ON THE RECURSIVE AND EXPLICIT FORM OF THE GENERAL J.C.P. MILLER FORMULA WITH APPLICATIONS

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ABSTRACT. The famous J.C.P. Miller formula provides a recurrence algorithm for the composition $B_a \circ f$, where B_a is the formal binomial series and f is a formal power series, however it requires that f has to be a nonunit.

In this paper we provide the general J.C.P. Miller formula which eliminates the requirement of nonunitness of f and, instead, we establish a necessary and sufficient condition for the existence of the composition $B_a \circ f$. We also provide the general J.C.P. Miller recurrence algorithm for computing the coefficients of that composition, if $B_a \circ f$ is well defined, obviously. Our generalizations cover both the case in which f is a one-variable formal power series and the case in which f is a multivariable formal power series.

In the central part of this article we state, using some combinatorial techniques, the explicit form of the general J.C.P. Miller formula for one-variable case.

As applications of these results we provide an explicit formula for the inverses of polynomials and formal power series for which the inverses exist, obviously. We also use our results to investigation of approximate solution to a differential equation which cannot be solved in an explicit way.

1. INTRODUCTION

In the well-known [12], P. Henrici introduced the following J.C.P. Miller formula, that is, for any nonunit formal power series

$$f(z) = b_1 z + b_2 z^2 + b_3 z^3 + \dots$$

over \mathbb{C} , if we write

$$B_a \circ f(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots,$$

where $a \in \mathbb{C}, B_a$ is a formal binomial series, that is

$$B_a(z) = 1 + \binom{a}{1}z + \binom{a}{2}z^2 + \binom{a}{3}z^3 + \dots, a \in \mathbb{C},$$

then $c_0 = 1$, and

(1.1)
$$c_n = \frac{1}{n} \sum_{k=0}^{n-1} \left[a(n-k) - k \right] c_k b_{n-k} = \frac{1}{n} \sum_{k=1}^n \left[(a+1)k - n \right] c_{n-k} b_k$$

for all $n \in \mathbb{N}$.

J.P.C. Miller formula has found many applications in algorithm theory, analysis, combinatorics, formal analysis, number theory, the theory of differential equations and other fields of mathematics and physics, like fluid mechanics (see e.g. [1], [2], [6], [16] and [20]). A recent application of formulas (1.1) to an algorithm of computing the real exponent of formal power series can be found in [8], p. 311.

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In this paper we extend the J.C.P. Miller formula beyond nonunit formal power series. More precisely, we establish a necessary and sufficient condition for the existence of the composition $B_a \circ f$, where f is a formal power series. We also provide the general J.C.P. Miller recurrence algorithm for computing the coefficients of that composition, if $B_a \circ f$ is well–defined, obviously.

In the central part of this article, using some combinatorial techniques, in Lemma 4.3 we propose formula 4.2, the Generalized Trudi Formula (cf. [19, p.214]; a formula for the determinant of a general Hessenberg matrix) and next, applying that formula, we provide the explicit form of the general J.C.P. Miller formula. It is worth mentioning that some already provided methods (see e.g. [18], [22]) can be used only for integer exponents. However, because we are going to deal with binomial series for arbitrary $a \in \mathbb{C}$, some new techniques must have been established from the ground up. The crucial result for our investigations for an explicit form of the J.C.P.Miller formula is the generalized Trudi formula (4.2). As far as we know, the formula for Toeplitz–Hessenberg matrices (cf. [13], [15] or see the introduction in [7]). Our results were obtained in the framework of finite dimensional spaces. In particular, we used the classic combinatorial definition of the determinant. Nevertheless, let us observe that we could achieve the same ends with help of notions connected to infinite matrices - this observation seems to be of some importance because applications of infinite matrices in the theory of formal Laurent series (see e.g. [5] and references therein).

In the final section of the paper we apply the above results to provide an explicit formula for the inverses of polynomials and formal power series for which the inverses exist, obviously. We also use our results to investigate approximate solutions to a differential equation which cannot be solved in an explicit way.

2. Preliminaries

In this section we are going to collect some definitions and facts which will be needed in the sequel. The interested reader can find basic definitions and results concerning the composition of formal power series e.g. in the monograph [8]. Let us introduce some notations and conventions first.

For any finite set A by #A we denote the number of its elements. By $\{\ldots\}_{mset}$ we denote multisets with elements to be listed within the curly brackets. \mathbb{C} is the field of complex numbers, \mathbb{R} is the field of reals. For $a \in \mathbb{C}$, $\operatorname{Re}(a)$ is the real part of a. By \mathbb{N} we denote the set of all positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We put [0] := 0 and, for $n \in \mathbb{N}$, $[n] := \{1, \ldots, n\}, [n]_0 := \{0, 1, \ldots, n\}$. We take on the convention that $0^0 := 1$ and $\Sigma_{\emptyset} := 0$. If A is a square complex (real) matrix, by |A| we denote its determinant. For $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$ we write $\mathbb{X}(\mathbb{K})$ (or $\mathbb{X}_1(\mathbb{K})$) to denote the set of one-variable formal power series with coefficients in \mathbb{K} .

Let us emphasize that an essential role in this topic plays the following

Theorem 2.1. ([10]) (General Composition Theorem) Let there be given $f, g \in \mathbb{X}(\mathbb{C})$:

$$f(x) = a_0 + a_1 z + \dots + a_n z^n + \dots,$$

$$g(x) = b_0 + b_1 z + \dots + b_n z^n + \dots$$

If $\deg(f) \neq 0$, then the composition $q \circ f$ exists if and only if

(2.1)
$$\sum_{n=k}^{\infty} \binom{n}{k} b_n a_0^{n-k} \in \mathbb{C} \quad for \ all \ k \in \mathbb{N}_0,$$

where $\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$. If $\deg(f) = 0$, then the existence of $g \circ f$ is equivalent to the existence of $g(a_0)$.

In what follows we will need a consequence of the above result, namely

Theorem 2.2. ([10]) Let be given $f, g \in \mathbb{X}(\mathbb{C})$:

$$f(x) = a_0 + a_1 z + \dots + a_n z^n + \dots,$$

$$g(x) = b_0 + b_1 z + \dots + b_n z^n + \dots$$

If the series $\sum_{n=0}^{\infty} b_n R^n$ converges for some real number $R > |a_0|$, then $g \circ f$ exists.

Definition 2.3. ([8], p.147) Let $g \in \mathbb{X}(\mathbb{C})$. The formal power series g is said to be formally analytic at $a \in \mathbb{C}$, or a is in the composition domain of g, if

 $q^{(n)}(a) \in \mathbb{C}$, for infinitely many $n \in \mathbb{N}$.

By [8, Theorem 5.4.6] it follows that the condition " $q^{(n)}(a) \in \mathbb{C}$, for infinitely many $n \in \mathbb{N}$ " is equivalent to the following " $q^{(n)}(a) \in \mathbb{C}$, for all $n \in \mathbb{N}$ ". Let us observe that this equivalence can be easily derived from the following property which is a simple consequence of [3, Lemma 1, Lemma 2]: for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, if $g^{(n)}(a)$ does not absolutely converge, then $g^{(n+1)}(a)$ is not absolutely convergent and $q^{(n+2)}(a)$ is divergent. Therefore, either $q^{(n)}(a) \in \mathbb{C}$ for all $n \in \mathbb{N}$ or $q^{(n)}(a) \in \mathbb{C}$ for a finite number of $n \in \mathbb{N}$. But this implies the equivalence of the aforementioned statements.

Proposition 2.4. ([8]) Let $q \in \mathbb{X}(\mathbb{C})$ be given. Then q is formally analytic at $a \in \mathbb{C}$ if and only if for any $f \in \mathbb{X}(\mathbb{C})$ such that $f(0) = a, g \circ f \in \mathbb{X}(\mathbb{C})$

Corollary 2.5. ([9]) Let $g \in \mathbb{X}(\mathbb{C})$ and $a \in \mathbb{C}$ be given. If $a \in \mathbb{C}$ is a formally analytic point of q, then z is a formally analytic point of q for all $z \in \mathbb{C}$ with |z| = |a|.

As regards formal power series of multiple variables we stick to notations given in [4]. However, for convenience of the reader, below we are going to recall some of those notions.

Let us fix $q \in \mathbb{N}, k \in \mathbb{N}_0$ and denote by C_k the set of all nonnegative integer solutions c_1, \ldots, c_q of the equation $c_1 + \ldots + c_q = k$ for $k \in \mathbb{N}_0, q \in \mathbb{N}$, that is

$$C_k := \{ c = (c_1, \dots, c_q) \in \mathbb{N}_0^q : c_1 + \dots + c_q = k \}$$

Obviously, $C := \mathbb{N}_0^q = \bigcup_{k \in \mathbb{N}_0} C_k$ and for each $c \in C$ there is exactly one $k \in \mathbb{N}_0$ for which $c \in C_k$. Let K stand for the field of real (or complex) numbers, that is, $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$. Now, let us state a definition of a formal power series of multiple variables (see also [11] or [21]).

Definition 2.6. A formal power series f in q-variables $x := (x_1, \ldots, x_q) \in \mathbb{K}^q$ (for short: q-fps) is a formal sum of the form $f(x) := \sum_{c \in C} f_c X^c$, where $f_c \in \mathbb{K}$ and $X^c := x_1^{c_1} \dots x_q^{c_q}$ for all $c \in C$. An element f_c is called the c-th coefficient of the q-fps $f, c \in C$. The set of all q-fps is denoted by $\mathbb{X}_q(\mathbb{K})$ or just by \mathbb{X}_q .

Remark 1. Let us observe that a q-fps f can be uniquely identified with the mapping $C \ni c \mapsto f_c \in \mathbb{K}$.

Let us recall the definition of the Cauchy product of q-fps.

Definition 2.7. For q-fps $f(x) = \sum_{c \in C} f_c X^c$, $g(x) = \sum_{c \in C} g_c X^c \in \mathbb{X}_q$, $q \in \mathbb{N}$, the q-dimensional Cauchy product of f and g is a q-fps h = fg defined as

$$h(x) := \sum_{c \in C} \underbrace{\left(\sum_{a, b \in C: a+b=c} f_a g_b\right)}_{h_c :=} X^c$$

Obviously, there are a finite number of pairs (a, b) such that $a, b \in C$ and a + b = c for a given $c \in C$. It is also clear that for such a pair we have $a \leq c$ and $b \leq c$, where the inequality \leq is taken coordinatewise.

We are now ready to define the composition of a 1-fps g with a q-fps f.

Definition 2.8. For $f(x) = \sum_{c \in C} f_c X^c \in \mathbb{X}_q$, $q \in \mathbb{N}$, and $g(y) = \sum_{n=0}^{\infty} g_n y^n \in \mathbb{X}_1$, the composition of g with f is a q-fps $h = g \circ f$ defined by

$$h(x) := \sum_{c \in C} \underbrace{\left(\sum_{n=0}^{\infty} g_n f_c^n\right)}_{h_c :=} X^c,$$

provided that the coefficients h_c exist, that is, if the series defining h_c converge for every $c \in C$, where $f^n \in \mathbb{X}_q$ is the n-th power of f, that is, $f^n := \underbrace{ff \dots f}_{n \times}, f^n_c := (f^n)_c, n \in \mathbb{N}, c \in C$, and $f^0 := (1, 0, 0, \ldots)$, where 1 is the multiplicative identity of the field \mathbb{K} .

3. The generalized J.C.P. Miller formula

Let us notice that if a formal power series f is a constant formal power series or, equivalently, if $\deg(f) = 0$, then the problem of the existence of the composition $B_a \circ f$ reduces to the problem of the convergence of the binomial series. Moreover, let us notice that if $a \in \mathbb{N}$, then we have

$$\binom{a}{n} = \frac{a(a-1)\dots(a-n+1)}{n!} = 0,$$

when n > a. In such a case $B_a(z)$ is a polynomial and therefore $B_a(f) \in \mathbb{X}(\mathbb{C})$ for all $f \in \mathbb{X}(\mathbb{C})$. Both these cases are trivial. In what follows, we suppose that $\deg(f) \neq 0$ and $a \in \mathbb{C} \setminus \mathbb{N}$ unless we indicate otherwise.

Now, let us investigate the composition of the formal binomial series B_a with a formal power series over $\mathbb{X}(\mathbb{C})$.

Theorem 3.1. Let $f \in \mathbb{X}(\mathbb{C})$ be a formal power series over \mathbb{C} with deg $(f) \neq 0$:

$$f(z) = b_0 + b_1 z + b_2 z^2 + \dots,$$

Let B_a be a formal binomial series with $a \in \mathbb{C} \setminus \mathbb{N}$. Then $B_a \circ f \in \mathbb{X}(\mathbb{C})$ if and only if $|b_0| < 1$.

Proof. It is well-known that the radius of convergence of the binomial power series is equal to 1. Thus applying Theorem 2.2, we infer that $B_a \circ f \in \mathbb{X}(\mathbb{C})$ if $|b_0| < 1$.

By Theorem 2.1, we need only to show that b_0 is not a formally analytic point of B_a for a formal power series f with $|b_0| = 1$. By [14, Theorem 247, p. 426], a binomial series B_a with $a \in C$, $\operatorname{Re}(a) < 0$, diverges at z = -1. Hence, by Corollary 2.5, if $\operatorname{Re}(a) < 0$, an element $z \in \mathbb{C}$ is a formally analytic point of B_a only if |z| < 1.

Now, let $a \in \mathbb{C} \setminus \mathbb{N}$ and $\operatorname{Re}(a) \geq 0$. By [8, Proposition 2.2.7], for $k \in \mathbb{N} : k > \operatorname{Re}(a)$, we have

$$B_a^{(k)}(z) = a(a-1)\dots(a-k+1)B_{a-k}(z),$$

where $B_a^{(k)}$ is the kth formal derivative of B_a . Hence, since $\operatorname{Re}(a) - k < 0$, any $z \in \mathbb{C}$ such that |z| = 1 is not a formally analytic point of B_{a-k} . Therefore $B_a(f) \in \mathbb{X}(\mathbb{C})$ with $\operatorname{Re}(a) > 0$ if and only if $|b_0| < 1$.

Thus, $B_a(f) \in \mathbb{X}(\mathbb{C})$ with all $a \in \mathbb{C} \setminus \mathbb{N}$ if and only if $|b_0| < 1$.

Now, we are going to establish the general J.C.P. Miller formula.

Theorem 3.2. Let $f \in \mathbb{X}(\mathbb{C})$ be a formal power series over \mathbb{C} with deg $(f) \neq 0$:

$$f(z) = b_0 + b_1 z + b_2 z^2 + \dots, \quad |b_0| < 1.$$

Then

$$B_a \circ f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \dots, \quad a \in \mathbb{C} \setminus \mathbb{N},$$

is a formal power series over \mathbb{C} , where $c_0 = (1+b_0)^a$, $c_1 = \frac{ac_0b_1}{1+b_0}$,

(3.1)
$$c_n = \frac{1}{n(1+b_0)} \left[nac_0 b_n + \sum_{k=1}^{n-1} b_k c_{n-k} [ka - (n-k)] \right] = \frac{1}{n(1+b_0)} \sum_{k=1}^n b_k c_{n-k} [ka - (n-k)],$$

for all $n \in \mathbb{N}$, $n \geq 2$.

Proof. Suppose that $|b_0| < 1$. By Theorem 3.1, $B_a \circ f \in \mathbb{X}(\mathbb{C})$ for any $a \in \mathbb{C}$ (the case when $a \in \mathbb{N}$ is obvious). Put

(3.2)
$$B_a \circ f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \dots,$$

We also have

$$B_a \circ f(z) = 1 + \binom{a}{1} f(z) + \binom{a}{2} f^2(z) + \binom{a}{3} f^3(z) + \dots$$

Considering the constant term of $B_a \circ f(z)$, we have

$$c_0 = 1 + {a \choose 1} b_0 + {a \choose 2} b_0^2 + {a \choose 3} b_0^3 + \ldots = (1 + b_0)^a.$$

By the Generalized Chain Rule for the formal composition, we get

$$(B_a \circ f)' = \mathop{a(B_{a-1} \circ f)}_{5} f'.$$

Multiplying by $(B_1 \circ f)$ both sides of the above equality and applying the Right Distributive Law, we obtain

$$(3.3) \qquad (B_1 \circ f)(B_a \circ f)' = a(B_1 \circ f)(B_{a-1} \circ f)f' = a[(B_1B_{a-1}) \circ f]f' = a(B_a \circ f)f'.$$

Since $B_1(z) = 1 + z$, we have
 $(1 + b_0 + b_1z + b_2z^2 + \dots)(c_1 + 2c_2z + 3c_3z^2 + \dots) =$
 $a(c_0 + c_1z + c_2z^2 + \dots)(b_1 + 2b_2z + 3b_3z^2 + \dots).$

Applying the formulas of Cauchy product on both sides of the above equality, and then equating the coefficients of the term z^{n-1} , we obtain

$$(1+b_0)nc_n + b_1(n-1)c_{n-1} + b_2(n-2)c_{n-2} + \ldots + b_{n-2}2c_2 + b_{n-1}c_1 = a\left[nb_nc_0 + b_{n-1}(n-1)c_1 + b_{n-2}(n-2)c_2 + \ldots + b_22c_{n-2} + b_1c_{n-1}\right].$$

If n = 1, the above equality provides

$$(1+b_0)\cdot 1\cdot c_1 = ac_0\cdot 1\cdot b_1,$$

and we get $c_1 = \frac{ac_0b_1}{1+b_0} = ab_1(1+b_0)^{a-1}$. For every $n \ge 2$,

$$(1+b_0)nc_n = a \left[nb_nc_0 + b_{n-1}(n-1)c_1 + b_{n-2}(n-2)c_2 + \dots + b_1c_{n-1} \right] - \left[b_1(n-1)c_{n-1} + b_2(n-2)c_{n-2} + \dots + b_{n-1}c_1 \right] = anb_nc_0 + \left[ab_1c_{n-1} - b_1(n-1)c_{n-1} \right] + \left[ab_2c_{n-2}2 - b_2(n-2)c_{n-2} \right] + \dots + \left[ab_{n-2}c_2(n-2) - b_{n-2}2c_2 \right] + \left[ab_{n-1}c_1(n-1) - b_{n-1}c_1 \right] = nac_0b_n + \sum_{k=1}^{n-1} b_kc_{n-k} \left[ka - (n-k) \right].$$

Thus

$$c_{n} = \frac{1}{n(1+b_{0})} \left[nac_{0}b_{n} + \sum_{k=1}^{n-1} b_{k}c_{n-k}[ka - (n-k)] \right] = ab_{n}(1+b_{0})^{a-1} + \frac{1}{n(1+b_{0})} \sum_{k=1}^{n-1} b_{k}c_{n-k}[ka - (n-k)] = \frac{1}{n(1+b_{0})} \sum_{k=1}^{n} b_{k}c_{n-k}[ka - (n-k)].$$

Remark 3.3. Theorems 3.1 and 3.2 and their proofs remain valid if we replace \mathbb{C} with \mathbb{R} . Remark 3.4. Let us notice that putting $b_0 = 0$ in formula (3.1), we get

$$c_n = \frac{1}{n} \left[\sum_{k=1}^n b_k c_{n-k} [k(a+1) - n] \right], \ n \ge 2,$$

which is the original J.C.P. Miller formula (1.1).

4. The explicit form of the general J.C.P. Miller formula

The main goal of this section is to establish the explicit form of the general J.C.P. Miller formula. For that goal we will need a few lemmas. Among them, the most important one is Lemma 4.3 containing the generalized Trudi formula - a new and useful formula for the determinant of any Hessenberg (almost lower-triangular) matrix.

Lemma 4.1. Let $n \in \mathbb{N}$ and σ be a permutation of [n] satisfying the condition

(4.1)
$$\sigma(k) \le k+1 \text{ for all } k \in [n],$$

and define $X(\sigma) := \{k \in [n] : \sigma(k) \le k\} = \{s_1, ..., s_l\} \subseteq [n]$, where $s_1 < ... < s_l$, $s_l = n$, for some $l \in [n]$. Then

(1) $\sigma(k) = k + 1$ for $k \in [n] \setminus X(\sigma)$, (2) $\sigma(s_{i+1}) = s_i + 1$ for all $i \in \{0, \dots, l-1\}$, where $s_0 := 0$.

Moreover, the sign of the permutaton σ is $(-1)^{n-l}$.

Proof. Denote $X := \{s_1, \ldots, s_l\}$. It is obvious that $n \in X$ and $s_l = n$. By the condition (4.1) $\sigma(k) = k + 1$ for all $k \in [n] \setminus X$, so $\sigma(X) = \{\sigma(s_i) : i \in [l]\} = [n] \setminus \{k + 1 : k \in [n]\}$ $[n] \setminus X$ = {1, $s_1 + 1, \ldots, s_{l-1} + 1$ }. The last equality stems from the fact σ is a bijection, $1 \notin \sigma([n] \setminus X)$, and because if, for some $i \in [l-1]$, $s_i + 1 \notin \sigma(X)$, then $s_i \in [n] \setminus X$ which is impossible. Observe that for any $k < s_1, k \in [n]$, we have $\sigma(k) = k + 1$ which implies $\sigma(\{k \in [n] : k < s_1\}) = \{2, 3, \dots, s_1\}$ provided that $s_1 > 1$. It is now clear that $\sigma(s_1) = 1$ because $\sigma(s_1) \leq s_1$. Arguing similarly we obtain $\sigma(s_2) = s_1 + 1$, and then $\sigma(s_3) = s_2 + 1$, and further up to $\sigma(s_l) = s_{l-1} + 1$. So, formulas (1) and (2) are valid. Now, by formulas (1) and (2) we have, for $k < s_1$, $\sigma(k) = k+1$ and $\sigma(s_1) = 1$. So the number of inversions in the sequence $(\sigma(1), \ldots, \sigma(s_1))$ is $s_1 - 1$. Analogously, since $\sigma(k) = k + 1$ for $s_1 < k < s_2$ and $\sigma(s_2) = s_1 + 1$, the number of inversions in the sequence $(\sigma(s_1 + 1), \ldots, \sigma(s_2))$ equals $s_2 - s_1 - 1$. Moreover, because $\sigma(k) < \sigma(l)$ for $k \le s_1 < l \le s_2$, the number of inversions in the sequence $(\sigma(1),\ldots,\sigma(s_2))$ is $s_1-1+s_2-s_1-1=s_2-2$. Continuing this way we conclude that the number of inversions in the sequence $(\sigma(1), \ldots, \sigma(n))$ is $(s_1 - 1) + (s_2 - 1)$ $(1-s_1)+\ldots+(s_l-1-s_{l-1})=s_l-l=n-l$. Therefore, the sign of the permutation σ is $(-1)^{n-l}$.

Remark 4.2. It easily follows from the above proof that the mapping $\sigma \mapsto X(\sigma)$ is a bijection between the set of permutations of [n] satisfying condition (4.1) and subsets of [n], containing n.

In what follows, whenever we see $\{s_1, \ldots, s_l\} \subseteq [k]$ under a summation symbol we mean that the sum extends over all *l*-element subsets $\{s_1, \ldots, s_l\} \subseteq [k]$ for which $s_1 < \ldots < s_l$ and $s_l = k$; it may happen that the family of such subsets is empty for given values of k, l. We also assume $s_0 := 0$. **Lemma 4.3.** (Generalized Trudi Formula) Let $n \in \mathbb{N}$ and

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & 0 & \dots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & a_{n-2,n-1} & 0 \\ \vdots & & & & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & \dots & & & a_{n,n-1} & a_{n,n} \end{bmatrix}$$

be a Hessenberg matrix. Then

(4.2)
$$|A| = \sum_{l=1}^{n} (-1)^{n-l} \sum_{\{s_1, \dots, s_l\} \subseteq [n]} \left(\prod_{q=1}^{l} a_{s_q, s_{q-1}+1} \prod_{k \in [n] \setminus \{s_i: i \in [l]\}} a_{k,k+1} \right).$$

In particular, if $a_{k,k+1} = a \in \mathbb{C}$ for $k \in [n-1]$, then

(4.3)
$$|A| = \sum_{l=1}^{n} (-a)^{n-l} \sum_{\{s_1, \dots, s_l\} \subseteq [n]} \prod_{q=1}^{l} a_{s_q, s_{q-1}+1}.$$

Proof. By definition of the determinant of a square matrix we have

$$|A| = \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n,\sigma(n)},$$

where the sum is taken over permutations σ of the set [n] and $\operatorname{sgn}(\sigma)$ is the sign of σ . However, if there exists $k \in [n]$ such that $\sigma(k) > k + 1$, then $a_{k,\sigma(k)} = 0$, and consequently we can assume the summation runs only over permutations σ of [n] for which $\sigma(k) \leq k + 1$, $k \in [n]$, that is, over σ satisfying condition (4.1). By Lemma 4.1, for any permutation σ satisfying condition (4.1), we get

$$\operatorname{sgn}(\sigma)a_{1,\sigma(1)}\dots a_{n,\sigma(n)} = (-1)^{n-l} \prod_{k \in X(\sigma)} a_{k,\sigma(k)} \prod_{k \in [n] \setminus X(\sigma)} a_{k,k+1} = (-1)^{n-l} \prod_{q=1}^{l} a_{s_q,s_{q-1}+1} \prod_{k \in [n] \setminus \{s_i: i \in [l]\}} a_{k,k+1},$$

where $X(\sigma) = \{s_1, ..., s_l\}, s_1 < ... < s_l, s_l = n \text{ and } s_0 = 0$. Therefore, by Remark 4.2,

$$|A| = \sum_{l=1}^{n} (-1)^{n-l} \sum_{\{s_1, \dots, s_l\} \subseteq [n]} \left(\prod_{q=1}^{l} a_{s_q, s_{q-1}+1} \prod_{k \in [n] \setminus \{s_i: i \in [l]\}} a_{k, k+1} \right),$$

which proves (4.2). Formula (4.3) is an obvious consequence of (4.2).

Now, we are going to prove the central result of this section.

Theorem 4.4. Let $a \in \mathbb{C} \setminus \mathbb{N}$ and $(b_n)_{n \in \mathbb{N}_0}$ be a sequence of complex numbers such that $|b_0| < 1$ and $b_n \neq 0$ for some $n \in \mathbb{N}$. Suppose that $(c_n)_{n \in \mathbb{N}_0}$ is a sequence defined by the following recursive J.C.P. Miller formula:

$$c_0 := (1+b_0)^a, \quad c_1 := \frac{ac_0b_1}{1+b_0} = ab_1(1+b_0)^{a-1},$$

and, for $n \geq 2$,

$$c_n := \frac{1}{n(1+b_0)} \left[nac_0 b_n + \sum_{k=1}^{n-1} b_k c_{n-k} [ka - (n-k)] \right] = \frac{1}{n(1+b_0)} \sum_{k=1}^n b_k c_{n-k} [ka - (n-k)].$$

Then

(4.4)
$$c_n = a(1+b_0)^{a-1} \times \left[b_n + \sum_{j=1}^{n-1} b_j \sum_{l=1}^{n-j} (1+b_0)^{-l} \sum_{\{s_1,\dots,s_l\} \subseteq [n-j]} \prod_{q=1}^l \frac{a(s_q - s_{q-1}) - s_{q-1} - j}{s_q + j} b_{s_q - s_{q-1}} \right].$$

Proof. Let us notice that $c_1 = ab_1(1+b_0)^{a-1}$ and for all $n \ge 2$

$$-\sum_{k=1}^{n-1} \frac{ka - (n-k)}{n(1+b_0)} b_k c_{n-k} + c_n = \frac{anc_0 b_n}{n(1+b_0)} = ab_n (1+b_0)^{a-1}.$$

Fix $n \geq 2$. By the above formulas we have (4.5)

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ -\frac{(a-1)b_1}{2(1+b_0)} & 1 & 0 & & \vdots \\ -\frac{(2a-1)b_2}{3(1+b_0)} & -\frac{(a-2)b_1}{3(1+b_0)} & 1 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ \vdots & & & \ddots & 1 & 0 \\ -\frac{((n-1)a-1)b_{n-1}}{n(1+b_0)} & -\frac{((n-2)a-2)b_{n-2}}{n(1+b_0)} & \dots & -\frac{(a-(n-1))b_1}{n(1+b_0)} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ \vdots \\ \vdots \\ c_n \end{bmatrix} = a(1+b_0)^{a-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ \vdots \\ c_n \end{bmatrix}.$$

Let us denote the lower-triangular matrix on the left-hand side as $B = [b_{i,j}]_{i,j \in [n]}$; then

$$b_{i,j} := \begin{cases} 0, & i < j, \\ 1, & i = j, \\ \frac{-((i-j)a-j)}{i(1+b_0)} b_{i-j}, & i > j. \end{cases}$$

It is clear that |B| = 1. Let B_i^j be the cofactor of the entry $b_{i,j}$, that is, the product of $(-1)^{i+j}$ and the determinant of the matrix obtained from B by crossing out its *i*th row and *j*th column, $i, j \in [n]$. It is not difficult to see that $B_i^j = 0$, for i > j, and $B_i^i = 1$.

Let us now consider the case i < j. We have

$$B_{i}^{j} = (-1)^{i+j} \begin{vmatrix} b_{i+1,i} & 1 & 0 & \dots & 0 \\ b_{i+2,i} & b_{i+2,i+1} & 1 & 0 & \vdots \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & 1 & 0 \\ \vdots & & & & b_{j-1,j-2} & 1 \\ b_{j,i} & \dots & & & b_{j,j-2} & b_{j,j-1} \end{vmatrix}$$

and denoting for convenience $a_{k,l} := b_{i+k,i+l-1}$, for $k, l \in [j-i]$ with k+1 > l, by Lemma 4.3 and the definition of $b_{i,j}$ we get

$$B_{i}^{j} = (-1)^{i+j} \sum_{l=1}^{j-i} \sum_{\{s_{1},\dots,s_{l}\}\subseteq[j-i]} (-1)^{j-i-l} \prod_{q=1}^{l} a_{s_{q},s_{q-1}+1} = \sum_{l=1}^{j-i} \sum_{\{s_{1},\dots,s_{l}\}\subseteq[j-i]} \left((-1)^{l} \prod_{q=1}^{l} b_{s_{q}+i,s_{q-1}+i} \right) = \sum_{l=1}^{j-i} \sum_{\{s_{1},\dots,s_{l}\}\subseteq[j-i]} (-1)^{l} \prod_{q=1}^{l} \frac{-\left[(s_{q}-s_{q-1})a-s_{q-1}-i\right]}{(s_{q}+i)(1+b_{0})} b_{s_{q}-s_{q-1}} = \sum_{l=1}^{j-i} (1+b_{0})^{-l} \sum_{\{s_{1},\dots,s_{l}\}\subseteq[j-i]} \prod_{q=1}^{l} \frac{(s_{q}-s_{q-1})a-s_{q-1}-i}{s_{q}+i} b_{s_{q}-s_{q-1}}.$$

Now, multiplying equation (4.5) from left by $B^{-1} = [b_{i,j}^{-1}]_{i,j \in [n]}$, where $b_{i,j}^{-1} = B_j^i$, we get

$$c_{n} = \sum_{j=1}^{n} B_{j}^{n} a (1+b_{0})^{a-1} b_{j} = a (1+b_{0})^{a-1} \left(b_{n} + \sum_{j=1}^{n-1} B_{j}^{n} b_{j} \right) =$$

$$= a (1+b_{0})^{a-1} \left[b_{n} + \sum_{j=1}^{n-1} b_{j} \sum_{l=1}^{n-j} (1+b_{0})^{-l} \sum_{\{s_{1},\dots,s_{l}\} \subseteq [n-j]} \prod_{q=1}^{l} \frac{a(s_{q} - s_{q-1}) - s_{q-1} - j}{s_{q} + j} b_{s_{q} - s_{q-1}} \right],$$
which completes the proof.

which completes the proof.

The usefulness of the above theorem can be illustrated by some examples included in Section 6 below.

Remark 4.5. We would like to draw the reader's attention to the paper [22], in which the Authors provide formulas for the coefficients of f^k , where f is a formal power series and k is an integer (some of which can also be found in [8] and references therein). The recursive and explicit forms of those formulas look quite analogous to those presented in this section. However, the methods used in [22] are strongly based on some results concerning integer powers of semicirculant matrices (see [18]) that cannot be easily extended to non-integer (real or complex) exponents, which are considered in this section.

5. Yet another generalization of the J.C.P.Miller formula

This part of the paper is to extend the J.C.P. Miller formula to the case of multivariable formal power series. We are going to follow the lines of proof of Theorem 3.2. To this end we need the following two simple results (cf. [8, Lemma 5.5.2 and Theorem 5.5.3 (Chain Rule)]).

Lemma 5.1. Let $g(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{X}_1(\mathbb{K})$ and $f(x) = \sum_{c \in C} f_c X^c \in \mathbb{X}_q(\mathbb{K})$, where $C := \mathbb{N}_0^q$, $\theta := (0, \ldots, 0) \in C$. Then $g^{(k)} \circ f$ exists, $k \in \mathbb{N}$, if and only if $g \circ f$ exists, where $g^{(0)} := g$ and $g^{(k)}$ is kth formal derivative of $g, k \in \mathbb{N}$.

Proof. By [4, Theorem 10] and the definition of formal derivative the theorem can be stated equivalently as: $g^{(k)}(b_0)$ is convergent in \mathbb{K} for every $k \in \mathbb{N}$ if and only if $g^{(k)}(b_0)$ is convergent in \mathbb{K} for every $k \in \mathbb{N}_0$, where $b_0 := f_{\theta}$. Sufficiency is obvious. So, let us assume that $g^{(0)}(b_0)$ diverges, that is, $g(b_0)$ diverges. Then, by [3, Lemma 1], $g^{(2)}(b_0)$ diverges (cf. the paragraph following Definition 2.3), which ends the proof.

Lemma 5.2 (Chain Rule for the composition of one-variable fps with multivariable fps). Let $g(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{X}_1(\mathbb{K})$ and $f(x) = \sum_{c \in C} f_c X^c \in \mathbb{X}_q(\mathbb{K})$ and suppose that $g \circ f$ exists. Then

$$D_i(g \circ f)(x) = (g' \circ f)(x)D_if(x),$$

where $D_i f$ denotes the formal partial derivative of f with respect to variable $x_i, i \in [q]$.

Proof. Recall that $C = \mathbb{N}_0^q$. Let $f^n(x) = \sum_{c \in C} f_c^n X^c$ denote the *n*th power of $f, f^0(x) := 1$, $n \in \mathbb{N}_0$. By Lemma 5.1, $g' \circ f$ exists. Without loss of generality let us assume that i = 1. We have

$$(g \circ f)(x) = \sum_{c \in C} (\sum_{n=0}^{\infty} g_n f_c^n) X^c,$$

$$(g' \circ f)(x) = \sum_{c \in C} (\sum_{n=0}^{\infty} (n+1)g_{n+1}f_c^n) X^c,$$

$$D_1 f(x) = \sum_{c \in C} (c_1+1)f_{c+e^1} X^c,$$

$$D_1 (g \circ f)(x) = \sum_{c \in C} (c_1+1) (\sum_{n=0}^{\infty} g_n f_{c+e^1}^n) X^c,$$

where $e^1 := (1, 0, \dots, 0) \in C$. Denote $(g' \circ f)(x)D_1f(x) = \sum_{c \in C} h_c X^c$. For $c \in C$, it holds

$$h_{c} = \sum_{a+b=c} (g' \circ f)_{a} (D_{1}f)_{b} = \sum_{a+b=c} \left(\sum_{n=0}^{\infty} (n+1)g_{n+1}f_{a}^{n} \right) (b_{1}+1)f_{b+e^{1}} = \sum_{n=0}^{\infty} (n+1)g_{n+1} \left(\sum_{a+b=c} f_{a}^{n} (b_{1}+1)f_{b+e^{1}} \right) = (\star),$$

where $a, b \in C$ under the summation symbols. Since $D_1 f^{n+1}(x) = (n+1)(f^n D_1 f)(x)$ [11, Theorem 4.2], for any $c \in C$ we get $(D_1 f^{n+1})_c = (n+1)(f^n D_1 f)_c$, that is, $(c_1 + 1)f_{c+e^1}^{n+1} = (n+1)\sum_{a+b=c} f_a^n (1+b_1) f_{b+e^1}$. Hence,

$$(\star) = \sum_{n=0}^{\infty} g_{n+1}(c_1+1) f_{c+e^1}^{n+1} = (c_1+1) \sum_{n=0}^{\infty} g_{n+1} f_{c+e^1}^{n+1} = (c_1+1) \sum_{n=1}^{\infty} g_n f_{c+e^1}^n.$$

Due to the equality $f_{c+e^1}^0 = 0$, $c \in C$, we obtain $h_c = (c_1 + 1) \sum_{n=0}^{\infty} g_n f_{c+e^1}^n$, $c \in C$, which proves the claim.

Let $e^i \in C$ be the point that has 1 on *i*th coordinate and 0 everywhere else, $i \in [q]$. We are now ready to prove the J.C.P.Miller formula for multivariable series.

Theorem 5.3. Let $f \in \mathbb{X}_q(\mathbb{K})$ be a formal power series over \mathbb{K} with $\deg(f) \neq 0$: $f(x) = \sum_{c \in C} f_c X^c$. Then

$$(B_r \circ f)(x) \in \mathbb{X}_q(\mathbb{K}), \quad r \in \mathbb{C} \setminus \mathbb{N}$$

if and only if $|f_{\theta}| < 1$.

Moreover, for $|f_{\theta}| < 1$, the coefficients of $(B_r \circ f)(x)$ denoted by $h_c, c \in C$, satisfy:

$$h_{\theta} = (1+b_0)^r, \qquad h_{e^i} = rf_{e^i}(1+f_{\theta})^{r-1}, \ i \in [q],$$

and for $c' = c + e^i \in C_{n+1}$, $i \in [q]$, $c \in C_n$, $n \ge 2$:

(5.1)
$$h_{c'} = \frac{1}{(c_i+1)(1+f_{\theta})} \times \left(r \sum_{b \le c: \ b_i = c_i} (b_i+1) f_{b+e^i} h_{c-b} + \sum_{b \le c-e^i} (b_i+1) [rf_{b+e^i} h_{c-b} - f_{c-b} h_{b+e^i}] \right), \quad i \in [q],$$

where $b \in C$.

Proof. That the composition $B_r \circ f$ exists if and only if $|f_{\theta}| < 1$ is clear by Theorem 3.1 above and [4, Theorem 10]). Moreover, the constant term h_{θ} is given by

$$h_{\theta} = \sum_{n=0}^{\infty} {\binom{r}{n}} f_{\theta}^n = \sum_{n=0}^{\infty} {\binom{r}{n}} (f_{\theta})^n = (1+f_{\theta})^r.$$

By the proof of Theorem 3.2, due to validity of the Chain Rule expressed in Lemma 5.2 and the Right Distributive Law for multivariable power series given in [4, Theorem 20], we see that adapted versions of equation (3.3) are valid under current assumptions, that is, we have

(5.2)
$$(B_i \circ f)(x)(D_i(B_r \circ f))(x) = r(B_r \circ f)(x)D_if(x), \quad i \in [q].$$

We shall now equate corresponding coefficients of the series on both sides of (5.2). Let $(B_a \circ f)(x) = \sum_{c \in C} h_c X^c$. Denoting $(B_1 \circ f)(x) = \sum_{c \in C} \overline{f_c} X^c$, we get

$$(B_1 \circ f)(x)(D_i(B_r \circ f))(x) = \left(\sum_{c \in C} \overline{f}_c X^c\right) \left(\sum_{c \in C} (c_i + 1)h_{c+e^i} X^c\right) = \sum_{c \in C} \left(\sum_{a+b=c} (b_i + 1)\overline{f}_a h_{b+e^i}\right) X^c$$

and

$$r(B_r \circ f)(x)D_i f(x) = r\left(\sum_{c \in C} h_c X^c\right) \left(\sum_{c \in C} (c_i + 1)f_{c+e^i} X^c\right) = \sum_{c \in C} \left(r\sum_{a+b=c} (b_i + 1)f_{b+e^i} h_a\right) X^c.$$

Now, for $c = \theta$,

$$\overline{f}_{\theta}h_{e^{i}} = rf_{e^{i}}h_{\theta} \Leftrightarrow (1+f_{\theta})h_{e^{i}} = rf_{e^{i}}h_{\theta},$$

which results in $h_{e^i} = r f_{e^i} h_{\theta} / (1 + f_{\theta}), i \in [q]$, and

$$h_{e^i} = rf_{e^i}(1+f_\theta)^{r-1}, \quad i \in [q].$$

Observe that we have just obtained h_c , for $c \in C_0 \cup C_1$. Let us assume that all $h_c, c \in$ $C_0 \cup \ldots \cup C_n$, are known. By (5.2), we have for $c \in C_n$, $i \in [q]$,

$$\sum_{a+b=c} (b_i+1)\overline{f}_a h_{b+e^i} = r \sum_{a+b=c} (b_i+1)f_{b+e^i} h_a \Leftrightarrow (c_i+1)(1+f_{\theta})h_{c+e^i} + \sum_{a+b=c:a_i \neq 0} (b_i+1)f_a h_{b+e^i} = r \sum_{a+b=c} (b_i+1)f_{b+e^i} h_a.$$

Therefore

$$(c_i+1)(1+f_{\theta})h_{c+e^i} = r\sum_{a+b=c} (b_i+1)f_{b+e^i}h_a - \sum_{a+b=c: a_i \neq 0} (b_i+1)f_ah_{b+e^i}h_{b+e^i}h_a - \sum_{a+b=c} (b_i+1)f_ah_{b+e^i}h_a - \sum_{a+$$

which is equivalent to

$$(c_i+1)(1+f_{\theta})h_{c+e^i} = r\sum_{a+b=c:\,a_i=0} (b_i+1)f_{b+e^i}h_a + r\sum_{a+b=c:\,a_i\neq 0} (b_i+1)f_{b+e^i}h_a - \sum_{a+b=c:\,a_i\neq 0} (b_i+1)f_ah_{b+e^i}h_a + r\sum_{a+b=c:\,a_i\neq 0} (b_i+1)f_{a+b+e^i}h_a - \sum_{a+b=c:\,a_i\neq 0} (b_i+1)f_ah_{b+e^i}h_a + r\sum_{a+b=c:\,a_i\neq 0} (b_i+1)f_{b+e^i}h_a - \sum_{a+b=c:\,a_i\neq 0} (b_i+1)f_ah_{b+e^i}h_a + r\sum_{a+b=c:\,a_i\neq 0} (b_i+1)f_ah_{b+e^i}h_a 0} (b_i+1$$

and, for $i \in [q]$,

$$h_{c+e^{i}} = \frac{1}{(c_{i}+1)(1+f_{\theta})} \times \left(r \sum_{a+b=c: a_{i}=0} (b_{i}+1)f_{b+e^{i}}h_{a} + r \sum_{a+b=c: a_{i}\neq0} (b_{i}+1)f_{b+e^{i}}h_{a} - \sum_{a+b=c: a_{i}\neq0} (b_{i}+1)f_{a}h_{b+e^{i}} \right).$$

Notice that all the coefficients in the sum on the right-hand side of the last equation are known by the assumption and equation (5.1) results from rearranging the sum.

Remark 5.4. All indexes of the coefficients of $B_r \circ f$ that appear on the right-hand side of formula (5.1) belong to $C_0 \cup \ldots \cup C_n$. Moreover, since any $c' \in C_{n+1}$ can be expressed as $c' = c + e^i$ for some $c \in C_n$, $i \in [q]$, the formula allows for the recursive computation of all coefficients of $B_r \circ f$.

6. Applications

6.1. Determinantion of coefficients by the explicit form of J.C.P. Miller formula. Let $f = b_0 + x^{\overline{n}} \in \mathbb{X}(\mathbb{C})$, where $|b_0| < 1$, $\overline{n} \in \mathbb{N}$, $\overline{n} > 2$. We will calculate all coefficients of the formal series $B_a \circ f = \sum_{n=0}^{\infty} c_n x^n$ with $a \in \mathbb{C} \setminus \mathbb{N}_0$. We have $c_0 = (1 + b_0)^a$, $c_1 = 0$ (because $b_1 = 0$), and, by (4.4), the following equalities

$$c_n = a(1+b_0)^{a-1} \left[b_n + \sum_{j=1}^{n-1} b_j \sum_{l=1}^{n-j} (1+b_0)^{-l} \sum_{\{s_1,\dots,s_l\} \subseteq [n-j]} \prod_{q=1}^l \frac{a(s_q - s_{q-1}) - s_{q-1} - j}{s_q + j} b_{s_q - s_{q-1}} \right],$$

for $n \ge 2$ (recall $s_0 = 0$). Therefore:

(1) $c_n = 0$ for $1 < n < \overline{n}$ (because then $b_1 = \dots = b_n = 0$),

(2) $c_{\overline{n}} = a(1+b_0)^{a-1}b_{\overline{n}} = a(1+b_0)^{a-1}$ (because $b_{\overline{n}} = 1$),

(3) for $n > \overline{n}$, we have

$$c_n = a(1+b_0)^{a-1} \sum_{l=1}^{n-\overline{n}} (1+b_0)^{-l} \sum_{\{s_1,\dots,s_l\}\subseteq [n-\overline{n}]} \prod_{q=1}^l \frac{a(s_q-s_{q-1})-s_{q-1}-\overline{n}}{s_q+\overline{n}} b_{s_q-s_{q-1}},$$

because if n > 0, then $b_n \neq 0$ if and only if $n = \overline{n}$. Moreover,

$$0 \neq \prod_{q=1}^{l} \frac{a(s_q - s_{q-1}) - s_{q-1} - \overline{n}}{s_q + 1} b_{s_q - s_{q-1}} \qquad (\star)$$

if and only if $s_q - s_{q-1} = \overline{n}$, $q \in [l]$ with $l \geq 1$, because $a \in \mathbb{C} \setminus \mathbb{N}_0$. In view of the equality $s_l = n - \overline{n}$, we have that condition (\star) is satisfied if and only if $s_l = n - \overline{n}, s_{l-1} = n - 2\overline{n}, \ldots, s_1 = n - l\overline{n}, s_0 = n - (l+1)\overline{n}$. However, $s_0 = 0$, so (\star) holds if and only if $l = \frac{n}{\overline{n}} - 1 \in \mathbb{N}$ and $s_j = j\overline{n}, j \in [l]$. Therefore, it is possible that $c_n \neq 0, n \geq 2$, if and only if $n = k\overline{n}$ for some $k \in \mathbb{N}$. By the preceding analysis we get for $k \in \mathbb{N}, k \geq 2$,

 $c_{k\overline{n}} =$

$$a(1+b_0)^{a-1} \sum_{l=1}^{(k-1)\overline{n}} (1+b_0)^{-l} \sum_{\{s_1,\dots,s_l\} \subseteq [(k-1)\overline{n}]} \prod_{q=1}^l \frac{a(s_q-s_{q-1})-s_{q-1}-\overline{n}}{s_q+\overline{n}} b_{s_q-s_{q-1}} = a(1+b_0)^{a-1} (1+b_0)^{-(k-1)} \prod_{q=1}^{k-1} \frac{a\overline{n}-q\overline{n}}{q\overline{n}+\overline{n}} = a(1+b_0)^{a-k} \prod_{q=1}^{k-1} \frac{a-q}{q+1} = \frac{a}{k} (1+b_0)^{a-k} \binom{a-1}{k-1},$$

It is obvious that $c_{\overline{n}} = a(1+b_0)^{a-1} = \frac{a}{1}(1+b_0)^{a-1} {a-1 \choose 0}$, so the above formula holds for k = 1 as well.

Therefore $B_a \circ f = \sum_{n=0}^{\infty} c_n x^n$, where

$$c_n = \begin{cases} 0, & \frac{n}{\overline{n}} \notin \mathbb{N}_0, \\ (1+b_0)^a, & n = 0, \\ \frac{a}{k}(1+b_0)^{a-k} {a-1 \choose k-1}, & n = k\overline{n}, \ k \in \mathbb{N}. \end{cases}$$

6.2. Multiplicative inverses of fps. We are now going to find a general formula for the inverse of a formal power series, provided it exists.

Let $f = b_0 + b_1 x + \ldots \in \mathbb{X}(\mathbb{C})$ with $b_0 = 1$. We will derive explicit formulas for the coefficients of f^{-1} using formula (4.4) (cf. [12, vol. 1, p.17], [17] or [22]).

By Theorem 3.1, the composition $B_{-1} \circ (f-1)$ exists. Denote $B_{-1} \circ (f-1) = c_0 + c_1 x + \dots$ By Theorem 3.2, we have $c_0 = \frac{1}{b_0} = 1$, $c_1 = -\frac{b_1}{b_0^2} = -b_1$ and $c_n = -\frac{1}{b_0} \sum_{k=1}^n b_k c_{n-k} = -\sum_{k=0}^{n-1} c_k b_{n-k}$ for n > 1, so by [8, Theorem 1.1.8], $B_{-1} \circ (f-1) = f^{-1}$. We will calculate c_n using Theorem 4.4. We have, for n > 1,

$$c_{n} = -\left[b_{n} + \sum_{j=1}^{n-1} b_{j} \sum_{l=1}^{n-j} \sum_{\{s_{1},\dots,s_{l}\} \subseteq [n-j]} \prod_{q=1}^{l} \frac{-(s_{q} - s_{q-1}) - s_{q-1} - j}{s_{q} + j} b_{s_{q} - s_{q-1}}\right] = -\left[b_{n} + \sum_{j=1}^{n-1} b_{j} \sum_{l=1}^{n-j} (-1)^{l} \sum_{\{s_{1},\dots,s_{l}\} \subseteq [n-j]} \prod_{q=1}^{l} b_{s_{q} - s_{q-1}}\right] = -b_{n} + \sum_{j=1}^{n-1} b_{j} \sum_{l=1}^{n-j} (-1)^{l+1} \sum_{\epsilon_{1} + \dots + \epsilon_{l} = n-j} \prod_{q=1}^{l} b_{\epsilon_{q}} \sum_{(\star\star)}^{l} (-1)^{l+1} \sum_{(\star\star)} \sum_{k=1}^{n-1} b_{k} \sum_{l=1}^{n-j} (-1)^{l+1} \sum_{k=1}^{n-j} \sum_{l=1}^{l} b_{k} \sum_{l=1}^{l} b_{k}$$

where $\epsilon_1, \ldots, \epsilon_l \in \mathbb{N}$. The last equality stems from the fact that there is a bijection between the set of sequences $s_1 < \ldots < s_l$ with $s_l = n - j$, $l \in [n - j]$, and the set of positive integer solutions to the equation $\epsilon_1 + \ldots + \epsilon_l = n - j$. The expression $(\star\star)$ can be written as the sum of $-b_n$ and expressions of the form $\gamma_{k_1,n_1,\ldots,k_m,n_m} b_{n_1}^{k_1} \ldots b_{n_m}^{k_m}$, where $n_1k_1 + \ldots + n_mk_m = n$, $n_i \in [n - 1]$, $i \in [m]$, $n_1 < n_2 < \ldots < n_m$, and $k_1 + \ldots + k_m = l + 1$, $k_1, \ldots, k_m \in \mathbb{N}$, $m \in [l + 1]$, $l \in [n - 1]$, $\gamma_{k_1,n_1,\ldots,k_m,n_m} \in \mathbb{Z}$. Observe that to each solution $\epsilon_1, \ldots, \epsilon_l \in \mathbb{N}$ of $\epsilon_1 + \ldots + \epsilon_l = n - j$ for some $j \in [n - 1]$, $l \in [n - j]$, there correspond $m := \#\{j, \epsilon_1, \ldots, \epsilon_l\}$, $\{n_1, \ldots, n_m\} := \{j, \epsilon_1, \ldots, \epsilon_l\}$ with $n_1 < \ldots < n_m$, exactly one $i' \in [m]$ for which $j = n_{i'}$, and $k_i := \#\{s \in [l] : \epsilon_s = n_i\}$, $i \neq i'$, and $k_{i'} := 1 + \#\{s \in [l] : \epsilon_s = n_{i'}\}$. It is clear that we have $\{n_i : i \in [m]\} \subseteq [n - 1]$, $k_1n_1 + \ldots + k_{i'}n_{i'} + \ldots + k_mn_m = n$ and $k_1 + \ldots + k_{i'} + \ldots + k_m = l + 1$. On the other hand, if $n_1k_1 + \ldots + n_mk_m = n$, $\{n_i : i \in [m]\} \subseteq [n - 1]$, $n_1 < n_2 < \ldots < n_m$, and $k_1 + \ldots + k_m = l + 1$, $k_1, \ldots, k_m \in \mathbb{N}$, $m \in [l + 1]$, $l \in [n - 1]$, and $j := n_{i'}$, where $i' \in [m]$ is fixed, then any multiset $\{\epsilon_1, \ldots, \epsilon_l\}$ must which is equal to the multiset $\{n_1, \ldots, n_1, \ldots, n_{i'}, \ldots, n_{i'}, \ldots, n_m\}$ must solves the equation $k_1 \times \ldots + \epsilon_l = n - j$. Thus, there are $\frac{(n_1k_1 + \ldots + n_{i'}(k_{i'}-1) + \ldots + n_mk_m!}{k_1 \dots (k_{i'}-1) \times k_m!} = \frac{(n - n_{i'})!}{k_1 \dots (k_{i'}-1) \dots k_{m_i}!}$ solutions $k_1 + \ldots + \epsilon_l = n - j$. Thus, there are $\frac{(n_1k_1 + \ldots + n_{i'}(k_{i'}-1) + \ldots + n_mk_m!}{k_1 \dots (k_{i'}-1) \dots k_m!}$ solutions

 $\epsilon_1 + \ldots + \epsilon_l = n - j$. Thus, there are $\frac{(n_1k_1 + \ldots + n_{i'}(k_{i'}-1) + \ldots + n_mk_m)!}{k_1! \ldots (k_{i'}-1)! \ldots k_m!} = \frac{(n - n_{i'})!}{k_1! \ldots (k_{i'}-1)! \ldots k_m!}$ solutions $\epsilon_1, \ldots, \epsilon_l \in \mathbb{N}$ corresponding to the given values of $n_i, k_i, i \in [m], m \in [n-1]$, and the fixed i'.

Therefore, by $(\star\star)$, we get

$$\gamma_{n_1,k_1,\dots,n_m,k_m} = (-1)^{k_1+\dots+k_m} \frac{(n-n_1)!}{(k_1-1)!\dots k_m!} + \dots + (-1)^{k_1+\dots+k_m} \frac{(n-n_m)!}{k_1!\dots (k_m-1)!} = (-1)^{k_1+\dots+k_m} \frac{k_1(n-n_1)!+\dots+k_m(n-n_m)!}{k_1!\dots k_m!}.$$

Observe that this formula is also true in the case m = 1, $k_1 = 1$, because $\gamma_{n,1} = (-1)^1 \frac{1 \cdot 0!}{1!} = -1$. Hence, we have, for every n > 1,

$$c_n = \sum_{n_1k_1 + \dots + n_mk_m = n} (-1)^{k_1 + \dots + k_m} \frac{k_1(n - n_1)! + \dots + k_m(n - n_m)!}{k_1! \dots k_m!} b_{n_1}^{k_1} \dots b_{n_m}^{k_m}$$

where the sum runs over all $m, n_1, \ldots, n_m, k_1, \ldots, k_m \in \mathbb{N}$ satisfying the equation $n_1k_1 + \ldots + n_mk_m = n$ with $n_1 < \ldots < n_m$.

TABLE 1.

n	c_n	a_n	n	c_n	a_n
0	1.2247448713915900	1.000000000000000000000000000000000000	16	0.0000035796625374	0.0000277881868848
1	0.00000000000000000	1.2247448713915900	17	0.00000000000000000	0.0000125869403043
2	0.2041241452319310	0.7500000000000000000000000000000000000	18	-0.0000003602232440	0.0000056387449858
3	0.00000000000000000	0.3742275995918740	19	0.00000000000000000	0.0000024442987887
4	0.0850517271799714	0.17708333333333333	20	-0.0000002793002863	0.0000010450469782
5	0.00000000000000000	0.0910053480825694	21	0.00000000000000000	0.0000004327886893
6	0.0198454030086600	0.0486689814814815	22	-0.000000039576999	0.0000001772501339
7	0.00000000000000000	0.0256268954157747	23	0.00000000000000000	0.000000741299381
8	0.0022444205783604	0.0132621321097884	24	0.000000200040961	0.000000312378594
9	0.00000000000000000	0.0064852176354488	25	0.00000000000000000	0.000000139531465
10	-0.0000511885395065	0.0031089507321061	26	0.000000025459408	0.000000063022832
11	0.00000000000000000	0.0014364173446264	27	0.00000000000000000	0.000000027839502
12	-0.0000246098747627	0.0006538262527318	28	-0.000000012350587	0.000000012100920
13	0.00000000000000000	0.0002972040963190	29	0.00000000000000000	0.000000004632152
14	0.0000117893019101	0.0001337370698301	30	-0.000000003410592	0.000000001648688
15	0.0000000000000000000000000000000000000	0.0000612246949009	-	-	-

Let us notice that one can easily extend the above method to any series with $b_0 \in \mathbb{C} \setminus \{0\}$, because $(\alpha f)^{-1} = \alpha^{-1} f^{-1}$ for any $\alpha \in \mathbb{C}$, $\alpha \neq 0$.

6.3. Approximate solution to a differential equation by the general J.C.P.Miller formula. Let us consider the following initial value problem:

$$y' = (1 + \frac{1}{2}e^{x^2})^{1/2}y, \qquad y(0) = 1.$$

We are going to find an ε -solution to this problem with help of formal power series. To this end we treat y as a formal power series of an indeterminate x. Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$. We are interested in determining the coefficients a_n , $n \in \mathbb{N}_0$, so that the above initial value problem would be satisfied. Hence the initial value problem treated as a formal differential initial value problem gives, by the initial condition, $a_0 = 1$, and

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n = \underbrace{\left(1 + \sum_{n=0}^{\infty} \frac{1}{2 \cdot n!} x^{2n}\right)^{1/2}}_{F(x)} \sum_{n=0}^{\infty} a_n x^n. \quad (\star)$$

Notice that F(x) is the composition of $B_{1/2}$ with $f(x) := \frac{1}{2}e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{2\cdot n!}x^{2n}$ and since f(0) = 1/2 we can apply the generalized J.C.P. Miller formula Theorem 3.2 to compute coefficients of $F(x) = \sum_{n=0}^{\infty} c_n x^n$. Then the right hand side is the product of two formal power series F(x) and y(x) and we obtain the coefficients of y(x) by equating corresponding coefficients of y'(x) and F(x)y(x). Thus,

$$a_0 = 1, a_1 = c_0, a_2 = \frac{1}{2}(a_0c_1 + a_1c_0), \dots, a_n = \frac{1}{n}\sum_{i=0}^{n-1}a_ic_{n-1}$$

Values of coefficients c_n and a_n are presented in Table 1. Let us now fix the degree n = 20of ε -solution: $y_{20}(x) := \sum_{n=0}^{20} a_n x^n$, $x \in [0, 1]$. Table 2 contains values of y_{20} and differences

TABLE 2.

x	$y_{20}(x)$	Difference at x	x	$y_{20}(x)$	Difference at x
0	1.0000000000000000	0.00000000000000000	0.4	1.63956586946161	-0.000000000001172
0.01	1.01232282472150	0.00000000000000000	0.41	1.66036516236899	-0.000000000001958
0.02	1.02479791987633	0.00000000000000000	0.42	1.68146063815948	-0.000000000003162
			0.43	1.70285790196189	-0.0000000000005080
0.33	1.50182164480892	-0.0000000000000020	0.32	1.48320605611964	0.00000000000000000
0.44	1.72456270488050	-0.000000000008091	0.45	1.74658094894847	-0.000000000012723
0.34	1.52069329521405	-0.000000000000044	0.46	1.76891869227989	-0.000000000019833
0.35	1.53982560178267	-0.0000000000000075	0.47	1.79158215442945	-0.000000000030655
0.36	1.55922327151806	-0.000000000000144	0.48	1.81457772196906	-0.000000000046909
0.37	1.57889112761902	-0.000000000000246	0.49	1.83791195429131	-0.0000000000071201
0.38	1.59883411326048	-0.0000000000000417	0.5	1.86159158965003	-0.0000000000107172
0.39	1.61905729552070	-0.0000000000000706	-	-	-

between the left-hand and right-hand sides of our initial value problem (with y replaced with y_{20}) at grid points of [0, 1], grid size is 0.01.

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