Thus, we need to compute the number of elements in

$$K_{\tau}B_K \cap s_i X_i^+ s_i B_K = K_{\tau}B_K \cap X_i^- B_K.$$

## Lemma 7.4.

$$\beta = \#\{v \in X_i^- \mid v\tau \in B_K\tau\} = \begin{cases} \boldsymbol{q} & \text{if } \tau \text{ is in Case (II),} \\ 1 & \text{if } \tau \text{ is in Case (III).} \end{cases}$$

*Proof.* Let  $V = \{v \in X_i^- \mid v \in K_\tau B_K\}$ . Note that the mapping

$$V \times B_K \to K_\tau B_K \cap X_i^- B_K, \ (v, b) \mapsto vb$$

is bijective. (It is clearly well defined.) It is injective since, if vb = v'b' for  $vv' \in V$  and  $bb' \in B_K$ , then we get  $v'^{-1}v = b'b^{-1} \in X_i^- \cap B_K = \{e\}$  hence (v,b) = (v',b'). It is surjective since any element in  $K_\tau B_K \cap X_i^- B_K$  can be written vb with  $v \in X_i^-$  and  $b \in B_K$ , and we have  $v = (vb)b^{-1} \in K_\tau B_K$ , hence  $v \in V$ . This observation combined with Theorem 5.1 (and the discussion above the statement of this lemma) implies that

$$\beta = \frac{\#(V \times B_K)}{\#B_K} = \#V.$$

Next, for  $v \in X_i^-$ , we note that

$$v \in V \Leftrightarrow v \in K_{\tau}B_K \Leftrightarrow v^{-1} \in B_KK_{\tau} \Leftrightarrow v^{-1}\tau \in B_K\tau.$$

This yields a well-defined bijection  $V \to \{v \in X_i^- \mid v\tau \in B_K\tau\}, v \mapsto v^{-1}$ . Hence

$$\beta = \#\{v \in X_i^- \mid v\tau \in B_K\tau\}$$

as asserted in the lemma.

It remains to show the second equality in Lemma 7.4. First assume that  $\tau$  is in Case (II). In this case, as recalled in Section 6 we have  $X_i^- \subset K_\tau$ , hence  $v\tau = \tau \in B_K\tau$  for all  $v \in X_i^-$ . This implies that  $\{v \in X_i^- \mid v\tau \in B_K\tau\} = X_i^-$ , hence  $\beta = \#X_i^- = \mathbf{q}$  in this case.

Finally assume that  $\tau$  is in Case (III). In this case, we claim that  $\{v \in X_i^- \mid v\tau \in B_K\tau\} = \{e\}$ , and this will imply that  $\beta = 1$  as asserted. Thus it remains to establish the claim. To this end, let  $v \in X_i^-$  be such that  $v\tau \in B_K\tau$ . Let us write  $v\tau = b\tau$  with  $b \in B_K$ . This implies that  $v^{-1}b \in P_i^{\tau}$  where  $P_i = B_K s_i B_K \sqcup B_K$  and  $P_i^{\tau} = P_i \cap K_{\tau}$  (notation of Section 6). Hence  $\pi_i(v^{-1}b) \in \pi_i(P_i^{\tau})$  (where, as in Section 6,  $\pi_i$  denotes the projection to the (i, i+1)-block). As used in Section 6, the fact that  $\tau$  is in Case (III) implies that  $\pi_i(P_i^{\tau})$  is formed by upper-triangular matrices. But  $\pi_i(v^{-1}b)$  is upper triangular if and only if v = e. Whence v = e, and the claim is established.

# 7.3. Action of simple reflections. Let us recall Cases (I)–(III) from §6.

**Theorem 7.5.** The Hecke algebra  $\mathcal{H} = \mathcal{H}(K, B_K)$  acts on the space of K-orbits  $\mathbb{C}\mathfrak{X}/K$  and the action is explicitly given by the formula:

$$T_{i} * \xi_{\tau} = \begin{cases} \boldsymbol{q}\xi_{\tau} & (s_{i}\tau = \tau) \text{ in } Case \text{ (I)}, \\ (\boldsymbol{q} - 1)\xi_{\tau} + \boldsymbol{q}\xi_{s_{i}\tau} & (s_{i}\tau \neq \tau) \text{ in } Case \text{ (II)}, \\ \xi_{s_{i}\tau} & (s_{i}\tau \neq \tau) \text{ in } Case \text{ (III)}, \end{cases}$$

$$(7.1)$$

where  $\{T_i\}$  are the generators of  $\mathscr{H}$  corresponding to the simple reflections.

Note that the Borel-Moore homology of the conormal variety  $\mathcal{Y}$  has its basis consisting of the closures of conormal bundles of K-orbits on  $\mathfrak{X}$ . So the above theorem tells that the space of top Borel-Moore homology has a natural Hecke module structure.

# 8. Representation of the Weyl Group

We get the action of Hecke algebra in terms of generators  $T_{s_i}$ 's. If we specialize the action by putting q = 1, then we get an action of the Weyl group  $W_K = \mathfrak{S}_p \times \mathfrak{S}_q$ .

From Theorem 7.5, a simple reflection  $s_i \in W_K$  acts on  $\tau$  simply by the multiplication  $s_i\tau$ , which causes the transposition of *i*-th and (i+1)-th rows of the  $(p+q) \times r$ -matrix  $\tau$ . So the action of the Weyl group on  $\tau$  is simply by the multiplication of permutation matrices from the left on the space of partial permutations.

In the graphical notation of  $\tau$ ,  $w \in W_K$  acts on  $\tau$  as a permutation of  $\mathcal{V}_p^+ \times \mathcal{V}_q^-$ . Thus we can easily see what kind of representations of  $W_K$  we get.

**Theorem 8.1.** The Weyl group  $W_K = \mathfrak{S}_p \times \mathfrak{S}_q$  acts on the orbit space  $\mathbb{C}\mathfrak{X}/K$ , and we have the following equivalence as representations of  $W_K$ .

$$\mathbb{C}\,\mathfrak{X}/K\simeq\bigoplus_{(k,s,t)}\mathrm{Ind}_{H_{k,s,t}}^{\mathfrak{S}_p\times\mathfrak{S}_q}\,\mathbf{1},$$

where the sums are running over triples (k, s, t) given in §3.5, and the subgroup  $H_{k,s,t} = \Delta \mathfrak{S}_k \times \mathfrak{S}_{s'} \times \mathfrak{S}_{t'} \times \mathfrak{S}_{t'}$  is defined in the same place.

Since the dimension of the representation coincides with the number of orbits, we retrieve the formula of the number of orbits (Theorem 3.7).

# 9. Appendix: Calculation of the stabilizer

Let  $\tau \in \overline{\mathfrak{T}}$  and we consider the orbit  $B_K \cdot [\tau] \subset \operatorname{Gr}_r(V)$ . Let  $P_\alpha \subset K$  be a standard minimal parabolic subgroup associated to a simple root  $\alpha$ . Then  $P_\alpha/B_K$  can be identified with  $\mathbb{P}^1$ , in fact  $B_K = \operatorname{Stab}(\mathcal{F}_0^+, \mathcal{F}_0^-)$ , where  $\mathcal{F}_0^{\pm}$  are the standard flags of  $V^{\pm}$  respectively.

Let us follow the notation of Bourbaki for root systems ([1]). In our case, the root system of K is  $A_{p-1} + A_{q-1}$ , and thus  $\alpha = \alpha_i = \varepsilon_i - \varepsilon_{i+1}$  (0 < i < p or p < i < p + q).

If 
$$\alpha = \alpha_i$$
  $(0 < i < p)$  then writing  $\mathcal{F}_0^+ = (F_{0,0}^+, \dots, F_{0,p}^+)$ , we have:

$$P_{\alpha}/B_K \simeq \{W \mid F_{0,i-1}^+ \subset W \subset F_{0,i+1}^+\} \simeq \mathbb{P}(F_{0,i+1}^+/F_{0,i-1}^+) \simeq \mathbb{P}^1.$$

Thus we conclude

$$Aut(P_{\alpha}/B_K) = PGL(V_{\alpha}), \quad \text{where} \quad V_{\alpha} := F_{0,i+1}^+/F_{0,i-1}^+.$$

Any element  $g \in P_{\alpha}$  determines  $\overline{g} \in \operatorname{PGL}(V_{\alpha})$ .

Let us write this more precisely. We have a Levi decomposition  $P_{\alpha} = L_{\alpha}U_{\alpha}$ , where  $U_{\alpha}$  denotes the unipotent radical, and  $L_{\alpha}$  is the standard Levi subgroup isomorphic to  $\operatorname{GL}(V_{\alpha}) \times \mathbb{G}_m^{p-2} \times \mathbb{G}_m^q$ . Thus any  $g \in P_{\alpha}$  can be written in the form

$$g = (g_{\alpha}, t_1, t_2) \cdot u \in (\mathrm{GL}(V_{\alpha}) \times \mathbb{G}_m^{p-2} \times \mathbb{G}_m^q) \ltimes U_{\alpha}.$$

We define  $\varphi_{\alpha}(g) = g_{\alpha} \in GL(V_{\alpha})$ , the projection to the  $GL(V_{\alpha})$ -component.

For  $\tau \in \overline{\mathfrak{T}}$ , we have to consider the stabilizer  $P_{\alpha}^{\tau}$  of  $[\tau]$  in  $P_{\alpha}$  and its Lie algebra  $\mathfrak{p}_{\alpha}^{\tau}$ , and their images by  $\varphi_{\alpha}$  and  $d\varphi_{\alpha}$  respectively.

Lemma 9.1. (1) If  $s_{\alpha}\tau = \tau$  then  $d\varphi_{\alpha}(\mathfrak{p}_{\alpha}^{\tau}) = \mathfrak{gl}_2$  holds.

(2) If  $s_{\alpha}\tau \neq \tau$  then  $d\varphi_{\alpha}(\mathfrak{p}_{\alpha}^{\tau})$  is a Borel subalgebra of  $\mathfrak{gl}_{2}$ .

*Proof.* We use the notation of §3.3.

(1) If  $s_{\alpha}\tau = \tau$  then either the vertices i and i+1 are both in the set L' of unmarked vertices; or i and i+1 are both marked belonging to the set L.

In the first case, we have  $[\tau] \subset \langle e_s^+ \mid s \notin \{i, i+1\} \rangle \oplus V^-$ . In the second case,  $\langle e_i^+, e_{i+1}^+ \rangle \subset [\tau]$ . In both cases, for any  $h \in \mathrm{GL}_2$ ,  $g := \mathrm{diag}(1, \ldots, 1, h, 1, \ldots, 1) \in P_{\alpha}^{\tau}$ , where h appears in the diagonal block of i-th and (i+1)-th rows. Whence  $\varphi_{\alpha}(P_{\alpha}^{\tau}) = \mathrm{GL}_2$  in this case.

(2) Assume  $s_{\alpha}\tau \neq \tau$ . A general description of the Lie algebra of the stabilizer tells

$$\mathfrak{k}^{\tau} = \{ x \in \mathfrak{k} \mid x([\tau]) \subset [\tau] \}.$$

Write  $x \in \mathfrak{k}$  as  $x = \operatorname{diag}(x^+, x^-)$  and

$$[\tau] = \langle \boldsymbol{e}_s^+ \mid s^+ \text{ is marked} \rangle \oplus \langle \boldsymbol{e}_t^- \mid t^- \text{ is marked} \rangle$$
$$\oplus \langle \boldsymbol{e}_s^+ + \boldsymbol{e}_t^- \mid \text{there is an edge } (s^+, t^-) \rangle.$$

Note the followings hold for  $x \in \mathfrak{k}^{\tau}$ .

- If  $s \in L'$  and  $t \in M'$ , then we have  $x_{s,k}^+ = 0 \ (k \in L \cup I)$  and  $x_{t,\ell}^- = 0 \ (\ell \in M \cup J)$ .
- If  $s \in I$  and  $t \in J$ , then we have  $x_{s,k}^+ = 0$   $(k \in L)$  and  $x_{t,\ell}^- = 0$   $(\ell \in M)$ .
- If  $(s^+, t^-)$  and  $(k^+, \ell^-)$  are edges, then we have  $x_{s,k}^+ = x_{t,\ell}^-$ .

In fact these conditions exactly characterizes the stabilizer  $\mathfrak{k}^{\tau}$ .

Based on these conditions, we can compute  $d\varphi_{\alpha}(\mathfrak{p}_{\alpha}^{\tau})$  explicitly. We divide the cases into eight, and examine each case. These eight cases are listed in Figure 9 below, where we denote the upper/lower triangular Borel subalgebras by  $\mathfrak{b}_{2}^{\pm}$ .

FIGURE 2. Table:  $\alpha$ -component of the stabilizer in Case (2), where  $\alpha = \varepsilon_i - \varepsilon_{i+1}$ .

	condition	graphical notation	$d\varphi_{\alpha}(\mathfrak{p}_{\alpha}^{\tau})$
(1)	$i \in L', i+1 \in L$	$i  i+1$ $\bullet  \odot$	$\mathfrak{b}_2^-$
(2)	$i \in L, i+1 \in L'$	$i$ $i+1$ $\bullet$	$\mathfrak{b}_2^+$
(3)	$i \in L', i+1 \in I$	<i>i i</i> +1	$\mathfrak{b}_2^-$
(4)	$i \in I, i+1 \in L'$	<i>i i</i> +1	$\mathfrak{b}_2^+$
(5)	$i \in L, i+1 \in I$	<i>i i</i> +1	$\mathfrak{b}_2^+$
(6)	$i \in I, i+1 \in L$	$i$ $i+1$ $\odot$	$\mathfrak{b}_2^-$
(7)	$i, i+1 \in I, \ k < \ell$ $(i, k)$ and $(i+1, \ell)$ are edges	$i  i+1$ $ \downarrow \qquad \qquad \downarrow \qquad \qquad$	$\mathfrak{b}_2^+$
(8)	$i, i+1 \in I, \ \ell < k$ $(i, k)$ and $(i+1, \ell)$ are edges	$i$ $i+1$ $\ell < k$	$\mathfrak{b}_2^-$

Note that in Cases (7) and (8), we must have  $x_{i,i+1}^+ = x_{k,\ell}^-$  and  $x_{i+1,i}^+ = x_{\ell,k}^-$ , respectively. Moreover, in Case (7), we have  $x_{\ell,k}^- = 0$ , and in Case (8),  $x_{k,\ell}^- = 0$ .

#### References

- [1] N. Bourbaki, Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines, Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], No. 1337, Hermann, Paris, 1968. MR 0240238
- [2] Neil Chriss and Victor Ginzburg, Representation theory and complex geometry, Birkhäuser Boston Inc., Boston, MA, 1997. MR MR1433132 (98i:22021)
- [3] Lucas Fresse and Kyo Nishiyama, On the exotic Grassmannian and its nilpotent variety, Represent. Theory **20** (2016), 451–481, [Paging previously given as 1–31]. MR 3576071
- [4] \_\_\_\_\_\_, A Generalization of Steinberg Theory and an Exotic Moment Map, International Mathematics Research Notices (2020), rnaa080.
- [5] Lucas Fresse and Kyo Nishiyama, On generalized Steinberg theory for type AIII, 2021, arXiv:2103.08460.
- [6] Lucas Fresse and Kyo Nishiyama, Orbit embedding for double flag varieties and Steinberg map, Contemp. Math. **768** (2021), 21–42.
- [7] Xuhua He, Kyo Nishiyama, Hiroyuki Ochiai, and Yoshiki Oshima, On orbits in double flag varieties for symmetric pairs, Transf. Groups 18 (2013), 1091–1136.
- [8] Nagayoshi Iwahori, On the structure of a Hecke ring of a Chevalley group over a finite field, J. Fac. Sci. Univ. Tokyo Sect. I 10 (1964), 215–236 (1964). MR 165016
- [9] Friedrich Knop, On the set of orbits for a Borel subgroup, Comment. Math. Helv. 70 (1995), no. 2, 285–309. MR 1324631
- [10] \_\_\_\_\_, Some remarks on multiplicity free spaces, Representation theories and algebraic geometry (Montreal, PQ, 1997), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 514, Kluwer Acad. Publ., Dordrecht, 1998, pp. 301–317. MR 1653036 (99i:20056)
- [11] J. G. M. Mars and T. A. Springer, Hecke algebra representations related to spherical varieties, Represent. Theory 2 (1998), 33–69. MR 1600804
- [12] Kyo Nishiyama and Hiroyuki Ochiai, Double flag varieties for a symmetric pair and finiteness of orbits, J. Lie Theory 21 (2011), no. 1, 79–99. MR 2797821

Université de Lorraine, CNRS, Institut Élie Cartan de Lorraine, UMR 7502, Vandoeuvre-lès-Nancy, F-54506, France

Email address: lucas.fresse@univ-lorraine.fr

Department of Mathematics, Aoyama Gakuin University, Fuchinobe 5-10-1, Chuo-ku, Sagamihara 252-5258, Japan

Email address: kyo@math.aoyama.ac.jp