Equidistribution of set-valued statistics on standard Young tableaux and transversals

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Abstract. As a natural generalization of permutations, transversals of Young diagrams play an important role in the study of pattern avoiding permutations. Let $\mathcal{T}_{\lambda}(\tau)$ and $\mathcal{ST}_{\lambda}(\tau)$ denote the set of τ -avoiding transversals and τ -avoiding symmetric transversals of a Young diagram λ , respectively. In this paper, we are mainly concerned with the distribution of the peak set and the valley set on standard Young tableaux and pattern avoiding transversals. In particular, by introducing Knuth transformations on standard Young tableaux, we prove that the peak set and the valley set are equidistributed on the standard Young tableaux of shape λ/μ for any skew diagram λ/μ . The equidistribution enables us to show that the peak set is equidistributed over $\mathcal{T}_{\lambda}(12\cdots k\tau)$ (resp. $\mathcal{ST}_{\lambda}(12\cdots k\tau)$) and $\mathcal{T}_{\lambda}(k\cdots 21\tau)$ (resp. $\mathcal{ST}_{\lambda}(k\cdots 21\tau)$) for any Young diagram λ and any permutation τ of $\{k+1,k+2,\ldots,k+m\}$ with $k,m \geq 1$. Our results are refinements of the result of Backelin-West-Xin which states that $|\mathcal{T}_{\lambda}(12\cdots k\tau)| = |\mathcal{T}_{\lambda}(k\cdots 21\tau)|$ and the result of Bousquet-Mélou and Steingrímsson which states that $|\mathcal{ST}_{\lambda}(12\cdots k\tau)| = |\mathcal{ST}_{\lambda}(k\cdots 21\tau)|$. As applications, we are able to

- confirm a recent conjecture posed by Yan-Wang-Zhou which asserts that the peak set is equidistributed over $12\cdots k\tau$ -avoiding involutions and $k\cdots 21\tau$ -avoiding involutions;
- prove that alternating involutions avoiding the pattern $12\cdots k\tau$ are equinumerous with alternating involutions avoiding the pattern $k\cdots 21\tau$, paralleling the result of Backelin-West-Xin for permutations, the result of Bousquet-Mélou and Steingrímsson for involutions, and the result of Yan for alternating permutations.

Keywords: pattern avoidance, standard Young tableau, transversal, equidistribution.

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1 Introduction

As a natural generalization of permutations, transversals of Young diagrams play an important role in the study of pattern avoiding permutations and various interesting results on them have been obtained in the literature (see e.g. [2, 3, 12, 17, 19, 38, 39]). This paper is devoted to the investigation of the distribution of the peak set and the valley set on standard Young tableaux and pattern avoiding transversals.

Let us first review some necessary definitions before we state our original motivation and main results. Let S_n denote the set of permutations of $[n]=\{1,2,\ldots,n\}$, which we will always write as words $\pi=\pi_1\pi_2\cdots\pi_n$. An index i ($2 \le i \le n-1$) is said to be a peak (resp. valley) of π if $\pi_{i-1} < \pi_i > \pi_{i+1}$ (resp. $\pi_{i-1} > \pi_i < \pi_{i+1}$). Let Peak(π) and Val(π) denote the set of peaks and the set of valleys of π , respectively. For example, let $\pi=561943728$. Then we have Peak(π) = $\{2,4,7\}$ and Val(π) = $\{3,6,8\}$. There are many research articles devoted to the combinatorics of peaks on permutations (see e.g. [1,6,25,26,28,30,31,33]).

Let $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathcal{S}_n$. A permutation π is said to be alternating if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$. A permutation π is said to be an involution if $\pi = \pi^{-1}$, where π^{-1} denotes the inverse permutation of π . For example, the permutation $\pi = 795634182$ is an alternating involution. Let \mathcal{A}_n (resp. \mathcal{I}_n and $\mathcal{A}\mathcal{I}_n$) denote the set of alternating permutations (resp. involutions and alternating involutions) of length n.

Given a permutation $\pi \in \mathcal{S}_n$ and a permutation $\sigma \in \mathcal{S}_k$, an occurrence of σ in π is a subsequence $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ of π that is order isomorphic to σ . We say π contains the pattern σ if π contains an occurrence of σ . Otherwise, we say π avoids the pattern σ and π is σ -avoiding. For instance, the permutation 57243618 is 1243-avoiding while it contains a pattern 1234 corresponding to the subsequence 2468. Let $\mathcal{S}_n(\sigma)$ denote the set of σ -avoiding permutations of length n. We will keep this notation also when \mathcal{S}_n is replaced by other subsets of permutations such as $\mathcal{A}_n, \mathcal{I}_n, \mathcal{A}\mathcal{I}_n$, etc.

Pattern avoiding permutations were introduced by Knuth [20] in 1970 and first systematically studied by Simion-Schmidt [27]. The theory of pattern avoidance has been extensively exploited over particular subsets of permutations. For example, various results have been obtained for pattern avoiding alternating permutations (see e.g. [10, 15, 21, 22, 23, 24, 28, 35, 36]) and pattern avoiding involutions (see e.g. [4, 11, 12, 17, 18, 19]).

Barnabei-Bonetti-Castronuovo-Silimbani [5] initiated the enumeration of pattern avoiding alternating involutions [5]. They enumerated and characterized some classes of alternating involutions avoiding a single pattern of length 4. Furthermore, Barnabei-Bonetti-Castronuovo-Silimbani [5] posed several conjectures concerning pattern avoiding alternating involutions, which have been confirmed by Yan-Wang-

Zhou [38] and Zhou-Yan [39].

Given an integer n, a partition of n is a sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ such that $n = \lambda_1 + \lambda_2 + \dots + \lambda_k$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$. We denote $\lambda \vdash n$ or $|\lambda| = n$. Here k is called the length of λ , denoted by $\ell(\lambda)$. A Young diagram of shape λ is defined to be a left-justified array of n squares with λ_1 squares in the first row, λ_2 squares in the second row and so on. The conjugate of a partition λ , denoted by λ^T , is the partition whose Young diagram is the reflection along the main diagonal of λ 's Young diagram, and λ is said to be self-conjugate if $\lambda = \lambda^T$. In this paper, a square (i,j) in a Young diagram λ is referred to the square in the i-th column and j-th row of λ , where we number the rows from top to bottom, and the columns from left to right. In what follows, we always treat a partition λ and its Young diagram as identical.

Let μ and λ be two Young diagrams with $\mu \subseteq \lambda$ (i.e. $\mu_i \leq \lambda_i$ for all i) and $|\lambda| - |\mu| = n$. The skew diagram λ/μ is defined to be the diagram obtained from the Young diagram λ by removing the Young diagram μ at the top-left corner. In this context, n is said to be the size of λ/μ . In what follows, we treat λ/\emptyset and λ as identical. A standard Young tableau (SYT) T of (skew) shape λ/μ is a filling of the skew diagram λ/μ with the numbers $1, 2, \ldots, n$ such that the entries are increasing alone each rows and each columns. Figure 1 (left) illustrates an SYT of shape (4,3,2) and Figure 1 (right) illustrates an SYT of shape (5,4,2)/(2,1). Let $SYT(\lambda/\mu)$ denote the set of SYT's of shape λ/μ . Let $f^{\lambda/\mu}$ denote the cardinality of $SYT(\lambda/\mu)$. Note that $f^{\lambda/\mu}$ can be given by the famous hook-length formula.



Figure 1: Examples of standard Young tableaux.

For an SYT T of shape λ/μ with n entries, an index i $(1 \le i \le n-1)$ is said to be a descent of T if i+1 appears in a lower row of T than i, otherwise, i is said to be an ascent of T. An index i $(2 \le i \le n-1)$ is said to be a peak of T if i-1 is an ascent and i is a descent, whereas an index i $(2 \le i \le n-1)$ is said to be a valley of T if i-1 is a descent and i is an ascent. Let Peak(T) and Val(T) denote the set of peaks and the set of valleys of the SYT T, respectively. For example, let T be the SYT as shown in Figure 1 (left), we have Peak $(T) = \{4,6\}$ and Val $(T) = \{3,5,7\}$.

The Robinson-Schensted-Knuth algorithm (RSK algorithm) [29] sets up a bijection between symmetric group S_n and pairs (P,Q) of SYT's of the same shape $\lambda \vdash n$. We denote this correspondence by $\pi \xrightarrow{\text{RSK}} (P,Q)$, where $\pi \in S_n$. We call P the insertion tableau and Q the recording tableau of π . The shape of π , denoted by

 $\operatorname{sh}(\pi)$, is defined to be the shape of P (or Q). By the insertion rule of the RSK algorithm, it can be easily seen that π has the same peak set and the same valley set as its recording tableau Q.

For a set A, write $t^A = \prod_{i \in A} t_i$. Set $t^A = 1$ when $A = \emptyset$. Our first main result is concerned with the equidistribution of the set-valued statistics Peak and Val on the SYT's of the same shape.

Theorem 1.1 Let $n \ge 1$. Then for any skew diagram λ/μ of size n, the set-valued statistics Peak and Val are equidistributed over $SYT(\lambda/\mu)$, that is,

$$\sum_{T \in \mathrm{SYT}(\lambda/\mu)} t^{\mathrm{Peak}(T)} = \sum_{T \in \mathrm{SYT}(\lambda/\mu)} t^{\mathrm{Val}(T)}.$$

It turns out that Theorem 1.1 can be used to study the distribution of the setvalued statistics on pattern avoiding transversals. This is actually the motivation and original intention behind our writing of this article. The equiditribution of permutation statistics on pattern avoiding permutations and transversals has been investigated in recent literature, see [7, 8, 9, 13, 16, 37, 38, 39] and references therein.

A transversal T of shape λ is a 01-filling of the squares of the Young diagram λ such that each row and column contains exactly one 1. A permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ can be regarded as a transversal of the n by n square diagram, in which the square (i, π_i) is filled with a 1 for all $1 \le i \le n$ and all the other squares are filled with 0's. The transversal corresponding to the permutation π is also called the permutation matrix of π . Let \mathcal{T}_{λ} denote the set of transversals of the Young diagram λ and let \mathcal{T}_n denote the set of transversals of all Young diagrams with n columns. A transversal T of shape λ will be written as $T = t_1 t_2 \cdots t_n$ if the square (i, t_i) in T contains a 1 for $1 \le i \le n$ and other squares of T contains 0's. For example, Figure 2 illustrates the transversal T = 218976534 of shape $\lambda = (9, 9, 9, 9, 7, 7, 7, 4, 4)$ in which each \bullet represents a 1 and each empty square represents a 0.

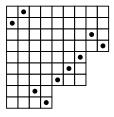


Figure 2: A transversal of the Young diagram $\lambda = (9, 9, 9, 9, 7, 7, 7, 4, 4)$.

A transversal $T = t_1 t_2 \cdots t_n$ of shape λ is said to be *symmetric* if λ is self-conjugate and the corresponding permutation $t_1 t_2 \cdots t_n$ is an involution. More precisely, in

a symmetric transversal T of a self-conjugate Young diagram λ , the square (i,j) contains a 1 if and only if the square (j,i) contains a 1. The transversal T as shown in Figure 2 gives an example of symmetric transversals. Let \mathcal{ST}_{λ} denote the set of symmetric transversals of the self-conjugate Young diagram λ and let \mathcal{ST}_n denote the set of symmetric transversals of all self-conjugate Young diagrams with n columns.

For a Young diagram λ , let $c_j(\lambda)$ denote the length of the j-th column of λ . Let T be a transversal of shape λ . An index i $(2 \le i \le n-1)$ with $c_{i-1}(\lambda) = c_i(\lambda) = c_{i+1}(\lambda)$ is said to be a peak (resp. valley) of T if $t_{i-1} < t_i > t_{i+1}$ (resp. $t_{i-1} > t_i < t_{i+1}$). Let Peak(T) and Val(T) denote the set of peaks and the set of valleys of T, respectively. It is easily seen that when restricted to permutations, the peaks (resp. valleys) of transversals coincide with the peaks (resp. valleys) of permutations.

The notion of pattern avoidance on permutations can be extended to transversals. Let T be a transversal of shape λ . We say T contains the pattern σ if there exist $R = \{r_1 < r_2 < \dots < r_k\}$ and $C = \{c_1 < c_2 < \dots < c_k\}$, such that each of the squares (c_i, r_j) falls within the board of λ and the matrix restricted on rows R and columns C is the permutation matrix of σ . Otherwise, we say T avoids σ and T is σ -avoiding. Let $T_n(\sigma)$ denote the set of σ -avoiding transversals in T_n . We will keep this notation also when T_n is replaced by other subsets of T_n . For example, the transversal as shown in Figure 2 is 4321-avoiding while it contains the pattern 2134.

Given two permutations $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathcal{S}_n$ and $\sigma = \sigma_1 \sigma_2 \cdots \sigma_m \in \mathcal{S}_m$, the direct sum of π and σ , denoted by $\pi \oplus \sigma$, is the permutation $\pi_1 \pi_2 \cdots \pi_n (\sigma_1 + n)(\sigma_2 + n) \cdots (\sigma_m + n)$. Let ϵ denote the empty permutation. Throughout the paper, we treat $\pi \oplus \epsilon$ and π as identical. Let $I_k = 12 \cdots k$ and $J_k = k \cdots 21$. Our second main result is concerned with the following equidistribution of set-valued statistics on pattern avoiding transversals.

Theorem 1.2 For any Young diagram λ and any positive integer k, we have

$$\sum_{T \in \mathcal{T}_{\lambda}(I_k)} t^{\operatorname{Peak}(T)} = \sum_{T \in \mathcal{T}_{\lambda}(I_k)} t^{\operatorname{Val}(T)} = \sum_{T \in \mathcal{T}_{\lambda}(J_k)} t^{\operatorname{Val}(T)} = \sum_{T \in \mathcal{T}_{\lambda}(J_k)} t^{\operatorname{Peak}(T)}.$$

Theorem 1.3 Let $k \geq 1$. For any Young diagram λ and any permutation τ , the set-valued statistic Peak is equidistributed over $\mathcal{T}_{\lambda}(I_k \oplus \tau)$ and $\mathcal{T}_{\lambda}(J_k \oplus \tau)$, that is,

$$\sum_{T \in \mathcal{T}_{\lambda}(I_k \oplus \tau)} t^{\operatorname{Peak}(T)} = \sum_{T \in \mathcal{T}_{\lambda}(J_k \oplus \tau)} t^{\operatorname{Peak}(T)}.$$

Setting $t_i = 1$ for all $i \ge 1$ in Theorem 1.3 recovers the result of Backelin-West-Xin [3] which states that $|\mathcal{T}_{\lambda}(I_k \oplus \tau)| = |\mathcal{T}_{\lambda}(J_k \oplus \tau)|$. Hence, Theorem 1.3 can be

viewed as a refinement of the result of Backelin-West-Xin. We remark that some examples show that the bijection between $\mathcal{T}_{\lambda}(I_k \oplus \tau)$ and $\mathcal{T}_{\lambda}(J_k \oplus \tau)$ established by Backelin-West-Xin does not preserve the peak set.

Recall that when we set λ to be the n by n square diagram, the transversal $T \in \mathcal{T}_{\lambda}$ becomes a permutation π in \mathcal{S}_n with $\operatorname{Peak}(\pi) = \operatorname{Peak}(T)$ and $\operatorname{Val}(\pi) = \operatorname{Val}(T)$. Then the following result follows directly from Theorem 1.3.

Corollary 1.4 Let $n, k \ge 1$. For any permutation τ , the peak set is equidistributed over $S_n(I_k \oplus \tau)$ and $S_n(J_k \oplus \tau)$.

For symmetric transversals, we have the following analogues of Theorems 1.2 and 1.3.

Theorem 1.5 Let $k \ge 1$. For any self-conjugate Young diagram λ , we have

$$\sum_{T \in \mathcal{ST}_{\lambda}(I_k)} t^{\operatorname{Peak}(T)} = \sum_{T \in \mathcal{ST}_{\lambda}(I_k)} t^{\operatorname{Val}(T)} = \sum_{T \in \mathcal{ST}_{\lambda}(J_k)} t^{\operatorname{Val}(T)} = \sum_{T \in \mathcal{ST}_{\lambda}(J_k)} t^{\operatorname{Peak}(T)}.$$

Theorem 1.6 Let $k \geq 1$. For any self-conjugate Young diagram λ and any pattern τ , the set-valued statistic Peak is equidistributed over $\mathcal{ST}_{\lambda}(I_k \oplus \tau)$ and $\mathcal{ST}_{\lambda}(J_k \oplus \tau)$, that is,

$$\sum_{T \in \mathcal{ST}_{\lambda}(I_k \oplus \tau)} t^{\operatorname{Peak}(T)} = \sum_{T \in \mathcal{ST}_{\lambda}(J_k \oplus \tau)} t^{\operatorname{Peak}(T)}.$$

Note that the case k = 3 of Theorem 1.6 has been verified by Yan-Wang-Zhou [38] by establishing a peak set preserving bijection between $\mathcal{ST}_{\lambda}(J_3 \oplus \tau)$ and $\mathcal{ST}_{\lambda}(I_3 \oplus \tau)$.

Setting $t_i = 1$ for all $i \ge 1$ in Theorem 1.6 recovers the result of Bousquet-Mélou and Steingrímsson [12] which states that $|\mathcal{ST}_{\lambda}(I_k \oplus \tau)| = |\mathcal{ST}_{\lambda}(J_k \oplus \tau)|$. Therefore, Theorem 1.6 can be viewed as a refinement of the result of Bousquet-Mélou and Steingrímsson. We remark that some examples show that the bijection between $\mathcal{ST}_{\lambda}(I_k \oplus \tau)$ and $\mathcal{ST}_{\lambda}(J_k \oplus \tau)$ established by Bousquet-Mélou and Steingrímsson does not preserve the peak set.

Recall that when we set λ to be the n by n square diagram, the transversal $T \in \mathcal{ST}_{\lambda}$ becomes an involution π in \mathcal{I}_n with $\operatorname{Peak}(\pi) = \operatorname{Peak}(T)$ and $\operatorname{Val}(\pi) = \operatorname{Val}(T)$. Hence, the following result follows directly from Theorem 1.6, confirming a recent conjecture posed by Yan-Wang-Zhou [38].

Corollary 1.7 ([38], Conjecture 4.1) Let $n, k \ge 1$. For any permutation τ , the set-valued statistic Peak is equidistributed over $\mathcal{I}_n(I_k \oplus \tau)$ and $\mathcal{I}_n(J_k \oplus \tau)$, that is,

$$\sum_{\pi \in \mathcal{I}_n(I_k \oplus \tau)} t^{\operatorname{Peak}(\pi)} = \sum_{\pi \in \mathcal{I}_n(J_k \oplus \tau)} t^{\operatorname{Peak}(\pi)}.$$

Recently, Yan-Wang-Zhou [38] proved that $|\mathcal{AI}_n(I_3 \oplus \tau)| = |\mathcal{AI}_n(J_3 \oplus \tau)|$ for any nonempty permutation τ as conjectured by Barnabei-Bonetti-Castronuovo-Silimbani [5]. In this paper, we shall obtain the following extension of the result of Yan-Wang-Zhou to general k.

Theorem 1.8 Let $n, k \ge 1$. For any nonempty permutation τ , we have

$$|\mathcal{AI}_n(I_k \oplus \tau)| = |\mathcal{AI}_n(J_k \oplus \tau)|.$$

Note that Backelin-West-Xin [3] proved that $|S_n(I_k \oplus \tau)| = |S_n(J_k \oplus \tau)|$, which has been extended to involutions by Bousquet-Mélou and Steingrímsson [12] and to alternating permutations by Yan [36]. Hence, Theorem 1.8 can be viewed as a parallel work of the above results.

The rest of the paper is organized as follows. Section 2 is devoted to the investigation of the equidistribution of the set-valued statistics Peak and Val on standard Young tableaux of a skew shape. By introducing Knuth transformations on standard Young tableaux, we prove Theorem 1.1. In Section 3, relying on Theorem 1.1, we shall investigate the distribution of Peak and Val on transversals, thereby proving Theorems 1.2, 1.3, 1.5, 1.6 and 1.8.

2 Peaks and valleys on standard Young tableaux

This section is devoted to the proof of Theorem 1.1. To this end, we shall associate each SYT with a permutation and show that all the permutations corresponding to the SYT's of a given skew shape can be divided into several Knuth equivalence classes.

Given an SYT T of shape λ/μ with n entries, we associate T with a word $y = y_1y_2\cdots y_n = \alpha(T)$ of length n, where y_i is the row index of the square of T containing the number i. The word y is known as $Yamanouchi\ word$. In this context, we say that y is a Yamanouchi word compatible to λ/μ .

For a word w, we denote $|w|_i$ to be the number of occurrences of i in w. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_\ell)$ be two partitions satisfying that $k \ge \ell$ and $\lambda_i \ge \mu_i$ for all i with the assumption $\mu_i = 0$ when $i > \ell$. In fact, a word $y = y_1 y_2 \cdots y_n$ is a Yamanouchi word compatible to λ/μ if and only if

- $y_j \in [k]$ for all $1 \le j \le n$;
- $|y|_i = \lambda_i \mu_i$ for all $1 \le i \le k$;

• for all $1 \le i \le k-1$ and $1 \le j \le n$, we have $|y^{(j)}|_i + \mu_i \ge |y^{(j)}|_{i+1} + \mu_{i+1}$, where $y^{(j)} = y_1 y_2 \cdots y_j$.

Denote by $\mathcal{Y}(\lambda/\mu)$ the set of Yamanouchi words compatible to the skew diagram λ/μ .

By the above analysis, it is straightforward to recover the corresponding SYT of shape λ/μ from its the Yamanouchi word y by letting the i-th row of T contain the indices of the letters of y which are equal to i. For example, the Yamanouchi word of the SYT T in Figure 1 (left) is given by 123121321 and the Yamanouchi word of the SYT S in Figure 1 (right) is given by 11223123.

For a Yamanouchi word $y = y_1y_2\cdots y_n$ of an SYT T, an index i $(2 \le i \le n-1)$ is said to be a *peak* of y if $y_{i-1} < y_i \ge y_{i+1}$, and i is said to be a *valley* of y if $y_{i-1} \ge y_i < y_{i+1}$. Let Peak(y) and Val(y) denote the set of peaks and the set of valleys of the Yamanouchi word y, respectively. The following result follows directly from the definition of the Yamanouchi word of an SYT.

Lemma 2.1 For any skew diagram λ/μ , the map α induces a bijection between $\operatorname{SYT}(\lambda/\mu)$ and $\mathcal{Y}(\lambda/\mu)$ such that for any $T \in \operatorname{SYT}(\lambda/\mu)$, we have $\operatorname{Peak}(T) = \operatorname{Val}(\alpha(T))$ and $\operatorname{Val}(T) = \operatorname{Peak}(\alpha(T))$.

Let λ/μ be a skew diagram of size n such that $\lambda_i - \mu_i = a_i$ for $1 \le i \le \ell(\lambda)$ with the convention $\mu_i = 0$ for $i > \ell(\mu)$. Given an SYT T of skew shape λ/μ , let $y = y_1 y_2 \cdots y_n$ be its corresponding Yamanouchi word. We associate T with a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n = \beta(T)$ of length n, where π is obtained from y be converting the a_i i's in y to the numbers $a_1 + a_2 + \cdots + a_{i-1} + 1$, $a_1 + a_2 + \cdots + a_{i-1} + 2, \ldots, a_1 + a_2 + \cdots + a_{i-1} + a_i$ from left to right in decreasing order. We call the permutation π the Yamanouchi permutation. It is easily seen that β gives an injective map from SYT(λ/μ) to \mathcal{S}_n . On the other hand, given a Yamanouchi permutation π , we can recover the Yamanouchi word $y \in \mathcal{Y}(\lambda/\mu)$ from π by converting $a_1 + a_2 + \cdots + a_{i-1} + 1$, $a_1 + a_2 + \cdots + a_{i-1} + 2$, ..., $a_1 + a_2 + \cdots + a_{i-1} + a_i$ in π to i, thereby recovering the SYT T of shape λ/μ from the Yamanouchi word y. For example, the Yamanouchi permutation of the SYT T in Figure 1 (left) is given by 479362851 and the Yamanouchi permutation of the SYT T in Figure 1 (right) is given by 32658147. One can easily check that Peak(y) = Peak(π) and Val(y) = Val(π). Combining Lemma 2.1, we immediately obtain the following result.

Lemma 2.2 Let T be a standard Young tableau and let π be its corresponding Yamanouchi permutation. Then we have $\operatorname{Peak}(T) = \operatorname{Val}(\pi)$ and $\operatorname{Val}(T) = \operatorname{Peak}(\pi)$.

We proceed to show that all the Yamanouchi permutations corresponding to the SYT's of a given skew shape can be divided into several Knuth equivalence classes. To this end, we need to define shape preserving transformations on SYT's, which turn out to be equivalent to the Knuth transformations on their corresponding Yamanouchi permutations. Hence we will also call the transformation the Knuth transformations on SYT's. Let us first recall the Knuth transformations on permutations.

The Knuth transformations on permutations

Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a permutation in S_n and let a < b < c be three adjacent numbers $\pi_{i-1} \pi_i \pi_{i+1}$ in the permutation π . A Knuth transformation κ_i of the permutation π is its transformation into another permutation σ that has one of the following four forms:

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(i) \pi = \cdots acb \cdots \longrightarrow \sigma = \cdots cab \cdots;
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(ii)
$$\pi = \cdots bca \cdots \longrightarrow \sigma = \cdots bac \cdots$$
;

(iii)
$$\pi = \cdots cab \cdots \longrightarrow \sigma = \cdots acb \cdots$$
;

(iv)
$$\pi = \cdots bac \cdots \longrightarrow \sigma = \cdots bca \cdots$$
.

Thus each Knuth transformation switches two adjacent entries a and c provided an entry b satisfying a < b < c is located next to a or c. From the definition of the transformations, one can easily observe that κ_i is well-defined if and only if i is a peak or a valley of the permutation π and κ_i changes a peak to a valley, and vise versa. Two permutations π and σ are said to be Knuth-equivalent, denoted by $\pi \stackrel{K}{\sim} \sigma$, if one of them can be obtained from another by a sequence of Knuth transformations. Let $[\pi]$ denote the Knuth equivalence class that contains the permutation π . For example, the five permutations as shown in Figure 3 form a Knuth equivalence class, where the ones that differ by a single Knuth transformation are connected by an edge.

$$32154 - 32514 - 35214 - 35241 - 32541$$

Figure 3: An example of Knuth equivalence class of permutations

Theorem 2.3 ([29], Theorem A1.1.4) Two permutations are Knuth-equivalent if and only if their insertion tableaux coincide.

Recall that a permutation π has the same peak set and the same valley set with its recording tableau Q. Then the following lemma follows directly from the Theorem 2.3 and RSK algorithm.

Lemma 2.4 Let π be a permutation of length n. Then we have

$$\sum_{\tau \in [\pi]} t^{\operatorname{Peak}(\tau)} = \sum_{Q \in \operatorname{SYT}(\operatorname{sh}(\pi))} t^{\operatorname{Peak}(Q)}$$

and

$$\sum_{\tau \in [\pi]} t^{\operatorname{Val}(\tau)} = \sum_{Q \in \operatorname{SYT}(\operatorname{sh}(\pi))} t^{\operatorname{Val}(Q)}.$$

The Knuth transformations on standard Young tableaux

Let T be an SYT with $n \geq 3$ entries and let i-1, i, i+1 be three entries in T. A Knuth transformation κ_i of the SYT T is its transformation into another SYT S that has one of the following four forms:

- (i) If $i \in Val(T)$ and i+1 appears in a lower row in T than i-1, then S is obtained from T by switching i-1 and i in T and preserving all the other entries;
- (ii) If $i \in Val(T)$ and i + 1 does not appear in a lower row in T than i 1, then S is obtained from T by switching i and i + 1 in T and preserving all the other entries;
- (iii) If $i \in \text{Peak}(T)$ and i+1 does not appear in a lower row in T than i-1, then S is obtained from T by switching i-1 and i in T and preserving all the other entries;
- (iv) If $i \in \text{Peak}(T)$ and i+1 appears in a lower row in T than i-1, then S is obtained from T by switching i and i+1 in T and preserving all the other entries.

We remark that one can distinguish from the context whether κ_i is defined on permutations or SYT's. Two SYT's T and S are said to be Knuth-equivalent, denoted by $T \overset{K}{\sim} S$, if one of them can be obtained from another by a sequence of Knuth transformations. Let [T] denote the Knuth equivalence class that contains the SYT T. For example, the five SYT's as shown in Figure 4 form a Knuth equivalence class, where the ones that differ by a single Knuth transformation are connected by an edge.

Figure 4: An example of Knuth equivalence class of SYT's

The following basic facts are fairly straightforward and we omit the detailed proofs here.

Fact #1 Let T be an SYT. Then the two modified consecutive entries of T under the Knuth transformation κ_i are always not in the same row or in the same column, which implies that $\kappa_i(T)$ is an SYT of the same shape with T.

Fact #2 The Knuth transformation κ_i is well-defined on T if and only if i is a peak or a valley of the SYT T and κ_i changes a peak to a valley, and vise versa. Moreover, κ_i is an involution, namely, $\kappa_i^2 = \mathrm{id}$, where id is the identity transformation.

Fact #3 Let T be an SYT of n entries with k and k+1 in different rows and columns and let S be the SYT obtained from T by switching k and k+1 in T and preserving other entries. Then we have that the Yamanouchi word $\alpha(S)$ of S is the word obtained from the Yamanouchi word $\alpha(T) = y_1 y_2 \cdots y_n$ of T by switching y_k and y_{k+1} . Since $y_k \neq y_{k+1}$, we deduce that the Yamanouchi permutation $\beta(S)$ of S is the permutation obtained from the Yamanouchi permutation $\beta(T) = \pi_1 \pi_2 \cdots \pi_n$ of T by switching π_k and π_{k+1} .

Lemma 2.5 Let T be a standard Young tableau of skew shape λ/μ and $i \in \operatorname{Peak}(T) \cup \operatorname{Val}(T)$. Then we have

$$\beta \circ \kappa_i(T) = \kappa_i \circ \beta(T).$$

Proof. Let $\pi = \beta(T)$, $S = \kappa_i(T)$ and $\sigma = \kappa_i(\pi)$. For a better view, we outlines the sets and relationships as shown in Figure 5. We need to show that $\sigma = \beta(S)$. We shall consider four cases according to the four forms of the Knuth transformations on SYT's.

Case (i): If $i \in Val(T)$ and i + 1 appears in a lower row in T than i - 1, then S is obtained from T by switching i - 1 and i in T and preserving all the other entries. Fact #1 tells us that i - 1 and i are in different rows and columns of T. Then by Fact #3, $\beta(S)$ is the permutation obtained from π by switching π_{i-1} and π_i in π . On the other hand, as $i \in Val(T)$, then from Lemma 2.2, we have $i \in Peak(\pi)$. Note that i + 1 appears in a lower row in T than i - 1. This implies that $\pi_{i-1} < \pi_{i+1}$. Thus $\pi_{i-1}\pi_i\pi_{i+1} = acb$ for some a < b < c. Then by the definition of κ_i , σ is obtained from π by switching π_{i-1} and π_i in π , namely, $\sigma = \beta(S)$.

Case (ii): If $i \in Val(T)$ and i + 1 does not appear in a lower row in T than i - 1, then S is obtained from T by switching i and i + 1 in T and preserving all the other entries. Fact #1 tells us that i and i + 1 are in different rows and columns of T. Then by Fact #3, $\beta(S)$ is the permutation obtained from π by switching π_i and π_{i+1} in π . On the other hand, as $i \in Val(T)$, then from Lemma 2.2, we have $i \in Peak(\pi)$. Note that i + 1 does not appear in a lower row in T than i - 1. This implies that $\pi_{i-1} > \pi_{i+1}$. Thus $\pi_{i-1}\pi_i\pi_{i+1} = bca$ for some a < b < c. Then by the definition of κ_i , σ is obtained from π by switching π_i and π_{i+1} in π , namely, $\sigma = \beta(S)$.

Case (iii): If $i \in \text{Peak}(T)$ and i + 1 does not appear in a lower row in T than i - 1, then S is obtained from T by switching i - 1 and i in T and preserving all the other entries. Fact #1 tells us that i - 1 and i are in different rows and columns of T. Then by Fact #3, $\beta(S)$ is the permutation obtained from π by switching π_{i-1} and π_i in π . On the other hand, as $i \in \text{Peak}(T)$, then from Lemma 2.2, we have $i \in \text{Val}(\pi)$. Note that i + 1 does not appear in a lower row in T than i - 1. We derive that $\pi_{i-1} > \pi_{i+1}$. Thus $\pi_{i-1}\pi_i\pi_{i+1} = cab$ for some a < b < c. Then by the definition of κ_i , σ is obtained from π by switching π_{i-1} and π_i in π , namely, $\sigma = \beta(S)$.

Case (iv): If $i \in \text{Peak}(T)$ and i+1 appears in a lower row in T than i-1, then S is obtained from T by switching i and i+1 in T and preserving all the other entries. Fact #1 tells us that i and i+1 are in different rows and columns of T. Then by Fact #3, $\beta(S)$ is the permutation obtained from π by switching π_i and π_{i+1} in π . On the other hand, as $i \in \text{Peak}(T)$, then from Lemma 2.2, we have $i \in \text{Val}(\pi)$. Note that i+1 appears in a lower row in T than i-1. This yields that $\pi_{i-1} < \pi_{i+1}$. Thus $\pi_{i-1}\pi_i\pi_{i+1} = bac$ for some a < b < c. Then by the definition of κ_i , σ is obtained from π by switching π_i and π_{i+1} in π , namely, $\sigma = \beta(S)$.

So far, we have concluded that $\sigma = \beta(S)$ for any case, completing the proof.

$$T \xrightarrow{\kappa_i} S$$

$$\beta \downarrow \qquad \qquad \downarrow \beta$$

$$\pi \xleftarrow{\kappa_i} \sigma$$

Figure 5: The commutativity of the maps β and κ_i .

Lemma 2.6 Let T and S be two standard Young tableaux of the shape λ/μ of size n and let π and σ be the corresponding Yamanouchi permutations of T and S, respectively. Then we have

$$T \stackrel{K}{\sim} S \Leftrightarrow \pi \stackrel{K}{\sim} \sigma.$$

Proof. If $T \stackrel{K}{\sim} S$, then there exists a sequence of Knuth transformations $\kappa_{i_1}, \kappa_{i_2}, \ldots, \kappa_{i_j}$ such that $S = \kappa_{i_j} \circ \cdots \circ \kappa_{i_2} \circ \kappa_{i_1}(T)$. By Lemma 2.5, we have $\beta(S) = \kappa_{i_j} \circ \cdots \circ \kappa_{i_2} \circ \kappa_{i_1} \circ \beta(T)$, that is, $\sigma = \kappa_{i_j} \circ \cdots \circ \kappa_{i_2} \circ \kappa_{i_1}(\pi)$. Hence $\pi \stackrel{K}{\sim} \sigma$. See Figure 6 for a better view.

Conversely, if $\pi \overset{K}{\sim} \sigma$, then there exists a sequence of Knuth transformations $\kappa_{i_1}, \kappa_{i_2}, \ldots, \kappa_{i_j}$ such that $\sigma = \kappa_{i_j} \circ \cdots \circ \kappa_{i_2} \circ \kappa_{i_1}(\pi)$. Let $S' = \kappa_{i_j} \circ \cdots \circ \kappa_{i_2} \circ \kappa_{i_1}(T)$. Then we have $\beta(S') = \kappa_{i_j} \circ \cdots \circ \kappa_{i_2} \circ \kappa_{i_1}(\pi) = \sigma$. By the definition of the Knuth transformations on SYT's, S' is of shape λ/μ . Since $\beta(S) = \sigma$ and β is an injection from SYT(λ/μ) to S_n , we have S' = S. Then $T \overset{K}{\sim} S$ follows directly.

$$T \stackrel{\kappa_{i_1}}{\longleftrightarrow} T_1 \stackrel{\kappa_{i_2}}{\longleftrightarrow} T_2 \quad \cdots \quad T_{j-1} \stackrel{\kappa_{i_j}}{\longleftrightarrow} S$$

$$\beta \downarrow \qquad \beta \downarrow \qquad \beta \downarrow \qquad \beta \downarrow \qquad \beta \downarrow$$

$$\pi \stackrel{\kappa_{i_1}}{\longleftrightarrow} \pi^1 \stackrel{\kappa_{i_2}}{\longleftrightarrow} \pi^2 \quad \cdots \qquad \pi^{j-1} \stackrel{\kappa_{i_j}}{\longleftrightarrow} \sigma$$

Figure 6: A better view for Lemma 2.6.

Actually, the proof of Lemma 2.6 enables us to obtain the following lemma directly.

Lemma 2.7 Let T be a standard Young tableau of skew shape λ/μ and let π be the Yamanouchi permutation of T. Then the map β induces a bijection between the sets [T] and $[\pi]$.

The following lemma plays an essential role in the proof of Theorem 1.1.

Lemma 2.8 If Peak and Val are equidistributed over $SYT(\nu)$ for any Young diagram ν of size n, then Peak and Val are also equidistributed over $SYT(\lambda/\mu)$ for any skew diagram λ/μ of size n.

Proof. Given any skew shape λ/μ of size n, assume that $\mathrm{SYT}(\lambda/\mu)$ can be divided into k Knuth equivalence classes $[T_1], [T_2], \ldots, [T_k]$. Let π^i be the corresponding Yamanouchi permutations of T_i and let v^i be the shape of π^i for $1 \le i \le k$. Then we

have

$$\begin{split} \sum_{T \in \mathrm{SYT}(\lambda/\mu)} t^{\mathrm{Peak}(T)} &= \sum_{i=1}^k \sum_{T \in [T_i]} t^{\mathrm{Peak}(T)} \\ &= \sum_{i=1}^k \sum_{\tau \in [\pi^i]} t^{\mathrm{Val}(\tau)} \quad \text{(by Lemmas 2.2 and 2.7)} \\ &= \sum_{i=1}^k \sum_{Q \in \mathrm{SYT}(\upsilon^i)} t^{\mathrm{Val}(Q)} \quad \text{(by Lemma 2.4)} \\ &= \sum_{i=1}^k \sum_{Q \in \mathrm{SYT}(\upsilon^i)} t^{\mathrm{Peak}(Q)} \quad \text{(by the hypothesis)} \\ &= \sum_{i=1}^k \sum_{\tau \in [\pi^i]} t^{\mathrm{Peak}(\tau)} \quad \text{(by Lemma 2.4)} \\ &= \sum_{i=1}^k \sum_{T \in [T_i]} t^{\mathrm{Val}(T)} \quad \text{(by Lemmas 2.2 and 2.7)} \\ &= \sum_{T \in \mathrm{SYT}(\lambda/\mu)} t^{\mathrm{Val}(T)}. \end{split}$$

This completes the proof.

For sets of positive integers, we define a *lexicographic order* on them, denoted by \leq . Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_r\}$ be two sets of positive integers where $a_1 < a_2 < \cdots < a_k$ and $b_1 < b_2 < \cdots < b_r$. Throughout the paper, we always list a set of positive integers in increasing order. Define $A \leq B$ if either A = B, or else for some i,

$$a_1 = b_1, a_2 = b_2, \dots, a_i = b_i, a_{i+1} < b_{i+1},$$

with the convention that $a_i = 0$ if i > k and $b_i = 0$ if i > r.

Given a skew diagram λ/μ , define the multisets

$$\mathcal{P}(\lambda/\mu) = \{ \text{Peak}(T) \mid T \in \text{SYT}(\lambda/\mu) \}$$

and

$$V(\lambda/\mu) = {\operatorname{Val}(T) \mid T \in \operatorname{SYT}(\lambda/\mu)}.$$

For example, we have

where the elements are listed in lexicographical order. See Table 1 for details.

T	Peak(T)	Val(T)	T	Peak(T)	Val(T)
1 3 2 4 5	{3}	$\{2,4\}$	2 3 4 1 5	{3}	Ø
1 4 2 3 5	{4}	{3}	2 4 3 1 5	$\{2,4\}$	{3}
1 5 2 3 4	Ø	{3}	2 5 3 1 4	{2}	{4}
1 2 4 3 5	$\{2, 4\}$	{3}	1 3 4 2 5	{3}	{2}
1 2 3 4 5	{2}	{4}	1 4 3 2 5	{4}	{2}
1 5	6.3				

Table 1: Equidistribution of Peak and Val over SYT((3,2,2)/(1,1)).

Let $M = \{M_1, M_2, \dots, M_k\}$ be a multiset in which each element M_i is a set of positive integers and let A be a set of positive integers. Define

$$M[A] = \{M_1 \cap A, M_2 \cap A, \dots, M_k \cap A\}.$$

We call M[A] the restriction of M into A. For example, we have

$$\mathcal{P}((3,2,2)/(1,1))[\{2,3\}] = \{\emptyset,\emptyset,\emptyset,\{2\},\{2\},\{2\},\{2\},\{3\},\{3\},\{3\},\{3\}\}.$$

For positive integers a, b with $a \le b$, we denote by [a, b] the set of all integers j with $a \le j \le b$. For a set A of positive integers and an positive integer k, we denote by A + k the set obtained from A by adding k to each element of A. Similarly, for an SYT T and a positive integer k, we denote by T + k the tableau obtained from T by increasing each entry in T by k. The following lemma will play an essential role in the proof of Theorem 1.1.

Lemma 2.9 Let n be a positive integer and let A be any subset of [n] such that $k, k+1 \notin A$ for some positive integer k $(1 \le k \le n-1)$. If Peak and Val are equidistributed over $\operatorname{SYT}(\lambda'/\mu')$ for any skew diagram λ'/μ' of size less than n, then for any skew diagram λ/μ of size n, we have

$$\mathcal{P}(\lambda/\mu)[A] = \mathcal{V}(\lambda/\mu)[A].$$

Proof. Let $B = [n] \setminus [k, k+1]$. Note that $k, k+1 \notin A$, we have $A \subseteq B$. One can easily verify that M[A] = M[B][A] for any multiset M. Hence it is sufficient to prove $\mathcal{P}(\lambda/\mu)[B] = \mathcal{V}(\lambda/\mu)[B]$.

We proceed to construct a bijection $\omega: \operatorname{SYT}(\lambda/\mu) \to \operatorname{SYT}(\lambda/\mu)$. For an SYT $T \in \operatorname{SYT}(\lambda/\mu)$, let P be the SYT which is obtained from T by reading the entries in [k]. And let Q be the SYT which is obtained from T by reading the entries in [k+1,n] and decreasing each entry by k. Assume that Q is of the skew shape λ/μ' . Then the shape of P is given by μ'/μ . By the hypothesis, there exist a shape preserving bijection, say $\omega_1:\operatorname{SYT}(\mu'/\mu)\to\operatorname{SYT}(\mu'/\mu)$, such that $\operatorname{Peak}(R)=\operatorname{Val}(\omega_1(R))$ for any $R\in\operatorname{SYT}(\mu'/\mu)$ and a shape preserving bijection, say $\omega_2:\operatorname{SYT}(\lambda/\mu')\to\operatorname{SYT}(\lambda/\mu')$, such that $\operatorname{Peak}(R)=\operatorname{Val}(\omega_2(R))$ for any $R\in\operatorname{SYT}(\lambda/\mu')$. Then $\omega(T)$ is defined to be the SYT by merging $\omega_1(P)$ and $\omega_2(Q)+k$. See Figure 7 for an illustration. Since both ω_1 and ω_2 are shape preserving bijections, ω is a bijection. It is easily checked that $\operatorname{Peak}(T)\cap B=\operatorname{Peak}(P)\cup(\operatorname{Peak}(Q)+k)=\operatorname{Val}(\omega_1(P))\cup(\operatorname{Val}(\omega_2(Q))+k)=\operatorname{Val}(\omega(T))\cap B$. To conclude, we construct a bijection ω from $\operatorname{SYT}(\lambda/\mu)$ to itself such that $\operatorname{Peak}(T)\cap B=\operatorname{Val}(\omega(T))\cap B$ for any $T\in\operatorname{SYT}(\lambda/\mu)$. Thus $\mathcal{P}(\lambda/\mu)[B]=\mathcal{V}(\lambda/\mu)[B]$, completing the proof.

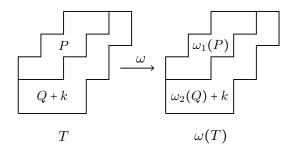


Figure 7: A better view for the bijection ω .

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be an integer partition. The rank of λ , denoted by $rank(\lambda)$, is defined to be the largest i for which $\lambda_i \geq i$. Equivalently, $rank(\lambda)$ is the length of the main diagonal in the Young diagram of λ .

Lemma 2.10 Let λ be a Young diagram of size n.

- (i) If $\operatorname{rank}(\lambda) \geq 2$, then there does not exist $T \in \operatorname{SYT}(\lambda)$ with $\operatorname{Peak}(T) = \emptyset$ and there does not exist $T \in \operatorname{SYT}(\lambda)$ with $\operatorname{Val}(T) = \emptyset$.
- (ii) If $\operatorname{rank}(\lambda) = 1$, then there exists exactly one $T \in \operatorname{SYT}(\lambda)$ with $\operatorname{Peak}(T) = \emptyset$ and there exists exactly one $T \in \operatorname{SYT}(\lambda)$ with $\operatorname{Val}(T) = \emptyset$.

Proof. Assume that $\operatorname{rank}(\lambda) \geq 2$. We proceed to show that there does not exist $T \in \operatorname{SYT}(\lambda)$ with $\operatorname{Peak}(T) = \emptyset$. If not, let T be the SYT in $\operatorname{SYT}(\lambda)$ such that $\operatorname{Peak}(T) = \emptyset$ and let $y = y_1 y_2 \cdots y_n$ be the corresponding Yamanouchi word of T. Then by Lemma 2.1, we have $y \in \mathcal{Y}(\lambda)$ with $\operatorname{Val}(y) = \emptyset$. Since $\operatorname{rank}(\lambda) \geq 2$, there exist at least two occurrences of 1 and at least two occurrences of 2 in y. If follows that there exists some i such that $y_i \geq y_{i+1}$. Let i be the smallest such integer. As $y \in \mathcal{Y}(\lambda)$, we deduce that $y_j = j$ for all $1 \leq j \leq i$ and $y_{i+1} = 1$. Note that y contains at least two occurrences of 2. It yields that there exists some $k \geq i+1$ such that $y_k < y_{k+1}$. Let k be the smallest such integer. Then we have $y_{k-1} \geq y_k < y_{k+1}$, namely, $k \in \operatorname{Val}(y)$, a contradiction with the fact $\operatorname{Val}(y) = \emptyset$. Hence there does not exist $T \in \operatorname{SYT}(\lambda)$ with $\operatorname{Peak}(T) = \emptyset$.

Now we prove that there does not exist $T \in \operatorname{SYT}(\lambda)$ with $\operatorname{Val}(T) = \emptyset$. If not, let T be the SYT in SYT(λ) such that $\operatorname{Val}(T) = \emptyset$ and let $y = y_1 y_2 \cdots y_n$ be the corresponding Yamanouchi word of T. Then by Lemma 2.1, we have $y \in \mathcal{Y}(\lambda)$ with $\operatorname{Peak}(y) = \emptyset$. Note that $y_1 = 1$ and y contains at least two occurrences of 2. If follows that there exists some i such that $y_i < y_{i+1}$. Let i be the smallest such integer. As $y \in \mathcal{Y}(\lambda)$, we deduce that $y_j = 1$ for all $1 \le j \le i$ and $y_{i+1} = 2$. Since y contains at least two occurrences of 2, there exists some $k \ge i + 1$ such that $y_k \ge y_{k+1}$. Let k be the smallest such integer. Then we have $y_{k-1} < y_k \ge y_{k+1}$, namely, $k \in \operatorname{Peak}(y)$, a contradiction with the fact $\operatorname{Peak}(y) = \emptyset$. Hence there does not exist $T \in \operatorname{SYT}(\lambda)$ with $\operatorname{Val}(T) = \emptyset$, completing the proof of (i).

Now we proceed to prove (ii). Assume that $\operatorname{rank}(\lambda) = 1$. By similar arguments as the proof of (i), one can easily derive that there exists exactly one Yamanouchi word of the form $y = 12 \cdots k11 \cdots 1 \in \mathcal{Y}(\lambda)$ with $\operatorname{Val}(y) = \emptyset$ and there exists exactly one Yamanouchi word of the form $y = 11 \cdots 12 \cdots k \in \mathcal{Y}(\lambda)$ with $\operatorname{Peak}(y) = \emptyset$. Then (ii) follows directly from Lemma 2.1.

Now we are ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.8, it is sufficient to prove that Peak and Val are equidistributed over $\operatorname{SYT}(\lambda)$ for any Young diagram λ of size n. We prove the assertion by induction on n. It is routine to check the assertion for $n \leq 3$. Assume that the assertion holds for all m < n with $n \geq 4$. We proceed to prove $\mathcal{P}(\lambda) = \mathcal{V}(\lambda)$ for any Young diagram λ of size n.

If we let A = [1, n-2], then by the induction hypothesis, Lemma 2.8 and Lemma 2.9, we have $\mathcal{P}(\lambda)[A] = \mathcal{V}(\lambda)[A]$. Let $\mathcal{P} = \mathcal{P}(\lambda) = \{P_1, P_2, \dots, P_t\}$ where $t = f^{\lambda}$ and P_i $(1 \le i \le t)$ are sorted according to the following rules:

- $P_1 \cap A \leq P_2 \cap A \leq \cdots \leq P_t \cap A$;
- for j > i, if $P_i \cap A = P_j \cap A$ and $n 1 \in P_i$, then $n 1 \in P_j$.

Similarly, let $\mathcal{V} = \mathcal{V}(\lambda) = \{V_1, V_2, \dots, V_t\}$ where V_i $(1 \le i \le t)$ are sorted according to the following rules:

- $V_1 \cap A \leq V_2 \cap A \leq \cdots \leq V_t \cap A$;
- for j > i, if $V_i \cap A = V_j \cap A$ and $n 1 \in V_i$, then $n 1 \in V_j$.

It follows that $P_i \cap A = V_i \cap A$ for $1 \le i \le t$. Combining the fact that $P_i, V_i \subseteq [2, n-1]$, we have either $P_i = V_i$ or P_i and V_i differ by one element n-1.

We shall prove that $P_i = V_i$ for $1 \le i \le t$. Assume on the contrary there exists some i such that $P_i \ne V_i$. Choose i to be the smallest such integer. Since P_i and V_i differ by one element n-1, without loss of generality, assume that $P_i = V_i \cup \{n-1\}$ and $n-1 \notin V_i$.

Claim 1. $V_i \neq \emptyset$.

If not, then we have $V_i = \emptyset$ and $P_i = \{n-1\}$. Notice that $P_i \cap A = \emptyset$. By the ordering rules of the elements in \mathcal{P} , it is easily checked that $P_j \neq \emptyset$ for all $j \geq i$. Note that $P_j = V_j$ for all j < i. One can easily check that \mathcal{V} contains more empty sets than \mathcal{P} , which contradicts Lemma 2.10. Hence we have $V_i \neq \emptyset$.

By Claim 1, we assume that $V_i = \{b_1, b_2, \dots, b_k\}$ with $2 \le b_1 < b_2 < \dots < b_k < n-1$. Claim 2. $b_1 = 2$.

If not, then $P_i \cap B = \{b_1, b_2, \dots, b_k, n-1\}$ and $V_i \cap B = \{b_1, b_2, \dots, b_k\}$, where B = [3, n-1]. Again by the induction hypothesis, Lemma 2.8 and Lemma 2.9, we have $\mathcal{P}[B] = \mathcal{V}[B]$. We assert that $P_j \cap B \neq \{b_1, b_2, \dots, b_k\}$ for j > i. If not, we have either $P_j = \{b_1, b_2, \dots, b_k\}$ or $P_j = \{2, b_1, b_2, \dots, b_k\}$. In both cases, we have $P_j \cap A = P_j$. It yields a contradiction with the ordering rules of the elements in \mathcal{P} as $P_i \cap A = \{b_1, b_2, \dots, b_k\}$ and $b_1 \geq 3$. Hence the assertion holds. Recall that $P_j = V_j$ for all j < i. Hence $\mathcal{V}[B]$ contains more $\{b_1, b_2, \dots, b_k\}$'s than $\mathcal{P}[B]$, a contradiction. Hence, the claim is proved.

By Claim 2, there exists at least one integer ℓ $(1 \le \ell \le k)$ such that $b_j = 2j$ for all $j \le \ell$. Choose ℓ to be the largest such integer.

Claim 3. $\ell < k$.

If not, we have $\ell = k$ and $b_k = 2k$, implying that $P_i = \{2, 4, ..., 2k, n-1\}$ and $V_i = \{2, 4, ..., 2k\}$. Let $C = [2, n-1] \setminus [2k-1, 2k]$. Again by the induction hypothesis, Lemma 2.8 and Lemma 2.9, we have $\mathcal{P}[C] = \mathcal{V}[C]$. We consider two cases for k. Case 1. k > 1.

It is easily seen that $P_i \cap C = \{2, 4, ..., 2k - 2, n - 1\}$ and $V_i \cap C = \{2, 4, ..., 2k - 2\}$. We assert that $P_j \cap C \neq \{2, 4, ..., 2k - 2\}$ for j > i. If not, we have either $P_j = \{2, 4, ..., 2k - 2\}$ or $P_j = \{2, 4, ..., 2k - 2, 2k\}$. For both cases of P_j , it will lead to a contradiction with the ordering rules of the elements in \mathcal{P} . Hence the assertion

holds. Recall that $P_j = V_j$ for all j < i. Hence $\mathcal{V}[C]$ contains more $\{2, 4, \dots, 2k-2\}$'s than $\mathcal{P}[C]$, a contradiction.

Case 2. k = 1.

Then we have $P_i = \{2, n-1\}$ and $V_i = \{2\}$. It yields that $P_i \cap C = \{n-1\}$ and $V_i \cap C = \emptyset$. Similarly, by the ordering rules of the elements in \mathcal{P} , we derive that $P_j \cap C \neq \emptyset$ for j > i. Recall that $P_j = V_j$ for all j < i. Hence $\mathcal{V}[C]$ contains more empty sets than $\mathcal{P}[C]$, a contradiction. Hence the claim is proved.

Claim 4. $b_{\ell+1} = 2\ell + 2$.

If not, we have $b_{\ell+1} > 2\ell + 2$ and $n-1 > 2\ell + 4$. Let $D = [2, n-1] \setminus [2\ell + 1, 2\ell + 2]$. Again by the induction hypothesis, Lemma 2.8 and Lemma 2.9, we have $\mathcal{P}[D] = \mathcal{V}[D]$. One can easily check that $P_i \cap D = P_i = \{2, 4, \dots, 2\ell, b_{l+1}, \dots, b_k, n-1\}$ and $V_i \cap D = V_i = \{2, 4, \dots, 2\ell, b_{l+1}, \dots, b_k\}$. Again by the ordering rules of the elements in \mathcal{P} , we derive that $P_j \cap D \neq V_i = \{2, 4, \dots, 2\ell, b_{l+1}, \dots, b_k\}$ for j > i. Recall that $P_j = V_j$ for all j < i. This implies that $\mathcal{V}[D]$ contains more $\{2, 4, \dots, 2\ell, b_{l+1}, \dots, b_k\}$'s than $\mathcal{P}[D]$, a contradiction. Hence, we have $b_{\ell+1} = 2\ell + 2$.

Combining Claims 3 and 4, we have $b_j = 2j$ for all $j \le \ell + 1$ and $\ell + 1 \le k$. This yields a contradiction with the choice of ℓ . Hence, we have concluded that $P_i = V_i$ for $1 \le i \le t$ as desired, completing the proof.

We remark that the joint distribution of Peak and Val is in general not symmetric over $SYT(\lambda/\mu)$. For example, we have

$$\sum_{T \in \mathrm{SYT}((3,2,2)/(1,1))} t^{\mathrm{Peak}(T)} q^{\mathrm{Val}(T)} = 2t_3q_2q_4 + 2t_2t_4q_3 + t_4q_3 + q_3 + t_3 + 2t_2q_4 + t_3q_2 + t_4q_2.$$

The following theorem follows directly from Theorem 1.1 and Lemma 2.1.

Theorem 2.11 For a skew diagram λ/μ of size n, Peak and Val are equidistributed over $\mathcal{Y}(\lambda/\mu)$, that is,

$$\sum_{y \in \mathcal{Y}(\lambda/\mu)} t^{\mathrm{Peak}(y)} = \sum_{y \in \mathcal{Y}(\lambda/\mu)} t^{\mathrm{Val}(y)}.$$

We conclude this section with the equidistribution of the peak set and the valley set on permutations and involutions with given shape. The following theorem follows directly from Theorem 1.1 and Lemma 2.4.

Theorem 2.12 Let π be a permutation of length n. Then Peak and Val are equidistributed over $[\pi]$, namely,

$$\sum_{\tau \in \lceil \pi \rceil} t^{\operatorname{Peak}(\tau)} = \sum_{\tau \in \lceil \pi \rceil} t^{\operatorname{Val}(\tau)}.$$

From Theorem 2.3, we deduce that the set of permutations with a given shape λ can be divided into f^{λ} Knuth equivalence classes, in which any two different Knuth equivalence classes have the same distribution of peak set (or valley set). Then the following theorem follows directly.

Theorem 2.13 Let λ be a Young diagram of size n and let π be any permutation of shape λ . Then Peak and Val are equidistributed over the set of permutations of shape λ , namely,

$$\sum_{\tau \in \mathcal{S}_n : \operatorname{sh}(\tau) = \lambda} t^{\operatorname{Peak}(\tau)} = \sum_{\tau \in \mathcal{S}_n : \operatorname{sh}(\tau) = \lambda} t^{\operatorname{Val}(\tau)} = f^{\lambda} \sum_{\tau \in [\pi]} t^{\operatorname{Peak}(\tau)} = f^{\lambda} \sum_{\tau \in [\pi]} t^{\operatorname{Val}(\tau)}.$$

Notice that $\pi \xrightarrow{\text{RSK}} (P, Q)$ if only if $\pi^{-1} \xrightarrow{\text{RSK}} (Q, P)$. It follows that RSK algorithm associates an involution π with an SYT T such that $\text{Peak}(\pi) = \text{Peak}(T)$ and $\text{Val}(\pi) = \text{Val}(T)$. Hence the following theorem follows directly from Theorem 1.1.

Theorem 2.14 Let λ be a Young diagram of size n. Then Peak and Val are equidistributed over the set of involutions of shape λ , namely,

$$\sum_{\tau \in \mathcal{I}_n : \operatorname{sh}(\tau) = \lambda} t^{\operatorname{Peak}(\tau)} = \sum_{\tau \in \mathcal{I}_n : \operatorname{sh}(\tau) = \lambda} t^{\operatorname{Val}(\tau)}.$$

3 Peaks and valleys on transversals

In this section, we aim to investigate the distribution of the set-valued statistics Peak and Val on transversals, thereby proving Theorems 1.2, 1.3, 1.5, 1.6 and 1.8. To this end, we shall establish an involution Φ on \mathcal{T}_n and a bijection Ψ from \mathcal{T}_n to itself via matchings and oscillating tableaux as intermediate structures. The involution Φ is achieved by employing the conjugate map γ on $\mathcal{O}(n)$, while the bijection Ψ is accomplished by constructing a bijection ψ from $\mathcal{O}(n)$ to itself based on Theorem 1.1. Figure 8 outlines our sets and maps of interest.

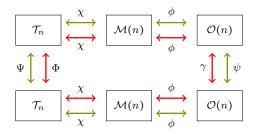


Figure 8: A diagrammatic summary of the sets and bijections.

3.1 The bijection χ from transversals to matchings

A (perfect) matching M of [2n] is a collection of n pairs $\{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\}$ with $i_k < j_k$ $(1 \le k \le n)$ such that each number of [2n] appears exactly once. A pair (i_k, j_k) $(1 \le k \le n)$ is called an arc of M, where i_k is called an opener while j_k is called a closer. Let $\mathcal{M}(n)$ denote the set of matchings of [2n]. Given a matching M of [2n], it can be represented by a graph G with the vertex set [2n] whose edge set consists of arcs of M. We usually draw the vertices of G on a horizontal line in increasing order and draw the arcs of G above the horizontal line. Such a graph is called the linear representation of the matching M. For example, Figure 9 is the linear representation of the matching $\{(1,3),(2,5),(4,6)\}$.

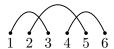


Figure 9: The linear representation of the matching $\{(1,3),(2,5),(4,6)\}$.

A k-crossing of a matching M is a k-subset $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ of the arcs of M such that $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$. Similarly, a k-nesting is a k-subset $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ of the arcs of M such that $i_1 < i_2 < \cdots < i_k < j_k < \cdots < j_2 < j_1$. A matching without any k-crossing is called k-noncrossing matching and a matching without any k-nesting is called k-nonnesting matching. Let $\mathcal{CM}_k(n)$ and $\mathcal{NM}_k(n)$ denote the sets of k-noncrossing and k-nonnesting matchings of [2n], respectively.

Given a matching $M = \{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\}$ of [2n], define $M^r = \{(2n + 1 - j_1, 2n + 1 - i_1), (2n + 1 - j_2, 2n + 1 - i_2), \dots, (2n + 1 - j_n, 2n + 1 - i_n)\}$. We call M^r the reverse of the matching M. If $M = M^r$, we say M is bilaterally symmetric. It can be easily seen that M is bilaterally symmetric if and only if its linear representation is symmetric along the vertical line $x = \frac{2n+1}{2}$. Figure 9 illustrates a bilaterally symmetric matchings of [2n]. Denote by $\mathcal{SCM}_k(n)$ denote the set of bilaterally symmetric matchings of [2n]. Denote by $\mathcal{SCM}_k(n)$ and $\mathcal{SNM}_k(n)$ the sets of k-noncrossing and k-nonnesting bilaterally symmetric matchings of $\mathcal{SM}(n)$, respectively.

Given a matching $M \in \mathcal{M}(n)$ which contains three arcs (i-1,a), (i,b), (i+1,c), the index i is said to be a peak of M if a < b > c, whereas i is said to be a valley of M if a > b < c. Denote by Peak(M) and Val(M) the set of peaks and the set of valleys of M, respectively. For example, let M be the matching whose linear representation is shown in Figure 10 (right). Then we have $Peak(M) = \{3\}$ and $Val(M) = \{2, 13\}$. The type of M, denoted by type(M), is the sequence obtained from M by tracing

from 1 to 2n and writing u (resp. d) whenever we encounter an opener (resp. a closer).

Let T be a transversal in \mathcal{T}_{λ} . The type of T, denoted by type(T), is the sequence obtained from T by tracing the south-east border of λ from south-west to north-east and writing u (resp. d) whenever we encounter a east step (resp. a north step). Notice that the type of a transversal T is uniquely determined by its shape λ .

In the following, we give a description of the bijection χ from transversals to matchings constructed in [38].

The bijection χ from \mathcal{T}_n to $\mathcal{M}(n)$.

Given a transversal $T \in \mathcal{M}(n)$, $M = \chi(T)$ is defined to be the unique matching satisfying that:

- type(M) = type(T);
- There is an arc connecting the *i*-th left-to-right opener and the *j*-th right-to-left closer if and only if the square (i, j) is filled with a 1.

For example, let $T \in \mathcal{T}_9$ be the transversal of the Young diagram $\lambda = (9, 9, 9, 9, 6, 6, 4, 4, 4)$ as shown in Figure 10 (left). By applying the map χ , we get a matching $M \in \mathcal{M}(9)$ as shown in Figure 10 (right). It is easily checked that both the types of T and M are uuuuddduuddduuddduuddddd.

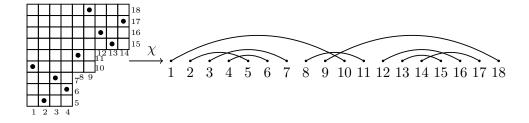


Figure 10: An example of the bijection χ between \mathcal{T}_n and $\mathcal{M}(n)$.

Assume that T is a transversal in \mathcal{T}_{λ} with $\operatorname{Peak}(T) = \{p_1, p_2, \dots, p_k\}$ and $\operatorname{Val}(T) = \{v_1, v_2, \dots, v_\ell\}$. We assign labels to the steps in the south-east border of λ with $1, 2, \dots, 2n$ from south-west to north-east. If the border corresponding to the bottom of column p_i (resp. v_i) receives the label p_i' (resp. v_i'), define $\widetilde{\operatorname{Peak}}(T) = \{p_1', p_2', \dots, p_k'\}$ and $\widetilde{\operatorname{Val}}(T) = \{v_1', v_2', \dots, v_\ell'\}$. For example, for the transversal T in Figure 10 (left), we have $\operatorname{Peak}(T) = \{2, 8\}$, $\widetilde{\operatorname{Peak}}(T) = \{2, 13\}$, $\operatorname{Val}(T) = \{3\}$ and $\widetilde{\operatorname{Val}}(T) = \{3\}$. It is apparent that for a transversal T of a given shape, $\widetilde{\operatorname{Peak}}(T)$

(resp. $\widetilde{\mathrm{Val}}(T)$) is uniquely determined by $\mathrm{Peak}(T)$ (resp. $\mathrm{Val}(T)$), and vise versa. Then the following theorem can be easily summarized from the relevant work in [38, 39].

Theorem 3.1 ([38, 39]) The map χ is a bijection between \mathcal{T}_n and $\mathcal{M}(n)$ such that for any $T \in \mathcal{T}_n$, its corresponding matching $M = \chi(T)$ verifies that

- (i) type(T) = type(M);
- (ii) $\widetilde{\operatorname{Peak}}(T) = \operatorname{Val}(M);$
- (iii) $\widetilde{\text{Val}}(T) = \text{Peak}(M)$;
- (iv) $T \in \mathcal{ST}_n$ if and only if $M \in \mathcal{SM}(n)$;
- (v) $T \in \mathcal{T}_n(J_k)$ if and only if $M \in \mathcal{CM}_k(n)$;
- (vi) $T \in \mathcal{T}_n(I_k)$ if and only if $M \in \mathcal{NM}_k(n)$.

3.2 The bijection ϕ from matchings to oscillating tableaux

An oscillating tableau of shape λ and length n is a sequence $(\emptyset = \lambda^0, \lambda^1, \dots, \lambda^n = \lambda)$ of integer partitions such that λ^i is obtained from λ^{i-1} by either adding a square or deleting a square. In what follows, oscillating tableaux are always of shape \emptyset unless specified otherwise.

In [32], Sundaram constructed a bijection ϕ between matchings and oscillating tableaux. The explicit description of the bijection between matchings and oscillating tableaux can also be found in [14] and [29]. Chen-Deng-Du-Stanley-Yan [14] proved that the bijection ϕ verifies the following celebrated property.

Lemma 3.2 ([14], Theorem 6) Let M be a matching of [2n] and $\phi(M) = (\lambda^0, \lambda^1, \ldots, \lambda^{2n})$. Then M is k-noncrossing (resp. k-nonnesting) if and only if any λ^i contains at most k-1 rows (resp. columns).

For a sequence of integer partitions $U = (\mu^1, \mu^2, \dots, \mu^k)$, define $U^r = (\mu^k, \dots, \mu^2, \mu^1)$. We call U^r the reverse of U. Given an oscillating tableau O, O^r is still an oscillating tableau. If $O^r = O$, we say that O is symmetric. By using Schützenberger's theorem (see [29], Chapter 7.11) for the RSK correspondence (see [29], Chapter 7.13), Xin-Zhang [34] proved that the bijection ϕ has the following celebrated property.

Lemma 3.3 ([34], Theorem 1) For any given matching M and oscillating tableau O, $\phi(M^r) = O^r$ if and only if $\phi(M) = O$.

Let $\mathcal{O}(n)$ and $\mathcal{SO}(n)$ denote the set of oscillating tableaux and symmetric oscillating tableaux of length 2n, respectively. Let $\mathcal{OR}_k(n)$ (resp. $\mathcal{OC}_k(n)$) denote the set of oscillating tableaux $(\lambda^0, \lambda^1, \ldots, \lambda^{2n})$ of length 2n in which any λ^i has at most k-1 rows (resp. columns). Let $\mathcal{SOR}_k(n)$ (resp. $\mathcal{SOC}_k(n)$) denote the set of symmetric oscillating tableaux of $\mathcal{OR}_k(n)$ (resp. $\mathcal{OC}_k(n)$).

The bijection ϕ from matchings to oscillating Tableaux.

Given a matching $M \in \mathcal{M}(n)$, we will recursively define a sequence of SYT's T_0, T_1, \ldots, T_{2n} as follows: Start from the empty SYT by letting $T_{2n} = \emptyset$, read the number $j \in [2n]$ one by one from 2n to 1, and let T_{j-1} be the SYT obtained from T_j for each j by the following procedure.

- If j is the closer of an arc (i, j), then insert i (by the RSK algorithm) into the tableau T_i .
- If j is the opener of an arc (j,k), then remove j from the tableau T_j .

Then the oscillating tableau $\phi(M)$ is the sequence of shapes of the above SYT's.

For example, let M be the matching as shown in Figure 10 (right). Table 2 describes in detail the processes when applying ϕ to M to obtain the corresponding oscillating tableau $O = (\lambda^0, \lambda^1, \dots, \lambda^{18})$.

Table 2: An oscillating tableau $\phi(M)$ corresponding to the matching $M = \{(1,10),(2,5),(3,7),(4,6),(8,11),(9,18),(12,16),(13,15),(14,17)\}.$

i	T_i	λ^i	i	T_i	λ^i
0	Ø	Ø	10	8 9	(1,1)
1	1	(1)	11	9	(1)
2	1 2	(2)	12	9 12	(2)
3	1 2 3	(2,1)	13	9 12 13	(3)
4	1 2 4 3	(3,1)	14	9 12 13 14	(3,1)
5	1 3 4	(3)	15	9 12 14	(2,1)
6	1 3	(2)	16	9 14	(2)
7	1	(1)	17	9	(1)
8	1 8	(1,1)	18	Ø	Ø
9	1 8 9	(1,1,1)			

Given an oscillating tableau $O = (\lambda^0, \lambda^1, \dots, \lambda^n)$, we can associate it with a sequence of u's and d's obtained from O by reading O forward and writing u (resp. d) whenever λ^i is obtained from λ^{i-1} by adding (resp. deleting) a square. Such a sequence is called the type of O, denoted by type(O). Let $O = (\lambda^0, \lambda^1, \dots, \lambda^n)$ be the oscillating tableau as shown in Table 2. Then we have type(O) = uuuuddduudduuudddd.

For an oscillating tableau $O = (\lambda^0, \lambda^1, ..., \lambda^{2n}) \in \mathcal{O}(n)$, we associate O with a word $y = y_1 y_2 ... y_{2n} = \bar{\alpha}(O)$ of length 2n where $y_i = i$ (resp. \bar{i}) if λ^i is obtained from λ^{i-1} by adding (resp. removing) one square in row i. We also call y the Yamanouchi word of O as we will see later it is closely related to the Yamanouchi words of SYT's.

Let O be an oscillating tableau of length 2n and let $y = y_1y_2\cdots y_{2n}$ be the corresponding Yamanouchi word of O. An index i (1 < i < 2n) is said to be a peak of O if y_{i-1}, y_i, y_{i+1} have no bars above them and $y_{i-1} < y_i \ge y_{i+1}$. Similarly, an index i (1 < i < 2n) is said to be a valley of O if y_{i-1}, y_i, y_{i+1} have no bars above them and $y_{i-1} \ge y_i < y_{i+1}$. Let Peak(O) and Val(O) denote the set of peaks and the set of valleys of O, respectively. For instance, let $O = (\emptyset, (1), (1, 1), (2, 1), (2, 2), (3, 2), (3, 1), (2, 1), (2), (1), \emptyset)$ be an oscillating tableau of length 10. Then we have $y = \bar{\alpha}(O) = 12121\bar{2}1\bar{2}1\bar{1}$, Val $(O) = \{3\}$ and Peak $(O) = \{2, 4\}$.

To establish a correspondence between the peak set (or valley set) of matchings and the peak set (or valley set) of oscillating tableaux, we shall introduce an equivalent description of ϕ .

The Bijection $\bar{\phi}$ from Matchings to Oscillating Tableaux.

Given a matching $M \in \mathcal{M}(n)$, we will recursively define a sequence of SYT's T_0, T_1, \ldots, T_{2n} as follows: Start from the empty SYT by letting $T_0 = \emptyset$, read the number $j \in [2n]$ one by one from 1 to 2n, and let T_j be the SYT obtained from T_{j-1} for each j by the following procedure.

- If j is the opener of an arc (j, k), then insert 2n+1-k (by the RSK algorithm) into the tableau T_{j-1} .
- If j is the closer of an arc (i,j), then remove 2n+1-j from the tableau T_{j-1} .

Then the oscillating tableau $\bar{\phi}(M)$ is the sequence of shapes of the above SYT's. One can easily check that $\bar{\phi}(M) = (\phi(M^r))^r$. Hence $\bar{\phi}$ is well-defined. Moreover, by Lemma 3.3, we derive that $\bar{\phi}(M) = (\phi(M^r))^r = (\phi(M)^r)^r = \phi(M)$. In what follows, we will treat ϕ and $\bar{\phi}$ as identical. Now we proceed to show that the bijection ϕ has the following desired properties.

Theorem 3.4 The map ϕ is a bijection between $\mathcal{M}(n)$ and $\mathcal{O}(n)$ such that for any $M \in \mathcal{M}(n)$, its corresponding oscillating tableau $O = \phi(M)$ verifies that

```
(i) type(M) = type(O);
```

(ii)
$$\operatorname{Peak}(M) = \operatorname{Peak}(O)$$
;

- (iii) Val(M) = Val(O);
- (iv) $M \in \mathcal{SM}(n)$ if and only if $O \in \mathcal{SO}(n)$;
- (v) $M \in \mathcal{CM}_k(n)$ if and only if $O \in \mathcal{OR}_k(n)$;
- (vi) $M \in \mathcal{NM}_k(n)$ if and only if $O \in \mathcal{OC}_k(n)$.

Proof. (i), (iv), (v) and (vi) follow directly from Lemmas 3.2 and 3.3. Now we will only deal with (ii), and (iii) can be deduced by similar arguments.

If $i \in \text{Peak}(M)$, then there exist three arcs (i-1,a), (i,b), (i+1,c) of M such that a < b > c. Let $y = y_1y_2\cdots y_{2n}$ be the corresponding Yamanouchi word of O. By the definition of $\bar{\phi}$ and $\bar{\alpha}$, we have that y_{i-1}, y_i, y_{i+1} record the added row indices when 2n+1-a, 2n+1-b, 2n+1-c are inserted one by one to a tableau. Note that a < b > c, we have 2n+1-a > 2n+1-b < 2n+1-c. By the property of RSK algorithm, we deduce that $y_{i-1} < y_i \ge y_{i+1}$, namely, $i \in \text{Peak}(O)$. Thus $\text{Peak}(M) \subseteq \text{Peak}(O)$. Since the above process is reversible, we have $\text{Peak}(O) \subseteq \text{Peak}(M)$. Hence Peak(M) = Peak(O), this completes the proof of (ii).

3.3 An involution Φ on transversals

In this subsection, we aim to establish an involution Φ on \mathcal{T}_n . To this end, we shall construct an involution γ on $\mathcal{O}(n)$.

Given an oscillating tableau $O = (\lambda^0, \lambda^1, \dots, \lambda^{2n})$, define $\gamma(O) = ((\lambda^0)^T, (\lambda^1)^T, \dots, (\lambda^{2n})^T)$. We call $\gamma(O)$ the *conjugate* of O. It is easily seen that $\gamma(O)$ is also an oscillating tableau.

Lemma 3.5 The conjugate map γ is an involution on $\mathcal{O}(n)$ such that for any $O \in \mathcal{O}(n)$, its corresponding oscillating tableau $O' = \gamma(O)$ verifies that

- (i) type(O) = type(O');
- (ii) Val(O) = Peak(O');
- (iii) $\operatorname{Peak}(O) = \operatorname{Val}(O');$
- (iv) $O \in \mathcal{SO}(n)$ if and only if $O' \in \mathcal{SO}(n)$;

- (v) $O \in \mathcal{OR}_k(n)$ if and only if $O' \in \mathcal{OC}_k(n)$;
- (vi) $O \in \mathcal{OC}_k(n)$ if and only if $O' \in \mathcal{OR}_k(n)$.

Proof. (i), (iv), (v) and (vi) follow directly from the definition of the conjugate of an oscillating tableau O.

Given an oscillating tableau $O = (\lambda^0, \lambda^1, \dots, \lambda^{2n})$ with its corresponding Yamanouchi word $y = y_1 y_2 \cdots y_{2n}$. Let $x = x_1 x_2 \cdots x_{2n}$ be the word of length 2n where $x_i = i$ (resp. \bar{i}) if λ^i is obtained from λ^{i-1} by adding (resp. removing) one square in column i. Then by the definition of the conjugate of an oscillating tableau O, we deduce that x is indeed the Yamanouchi word of O'. Let y_i, y_{i+1} be two adjacent numbers of y without bars above them. Then x_i, x_{i+1} also have no bars above them. It is easily seen that $y_i < y_{i+1}$ if and only if $x_i \ge x_{i+1}$, and $y_i \ge y_{i+1}$ if and only if $x_i < x_{i+1}$. Then the (ii) and (iii) follow directly from the definitions of peak and valley of oscillating tableaux.

Notice that for any two transversals T_1, T_2 of the same shape λ and any stat₁, stat₂ \in {Peak, Val}, we have $\widetilde{\text{stat}}_1(T_1) = \widetilde{\text{stat}}_2(T_2)$ if and only if $\operatorname{stat}_1(T_1) = \operatorname{stat}_2(T_2)$. Recall that the type of a transversal is uniquely determined by its shape. By Theorem 3.1, Theorem 3.4 and Lemma 3.5, we deduce the following result.

Theorem 3.6 The map $\Phi = \chi^{-1} \circ \phi^{-1} \circ \gamma \circ \phi \circ \chi$ induces an involution on \mathcal{T}_n such that for any $T \in \mathcal{T}_n$, its corresponding transversal $S = \Phi(T)$ verifies that

- (i) type(T) = type(S);
- (ii) $\operatorname{Peak}(T) = \operatorname{Val}(S);$
- (iii) Val(T) = Peak(S);
- (iv) $T \in \mathcal{ST}_n$ if and only if $S \in \mathcal{ST}_n$;
- (v) $T \in \mathcal{T}_n(J_k)$ if and only if $S \in \mathcal{T}_n(I_k)$;
- (vi) $T \in \mathcal{T}_n(I_k)$ if and only if $S \in \mathcal{T}_n(J_k)$.

When restricted to transversals in \mathcal{T}_{λ} for any given Young diagram λ , Φ induces an involution on \mathcal{T}_{λ} . Then we have the following corollary of Theorem 3.6.

Corollary 3.7 For any Young diagram λ , Peak and Val have a symmetric joint distribution over \mathcal{T}_{λ} , that is,

$$\sum_{T \in \mathcal{T}_{\lambda}} t^{\operatorname{Peak}(T)} q^{\operatorname{Val}(T)} = \sum_{T \in \mathcal{T}_{\lambda}} t^{\operatorname{Val}(T)} q^{\operatorname{Peak}(T)}.$$

When further restricted to transversals in \mathcal{ST}_{λ} for any self-conjugate Young diagram λ , Φ induces an involution on \mathcal{ST}_{λ} . Then we have the following corollary of Theorem 3.6.

Corollary 3.8 For any self-conjugate Young diagram λ , Peak and Val have a symmetric joint distribution over ST_{λ} , that is,

$$\sum_{T \in \mathcal{ST}_{\lambda}} t^{\mathrm{Peak}(T)} q^{\mathrm{Val}(T)} = \sum_{T \in \mathcal{ST}_{\lambda}} t^{\mathrm{Val}(T)} q^{\mathrm{Peak}(T)}.$$

When restricted to transversals in $\mathcal{T}_{\lambda}(J_k)$ for any given Young diagram λ and for any positive integer k, we have the following refinements of Corollaries 3.7 and 3.8, respectively.

Corollary 3.9 For any Young diagram λ and any positive integer k, we have

$$\sum_{T \in \mathcal{T}_{\lambda}(J_k)} t^{\operatorname{Peak}(T)} q^{\operatorname{Val}(T)} = \sum_{T \in \mathcal{T}_{\lambda}(I_k)} t^{\operatorname{Val}(T)} q^{\operatorname{Peak}(T)}.$$

Corollary 3.10 For any self-conjugate Young diagram λ and any positive integer k, we have

$$\sum_{T \in \mathcal{ST}_{\lambda}(J_k)} t^{\operatorname{Peak}(T)} q^{\operatorname{Val}(T)} = \sum_{T \in \mathcal{ST}_{\lambda}(I_k)} t^{\operatorname{Val}(T)} q^{\operatorname{Peak}(T)}.$$

Setting $q_i = 1$ for $i \ge 1$ in Corollaries 3.9 and 3.10, we derive that

$$\sum_{T \in \mathcal{T}_{\lambda}(J_k)} t^{\operatorname{Peak}(T)} = \sum_{T \in \mathcal{T}_{\lambda}(I_k)} t^{\operatorname{Val}(T)}$$
(3.1)

and

$$\sum_{T \in \mathcal{ST}_{\lambda}(J_k)} t^{\operatorname{Peak}(T)} = \sum_{T \in \mathcal{ST}_{\lambda}(I_k)} t^{\operatorname{Val}(T)}.$$
 (3.2)

Setting $t_i = 1$ for $i \ge 1$ in Corollaries 3.9 and 3.10, we derive that

$$\sum_{T \in \mathcal{T}_{\lambda}(J_k)} q^{\text{Val}(T)} = \sum_{T \in \mathcal{T}_{\lambda}(I_k)} q^{\text{Peak}(T)}$$
(3.3)

and

$$\sum_{T \in \mathcal{ST}_{\lambda}(J_k)} q^{\text{Val}(T)} = \sum_{T \in \mathcal{ST}_{\lambda}(I_k)} q^{\text{Peak}(T)}.$$
 (3.4)

3.4 A bijection Ψ from \mathcal{T}_n to itself

In this subsection, we aim to establish a bijection Ψ from \mathcal{T}_n to itself which enables us to prove Theorem 1.2 and Theorem 1.5. To this end, we shall construct a bijection ψ from $\mathcal{O}(n)$ to itself relying on Theorem 1.1.

For an oscillating tableau $O = (\lambda^0, \lambda^1, \dots, \lambda^{2n})$, it can be uniquely decomposed as $(A_1, D_1, A_2, D_2, \dots, A_k, D_k)$, where each A_i (resp. D_i) is called an addition run (resp. a deletion run) of O consisting of a maximal chain of consecutive partitions such that each partition is obtained from the one before it by adding (resp. removing) one square. In what follows, such a decomposition is called the addition-deletion decomposition of O. For example, let $O = (\lambda^0, \lambda^1, \dots, \lambda^{18})$ be the oscillating tableau as shown in Table 2. Then the addition-deletion decomposition of O is given by $((\lambda^0, \lambda^1, \lambda^2, \lambda^3, \lambda^4), (\lambda^4, \lambda^5, \lambda^6, \lambda^7), (\lambda^7, \lambda^8, \lambda^9), (\lambda^9, \lambda^{10}, \lambda^{11}), (\lambda^{11}, \lambda^{12}, \lambda^{13}, \lambda^{14}), (\lambda^{14}, \lambda^{15}, \lambda^{16}, \lambda^{17}, \lambda^{18})).$

Observation 3.11 Given an oscillating tableau $(\lambda^0, \lambda^1, ..., \lambda^{2n})$, assume that the addition-deletion decomposition of O is given by $(A_1, D_1, A_2, D_2, ..., A_k, D_k)$. Then $O \in \mathcal{SO}(n)$ if and only if $A_i = D_{k+1-i}^r$ for all $1 \le i \le k$.

Given a skew diagram λ/μ of size n, let $\mathcal{AR}(\lambda/\mu)$ denote the set of sequences $A = (\mu = \lambda^0, \lambda^1, \dots, \lambda^n = \lambda)$ of partitions in which λ^i is obtained from λ^{i-1} by adding a square. Let $\mathcal{DR}(\lambda/\mu)$ denote the set of sequences $D = (\lambda = \lambda^0, \lambda^1, \dots, \lambda^n = \mu)$ of partitions in which λ^i is obtained from λ^{i-1} by deleting a square.

Now we build a bijection $\theta: \mathcal{AR}(\lambda/\mu) \to \mathcal{Y}(\lambda/\mu)$. Given a sequence $A = (\mu = \lambda^0, \lambda^1, \dots, \lambda^n = \lambda) \in \mathcal{AR}(\lambda/\mu)$, assume that λ^i is obtained from λ^{i-1} by adding a square at row y_i for all $1 \le i \le n$, then set $\theta(A) = y_1 y_2 \dots y_n$. Clearly, $\theta(A) \in \mathcal{Y}(\lambda/\mu)$. Conversely, given a Yamanouchi word $y = y_1 y_2 \dots y_n \in \mathcal{Y}_{\lambda/\mu}$, one can easily recover a sequence $(\mu = \lambda^0, \lambda^1, \dots, \lambda^n = \lambda)$ of partitions, denoted by $\theta'(y)$, in which λ^i is obtained from λ^{i-1} by adding a square located at row y_i . It is easily seen that the maps θ and θ' are inverses of each other and hence θ is a bijection. Define $\bar{\theta}(D) = \theta(D^r)$ for any $D \in \mathcal{DR}(\lambda/\mu)$. It is easily checked that $\bar{\theta}$ induces a bijection between $\mathcal{DR}(\lambda/\mu)$ and $\mathcal{Y}(\lambda/\mu)$.

Lemma 3.12 Let $n \ge 1$. There exists a bijection $\psi : \mathcal{O}(n) \to \mathcal{O}(n)$ such that for any oscillating tableau $O \in \mathcal{O}(n)$, its corresponding oscillating tableau $O' = \psi(O)$ verifies that

- (i) type(O) = type(O');
- (ii) Val(O) = Peak(O');

- (iii) $O \in \mathcal{SO}(n)$ if and only $O' \in \mathcal{SO}(n)$;
- (iv) $O \in \mathcal{OR}_k(n)$ if and only if $O' \in \mathcal{OR}_k(n)$;
- (v) $O \in \mathcal{OC}_k(n)$ if and only if $O' \in \mathcal{OC}_k(n)$.

Proof. Now we first give a description of the map ψ . Given an oscillating tableau $O = (\lambda^0, \lambda^1, \dots, \lambda^{2n})$, assume that the addition-deletion decomposition of O is given by

$$(A_1, D_1, A_2, D_2, \ldots, A_k, D_k).$$

Let $W_i = \theta(A_i)$ and $V_i = \bar{\theta}(D_i)$ for all $1 \le i \le k$. By Theorem 2.11, for any skew shape λ/μ , there exists a bijection $\xi : \mathcal{Y}(\lambda/\mu) \to \mathcal{Y}(\lambda/\mu)$ such that for any $w \in \mathcal{Y}(\lambda/\mu)$, we have $Val(w) = Peak(\xi(w))$. Let $W_i' = \xi(W_i)$ and $V_i' = \xi(V_i)$ for all $1 \le i \le k$. Define $O' = \psi(O)$ to be the oscillating tableau whose addition-deletion decomposition is given by

$$(A'_1, D'_1, A'_2, D'_2, \dots, A'_k, D'_k),$$

where $A'_i = \theta^{-1}(W'_i)$ and $D'_i = (\bar{\theta}^{-1}(V'_i))$ for $1 \le i \le k$.

By the definitions of θ , $\bar{\theta}$ and ξ , one can easily find that $A_i \in \mathcal{AR}(\lambda/\mu)$ if and only if $A_i' \in \mathcal{AR}(\lambda/\mu)$ and $D_i \in \mathcal{DR}(\lambda/\mu)$ if and only if $D_i' \in \mathcal{DR}(\lambda/\mu)$ for all $1 \le i \le k$. This implies that O' is indeed an oscillating tableau. Moreover, by the construction of ψ , the map ψ verifies the properties (i), (iv) and (v). Notice that Val(O) is uniquely determined by the sets $Val(W_i)$ for all $1 \le i \le k$ and Peak(O') is uniquely determined by the sets $Peak(W_i')$ for all $1 \le i \le k$. Since $Val(W_i) = Peak(W_i')$ for all $1 \le i \le k$. This yields that Val(O) = Peak(O') as desired, completing the proof of (ii).

It is routine to check that

$$O \in \mathcal{SO}(n) \Leftrightarrow A_i = D_{k+1-i}^r$$

$$\Leftrightarrow W_i = \theta(A_i) = \theta(D_{k+1-i}^r) = V_{k+1-i}$$

$$\Leftrightarrow W_i' = \xi(W_i) = \xi(V_{k+1-i}) = V_{k+1-i}'$$

$$\Leftrightarrow A_i' = \theta^{-1}(W_i') = \theta^{-1}(V_{k+1-i}') = (D_{k+1-i}')^r$$

$$\Leftrightarrow O' \in \mathcal{SO}(n)$$

for all $1 \le i \le k$. Thus (iii) follows, completing the proof.

By Theorem 3.1, Theorem 3.4 and Lemma 3.12, we deduce the following result.

Theorem 3.13 The map $\Psi = \chi^{-1} \circ \phi^{-1} \circ \psi \circ \phi \circ \chi$ induces a bijection from \mathcal{T}_n to itself such that for any $T \in \mathcal{T}_n$, its corresponding transversal $S = \Psi(T)$ verifies that

- (i) type(T) = type(S);
- (ii) $\operatorname{Peak}(T) = \operatorname{Val}(S)$;
- (iii) $T \in \mathcal{ST}_n$ if and only if $S \in \mathcal{ST}_n$;
- (iv) $T \in \mathcal{T}_n(J_k)$ if and only if $S \in \mathcal{T}_n(J_k)$;
- (v) $T \in \mathcal{T}_n(I_k)$ if and only if $S \in \mathcal{T}_n(I_k)$.

Proof of Theorem 1.2. From (i), (ii) and (iv) in Theorem 3.13, we deduce that Peak and Val are equidistributed over the set $\mathcal{T}_{\lambda}(J_k)$, namely,

$$\sum_{T \in \mathcal{T}_{\lambda}(J_k)} t^{\operatorname{Val}(T)} = \sum_{T \in \mathcal{T}_{\lambda}(J_k)} t^{\operatorname{Peak}(T)}.$$

Combining (3.1) and (3.3), Theorem 1.2 follows.

Proof of Theorem 1.5. From (i), (ii), (iii) and (iv) in Theorem 3.13, we deduce that Peak and Val are equidistributed over the set $\mathcal{ST}_{\lambda}(J_k)$, namely,

$$\sum_{T \in \mathcal{ST}_{\lambda}(J_k)} t^{\operatorname{Val}(T)} = \sum_{T \in \mathcal{ST}_{\lambda}(J_k)} t^{\operatorname{Peak}(T)}.$$

Combining (3.2) and (3.4), Theorem 1.5 follows.

3.5 Proofs of Theorems 1.3 and 1.6

In [3], Backelin-West-Xin obtained the following lemma.

Lemma 3.14 ([3], Proposition 2.3) For any permutations α , β and τ , if $|\mathcal{T}_{\mu}(\alpha)| = |\mathcal{T}_{\mu}(\beta)|$ for any Young diagram μ , then $|\mathcal{T}_{\lambda}(\alpha \oplus \tau)| = |\mathcal{T}_{\lambda}(\beta \oplus \tau)|$ for any Young diagram λ .

In the spirit of Backelin-West-Xin's proof of Lemma 3.14, we deduce the following theorem, which will enable us to prove Theorem 1.3.

Theorem 3.15 For any permutations α , β and τ , if Peak is equidistributed over $\mathcal{T}_{\mu}(\alpha)$ and $\mathcal{T}_{\mu}(\beta)$ for any Young diagram μ , then there exists a peak set preserving bijection Θ between $\mathcal{T}_{\lambda}(\alpha \oplus \tau)$ and $\mathcal{T}_{\lambda}(\beta \oplus \tau)$ for any Young diagram λ .

Proof. For any Young diagram μ , let $\delta_{\mu}: \mathcal{T}_{\mu}(\alpha) \to \mathcal{T}_{\mu}(\beta)$ be the peak set preserving bijection implied by the hypothesis. Now we give a description of the bijection $\Theta: \mathcal{T}_{\lambda}(\alpha \oplus \tau) \to \mathcal{T}_{\lambda}(\beta \oplus \tau)$. Given a transversal $T = t_1 t_2 \cdots t_n \in \mathcal{T}_{\lambda}(\alpha \oplus \tau)$, color the square (c, r) by white if the board of λ lying below and to the right of it contains the pattern τ , or gray otherwise. Find out all the 1's colored by gray and color the corresponding rows and columns by gray. Delete all the squares colored by gray, as well as the fillings in them. Let μ be the resulting board and let $T' = t'_1 t'_2 \cdots t'_k$ be the corresponding 01-filling of μ . One can easily verify that μ is a Young diagram and T' is a transversal of shape μ . Notice that T avoids the pattern $\sigma \oplus \tau$ if and only if T' avoids the patter σ for any permutation σ . Thus we have T' avoids α . By applying the bijection δ_{μ} , we obtain a transversal $\delta_{\mu}(T') = R' = r'_1 r'_2 \cdots r'_k \in \mathcal{T}_{\mu}(\beta)$ with the same peak set as T'. Restoring the gray cells of λ and their contents, we obtain a transversal $R = \Theta(T)$ of shape λ which avoids the pattern $\beta \oplus \tau$. Figure 11 illustrates an example of $\Theta: \mathcal{T}_{\lambda}(\alpha \oplus \tau) \to \mathcal{T}_{\lambda}(\beta \oplus \tau)$ where $\alpha = 12, \beta = 21, \tau = 1$ and $\lambda = (9,9,9,9,9,8,8,8,5)$.

In order to show that the map Θ is a bijection, it suffices to show that the above procedure is invertible. It is obvious that Θ changes the 01-filling located at the white squares and leaves the 01-filling located at the gray squares fixed. Hence when applying the inverse map Θ^{-1} , the coloring of R will result in the same Young diagram μ and the same transversal R' such that when applying the inverse bijection δ_{μ}^{-1} to R', we will recover the same transversal T' and hence the same transversal T.

Now it is sufficient to show $\operatorname{Peak}(T) = \operatorname{Peak}(R)$. Assume that $i \in \operatorname{Peak}(T)$. If the square (i,t_i) is colored by gray, then by the coloring rule and the fact that $t_{i-1} < t_i > t_{i+1}$, the squares (i-1,j) and (i+1,j) with $j \ge t_i$ are also colored by gray. Then by the construction of Θ , one can easily check that $i \in \operatorname{Peak}(R)$. Now we assume that the square (i,t_i) is colored by white and the lowest white colored square in column i is (i,r). Assume that column i of λ corresponds column i' in μ . Then the lowest white colored square in column i-1 (resp. i+1) is (i-1,r) (resp. (i+1,r)) as $i \in \operatorname{Peak}(T)$. This implies that $c_{i'-1}(\lambda') = c_{i'}(\lambda') = c_{i'+1}(\lambda')$ and the relative positions of 1's in columns i'-1, i' and i'+1 in i' are the same as those of 1's in columns i-1, i and i+1 in i'. Hence we have $i' \in \operatorname{Peak}(T') = \operatorname{Peak}(R')$. By restoring the gray cells of λ and their contents, it is easily seen that i is a peak of i'. To conclude, we have that i' is an easily seen that i' is a peak of i'. To conclude, we have that i' is easily seen that i' is a peak of i'. Hence we have concluded that i' is i' encorporated that i' is an easily seen that i' is a peak of i' encorporated that i' is easily seen that i' is a peak of i' encorporated that i' is easily seen that i' is a peak of i' encorporated that i' encorporated

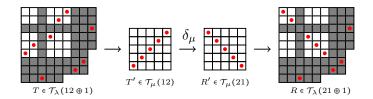


Figure 11: An example of the bijection Θ between $\mathcal{T}_{\lambda}(\alpha \oplus \tau)$ and $\mathcal{T}_{\lambda}(\beta \oplus \tau)$.

Combining Theorems 1.2 and 3.15, we are led to a proof of Theorem 1.3.

For symmetric transversals, one can deduce the following analogue of Theorem 3.15 by making a slight adaption of the coloring rule in the proof. More precisely, we color the square by white if the board lying below and to the right of it contains the patterns τ or τ^{-1} , or gray otherwise.

Theorem 3.16 Let α and β be any involutions and let τ be any permutation. If the set-valued statistic Peak is equidistributed over $\mathcal{ST}_{\mu}(\alpha)$ and $\mathcal{ST}_{\mu}(\beta)$ for any self-conjugate Young diagram μ , then there exists a peak set preserving bijection between $\mathcal{ST}_{\lambda}(\alpha \oplus \tau)$ and $\mathcal{ST}_{\lambda}(\beta \oplus \tau)$ for any self-conjugate Young diagram λ .

Combining Theorems 1.5 and 3.16, we are led to a proof of Theorem 1.6. To prove Theorem 1.8, we need the following lemma.

- **Lemma 3.17** (i) For any permutations α , β and any nonempty permutation τ , if the set-valued statistic Peak is equidistributed over $\mathcal{T}_{\mu}(\alpha)$ and $\mathcal{T}_{\mu}(\beta)$ for any Young diagram μ , then we have $|\mathcal{A}_n(\alpha \oplus \tau)| = |\mathcal{A}_n(\beta \oplus \tau)|$.
 - (ii) Let α and β be any involutions and let τ be any nonempty permutation. If the set-valued statistic Peak is equidistributed over $\mathcal{ST}_{\mu}(\alpha)$ and $\mathcal{ST}_{\mu}(\beta)$ for any self-conjugate Young diagram μ , then we have $|\mathcal{AI}_n(\alpha \oplus \tau)| = |\mathcal{AI}_n(\beta \oplus \tau)|$.

Proof. We shall only prove (i), as (ii) can be deduced by similar arguments. By the hypothesis and Theorem 3.15, the bijection Θ induces a peak set preserving between $S_n(\alpha \oplus \tau)$ and $S_n(\beta \oplus \tau)$. Let $\pi \in A_n(\alpha \oplus \tau)$ and $\Theta(\pi) = \sigma = \sigma_1 \sigma_2 \cdots \sigma_n$. Notice that the permutation matrix of an alternating permutation $\pi \in A_n$ is a transversal $T = t_1 t_2 \cdots t_n$ of a square Young diagram $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ with $\lambda_1 = \lambda_2 = \cdots = \lambda_n = n$ such that $Peak(T) = \{2, 4, \dots, 2\lfloor \frac{n-1}{2} \rfloor\}$ and when n is even, $t_{n-1} < t_n$. In order to show that $\Theta(\pi) \in A_n(\beta \oplus \tau)$, it suffices to show that $\sigma_{2k} > \sigma_{2k-1}$ when n = 2k. This is justified by the fact that, when applying the bijection Θ , column 2k is always colored by gray and all the squares colored by white (if any) in column 2k - 1 are positioned above

the 1 located in column 2k as τ is nonempty. Hence Θ induces a bijection between $\mathcal{A}_n(\alpha \oplus \tau)$ and $\mathcal{A}_n(\beta \oplus \tau)$. This completes the proof.

Combining Theorem 1.2 and Lemma 3.17, we derive that $|\mathcal{A}_n(I_k \oplus \tau)| = |\mathcal{A}_n(J_k \oplus \tau)|$ for any nonempty permutation τ and for any positive integer k which was first proved by Yan [36]. Similarly, by Theorem 1.5 and Lemma 3.17, we have $|\mathcal{AI}_n(I_k \oplus \tau)| = |\mathcal{AI}_n(J_k \oplus \tau)|$ for any nonempty pattern τ and for any positive integer k, completing the proof of Theorem 1.8.

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Data availability statements

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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