Partial-twuality polynomials of delta-matroids

Qi Yan

School of Mathematics China University of Mining and Technology P. R. China Xian'an Jin¹ School of Mathematical Sciences Xiamen University P. R. China

Email:qiyan@cumt.edu.cn; xajin@xmu.edu.cn

Abstract

Gross, Mansour and Tucker introduced the partial-twuality polynomial of a ribbon graph. Chumutov and Vignes-Tourneret posed a problem: it would be interesting to know whether the partial duality polynomial and the related conjectures would make sense for general delta-matroids. In this paper we consider analogues of partial-twuality polynomials for delta-matroids. Various possible properties of partial-twuality polynomials of set systems are studied. We discuss the numerical implications of partial-twualities on a single element and prove that the intersection graphs can determine the partialtwuality polynomials of bouquets and normal binary delta-matroids, respectively. Finally, we give a characterization of vf-safe delta-matroids whose partial-twuality polynomials have only one term.

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1. Introduction

In [16], Wilson found that the two long-standing duality operators δ (geometric duality) and τ (Petrie duality) generate a group of six ribbon graph operators, that is, every other composition of δ and τ is equivalent to one of

¹Corresponding author.

the five operators δ , τ , $\delta\tau$, $\tau\delta$, $\delta\tau\delta$, or to the identity operator. Abrams and Ellis-Monaghan [1] called the five operators twualities. The partial (geometric) dual with respect to a subset of edges of a ribbon graph was introduced by Chmutov [7] in order to unify various connections between the Jones-Kauffman and Bollobás-Riordan polynomials. Ellis-Monaghan and Moffatt [12] generalized this partial-duality construction to the other four operators, which they called partial-twualities.

Gross, Mansour and Tucker [13, 14] introduced the partial-twuality polynomial for $\delta, \tau, \delta\tau, \tau\delta$, and $\delta\tau\delta$. Various basic properties of partial-twuality polynomials were studied, including interpolation and log-concavity. Recently, Chumutov and Vignes-Tourneret [8] posed the following question:

Question 1. [8] Ribbon graphs may be considered from the point of view of delta-matroid. In this way the concepts of partial (geometric) duality and genus can be interpreted in terms of delta-matroids [9, 10]. It would be interesting to know whether the partial- δ polynomial and the related conjectures would make sense for general delta-matroids.

In [18], we showed that the partial- δ polynomials have delta-matroid analogues. We introduced the twist polynomials of delta-matroids and discussed their basic properties for delta-matroids. Chun et al. [9] showed that the loop complementation is the delta-matroid analogue of partial Petriality. In this paper we consider analogues of other partial-twuality polynomials for delta-matroids.

This paper is organised as follows. In Section 2 we recall the definition of partial-twuality polynomials of ribbon graphs. Analogously, we introduce the partial-twuality polynomials of set systems. In Section 3, various possible properties of partial-twuality polynomials of set systems are studied. In Section 4 we discuss the numerical implications of partial-twualities on a single element and the interpolation. In Section 5, we prove that the intersection graphs can determine the partial-twuality polynomials of bouquets and normal binary delta-matroids, respectively. Here we provide an answer to the question [17]: can one derive something from bouquets that could determine the partial-twuality polynomial completely. In Section 6 we give a characterization of vf-safe delta-matroids whose partial-twuality polynomials have only one term.

2. Preliminaries

2.1. Set systems and widths

A set system is a pair $D = (E, \mathcal{F})$ of a finite set E together with a collection \mathcal{F} of subsets of E. The set E is called the ground set and the elements of \mathcal{F} are the feasible sets. We often use $\mathcal{F}(D)$ to denote the set of feasible sets of D. D is proper if $\mathcal{F} \neq \emptyset$, and is normal (respectively, dual normal) if the empty set (respectively, the ground set) is feasible. The direct sum of two set systems $D = (E, \mathcal{F})$ and $\widetilde{D} = (\widetilde{E}, \widetilde{\mathcal{F}})$ with disjoint ground sets E and \widetilde{E} , written $D \oplus \widetilde{D}$, is defined to be

$$D \oplus \widetilde{D} := (E \cup \widetilde{E}, \{F \cup \widetilde{F} : F \in \mathcal{F} \text{ and } \widetilde{F} \in \widetilde{\mathcal{F}}\}).$$

As introduced by Bouchet in [3], a *delta-matroid* is a proper set system $D = (E, \mathcal{F})$ such that if $X, Y \in \mathcal{F}$ and $u \in X\Delta Y$, then there is $v \in X\Delta Y$ (possibly v = u) such that $X\Delta\{u, v\} \in \mathcal{F}$. Here

$$X\Delta Y := (X \cup Y) - (X \cap Y)$$

is the usual symmetric difference of sets. Note that the maximum gap in the collection of sizes of feasible sets of a delta-matroid is two [15].

For a set system $D = (E, \mathcal{F})$, let $\mathcal{F}_{max}(D)$ and $\mathcal{F}_{min}(D)$ be the collections of maximum and minimum cardinality feasible sets of D, respectively. Let $D_{max} := (E, \mathcal{F}_{max}(D))$ and $D_{min} := (E, \mathcal{F}_{min}(D))$. Let $r(D_{max})$ and $r(D_{min})$ denote the sizes of largest and smallest feasible sets of D, respectively. The width of D, denote by w(D), is defined by

$$w(D) := r(D_{max}) - r(D_{min}).$$

For all non-negative integers $i \leq w(D)$, let

$$\mathcal{F}_{max-i}(D) = \{F \in \mathcal{F} : |F| = r(D_{max}) - i\}$$

and

$$\mathcal{F}_{min+i}(D) = \{F \in \mathcal{F} : |F| = r(D_{min}) + i\}.$$

2.2. Partial-twualities of set systems

We will consider the operations of twisting and loop complementation on set systems. Twisting was introduced by Bouchet in [3], and loop complementation by Brijder and Hoogeboom in [5]. Let $D = (E, \mathcal{F})$ be a set system. For $A \subseteq E$, the *twist* of D with respect to A, denoted by $D^{*|A}$, is given by

$$(E, \{A\Delta X : X \in \mathcal{F}\}).$$

The *-dual of D, written D^* , is equal to $D^{*|E}$. Note that *-duality preserves width. Throughout the paper, we will often omit the set brackets in the case of a single element set. For example, we write $D^{*|e}$ instead of $D^{*|e}$.

Let $D = (E, \mathcal{F})$ be a set system and $e \in E$. Then $D^{\times |e|}$ is defined to be the set system (E, \mathcal{F}') , where

$$\mathcal{F}' = \mathcal{F}\Delta\{F \cup e : F \in \mathcal{F} \text{ and } e \notin F\}.$$

If $e_1, e_2 \in E$ then

$$(D^{\times|e_1})^{\times|e_2} = (D^{\times|e_2})^{\times|e_1}.$$

This means that if $A = \{e_1, \dots, e_m\} \subseteq E$ we can unambiguously define the *loop complementation* [5] of D on A, by

$$D^{\times|A} := (\cdots (D^{\times|e_1})^{\times|e_2} \cdots)^{\times|e_m}.$$

It is straightforward to show that the twist of a delta-matroid is a deltamatroid [3], but the set of delta-matroids is not closed under loop complementation (see, for example, [9]). Thus, we often restrict our attention to a class of delta-matroids that is closed under loop complementation. A delta-matroid $D = (E, \mathcal{F})$ is said to be *vf-safe* [9] if the application of every sequence of twists and loop complementations results in a delta-matroid.

In [5] it was shown that twists and loop complementations give rise to an action of the symmetric group S_3 , with the presentation

$$S_3 \cong \mathcal{B} := < *, \times | *^2, \times^2, (*\times)^3 >,$$

on set systems. If $D = (E, \mathcal{F})$ is a set system, $e \in E$ and $a = a_1 a_2 \cdots a_n$ is a word in the alphabet $\{*, \times\}$, then

$$D^{a|e} := (\cdots (D^{a_1|e})^{a_2|e} \cdots)^{a_n|e}$$

Note that the operators * and \times on different elements commute [5]. If $A = \{e_1, \dots, e_m\} \subseteq E$, we can unambiguously define

$$D^{a|A} := (\cdots (D^{a|e_1})^{a|e_2} \cdots)^{a|e_m}.$$

Let $D_1 = (E, \mathcal{F})$ and D_2 be set systems. For $\bullet \in \{*, \times, *\times, \times *, *\times *\}$, we say that D_2 is a *partial*- \bullet dual of D_1 if there exists $A \subseteq E$ such that $D_2 = D_1^{\bullet|A}$.

2.3. Partial-twualities of ribbon graphs

Ribbon graphs are well-known to be equivalent to cellularly embedded graphs. The reader is referred to [11, 12] for further details about ribbon graphs. A *quasi-tree* is a ribbon graph with one boundary component. Let G = (V, E) be a ribbon graph and let

 $\mathcal{F} := \{ F \subseteq E(G) : F \text{ is the edge set of a spanning quasi-tree of } G \}.$

We call $D(G) =: (E, \mathcal{F})$ the *delta-matroid* [10] of G. We say a delta-matroid is *ribbon-graphic* if it is equal to the delta-matroid of some ribbon graph. Note that ribbon-graphic delta-matroids are vf-safe [9].

For a ribbon graph G and a subset A of its edge-ribbons E(G), the partial dual $G^{\delta|A}$ [7] of G with respect to A is a ribbon graph obtained from G by gluing a disc to G along each boundary component of the spanning ribbon subgraph (V(G), A) (such discs will be the vertex-discs of $G^{\delta|A}$), removing the interiors of all vertex-discs of G and keeping the edge-ribbons unchanged.

Let G be a ribbon graph and $A \subseteq E(G)$. Then the partial Petrial $G^{\tau|A}$ [11] of G with respect to A is the ribbon graph obtained from G by adding a half-twist to each of the edges in A.

In [11] it was shown that the partial dual, δ , and the partial Petrial, τ , give rise to an action of the symmetric group S_3 , with the presentation

$$S_3 \cong \mathcal{R} := <\delta, \tau \mid \delta^2, \tau^2, (\delta\tau)^3 >,$$

on ribbon graphs. If G is a ribbon graph, $e \in E(G)$ and $a = a_1 a_2 \cdots a_n$ is a word in the alphabet $\{\delta, \tau\}$, then

$$G^{a|e} := (\cdots (G^{a_1|e})^{a_2|e} \cdots)^{a_n|e}.$$

Observe that the partial dual and the partial Petrial commute when applied to different edges [11]. If $A = \{e_1, \dots, e_m\} \subseteq E$, we define

$$G^{a|A} := (\cdots (G^{a|e_1})^{a|e_2} \cdots)^{a|e_m}$$

Let G_1 and G_2 be ribbon graphs. For $\bullet \in \{\delta, \tau, \delta\tau, \tau\delta, \delta\tau\delta\}$, we say that G_2 is a partial- \bullet dual [11] of G_1 if there exists $A \subseteq E(G_1)$ such that $G_2 = G_1^{\bullet|A}$.

2.4. Partial-twuality polynomials of ribbon graphs and set systems

Gross, Mansour and Tucker [14] introduced the concept of partial-twuality polynomials of ribbon graphs as follows.

Definition 2 ([14]). For $\bullet \in \mathcal{R}$, we define the partial- \bullet polynomial for any ribbon graph G to be the generating function

$${}^{\partial} \varepsilon_{G}^{\bullet}(z) := \sum_{A \subseteq E(G)} z^{\varepsilon(G^{\bullet|A})}$$

that enumerates all partial- \bullet duals of G by Euler genus.

Analogously, we define the partial-twuality polynomials of set systems as follows.

Definition 3. For $\bullet \in \mathcal{B}$, the *partial*- \bullet *polynomial* of any set system $D = (E, \mathcal{F})$ is defined to be the generating function

$${}^{\partial}w_D^{\bullet}(z) := \sum_{A \subseteq E} z^{w(D^{\bullet|A})}$$

that enumerates all partial- \bullet duals of D by width.

2.5. Binary and intersection graphs

For a finite set E, let C be a symmetric |E| by |E| matrix over GF(2), with rows and columns indexed, in the same order, by the elements of E. Let C[A] be the principal submatrix of C induced by the set $A \subseteq E$. We define the set system $D(C) = (E, \mathcal{F})$ with

$$\mathcal{F} := \{ A \subseteq E : C[A] \text{ is non-singular} \}.$$

By convention $C[\emptyset]$ is non-singular. Then D(C) is a delta-matroid [4]. A delta-matroid is said to be *binary* if it has a twist that is isomorphic to D(C) for some symmetric matrix C over GF(2).

Let $D = (E, \mathcal{F})$ be a normal binary delta-matroid. Then there exists a unique symmetric |E| by |E| matrix C over GF(2) such that D = D(C)[15, 18]. The *intersection graph* G_D of D is the graph with vertex set E and in which two vertices u and v of G_D are adjacent if and only if $C_{u,v} = 1$. A *bouquet* is a ribbon graph with only one vertex. If B is a bouquet, then D(B) is a normal binary delta-matroid [10]. The *intersection graph* I(B) of a bouquet B is the graph $G_{D(B)}$. Conversely, recall that a *looped simple graph* [15] is a graph obtained from a simple graph by adding (exactly) one loop to some of its vertices. The adjacency matrix A(G) of a looped simple graph G is the matrix over GF(2) whose rows and columns correspond to the vertices of G; and where, $A(G)_{u,v} = 1$ if and only if u and v are adjacent in G and $A(G)_{u,u} = 1$ if and only if there is a loop at u. Let D be a normal binary delta-matroid. It obvious that $D = D(A(G_D))$.

2.6. Primal and dual types

Let $D = (E, \mathcal{F})$ be a proper set system. An element $e \in E$ contained in no feasible set of D is said to be a *loop*.

Definition 4 ([10]). Let $D = (E, \mathcal{F})$ be a set system and $e \in E$. Then

- (1) e is a ribbon loop if e is a loop in D_{min} ;
- (2) A ribbon loop e is non-orientable if e is a ribbon loop in $D^{*|e}$ and is orientable otherwise.

Let $D = (E, \mathcal{F})$ be a set system and $e \in E$. The primal type of e is p, u, or t in D, if e is a non-ribbon loop, an orientable loop, or a non-orientable loop, respectively, in D. The primal type of e in D^* is called the *dual type* of e in D. In combination, the primal and dual types of e in D are called the *type* of e in D, which is denoted by a juxtaposed pair of letters representing the primal and dual types of e in D. For example, the type pu means that the primal and dual types of e are p and u, respectively, in D. We observe that

- (1) The primal type of e is p in D if and only if there exists $A \in \mathcal{F}_{min}(D)$ such that $e \in A$;
- (2) The dual type of e is p in D if and only if there exists $A \in \mathcal{F}_{max}(D)$ such that $e \notin A$;
- (3) The primal type of e is u in D if and only if for any $A \in \mathcal{F}_{min}(D) \cup \mathcal{F}_{min+1}(D), e \notin A;$
- (4) The dual type of e is u in D if and only if for any $A \in \mathcal{F}_{max}(D) \cup \mathcal{F}_{max-1}(D), e \in A;$

- (5) The primal type of e is t in D if and only if for any $A \in \mathcal{F}_{min}(D), e \notin A$, and there exists $B \in \mathcal{F}_{min+1}(D)$ such that $e \in B$;
- (6) The dual type of e is t in D if and only if for any $A \in \mathcal{F}_{max}(D), e \in A$, and there exists $B \in \mathcal{F}_{max-1}(D)$ such that $e \notin B$.

3. Some properties of partial-twuality polynomials

Various possible properties of partial-twuality polynomials of ribbon graphs were studied by Gross, Mansour and Tucker in [13, 14]. In this section we discuss the analogous results on set systems or delta-matroids.

Proposition 5. Let $D = (E, \mathcal{F})$ and $\widetilde{D} = (\widetilde{E}, \widetilde{\mathcal{F}})$ be set systems. Then for any $\bullet \in \mathcal{B}$,

- (1) $^{\partial}w_D^{\bullet}(1) = 2^{|E|};$
- (2) ${}^{\partial}w_D^{\bullet}(z)$ has degree at most |E|;
- (3) ${}^{\partial}w^{\bullet}_{D\oplus\widetilde{D}}(z) = {}^{\partial}w^{\bullet}_{D}(z){}^{\partial}w^{\bullet}_{\widetilde{D}}(z).$

Proof. For (1), the value ${}^{\partial}w_{D}^{\bullet}(1)$ counts the total number of partial- \bullet duals of D, which is $2^{|E|}$. For any subset $A \subseteq E$, if $B \in \mathcal{F}(D^{\bullet|A})$, then $\emptyset \subseteq B \subseteq E$. We have $r(D^{\bullet|A}_{min}) \geq 0$ and $r(D^{\bullet|A}_{max}) \leq |E|$. Thus $0 \leq w(D^{\bullet|A}) \leq |E|$ and (2) then follows. For any subset $C \subseteq E \cup \widetilde{E}$, we have

$$(D \oplus \widetilde{D})^{\bullet|C} = D^{\bullet|(C \cap E)} \oplus \widetilde{D}^{\bullet|(C \cap \widetilde{E})}.$$

Then

$${}^{\partial}w^{\bullet}_{D\oplus\widetilde{D}}(z) = {}^{\partial}w^{\bullet}_{D}(z){}^{\partial}w^{\bullet}_{\widetilde{D}}(z),$$

by the additivity of width over the direct sum, from which (3) follows. \Box

Proposition 6. Let $D = (E, \mathcal{F})$ be a set system and $A \subseteq E$. Then

$${}^{\partial}w_{D}^{\bullet}(z) = {}^{\partial}w_{D^{\bullet|A}}^{\bullet}(z)$$

for $\bullet \in \{*, \times, * \times *\}.$

Proof. This is because the set of all loop complementations of D is the same as that of $D^{\times|A}$. The same reasoning applies to the operators * and $* \times *$. \Box

Remark 7. Proposition 6 is not true for the operators $*\times$ and $\times *$. For example, let $D = (E, \mathcal{F})$ with $E = \{1\}$ and $\mathcal{F} = \{\emptyset, \{1\}\}$. Then $D^{*\times|1} = (\{1\}, \{\emptyset\})$ and $D^{\times*|1} = (\{1\}, \{\{1\}\})$. We have

$${}^{\partial}w_D^{*\times}(z) = {}^{\partial}w_D^{\times*}(z) = 1 + z$$

and

$${}^{\partial}w_{D^{*\times|1}}^{*\times}(z) = {}^{\partial}w_{D^{\times*|1}}^{\times*}(z) = 2.$$

Obviously, ${}^{\partial}w_D^{*\times}(z) \neq {}^{\partial}w_{D^{*\times|1}}^{*\times}(z)$ and ${}^{\partial}w_D^{\times*}(z) \neq {}^{\partial}w_{D^{\times*|1}}^{\times*}(z)$.

Lemma 8 ([5]). Let $D = (E, \mathcal{F})$ be a set system and $A \subseteq E$. Then

$$\mathcal{F}_{min}(D) = \mathcal{F}_{min}(D^{\times|A})$$

and

$$\mathcal{F}_{max}(D) = \mathcal{F}_{max}(D^{*\times*|A}) = \mathcal{F}_{max}(D^{\times*\times|A}).$$

Proposition 9. Let $D = (E, \mathcal{F})$ be a set system and $A \in \mathcal{F}_{min}(D), B \in \mathcal{F}_{min}(D^*)$. Then

- (1) $D^{\bullet|A}$ is normal for $\bullet \in \{*, *\times, \times *, *\times *\};$
- (2) $D^{\times|B}$ is dual normal.

Proof. (1) We may assume that $A \neq \emptyset$, otherwise the conclusion is trivial. For any $e \in A$, since $A \in \mathcal{F}_{min}(D)$, it follows that $A \in \mathcal{F}_{min}(D^{\times|e})$ by Lemma 8 and $A - e \in \mathcal{F}_{min}(D^{*|e})$. Then $A - e \in \mathcal{F}_{min}(D^{*\times|e})$ by Lemma 8 and $A - e \in \mathcal{F}_{min}(D^{\times *|e})$. Thus $A - e \in \mathcal{F}_{min}(D^{\times *\times|e})$ by Lemma 8. From the above, we have $A - e \in \mathcal{F}_{min}(D^{\bullet|e})$ for $\bullet \in \{*, *\times, \times *, *\times *\}$. In the same manner we can see that $\emptyset \in \mathcal{F}_{min}(D^{\bullet|A})$ for $\bullet \in \{*, *\times, \times *, *\times *\}$ and conclusion (1) then follows.

(2) Since $B \in \mathcal{F}_{min}(D^*)$, it follows that $E - B \in \mathcal{F}_{max}(D)$. Then $E - B \in \mathcal{F}_{max}(D^{\times * \times |B})$ by Lemma 8, that is, $E - B \in \mathcal{F}(D^{\times * \times |B})$. Thus $E \in \mathcal{F}(D^{\times * \times *|B})$. Obviously,

$$E \in \mathcal{F}_{max}(D^{\times * \times *|B}) = \mathcal{F}_{max}((D^{\times|B})^{* \times *|B}).$$

Then $E \in \mathcal{F}_{max}(D^{\times|B})$ by Lemma 8, that is, $E \in \mathcal{F}(D^{\times|B})$. Thus $D^{\times|B}$ is dual normal.

Remark 10. For investigation of partial-• polynomials of set systems for • \in {*, * × *}, Propositions 6 and 9 motivate us to focus on normal set systems, and for • = ×, to focus on dual normal set systems. But for *× or ×*, we cannot just focus on normal set systems. For example, let $D = (\{1\}, \{\{1\}\})$. We have $\partial w_D^{**}(z) = 2$. Observe that all normal set systems with ground set {1} are $D_1 = (\{1\}, \{\emptyset\})$ and $D_2 = (\{1\}, \{\emptyset, \{1\}\})$. Since $\partial w_{D_1}^{**}(z) = \partial w_{D_2}^{**}(z) = 1 + z$, it follows that there is no normal set system D' such that $\partial w_{D'}^{**}(z) = \partial w_D^{**}(z)$.

The following theorem provides a link between partial-* \bullet * and partial- \bullet polynomials of set systems.

Theorem 11. Let $D = (E, \mathcal{F})$ be a set system. Then for any $\bullet \in \mathcal{B}$, ${}^{\partial}w_{D}^{*\bullet*}(z) = {}^{\partial}w_{D*}^{\bullet}(z).$

Proof. For any $A \subseteq E$, we observe that doing partial-*•* on A is the same as first doing * to E, then doing • to A, and then doing * to E again, that is,

$$D^{*\bullet*|A} = ((D^*)^{\bullet|A})^*.$$

Since *-duality preserves width, it follows that

$$w(D^{*\bullet*|A}) = w(((D^*)^{\bullet|A})^*) = w((D^*)^{\bullet|A}).$$

Thus the partial-*•* polynomial of D is identical to the partial-• polynomial of D^* .

4. Partial-twuality for a single element

In this section, we discuss the numerical implications of partial-twualities on a single element e, depending on the type of e.

Lemma 12 ([6]). Let $D = (E, \mathcal{F})$ be a delta-matroid and $e \in E$ such that $r(D_{min}) = r(D^{*|e}_{min})$. Then $\mathcal{F}_{min}(D) = \mathcal{F}_{min}(D^{*|e})$.

Remark 13. Lemma 12 is not true for set systems. For example, let

$$D = (\{1, 2, 3\}, \{\{1\}, \{2, 3\}\}).$$

We know $r(D_{min}) = r(D^{*|2}_{min}) = 1$. But

$$\mathcal{F}_{min}(D) = \{\{1\}\}\$$

and

$$\mathcal{F}_{min}(D^{*|2}) = \{\{3\}\}.$$

Table 1. The difference $w(D^{-1}) = w(D)$ for any $\bullet \in D$.						
Type of e	*	×	* ×	$\times *$	$* \times *$	
pp	+2	+1	+2	+2	+1	
uu	-2	0	-1	-1	0	
pu	0	0	+1	0	+1	
up	0	+1	0	+1	0	
tp	+1	+1	+1	0	-1	
tu	-1	0	0	-2	-1	
pt	+1	-1	0	+1	+1	
ut	-1	-1	-2	0	0	
tt	0	-1	-1	-1	-1	

Table 1: The difference $w(D^{\bullet|e}) - w(D)$ for any $\bullet \in \mathcal{B}$.

Lemma 14. Let $D = (E, \mathcal{F})$ be a delta-matroid and $e \in E$. If e is a nonorientable loop, then for any $A \in \mathcal{F}_{min}(D)$, $A \cup e \in \mathcal{F}(D)$.

Proof. Since the primal type of e is t in D, it follows that $e \notin A$ and there exists $B \in \mathcal{F}_{min+1}(D)$ such that $e \in B$. Then $B - e \in \mathcal{F}_{min}(D^{*|e})$. We have $r(D_{min}) = r(D^{*|e}_{min})$ and hence $\mathcal{F}_{min}(D) = \mathcal{F}_{min}(D^{*|e})$ by Lemma 12. Then $A \in \mathcal{F}_{min}(D^{*|e})$, that is, $A \in \mathcal{F}(D^{*|e})$. Thus, $A \cup e \in \mathcal{F}(D)$.

Lemma 15 ([2]). If X is any feasible set in a delta-matroid D, then there exist $A \in \mathcal{F}_{min}(D)$ and $B \in \mathcal{F}_{max}(D)$ such that $A \subseteq X \subseteq B$.

Theorem 16. Let $D = (E, \mathcal{F})$ be a vf-safe delta-matroid and $e \in E$. Table 1 gives the value of $w(D^{\bullet|e}) - w(D)$ for any $\bullet \in \mathcal{B}$.

Proof. The three possible primal types (and dual types) of e in D are as follows:

Case 1. If the primal type of e is p in D, there exists $A \in \mathcal{F}_{min}(D)$ such that $e \in A$. Then $A - e \in \mathcal{F}_{min}(D^{*|e})$. Thus

$$r(D^{*|e}{}_{min}) = r(D_{min}) - 1$$

and the primal types of e are u and p in $D^{*|e}$ and $D^{\times|e}$, respectively.

Case 2. If the primal type of e is u in D, then for any $A \in \mathcal{F}_{min}(D) \cup \mathcal{F}_{min+1}(D), e \notin A$. Thus

$$r(D^{*|e}{}_{min}) = r(D_{min}) + 1$$

and the types of e are p and t in $D^{*|e}$ and $D^{\times|e}$, respectively.

Table 2: A summary of Cases 1, 2 and 3.				
Primal type of e			$- r(D^{* e}{}_{min})$	
D	$D^{* e}$	$D^{\times e }$	= T(D + min)	
p	u	p	$r(D_{min}) - 1$	
u	p	t	$r(D_{min}) + 1$	
t	t	u	$r(D_{min})$	

Case 3. If the primal type of e is t in D, then for any $A \in \mathcal{F}_{min}(D)$, $e \notin A$, and there exists $B \in \mathcal{F}_{min+1}(D)$ such that $e \in B$. Thus

$$r(D^{*|e}_{min}) = r(D_{min})$$

and the primal types of e is t in $D^{*|e}$. By Lemma 14, we have $A \cup e \in \mathcal{F}_{min+1}(D)$ for any $A \in \mathcal{F}_{min}(D)$. Then $A \cup e \notin \mathcal{F}(D^{\times|e})$. Furthermore, we know that for any $B \in \mathcal{F}_{min+1}(D)$ containing $e, B - e \in \mathcal{F}_{min}(D)$, otherwise there is no $A' \in \mathcal{F}_{min}(D)$ such that $A' \subseteq B$, contradicting Lemma 15. Since $\mathcal{F}_{min}(D^{\times|e}) = \mathcal{F}_{min}(D)$, it follows that there is no $B' \in \mathcal{F}_{min}(D^{\times|e}) \cup \mathcal{F}_{min+1}(D^{\times|e})$ such that $e \in B'$. Then the primal type of e is u in $D^{\times|e}$.

Here, we give a summary of Cases 1, 2 and 3 as shown in Table 2.

Case 4. If the dual type of e is p in D, there exists $A \in \mathcal{F}_{max}(D)$ such that $e \notin A$. Then $A \cup e \in \mathcal{F}_{max}(D^{*|e}) \cap \mathcal{F}_{max}(D^{\times|e})$. Thus

$$r(D^{*|e}_{max}) = r(D^{\times|e}_{max}) = r(D_{max}) + 1$$

and the dual types of e are u and t in $D^{*|e}$ and $D^{\times|e}$, respectively.

Case 5. If the dual type of e is u in D, then for any $A \in \mathcal{F}_{max}(D) \cup \mathcal{F}_{max-1}(D), e \in A$. Thus

$$r(D^{*|e}_{max}) = r(D_{max}) - 1$$

and

$$r(D^{\times|e}{}_{max}) = r(D_{max})$$

and the dual types of e are p and u in $D^{*|e}$ and $D^{\times|e}$, respectively.

Table 3: A summary of Cases 4, 5 and 6

Dual type of e			$- r(D^{* e}_{max})$	$r(D^{\times e}{}_{max})$	
D	$D^{* e}$	$D^{\times e }$	- I(D + max)	I(D + max)	
p	u	t	$r(D_{max}) + 1$	$r(D_{max}) + 1$	
u	p	u	$r(D_{max}) - 1$	$r(D_{max})$	
t	t	p	$r(D_{max})$	$r(D_{max}) - 1$	

Table 4: The the widths of $D^{*|e}$ and $D^{\times|e}$

Type of e	$r(D^{* e}{}_{min})$	$r(D^{* e}{}_{max})$	$r(D^{\times e}{}_{max})$	$w(D^{* e})$	$w(D^{\times e})$
pp	$r(D_{min}) - 1$	$r(D_{max}) + 1$	$r(D_{max}) + 1$	w(D) + 2	w(D) + 1
uu	$r(D_{min}) + 1$	$r(D_{max}) - 1$	$r(D_{max})$	w(D) - 2	w(D)
pu	$r(D_{min}) - 1$	$r(D_{max}) - 1$	$r(D_{max})$	w(D)	w(D)
up	$r(D_{min}) + 1$	$r(D_{max}) + 1$	$r(D_{max}) + 1$	w(D)	w(D) + 1
tp	$r(D_{min})$	$r(D_{max}) + 1$	$r(D_{max}) + 1$	w(D) + 1	w(D) + 1
tu	$r(D_{min})$	$r(D_{max}) - 1$	$r(D_{max})$	w(D) - 1	w(D)
pt	$r(D_{min}) - 1$	$r(D_{max})$	$r(D_{max}) - 1$	w(D) + 1	w(D) - 1
ut	$r(D_{min}) + 1$	$r(D_{max})$	$r(D_{max}) - 1$	w(D) - 1	w(D) - 1
tt	$r(D_{min})$	$r(D_{max})$	$r(D_{max}) - 1$	w(D)	w(D) - 1

Case 6. If the dual type of e is t in D, then for any $A \in \mathcal{F}_{max}(D)$, $e \in A$. Thus $E - A \in \mathcal{F}_{min}(D^*)$ and $(E - A) \cup e \in \mathcal{F}(D^*)$ by Lemma 14. It follows that $A - e \in \mathcal{F}_{max-1}(D)$. We have

$$r(D^{*|e}{}_{max}) = r(D_{max})$$

and the dual type of e is t in $D^{*|e}$. Moreover, we observe that for any $B \in \mathcal{F}_{max-1}(D)$ not containing $e, B \cup e \in \mathcal{F}_{max}(D)$, otherwise there is no $B' \in \mathcal{F}_{max}(D)$ such that $B \subseteq B'$, contradicting Lemma 15. It follows that

$$r(D^{\times|e}{}_{max}) = r(D_{max}) - 1$$

and the dual type of e is p in $D^{\times|e}$, respectively.

Here, we provide a summary of Cases 4, 5 and 6 as shown in Table 3. Then the the widths of $D^{*|e}$ and $D^{\times|e}$ can be calculated by Tables 2 and 3 as shown in Table 4. Hence, the columns 2 and 3 of Table 1 are computed. If the type of e is pp in D, then

$$w(D^{*|e}) = w(D) + 2$$

and

$$w(D^{\times|e}) = w(D) + 1,$$

and the types of e are uu and pt in $D^{*|e}$ and $D^{\times|e}$, respectively. Thus

$$w(D^{*\times|e}) = w((D^{*|e})^{\times|e}) = w(D^{*|e}) = w(D) + 2$$

and

$$w(D^{\times *|e}) = w((D^{\times|e})^{*|e}) = w(D^{\times|e}) + 1 = w(D) + 2,$$

and the type of e is tu in $D^{*\times|e}$. We have

$$w(D^{*\times *|e}) = w((D^{*\times |e})^{*|e}) = w(D^{*\times |e}) - 1 = w(D) + 1.$$

The other entries in columns 4, 5 and 6 of Table 1 are computed similarly. \Box

The polynomial $p(z) = \sum_{i=0}^{n} c_i z^i$ is said to have a gap of size k [14] at coefficient c_i if $c_{i-1}c_{i+k} \neq 0$ but $c_i = c_{i+1} = \cdots = c_{i+k-1} = 0$. If the polynomial p(z) is nonzero and has no gaps, we call it *interpolating*.

Proposition 17. For any vf-safe delta-matroid D, the following statements hold:

- (1) ${}^{\partial}w_D^{\bullet}(z)$ is interpolating for $\bullet = \times$ or $* \times *;$
- (2) ${}^{\partial}w_D^{\bullet}(z)$ has no gaps of size 2 or more for any $\bullet \in \mathcal{B}$.

Proof. For any element e and subset A of E, we observe that $w(D^{\bullet|A\Delta e})$ and $w(D^{\bullet|A})$ differ by at most one for $\bullet \in \{\times, \times \times \}$, and by at most two for $\bullet \in \{\ast, \times, \times \}$ by Theorem 16. This yields statements (1) and (2).

Remark 18. There exists a vf-safe delta-matroid D such that ${}^{\partial}w_{D}^{\bullet}(z)$ is not interpolating for $\bullet \in \{*, *\times, \times *\}$. For example, let

$$D_1 = (\{1, 2\}, \{\emptyset, \{1, 2\}\})$$

and

$$D_2 = (\{1, 2\}, \{\emptyset, \{1\}, \{1, 2\}\}).$$

We have

$${}^{\partial}w_{D_1}^*(z) = 2 + 2z^2$$

and

$${}^{\partial}w_{D_2}^{*\times}(z) = {}^{\partial}w_{D_2}^{\times*}(z) = 1 + 3z^2.$$

5. Partial-twuality polynomials and intersection graphs

In [17], we showed that two bouquets with the same intersection graph have the same partial- δ polynomial. In this section, we prove that the intersection graphs can determine the partial-twuality polynomials of bouquets and normal binary delta-matroids, respectively. Let $\eta : \mathcal{R} \to \mathcal{B}$ be the group isomorphism induced by $\eta(\delta) = *$, and $\eta(\tau) = \times$.

Lemma 19 ([10]). If G is a ribbon graph, $A \subseteq E$ and $\bullet \in \mathcal{R}$. Then

 $D(G^{\bullet|A}) = D(G)^{\eta(\bullet)|A}$

and

$$\varepsilon(G) = w(D(G)).$$

Proposition 20. Let G = (V, E) be a ribbon graph and $\bullet \in \mathcal{R}$. Then

$${}^{\partial}w_{D(G)}^{\eta(\bullet)}(z) = {}^{\partial}\varepsilon_{G}^{\bullet}(z)$$

Proof. By Lemma 19, for any $A \subseteq E$,

$$w(D(G)^{\eta(\bullet)|A}) = w(D(G^{\bullet|A})) = \varepsilon(G^{\bullet|A}).$$

Hence ${}^{\partial}w_{D(G)}^{\eta(\bullet)}(z) = {}^{\partial}\varepsilon_{G}^{\bullet}(z).$

Theorem 21. If two normal binary delta-matroids D and D have the same intersection graph, then ${}^{\partial}w^{\bullet}_{D}(z) = {}^{\partial}w^{\bullet}_{\widetilde{D}}(z)$ for any $\bullet \in \mathcal{B}$.

Proof. Since $G_D = G_{\widetilde{D}}$, $D = D(A_{G_D})$ and $\widetilde{D} = D(A_{G_{\widetilde{D}}})$, we have $D = \widetilde{D}$. Thus ${}^{\partial}w^{\bullet}_{D}(z) = {}^{\partial}w^{\bullet}_{\widetilde{D}}(z)$ for any $\bullet \in \mathcal{B}$.

Theorem 22. Let *B* and \widetilde{B} be two bouquets. If $G_{D(B)} = G_{D(\widetilde{B})}$, then ${}^{\partial} \varepsilon_{B}^{\bullet}(z) = {}^{\partial} \varepsilon_{\widetilde{B}}^{\bullet}(z)$ for any $\bullet \in \mathcal{R}$.

Proof. Since $G_{D(B)} = G_{D(\widetilde{B})}$, it follows that $D(B) = D(\widetilde{B})$. For any $A \subseteq E(B)$, we denote its corresponding subset of $E(\widetilde{B})$ by \widetilde{A} , then

$$D(B^{\bullet|A}) = D(B)^{\eta(\bullet)|A} = D(\widetilde{B})^{\eta(\bullet)|\widetilde{A}} = D(\widetilde{B}^{\bullet|\widetilde{A}}),$$

by Lemma 19. We have

$$w(D(B^{\bullet|A})) = w(D(\widetilde{B}^{\bullet|A})).$$

Since $w(D(B^{\bullet|A})) = \varepsilon(B^{\bullet|A})$ and $w(D(\widetilde{B}^{\bullet|\widetilde{A}})) = \varepsilon(\widetilde{B}^{\bullet|\widetilde{A}})$, it follows that $\varepsilon(B^{\bullet|A}) = \varepsilon(\widetilde{B}^{\bullet|\widetilde{A}})$. Thus ${}^{\partial}\varepsilon_{B}^{\bullet}(z) = {}^{\partial}\varepsilon_{\widetilde{B}}^{\bullet}(z)$.

6. Partial-twuality monomials

We [18, 19] showed that a normal binary delta-matroid whose partial-* polynomials have only one term if and only if each connected component of the intersection graph of the delta-matroid is either a complete graph of odd order or a single vertex with a loop. In this section, we give a characterization of vf-safe delta-matroids whose partial- \times and $* \times *$ polynomials have only one term.

Lemma 23 ([5]). Let $D = (E, \mathcal{F})$ be a set system and $X, Y \subseteq E$. We have $Y \in \mathcal{F}(D^{\times|X})$ if and only if $|\{Z \in \mathcal{F}(D) \mid Y - X \subseteq Z \subseteq Y\}|$ is odd.

Theorem 24. Let $D = (E, \mathcal{F})$ be a vf-safe delta-matroid. Then

(1)
$${}^{\partial}w_D^{\times}(z) = cz^m$$
 if and only if $\mathcal{F}(D) = \{E\};$

(2) ${}^{\partial}w_D^{*\times*}(z) = cz^m$ if and only if $\mathcal{F}(D) = \{\emptyset\}$.

Proof. (1) Suppose that ${}^{\partial}w_{D}^{\times}(z) = cz^{m}$. Then for any $e \in E$, the dual type of e is u in D, otherwise applying $\times | e$ changes the width according to Theorem 16. Then for any $A \in \mathcal{F}_{max}(D) \cup \mathcal{F}_{max-1}(D)$, we have $e \in A$. Thus $\mathcal{F}_{max}(D) = \{E\}$ and $\mathcal{F}_{max-1}(D) = \emptyset$. Suppose $\mathcal{F}_{max-2}(D) \neq \emptyset$. Let $B \in \mathcal{F}_{max-2}(D)$ and $f \in E - B$. Then $B \cup f, E \in \mathcal{F}(D^{\times | f})$ by Lemma 23. Observe that $B \cup f \in \mathcal{F}_{max-1}(D^{\times | f})$ and $E \in \mathcal{F}_{max}(D^{\times | f})$. Let $g \in E - (B \cup f)$. Then there exists $B \cup f \in \mathcal{F}_{max-1}(D^{\times | f}) \cup \mathcal{F}_{max}(D^{\times | f})$ such that $g \notin B \cup f$. Thus the dual type of g is not u in $D^{\times | f}$. We have $w(D^{\times | f}) \neq w((D^{\times | f})^{\times | g})$ by Theorem 16. Then ${}^{\partial}w_{D^{\times | f}}(z) \neq cz^{m}$. Note that ${}^{\partial}w_{D^{\times | f}}(z) = {}^{\partial}w_{D}^{\times}(z)$ by Proposition 6. It follows that ${}^{\partial}w_{D}^{\times}(z) \neq cz^{m}$, a contradiction. Then $\mathcal{F}_{max-2}(D) = \emptyset$. Since the maximum gap in the collection of sizes of feasible sets of a delta-matroid is two, it follows that $\mathcal{F}(D) = \{E\}$.

Conversely, for any $X \subseteq E$,

$$\mathcal{F}_{min}(D^{\times|X}) = \mathcal{F}_{min}(D) = \{E\}$$

by Lemma 8. Then $\mathcal{F}(D^{\times|X}) = \{E\}$. Thus $w(D^{\times|X}) = 0$ and ${}^{\partial}w_D^{\times}(z) = 2^{|E|}$. (2) For $* \times *$, by Theorem 11, ${}^{\partial}w_D^{*\times*}(z) = {}^{\partial}w_{D^*}(z) = cz^m$ if and only if

 $\mathcal{F}(D^*) = \{E\} \text{ if and only if } \mathcal{F}(D) = \{\emptyset\}.$

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