# Partial-twuality polynomials of delta-matroids 

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#### Abstract

Gross, Mansour and Tucker introduced the partial-twuality polynomial of a ribbon graph. Chumutov and Vignes-Tourneret posed a problem: it would be interesting to know whether the partial duality polynomial and the related conjectures would make sense for general delta-matroids. In this paper we consider analogues of partial-twuality polynomials for delta-matroids. Various possible properties of partial-twuality polynomials of set systems are studied. We discuss the numerical implications of partial-twualities on a single element and prove that the intersection graphs can determine the partialtwuality polynomials of bouquets and normal binary delta-matroids, respectively. Finally, we give a characterization of vf-safe delta-matroids whose partial-twuality polynomials have only one term.


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## 1. Introduction

In [16], Wilson found that the two long-standing duality operators $\delta$ (geometric duality) and $\tau$ (Petrie duality) generate a group of six ribbon graph operators, that is, every other composition of $\delta$ and $\tau$ is equivalent to one of

[^0]the five operators $\delta, \tau, \delta \tau, \tau \delta, \delta \tau \delta$, or to the identity operator. Abrams and Ellis-Monaghan [1] called the five operators twualities. The partial (geometric) dual with respect to a subset of edges of a ribbon graph was introduced by Chmutov [7] in order to unify various connections between the JonesKauffman and Bollobás-Riordan polynomials. Ellis-Monaghan and Moffatt [12] generalized this partial-duality construction to the other four operators, which they called partial-twualities.

Gross, Mansour and Tucker $[13,14]$ introduced the partial-twuality polynomial for $\delta, \tau, \delta \tau, \tau \delta$, and $\delta \tau \delta$. Various basic properties of partial-twuality polynomials were studied, including interpolation and log-concavity. Recently, Chumutov and Vignes-Tourneret [8] posed the following question:

Question 1. [8] Ribbon graphs may be considered from the point of view of delta-matroid. In this way the concepts of partial (geometric) duality and genus can be interpreted in terms of delta-matroids [9, 10]. It would be interesting to know whether the partial- $\delta$ polynomial and the related conjectures would make sense for general delta-matroids.

In [18], we showed that the partial- $\delta$ polynomials have delta-matroid analogues. We introduced the twist polynomials of delta-matroids and discussed their basic properties for delta-matroids. Chun et al. [9] showed that the loop complemenation is the delta-matroid analogue of partial Petriality. In this paper we consider analogues of other partial-twuality polynomials for delta-matroids.

This paper is organised as follows. In Section 2 we recall the definition of partial-twuality polynomials of ribbon graphs. Analogously, we introduce the partial-twuality polynomials of set systems. In Section 3, various possible properties of partial-twuality polynomials of set systems are studied. In Section 4 we discuss the numerical implications of partial-twualities on a single element and the interpolation. In Section 5, we prove that the intersection graphs can determine the partial-twuality polynomials of bouquets and normal binary delta-matroids, respectively. Here we provide an answer to the question [17]: can one derive something from bouquets that could determine the partial-twuality polynomial completely. In Section 6 we give a characterization of vf-safe delta-matroids whose partial-twuality polynomials have only one term.

## 2. Preliminaries

### 2.1. Set systems and widths

A set system is a pair $D=(E, \mathcal{F})$ of a finite set $E$ together with a collection $\mathcal{F}$ of subsets of $E$. The set $E$ is called the ground set and the elements of $\mathcal{F}$ are the feasible sets. We often use $\mathcal{F}(D)$ to denote the set of feasible sets of $D . D$ is proper if $\mathcal{F} \neq \emptyset$, and is normal (respectively, dual normal) if the empty set (respectively, the ground set) is feasible. The direct sum of two set systems $D=(E, \mathcal{F})$ and $\widetilde{D}=(\widetilde{E}, \widetilde{\mathcal{F}})$ with disjoint ground sets $E$ and $\widetilde{E}$, written $D \oplus \widetilde{D}$, is defined to be

$$
D \oplus \widetilde{D}:=(E \cup \widetilde{E},\{F \cup \widetilde{F}: F \in \mathcal{F} \text { and } \widetilde{F} \in \widetilde{\mathcal{F}}\})
$$

As introduced by Bouchet in [3], a delta-matroid is a proper set system $D=(E, \mathcal{F})$ such that if $X, Y \in \mathcal{F}$ and $u \in X \Delta Y$, then there is $v \in X \Delta Y$ (possibly $v=u$ ) such that $X \Delta\{u, v\} \in \mathcal{F}$. Here

$$
X \Delta Y:=(X \cup Y)-(X \cap Y)
$$

is the usual symmetric difference of sets. Note that the maximum gap in the collection of sizes of feasible sets of a delta-matroid is two [15].

For a set system $D=(E, \mathcal{F})$, let $\mathcal{F}_{\max }(D)$ and $\mathcal{F}_{\min }(D)$ be the collections of maximum and minimum cardinality feasible sets of $D$, respectively. Let $D_{\max }:=\left(E, \mathcal{F}_{\max }(D)\right)$ and $D_{\min }:=\left(E, \mathcal{F}_{\min }(D)\right)$. Let $r\left(D_{\max }\right)$ and $r\left(D_{\min }\right)$ denote the sizes of largest and smallest feasible sets of $D$, respectively. The width of $D$, denote by $w(D)$, is defined by

$$
w(D):=r\left(D_{\max }\right)-r\left(D_{\min }\right) .
$$

For all non-negative integers $i \leq w(D)$, let

$$
\mathcal{F}_{\max -i}(D)=\left\{F \in \mathcal{F}:|F|=r\left(D_{\max }\right)-i\right\}
$$

and

$$
\mathcal{F}_{\min +i}(D)=\left\{F \in \mathcal{F}:|F|=r\left(D_{\min }\right)+i\right\} .
$$

### 2.2. Partial-twualities of set systems

We will consider the operations of twisting and loop complementation on set systems. Twisting was introduced by Bouchet in [3], and loop complementation by Brijder and Hoogeboom in [5].

Let $D=(E, \mathcal{F})$ be a set system. For $A \subseteq E$, the twist of $D$ with respect to $A$, denoted by $D^{* \mid A}$, is given by

$$
(E,\{A \Delta X: X \in \mathcal{F}\})
$$

The $*$-dual of $D$, written $D^{*}$, is equal to $D^{* \mid E}$. Note that $*$-duality preserves width. Throughout the paper, we will often omit the set brackets in the case of a single element set. For example, we write $D^{* \mid e}$ instead of $D^{* \mid\{e\}}$.

Let $D=(E, \mathcal{F})$ be a set system and $e \in E$. Then $D^{\times \mid e}$ is defined to be the set system $\left(E, \mathcal{F}^{\prime}\right)$, where

$$
\mathcal{F}^{\prime}=\mathcal{F} \Delta\{F \cup e: F \in \mathcal{F} \text { and } e \notin F\} .
$$

If $e_{1}, e_{2} \in E$ then

$$
\left(D^{\times \mid e_{1}}\right)^{\times \mid e_{2}}=\left(D^{\times \mid e_{2}}\right)^{\times \mid e_{1}} .
$$

This means that if $A=\left\{e_{1}, \cdots, e_{m}\right\} \subseteq E$ we can unambiguously define the loop complementation [5] of $D$ on $A$, by

$$
D^{\times \mid A}:=\left(\cdots\left(D^{\times \mid e_{1}}\right)^{\times \mid e_{2}} \cdots\right)^{\times \mid e_{m}} .
$$

It is straightforward to show that the twist of a delta-matroid is a deltamatroid [3], but the set of delta-matroids is not closed under loop complementation (see, for example, [9]). Thus, we often restrict our attention to a class of delta-matroids that is closed under loop complementation. A delta-matroid $D=(E, \mathcal{F})$ is said to be vf-safe [9] if the application of every sequence of twists and loop complementations results in a delta-matroid.

In [5] it was shown that twists and loop complementations give rise to an action of the symmetric group $S_{3}$, with the presentation

$$
S_{3} \cong \mathcal{B}:=<*, \times \mid *^{2}, \times^{2},(* \times)^{3}>,
$$

on set systems. If $D=(E, \mathcal{F})$ is a set system, $e \in E$ and $a=a_{1} a_{2} \cdots a_{n}$ is a word in the alphabet $\{*, \times\}$, then

$$
D^{a \mid e}:=\left(\cdots\left(D^{a_{1} \mid e}\right)^{a_{2} \mid e} \cdots\right)^{a_{n} \mid e} .
$$

Note that the operators $*$ and $\times$ on different elements commute [5]. If $A=$ $\left\{e_{1}, \cdots, e_{m}\right\} \subseteq E$, we can unambiguously define

$$
D^{a \mid A}:=\left(\cdots\left(D^{a \mid e_{1}}\right)^{a \mid e_{2}} \cdots\right)^{a \mid e_{m}}
$$

Let $D_{1}=(E, \mathcal{F})$ and $D_{2}$ be set systems. For $\bullet \in\{*, \times, * \times, \times *, * \times *\}$, we say that $D_{2}$ is a partial $\bullet$ dual of $D_{1}$ if there exists $A \subseteq E$ such that $D_{2}=D_{1}^{\bullet}{ }^{\bullet}$.

### 2.3. Partial-twualities of ribbon graphs

Ribbon graphs are well-known to be equivalent to cellularly embedded graphs. The reader is referred to $[11,12]$ for further details about ribbon graphs. A quasi-tree is a ribbon graph with one boundary component. Let $G=(V, E)$ be a ribbon graph and let

$$
\mathcal{F}:=\{F \subseteq E(G): F \text { is the edge set of a spanning quasi-tree of } G\} .
$$

We call $D(G)=:(E, \mathcal{F})$ the delta-matroid [10] of $G$. We say a delta-matroid is ribbon-graphic if it is equal to the delta-matroid of some ribbon graph. Note that ribbon-graphic delta-matroids are vf-safe [9].

For a ribbon graph $G$ and a subset $A$ of its edge-ribbons $E(G)$, the partial dual $G^{\delta \mid A}$ [7] of $G$ with respect to $A$ is a ribbon graph obtained from $G$ by gluing a disc to $G$ along each boundary component of the spanning ribbon subgraph $(V(G), A)$ (such discs will be the vertex-discs of $G^{\delta \mid A}$ ), removing the interiors of all vertex-discs of $G$ and keeping the edge-ribbons unchanged.

Let $G$ be a ribbon graph and $A \subseteq E(G)$. Then the partial Petrial $G^{\tau \mid A}$ [11] of $G$ with respect to $A$ is the ribbon graph obtained from $G$ by adding a half-twist to each of the edges in $A$.

In [11] it was shown that the partial dual, $\delta$, and the partial Petrial, $\tau$, give rise to an action of the symmetric group $S_{3}$, with the presentation

$$
S_{3} \cong \mathcal{R}:=<\delta, \tau \mid \delta^{2}, \tau^{2},(\delta \tau)^{3}>
$$

on ribbon graphs. If $G$ is a ribbon graph, $e \in E(G)$ and $a=a_{1} a_{2} \cdots a_{n}$ is a word in the alphabet $\{\delta, \tau\}$, then

$$
G^{a \mid e}:=\left(\cdots\left(G^{a_{1} \mid e}\right)^{a_{2} \mid e} \cdots\right)^{a_{n} \mid e} .
$$

Observe that the partial dual and the partial Petrial commute when applied to different edges [11]. If $A=\left\{e_{1}, \cdots, e_{m}\right\} \subseteq E$, we define

$$
G^{a \mid A}:=\left(\cdots\left(G^{a \mid e_{1}}\right)^{a \mid e_{2}} \cdots\right)^{a \mid e_{m}} .
$$

Let $G_{1}$ and $G_{2}$ be ribbon graphs. For $\bullet \in\{\delta, \tau, \delta \tau, \tau \delta, \delta \tau \delta\}$, we say that $G_{2}$ is a partial- $\bullet$ dual [11] of $G_{1}$ if there exists $A \subseteq E\left(G_{1}\right)$ such that $G_{2}=G_{1} \bullet \mid A$.

### 2.4. Partial-twuality polynomials of ribbon graphs and set systems

Gross, Mansour and Tucker [14] introduced the concept of partial-twuality polynomials of ribbon graphs as follows.

Definition 2 ([14]). For $\bullet \in \mathcal{R}$, we define the partial- $\bullet$ polynomial for any ribbon graph $G$ to be the generating function

$$
{ }^{\partial} \varepsilon_{G}^{\bullet}(z):=\sum_{A \subseteq E(G)} z^{\varepsilon\left(G^{\bullet} \mid A\right)}
$$

that enumerates all partial- duals of $G$ by Euler genus.
Analogously, we define the partial-twuality polynomials of set systems as follows.

Definition 3. For $\bullet \in \mathcal{B}$, the partial- $\bullet$ polynomial of any set system $D=$ $(E, \mathcal{F})$ is defined to be the generating function

$$
{ }^{2} w_{D}^{\bullet}(z):=\sum_{A \subseteq E} z^{w\left(D^{\bullet} \mid A\right)}
$$

that enumerates all partial- duals of $D$ by width.

### 2.5. Binary and intersection graphs

For a finite set $E$, let $C$ be a symmetric $|E|$ by $|E|$ matrix over $G F(2)$, with rows and columns indexed, in the same order, by the elements of $E$. Let $C[A]$ be the principal submatrix of $C$ induced by the set $A \subseteq E$. We define the set system $D(C)=(E, \mathcal{F})$ with

$$
\mathcal{F}:=\{A \subseteq E: C[A] \text { is non-singular }\} .
$$

By convention $C[\emptyset]$ is non-singular. Then $D(C)$ is a delta-matroid [4]. A delta-matroid is said to be binary if it has a twist that is isomorphic to $D(C)$ for some symmetric matrix $C$ over $G F(2)$.

Let $D=(E, \mathcal{F})$ be a normal binary delta-matroid. Then there exists a unique symmetric $|E|$ by $|E|$ matrix $C$ over $G F(2)$ such that $D=D(C)$ $[15,18]$. The intersection graph $G_{D}$ of $D$ is the graph with vertex set $E$ and in which two vertices $u$ and $v$ of $G_{D}$ are adjacent if and only if $C_{u, v}=1$. A bouquet is a ribbon graph with only one vertex. If $B$ is a bouquet, then $D(B)$ is a normal binary delta-matroid [10]. The intersection graph $I(B)$ of a bouquet $B$ is the graph $G_{D(B)}$.

Conversely, recall that a looped simple graph [15] is a graph obtained from a simple graph by adding (exactly) one loop to some of its vertices. The adjacency matrix $A(G)$ of a looped simple graph $G$ is the matrix over $G F(2)$ whose rows and columns correspond to the vertices of $G$; and where, $A(G)_{u, v}=1$ if and only if $u$ and $v$ are adjacent in $G$ and $A(G)_{u, u}=1$ if and only if there is a loop at $u$. Let $D$ be a normal binary delta-matroid. It obvious that $D=D\left(A\left(G_{D}\right)\right)$.

### 2.6. Primal and dual types

Let $D=(E, \mathcal{F})$ be a proper set system. An element $e \in E$ contained in no feasible set of $D$ is said to be a loop.

Definition $4([10])$. Let $D=(E, \mathcal{F})$ be a set system and $e \in E$. Then
(1) $e$ is a ribbon loop if $e$ is a loop in $D_{\min }$;
(2) A ribbon loop $e$ is non-orientable if $e$ is a ribbon loop in $D^{* \mid e}$ and is orientable otherwise.

Let $D=(E, \mathcal{F})$ be a set system and $e \in E$. The primal type of $e$ is $p, u$, or $t$ in $D$, if $e$ is a non-ribbon loop, an orientable loop, or a non-orientable loop, respectively, in $D$. The primal type of $e$ in $D^{*}$ is called the dual type of $e$ in $D$. In combination, the primal and dual types of $e$ in $D$ are called the type of $e$ in $D$, which is denoted by a juxtaposed pair of letters representing the primal and dual types of $e$ in $D$. For example, the type $p u$ means that the primal and dual types of $e$ are $p$ and $u$, respectively, in $D$. We observe that
(1) The primal type of $e$ is $p$ in $D$ if and only if there exists $A \in \mathcal{F}_{\min }(D)$ such that $e \in A$;
(2) The dual type of $e$ is $p$ in $D$ if and only if there exists $A \in \mathcal{F}_{\max }(D)$ such that $e \notin A$;
(3) The primal type of $e$ is $u$ in $D$ if and only if for any $A \in \mathcal{F}_{\min }(D) \cup$ $\mathcal{F}_{\text {min }+1}(D), e \notin A ;$
(4) The dual type of $e$ is $u$ in $D$ if and only if for any $A \in \mathcal{F}_{\max }(D) \cup$ $\mathcal{F}_{\text {max }-1}(D), e \in A ;$
(5) The primal type of $e$ is $t$ in $D$ if and only if for any $A \in \mathcal{F}_{\min }(D), e \notin A$, and there exists $B \in \mathcal{F}_{\text {min }+1}(D)$ such that $e \in B$;
(6) The dual type of $e$ is $t$ in $D$ if and only if for any $A \in \mathcal{F}_{\max }(D), e \in A$, and there exists $B \in \mathcal{F}_{\text {max }-1}(D)$ such that $e \notin B$.

## 3. Some properties of partial-twuality polynomials

Various possible properties of partial-twuality polynomials of ribbon graphs were studied by Gross, Mansour and Tucker in [13, 14]. In this section we discuss the analogous results on set systems or delta-matroids.

Proposition 5. Let $D=(E, \mathcal{F})$ and $\widetilde{D}=(\widetilde{E}, \widetilde{\mathcal{F}})$ be set systems. Then for any $\bullet \in \mathcal{B}$,
(1) ${ }^{\partial} w_{D}^{\bullet}(1)=2^{|E|}$;
(2) ${ }^{\partial} w_{D}^{\bullet}(z)$ has degree at most $|E|$;
(3) ${ }^{\partial} w_{D \oplus \widetilde{D}}^{\bullet}(z)={ }^{\partial} w_{D}^{\bullet}(z)^{\partial} w_{\tilde{D}}^{\bullet}(z)$.

Proof. For (1), the value ${ }^{2} w_{D}^{\bullet}(1)$ counts the total number of partial- $\bullet$ duals of $D$, which is $2^{|E|}$. For any subset $A \subseteq E$, if $B \in \mathcal{F}\left(D^{\bullet \mid A}\right)$, then $\emptyset \subseteq B \subseteq E$. We have $r\left(D^{\bullet \mid A}{ }_{\text {min }}\right) \geq 0$ and $r\left(D^{\bullet \mid A}{ }_{\text {max }}\right) \leq|E|$. Thus $0 \leq w\left(D^{\bullet \mid A}\right) \leq|E|$ and (2) then follows. For any subset $C \subseteq E \cup \widetilde{E}$, we have

$$
(D \oplus \widetilde{D})^{\bullet \mid C}=D^{\bullet \mid(C \cap E)} \oplus \widetilde{D} \widetilde{D}^{\bullet \mid(C \cap \widetilde{E})}
$$

Then

$$
{ }^{\partial} w_{D \oplus \widetilde{D}}^{\bullet}(z)={ }^{\partial} w_{D}^{\bullet}(z)^{\partial} w_{\widetilde{D}}^{\bullet}(z),
$$

by the additivity of width over the direct sum, from which (3) follows.
Proposition 6. Let $D=(E, \mathcal{F})$ be a set system and $A \subseteq E$. Then

$$
{ }^{\partial} w_{D}^{\bullet}(z)={ }^{\partial} w_{D^{\bullet} \mid A}^{\bullet}(z)
$$

for $\bullet \in\{*, \times, * \times *\}$.
Proof. This is because the set of all loop complementations of $D$ is the same as that of $D^{\times \mid A}$. The same reasoning applies to the operators $*$ and $* \times *$.

Remark 7. Proposition 6 is not true for the operators $* \times$ and $\times *$. For example, let $D=(E, \mathcal{F})$ with $E=\{1\}$ and $\mathcal{F}=\{\emptyset,\{1\}\}$. Then $D^{* x \mid 1}=$ $(\{1\},\{\emptyset\})$ and $D^{\times * \mid 1}=(\{1\},\{\{1\}\})$. We have

$$
{ }^{\partial} w_{D}^{* \times}(z)={ }^{\partial} w_{D}^{\times *}(z)=1+z
$$

and

$$
{ }^{\partial} w_{D^{* \times \mid 1}}^{* \times}(z)={ }^{\partial} w_{D^{\times * \mid 1}}^{\times *}(z)=2 .
$$

Obviously, ${ }^{\partial} w_{D}^{* \times}(z) \not \boldsymbol{1}^{\partial} w_{D^{* \times \mid 1}}^{* \times}(z)$ and ${ }^{\partial} w_{D}^{\times *}(z) \not \neq^{\partial} w_{D^{\times * \mid 1}}^{\times *}(z)$.
Lemma $8([5])$. Let $D=(E, \mathcal{F})$ be a set system and $A \subseteq E$. Then

$$
\mathcal{F}_{\min }(D)=\mathcal{F}_{\min }\left(D^{\times \mid A}\right)
$$

and

$$
\mathcal{F}_{\max }(D)=\mathcal{F}_{\max }\left(D^{* \times * \mid A}\right)=\mathcal{F}_{\max }\left(D^{\times * \times \mid A}\right)
$$

Proposition 9. Let $D=(E, \mathcal{F})$ be a set system and $A \in \mathcal{F}_{\min }(D), B \in$ $\mathcal{F}_{\text {min }}\left(D^{*}\right)$. Then
(1) $D^{\bullet} \mid A$ is normal for $\bullet \in\{*, * \times, \times *, * \times *\}$;
(2) $D^{\times \mid B}$ is dual normal.

Proof. (1) We may assume that $A \neq \emptyset$, otherwise the conclusion is trivial. For any $e \in A$, since $A \in \mathcal{F}_{\min }(D)$, it follows that $A \in \mathcal{F}_{\min }\left(D^{\times 1 e}\right)$ by Lemma 8 and $A-e \in \mathcal{F}_{\text {min }}\left(D^{* \mid e}\right)$. Then $A-e \in \mathcal{F}_{\text {min }}\left(D^{* \times \mid e}\right)$ by Lemma 8 and $A-e \in \mathcal{F}_{\text {min }}\left(D^{\times * \mid e}\right)$. Thus $A-e \in \mathcal{F}_{\text {min }}\left(D^{\times * \times \mid e}\right)$ by Lemma 8 . From the above, we have $A-e \in \mathcal{F}_{\text {min }}\left(D^{\bullet \mid e}\right)$ for $\bullet \in\{*, * \times, \times *, * \times *\}$. In the same manner we can see that $\emptyset \in \mathcal{F}_{\text {min }}\left(D^{\bullet} \mid A\right)$ for $\bullet \in\{*, * \times, \times *, * \times *\}$ and conclusion (1) then follows.
(2) Since $B \in \mathcal{F}_{\min }\left(D^{*}\right)$, it follows that $E-B \in \mathcal{F}_{\max }(D)$. Then $E-$ $B \in \mathcal{F}_{\max }\left(D^{\times * \times \mid B}\right)$ by Lemma 8 , that is, $E-B \in \mathcal{F}\left(D^{\times * \times \mid B}\right)$. Thus $E \in$ $\mathcal{F}\left(D^{\times * \times * \mid B}\right)$. Obviously,

$$
E \in \mathcal{F}_{\max }\left(D^{\times * \times * \mid B}\right)=\mathcal{F}_{\max }\left(\left(D^{\times \mid B}\right)^{* \times * \mid B}\right)
$$

Then $E \in \mathcal{F}_{\text {max }}\left(D^{\times \mid B}\right)$ by Lemma 8, that is, $E \in \mathcal{F}\left(D^{\times \mid B}\right)$. Thus $D^{\times \mid B}$ is dual normal.

Remark 10. For investigation of partial- $\bullet$ polynomials of set systems for $\bullet \in$ $\{*, * \times *\}$, Propositions 6 and 9 motivate us to focus on normal set systems, and for $\bullet=\times$, to focus on dual normal set systems. But for $* \times$ or $\times *$, we cannot just focus on normal set systems. For example, let $D=(\{1\},\{\{1\}\})$. We have ${ }^{\partial} w_{D}^{\times *}(z)=2$. Observe that all normal set systems with ground set $\{1\}$ are $D_{1}=(\{1\},\{\emptyset\})$ and $D_{2}=(\{1\},\{\emptyset,\{1\}\})$. Since ${ }^{\partial} w_{D_{1}}^{\times *}(z)=$ ${ }^{\partial} w_{D_{2}}^{\times *}(z)=1+z$, it follows that there is no normal set system $D^{\prime}$ such that ${ }^{\partial} w_{D^{\prime}}^{\times{ }^{*}}(z)={ }^{\partial} w_{D}^{\times *}(z)$.

The following theorem provides a link between partial- $-\bullet *$ and partial $-\bullet$ polynomials of set systems.

Theorem 11. Let $D=(E, \mathcal{F})$ be a set system. Then for any $\bullet \in \mathcal{B}$,

$$
{ }^{\partial} w_{D}^{* * *}(z)={ }^{\partial} w_{D^{*}}^{\bullet}(z) .
$$

Proof. For any $A \subseteq E$, we observe that doing partial $-* \bullet *$ on $A$ is the same as first doing $*$ to $E$, then doing $\bullet$ to $A$, and then doing $*$ to $E$ again, that is,

$$
D^{* \bullet * \mid A}=\left(\left(D^{*}\right)^{\bullet \mid A}\right)^{*} .
$$

Since $*$-duality preserves width, it follows that

$$
w\left(D^{* \bullet * \mid A}\right)=w\left(\left(\left(D^{*}\right)^{\bullet \mid A}\right)^{*}\right)=w\left(\left(D^{*}\right)^{\bullet \mid A}\right) .
$$

Thus the partial $-* \bullet *$ polynomial of $D$ is identical to the partial $-\bullet$ polynomial of $D^{*}$.

## 4. Partial-twuality for a single element

In this section, we discuss the numerical implications of partial-twualities on a single element $e$, depending on the type of $e$.
Lemma $12([6])$. Let $D=(E, \mathcal{F})$ be a delta-matroid and $e \in E$ such that $r\left(D_{\text {min }}\right)=r\left(D^{* \mid e}{ }_{\text {min }}\right)$. Then $\mathcal{F}_{\text {min }}(D)=\mathcal{F}_{\text {min }}\left(D^{* \mid e}\right)$.
Remark 13. Lemma 12 is not true for set systems. For example, let

$$
D=(\{1,2,3\},\{\{1\},\{2,3\}\}) .
$$

We know $r\left(D_{\text {min }}\right)=r\left(D^{* \mid 2}{ }_{\text {min }}\right)=1$. But

$$
\mathcal{F}_{\min }(D)=\{\{1\}\}
$$

and

$$
\mathcal{F}_{\min }\left(D^{* \mid 2}\right)=\{\{3\}\} .
$$

Table 1: The difference $w\left(D^{\bullet \mid e}\right)-w(D)$ for any $\bullet \in \mathcal{B}$.

| Type of $e$ | $*$ | $\times$ | $* \times$ | $\times *$ | $* \times *$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $p p$ | +2 | +1 | +2 | +2 | +1 |
| $u u$ | -2 | 0 | -1 | -1 | 0 |
| $p u$ | 0 | 0 | +1 | 0 | +1 |
| $u p$ | 0 | +1 | 0 | +1 | 0 |
| $t p$ | +1 | +1 | +1 | 0 | -1 |
| $t u$ | -1 | 0 | 0 | -2 | -1 |
| $p t$ | +1 | -1 | 0 | +1 | +1 |
| $u t$ | -1 | -1 | -2 | 0 | 0 |
| $t t$ | 0 | -1 | -1 | -1 | -1 |

Lemma 14. Let $D=(E, \mathcal{F})$ be a delta-matroid and $e \in E$. If $e$ is a nonorientable loop, then for any $A \in \mathcal{F}_{\min }(D), A \cup e \in \mathcal{F}(D)$.

Proof. Since the primal type of $e$ is $t$ in $D$, it follows that $e \notin A$ and there exists $B \in \mathcal{F}_{\text {min }+1}(D)$ such that $e \in B$. Then $B-e \in \mathcal{F}_{\text {min }}\left(D^{* \mid e}\right)$. We have $r\left(D_{\text {min }}\right)=r\left(D^{* \mid e}{ }_{\text {min }}\right)$ and hence $\mathcal{F}_{\text {min }}(D)=\mathcal{F}_{\text {min }}\left(D^{* \mid e}\right)$ by Lemma 12. Then $A \in \mathcal{F}_{\text {min }}\left(D^{* \mid e}\right)$, that is, $A \in \mathcal{F}\left(D^{* \mid e}\right)$. Thus, $A \cup e \in \mathcal{F}(D)$.

Lemma 15 ([2]). If $X$ is any feasible set in a delta-matroid $D$, then there exist $A \in \mathcal{F}_{\min }(D)$ and $B \in \mathcal{F}_{\max }(D)$ such that $A \subseteq X \subseteq B$.

Theorem 16. Let $D=(E, \mathcal{F})$ be a vf-safe delta-matroid and $e \in E$. Table 1 gives the value of $w\left(D^{\bullet \bullet e}\right)-w(D)$ for any $\bullet \in \mathcal{B}$.

Proof. The three possible primal types (and dual types) of $e$ in $D$ are as follows:

Case 1. If the primal type of $e$ is $p$ in $D$, there exists $A \in \mathcal{F}_{\text {min }}(D)$ such that $e \in A$. Then $A-e \in \mathcal{F}_{\text {min }}\left(D^{* \mid e}\right)$. Thus

$$
r\left(D^{* \mid e}{ }_{\text {min }}\right)=r\left(D_{\text {min }}\right)-1
$$

and the primal types of $e$ are $u$ and $p$ in $D^{* \mid e}$ and $D^{\times \mid e}$, respectively.
Case 2. If the primal type of $e$ is $u$ in $D$, then for any $A \in \mathcal{F}_{\text {min }}(D) \cup$ $\mathcal{F}_{\text {min }+1}(D), e \notin A$. Thus

$$
r\left(D^{* \mid e}{ }_{\text {min }}\right)=r\left(D_{\text {min }}\right)+1
$$

and the types of $e$ are $p$ and $t$ in $D^{* \mid e}$ and $D^{\times \mid e}$, respectively.

Table 2: A summary of Cases 1,2 and 3 .

|  | Primal type of $e$ |  | $\left(D^{* \mid e}{ }_{\text {min }}\right)$ |  |
| :---: | :---: | :---: | :--- | :---: |
| $D$ | $D^{* \mid e}$ | $D^{\times \mid e}$ |  |  |
| $p$ | $u$ | $p$ | $r\left(D_{\text {min }}\right)-1$ |  |
| $u$ | $p$ | $t$ | $r\left(D_{\text {min }}\right)+1$ |  |
| $t$ | $t$ | $u$ | $r\left(D_{\text {min }}\right)$ |  |

Case 3. If the primal type of $e$ is $t$ in $D$, then for any $A \in \mathcal{F}_{\text {min }}(D), e \notin A$, and there exists $B \in \mathcal{F}_{\operatorname{min+1}}(D)$ such that $e \in B$. Thus

$$
r\left(D_{\min }^{* \mid e}\right)=r\left(D_{\min }\right)
$$

and the primal types of $e$ is $t$ in $D^{* \mid e}$. By Lemma 14, we have $A \cup e \in$ $\mathcal{F}_{\text {min }+1}(D)$ for any $A \in \mathcal{F}_{\text {min }}(D)$. Then $A \cup e \notin \mathcal{F}\left(D^{\times \mid e}\right)$. Furthermore, we know that for any $B \in \mathcal{F}_{\min +1}(D)$ containing $e, B-e \in \mathcal{F}_{\min }(D)$, otherwise there is no $A^{\prime} \in \mathcal{F}_{\min }(D)$ such that $A^{\prime} \subseteq B$, contradicting Lemma 15. Since $\mathcal{F}_{\text {min }}\left(D^{\times \mid e}\right)=\mathcal{F}_{\text {min }}(D)$, it follows that there is no $B^{\prime} \in \mathcal{F}_{\min }\left(D^{\times \mid e}\right) \cup \mathcal{F}_{\min +1}\left(D^{\times \mid e}\right)$ such that $e \in B^{\prime}$. Then the primal type of $e$ is $u$ in $D^{\times \mid e}$.

Here, we give a summary of Cases 1,2 and 3 as shown in Table 2.
Case 4. If the dual type of $e$ is $p$ in $D$, there exists $A \in \mathcal{F}_{\max }(D)$ such that $e \notin A$. Then $A \cup e \in \mathcal{F}_{\max }\left(D^{* \mid e}\right) \cap \mathcal{F}_{\max }\left(D^{\times \mid e}\right)$. Thus

$$
r\left(D^{* \mid e}{ }_{\max }\right)=r\left(D^{\times \mid e}{ }_{\max }\right)=r\left(D_{\max }\right)+1
$$

and the dual types of $e$ are $u$ and $t$ in $D^{* \mid e}$ and $D^{\times \mid e}$, respectively.
Case 5. If the dual type of $e$ is $u$ in $D$, then for any $A \in \mathcal{F}_{\max }(D) \cup$ $\mathcal{F}_{\text {max }-1}(D), e \in A$. Thus

$$
r\left(D^{* \mid e}{ }_{\text {max }}\right)=r\left(D_{\text {max }}\right)-1
$$

and

$$
r\left(D^{\times \mid e} \text { max }\right)=r\left(D_{\max }\right)
$$

and the dual types of $e$ are $p$ and $u$ in $D^{* \mid e}$ and $D^{\times \mid e}$, respectively.

Table 3: A summary of Cases 4,5 and 6

| Dual type of $e$ |  |  | $r\left(D^{* \mid e}{ }_{\text {max }}\right)$ |  |
| :--- | :--- | :--- | :--- | :--- |
| $D$ | $D^{* \mid e}$ | $D^{\times \mid e}$ |  | $r\left(D^{\times \mid e}{ }_{\text {max }}\right)$ |
| $p$ | $u$ | $t$ | $r\left(D_{\max }\right)+1$ | $r\left(D_{\max }\right)+1$ |
| $u$ | $p$ | $u$ | $r\left(D_{\max }\right)-1$ | $r\left(D_{\max }\right)$ |
| $t$ | $t$ | $p$ | $r\left(D_{\max }\right)$ | $r\left(D_{\max }\right)-1$ |


| Type of $e$ | $r\left(D^{* \mid}{ }_{\text {min }}\right)$ | $r\left(D^{* \mid e}{ }_{\text {max }}\right)$ | $r\left(D^{\times \times e}{ }_{\text {max }}\right)$ | $w\left(D^{* \mid e}\right)$ | $w\left(D^{\times \times e}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $p p$ | $r\left(D_{\text {min }}\right)-1$ | $r\left(D_{\max }\right)+1$ | $r\left(D_{\max }\right)+1$ | $w(D)+2$ | $w(D)+1$ |
| uu | $r\left(D_{\text {min }}\right)+1$ | $r\left(D_{\max }\right)-1$ | $r\left(D_{\text {max }}\right)$ | $w(D)-2$ | $w(D)$ |
| pu | $r\left(D_{\text {min }}\right)-1$ | $r\left(D_{\max }\right)-1$ | $r\left(D_{\text {max }}\right)$ | $w(D)$ | $w(D)$ |
| up | $r\left(D_{\text {min }}\right)+1$ | $r\left(D_{\max }\right)+1$ | $r\left(D_{\max }\right)+1$ | $w(D)$ | $w(D)+1$ |
| $t p$ | $r\left(D_{\text {min }}\right)$ | $r\left(D_{\text {max }}\right)+1$ | $r\left(D_{\max }\right)+1$ | $w(D)+1$ | $w(D)+1$ |
| tu | $r\left(D_{\text {min }}\right)$ | $r\left(D_{\text {max }}\right)-1$ | $r\left(D_{\text {max }}\right)$ | $w(D)-1$ | $w(D)$ |
| $p t$ | $r\left(D_{\text {min }}\right)-1$ | $r\left(D_{\text {max }}\right)$ | $r\left(D_{\max }\right)-1$ | $w(D)+1$ | $w(D)-1$ |
| ut | $r\left(D_{\text {min }}\right)+1$ | $r\left(D_{\max }\right)$ | $r\left(D_{\max }\right)-1$ | $w(D)-1$ | $w(D)-1$ |
| $t t$ | $r\left(D_{\text {min }}\right)$ | $r\left(D_{\max }\right)$ | $r\left(D_{\max }\right)-1$ | $w(D)$ | $w(D)-1$ |

Case 6. If the dual type of $e$ is $t$ in $D$, then for any $A \in \mathcal{F}_{\text {max }}(D), e \in A$. Thus $E-A \in \mathcal{F}_{\min }\left(D^{*}\right)$ and $(E-A) \cup e \in \mathcal{F}\left(D^{*}\right)$ by Lemma 14. It follows that $A-e \in \mathcal{F}_{\text {max }-1}(D)$. We have

$$
r\left(D^{* \mid e}{ }_{\max }\right)=r\left(D_{\max }\right)
$$

and the dual type of $e$ is $t$ in $D^{* \mid e}$. Moreover, we observe that for any $B \in \mathcal{F}_{\max -1}(D)$ not containing $e, B \cup e \in \mathcal{F}_{\max }(D)$, otherwise there is no $B^{\prime} \in \mathcal{F}_{\max }(D)$ such that $B \subseteq B^{\prime}$, contradicting Lemma 15 . It follows that

$$
r\left(D^{\times \mid e}{ }_{\text {max }}\right)=r\left(D_{\max }\right)-1
$$

and the dual type of $e$ is $p$ in $D^{\times \mid e}$, respectively.
Here, we provide a summary of Cases 4,5 and 6 as shown in Table 3. Then the the widths of $D^{* \mid e}$ and $D^{\times \mid e}$ can be calculated by Tables 2 and 3 as shown in Table 4. Hence, the columns 2 and 3 of Table 1 are computed. If the type of $e$ is $p p$ in $D$, then

$$
w\left(D^{* \mid e}\right)=w(D)+2
$$

and

$$
w\left(D^{\times \mid e}\right)=w(D)+1,
$$

and the types of $e$ are $u u$ and $p t$ in $D^{* \mid e}$ and $D^{\times \mid e}$, respectively. Thus

$$
w\left(D^{* \times \mid e}\right)=w\left(\left(D^{* \mid e}\right)^{\times \mid e}\right)=w\left(D^{* \mid e}\right)=w(D)+2,
$$

and

$$
w\left(D^{\times * \mid e}\right)=w\left(\left(D^{\times \mid e}\right)^{* \mid e}\right)=w\left(D^{\times \mid e}\right)+1=w(D)+2,
$$

and the type of $e$ is $t u$ in $D^{* \times \mid e}$. We have

$$
w\left(D^{* \times * \mid e}\right)=w\left(\left(D^{* \times \mid e}\right)^{* \mid e}\right)=w\left(D^{* \times \mid e}\right)-1=w(D)+1 .
$$

The other entries in columns 4, 5 and 6 of Table 1 are computed similarly.
The polynomial $p(z)=\sum_{i=0}^{n} c_{i} z^{i}$ is said to have a gap of size $k$ [14] at coefficient $c_{i}$ if $c_{i-1} c_{i+k} \neq 0$ but $c_{i}=c_{i+1}=\cdots=c_{i+k-1}=0$. If the polynomial $p(z)$ is nonzero and has no gaps, we call it interpolating.

Proposition 17. For any vf-safe delta-matroid $D$, the following statements hold:
(1) ${ }^{\partial} w_{D}^{\bullet}(z)$ is interpolating for $\bullet=\times$ or $* \times *$;
(2) ${ }^{\partial} w_{D}^{\bullet}(z)$ has no gaps of size 2 or more for any $\bullet \in \mathcal{B}$.

Proof. For any element $e$ and subset $A$ of $E$, we observe that $w\left(D^{\bullet \mid A \Delta e}\right)$ and $w\left(D^{\bullet} \mid A\right)$ differ by at most one for $\bullet \in\{\times, * \times *\}$, and by at most two for $\bullet \in\{*, * \times, \times *\}$ by Theorem 16. This yields statements (1) and (2).
Remark 18. There exists a vf-safe delta-matroid $D$ such that ${ }^{\partial} w_{D}^{\bullet}(z)$ is not interpolating for $\bullet \in\{*, * \times, \times *\}$. For example, let

$$
D_{1}=(\{1,2\},\{\emptyset,\{1,2\}\})
$$

and

$$
D_{2}=(\{1,2\},\{\emptyset,\{1\},\{1,2\}\}) .
$$

We have

$$
{ }^{\partial} w_{D_{1}}^{*}(z)=2+2 z^{2}
$$

and

$$
{ }^{\partial} w_{D_{2}}^{* \times}(z)={ }^{\partial} w_{D_{2}}^{\times *}(z)=1+3 z^{2} .
$$

## 5. Partial-twuality polynomials and intersection graphs

In [17], we showed that two bouquets with the same intersection graph have the same partial- $\delta$ polynomial. In this section, we prove that the intersection graphs can determine the partial-twuality polynomials of bouquets and normal binary delta-matroids, respectively. Let $\eta: \mathcal{R} \rightarrow \mathcal{B}$ be the group isomorphism induced by $\eta(\delta)=*$, and $\eta(\tau)=\times$.
Lemma 19 ([10]). If $G$ is a ribbon graph, $A \subseteq E$ and $\bullet \in \mathcal{R}$. Then

$$
D\left(G^{\bullet \mid A}\right)=D(G)^{\eta \bullet \bullet \mid A}
$$

and

$$
\varepsilon(G)=w(D(G))
$$

Proposition 20. Let $G=(V, E)$ be a ribbon graph and $\bullet \in \mathcal{R}$. Then

$$
{ }^{\partial} w_{D(G)}^{\eta(\bullet)}(z)={ }^{2} \varepsilon_{G}^{\bullet}(z) .
$$

Proof. By Lemma 19, for any $A \subseteq E$,

$$
w\left(D(G)^{\eta \bullet \bullet \mid A}\right)=w\left(D\left(G^{\bullet \mid A}\right)\right)=\varepsilon\left(G^{\bullet \mid A}\right)
$$

Hence ${ }^{\partial} w_{D(G)}^{\eta(\bullet)}(z)={ }^{\partial} \varepsilon_{G}^{\bullet}(z)$.
Theorem 21. If two normal binary delta-matroids $D$ and $\widetilde{D}$ have the same intersection graph, then ${ }^{\partial} w_{D}^{\bullet}(z)={ }^{\partial} w_{\tilde{D}}^{\bullet}(z)$ for any $\bullet \in \mathcal{B}$.
Proof. Since $G_{D}=G_{\widetilde{D}}, D=D\left(A_{G_{D}}\right)$ and $\widetilde{D}=D\left(A_{G_{\widetilde{D}}}\right)$, we have $D=\widetilde{D}$. Thus ${ }^{2} w_{D}^{\bullet}(z)={ }^{\partial} w_{\tilde{D}}^{\bullet}(z)$ for any $\bullet \in \mathcal{B}$.

Theorem 22. Let $B$ and $\widetilde{B}$ be two bouquets. If $G_{D(B)}=G_{D(\widetilde{B})}$, then ${ }^{\partial} \varepsilon_{B}^{\bullet}(z)={ }^{\partial} \varepsilon_{\stackrel{\rightharpoonup}{B}}^{\bullet}(z)$ for any $\bullet \in \mathcal{R}$.
Proof. Since $G_{D(B)}=G_{D(\widetilde{B})}$, it follows that $D(B)=D(\widetilde{B})$. For any $A \subseteq$ $E(B)$, we denote its corresponding subset of $E(\widetilde{B})$ by $\widetilde{A}$, then

$$
D\left(B^{\bullet \mid A}\right)=D(B)^{\eta \bullet \bullet \mid A}=D(\widetilde{B})^{\eta(\bullet) \mid \widetilde{A}}=D\left(\widetilde{B}^{\bullet \mid \widetilde{A}}\right),
$$

by Lemma 19. We have

$$
w\left(D\left(B^{\bullet \mid A}\right)\right)=w\left(D\left(\widetilde{B}^{\bullet \mid \widetilde{A}}\right)\right)
$$

Since $w\left(D\left(B^{\bullet} \mid A\right)\right)=\varepsilon\left(B^{\bullet} \mid A\right)$ and $w\left(D\left(\widetilde{B^{\bullet} \mid \widetilde{A}}\right)\right)=\varepsilon\left(\widetilde{B}^{\bullet} \mid \widetilde{A}\right)$, it follows that $\varepsilon\left(B^{\bullet} \mid A\right)=\varepsilon\left(\widetilde{B}^{\bullet} \mid \widetilde{A}\right)$. Thus ${ }^{\partial} \varepsilon_{B}^{\bullet}(z)={ }^{\partial} \varepsilon_{\widetilde{B}}^{\bullet}(z)$.

## 6. Partial-twuality monomials

We $[18,19]$ showed that a normal binary delta-matroid whose partial-* polynomials have only one term if and only if each connected component of the intersection graph of the delta-matroid is either a complete graph of odd order or a single vertex with a loop. In this section, we give a characterization of vf-safe delta-matroids whose partial $-\times$ and $* \times *$ polynomials have only one term.

Lemma 23 ([5]). Let $D=(E, \mathcal{F})$ be a set system and $X, Y \subseteq E$. We have $Y \in \mathcal{F}\left(D^{\times \mid X}\right)$ if and only if $|\{Z \in \mathcal{F}(D) \mid Y-X \subseteq Z \subseteq Y\}|$ is odd.

Theorem 24. Let $D=(E, \mathcal{F})$ be a vf-safe delta-matroid. Then
(1) ${ }^{\partial} w_{D}^{\times}(z)=c z^{m}$ if and only if $\mathcal{F}(D)=\{E\}$;
(2) ${ }^{\partial} w_{D}^{* \times *}(z)=c z^{m}$ if and only if $\mathcal{F}(D)=\{\emptyset\}$.

Proof. (1) Suppose that ${ }^{\partial} w_{D}^{\times}(z)=c z^{m}$. Then for any $e \in E$, the dual type of $e$ is $u$ in $D$, otherwise applying $\times \mid e$ changes the width according to Theorem 16. Then for any $A \in \mathcal{F}_{\max }(D) \cup \mathcal{F}_{\max -1}(D)$, we have $e \in A$. Thus $\mathcal{F}_{\max }(D)=\{E\}$ and $\mathcal{F}_{\max -1}(D)=\emptyset$. Suppose $\mathcal{F}_{\max -2}(D) \neq \emptyset$. Let $B \in \mathcal{F}_{\max -2}(D)$ and $f \in E-B$. Then $B \cup f, E \in \mathcal{F}\left(D^{\times \mid f}\right)$ by Lemma 23. Observe that $B \cup f \in \mathcal{F}_{\max -1}\left(D^{\times \mid f}\right)$ and $E \in \mathcal{F}_{\max }\left(D^{\times \mid f}\right)$. Let $g \in E-(B \cup f)$. Then there exists $B \cup f \in \mathcal{F}_{\max -1}\left(D^{\times \mid f}\right) \cup \mathcal{F}_{\max }\left(D^{\times \mid f}\right)$ such that $g \notin B \cup f$. Thus the dual type of $g$ is not $u$ in $D^{\times \mid f}$. We have $w\left(D^{\times \mid f}\right) \neq w\left(\left(D^{\times \mid f}\right)^{\times \mid g}\right)$ by Theorem 16. Then ${ }^{\partial} w_{D \times \mid f}^{\times}(z) \neq c z^{m}$. Note that ${ }^{\partial} w_{D \times \mid f}^{\times}(z)={ }^{\partial} w_{D}^{\times}(z)$ by Proposition 6. It follows that ${ }^{\partial} w_{D}^{\times}(z) \neq c z^{m}$, a contradiction. Then $\mathcal{F}_{\max -2}(D)=\emptyset$. Since the maximum gap in the collection of sizes of feasible sets of a delta-matroid is two, it follows that $\mathcal{F}(D)=\{E\}$.

Conversely, for any $X \subseteq E$,

$$
\mathcal{F}_{\min }\left(D^{\times \mid X}\right)=\mathcal{F}_{\min }(D)=\{E\}
$$

by Lemma 8. Then $\mathcal{F}\left(D^{\times \mid X}\right)=\{E\}$. Thus $w\left(D^{\times \mid X}\right)=0$ and ${ }^{\partial} w_{D}^{\times}(z)=2^{|E|}$.
(2) For $* \times *$, by Theorem 11, ${ }^{\partial} w_{D}^{* \times *}(z)={ }^{\partial} w_{D^{*}}^{\times}(z)=c z^{m}$ if and only if $\mathcal{F}\left(D^{*}\right)=\{E\}$ if and only if $\mathcal{F}(D)=\{\emptyset\}$.

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