# Modelling Mobility: A Discrete Revolution 

(Extended Abstract)

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#### Abstract

We introduce a new approach to model and analyze Mobility. It is fully based on discrete mathematics and yields a class of mobility models, called the Markov Trace Model. This model can be seen as the discrete version of the Random Trip Model: including all variants of the Random Way-Point Model 14.

We derive fundamental properties and explicit analytical formulas for the stationary distributions yielded by the Markov Trace Model. Such results can be exploited to compute formulas and properties for concrete cases of the Markov Trace Model by just applying counting arguments.

We apply the above general results to the discrete version of the Manhattan Random Way-Point over a square of bounded size. We get formulas for the total stationary distribution and for two important conditional ones: the agent spatial and destination distributions.

Our method makes the analysis of complex mobile systems a feasible task. As a further evidence of this important fact, we first model a complex vehicular-mobile system over a set of crossing streets. Several concrete issues are implemented such as parking zones, traffic lights, and variable vehicle speeds. By using a modular version of the Markov Trace Model, we get explicit formulas for the stationary distributions yielded by this vehicular-mobile model as well.


Keywords: Models of Mobility, Mobile Ad-Hoc Networks, Discrete Markov Chains.

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## 1 Introduction

A crucial issue in modeling mobility is to find a good balance between the goal of implementing important features of concrete scenarios and the possibility to study the model from an analytical point of view. Several interesting approaches have been introduced and studied over the last years [3, 11, 16, 14]. Among them, we focus on those models where agents move independently and according to some random process, i.e., random mobility models. Nice examples of such random models are the random-way point and the walker models [3, [5, 11, 14, 16] which are, in turn, special cases of a family of mobility models known as random trip model [14].

Mobile networks are complex dynamical systems whose analysis is far to be trivial. In particular, deriving explicit formulas of the relative stationary probabilistic distributions, such as the agent spatial one, requires very complex integral calculus and/or sophisticated tools like the Palm Calculus [4, 6, 14, 15].

A first goal of our study is to make the analysis of such dynamical systems more accessible to the (Theoretical) Computer Science Community by adopting concepts and methods which are typical of this Community. A possible solution for this issue could be that of considering agents that walk over (random) paths of a graph. Nice results on random walks over graphs are available [2, 11], however, such models are not suitable to consider agent speed variations, crossways, parking zones and other concrete aspects of mobile systems.

We propose a new approach to model and analyse mobility. This approach is based on a simple observation over concrete network scenarios:

It is not so important to record every position of the agents at every instant of time and it thus suffices to discretize the space into a set of cells and record the current agent cell at discrete time steps.

We exploit the above observation to get a class of fully-discrete mobility models based on agent movement-traces (in short, traces). A trace is the representation of an agent trajectory by means of the sequence of visited cells. Similarly to the Random Trip Model, our mobile model is defined by fixing the set of feasible traces and the criterium the agent adopts to select the next trace after arriving at the end of the current trace.

We define the (Discrete) Markov Trace Model (in short, MTM) where, at every time step, an agent either is (deterministically) following the selected trace or is choosing at random (according to a given probability distribution) the next trace over a set of feasible traces, all starting from the final cell of the previous trace.
It is important to observe that the same trajectory run at different speeds yields different traces (cell-sequences are in general ordered multi-sets): so, it is possible to model variable agent speeds that may depend on the specific area traffic or on other concrete issues.

A detailed description of the MTM is given in Section2. We here discuss its major features and benefits.

Any MTM $\mathcal{D}$ determines a discrete-time Markov chain $\mathcal{M}_{\mathcal{D}}$ whose generic state is a pair $\langle T, i\rangle$ : an agent has chosen trace $T$ and, at that time step, she is in position $T(i)$.

We first study the stationary distribution(s) of the Markov chain $\mathcal{M}_{\mathcal{D}}$ (in what follows we will say shortly: "stationary distribution of $\mathcal{D}$ "). We show evidence of the generality of our model, derive an explicit form of the stationary distribution(s), and establish existence and uniqueness conditions for the stationary distributions of an MTM.

Our last result for general MTM is a necessary and sufficient condition for uniformness of the stationary distribution of $\mathcal{D}$.

The above results for the stationary phase can be applied to get explicit formulas for the stationary (agent) spatial distribution and the stationary (agent) destination one. The former gives the probability that an agent lies in a given cell, while the latter gives the probability that an agent, conditioned to stay in a cell $v$, has destination cell $w$, for any choice of $v$ and $w$.

The knowledge of such distributions is crucial to achieve perfect simulation, to derive connectivity properties of Mobile Ad-hoc NETworkS (MANETS) defined over the mobility model, and for the study of information spreading over such MANETS [7, 8, 10].

We emphasize that all the obtained explicit formulas can be computed by counting arguments (it mainly concerns calculating the number of feasible traces passing over or starting from a cell). If the agent's behaviour can be described by using a limited set of typical traces (this happens in most of MANETS applications), then such formulas can be computed by a computer in few minutes.

Our MTM model can thus serve as a general framework that allows an analytical study of concrete mobility scenarios. We provide two examples that show its power and applicability.

In the first one, we consider the Manhattan Random Way-Point (MRWP) model 10, 5, 14. This version of the Random Way-Point model is motivated by scenarios where agents travel over an urban zone and try to minimize the number of turns while keeping the chosen route as short as possible. We then implement this model as a specific MTM and we derive explicit formulas for its stationary distributions. In particular, we provide the spatial and the destination distributions for any choice of the cell resolution parameter $\epsilon>0$. We observe that, by taking the limit for $\epsilon \rightarrow 0$, our explicit formula of the spatial distribution coincides to that computed by using rather complex integral calculus in [10 for the continuous space-time MRWP model (in terms of probability density functions).
Finally, we give, for the first time, the destination distribution of the continuous space-time MRWP model as well. Both these formulas have been recently used to derive the first analytical bounds on flooding time for the MRWP model 9 .

Our approach can make the analysis of complex scenarios much simpler: it is just a matter of modelling objects and events as ingredients of an MTM. After doing that, you do not need to prove new properties or new formulas, you can just apply ours.
As a second concrete example of this fact, we consider a more complex vehicular-mobility scenario: The Down-Town model where a set of horizontal and vertical streets cross each other and they alternate with building blocks (see Fig (2). Agents (i.e. vehicles) move over the streets according to Manhattan-like paths and park on the border of the streets (a detailed description of the model is given in Section 4.1). Different agent speeds and red and green events of traffic lights can be implemented by considering different traces over the same street path.

Thanks to a modular version of our MTM model, we are also able to analyze this more complex scenario. In fact, the main advantage of our approach is that a given scenario can be analyzed by simply modelling objects and events as "ingredient" of an MTM, thus obtaining the stationary probability distributions directly from our formulas.

## 2 The Markov Trace Model

The mobility model we are introducing is discrete with respect to time and space. The positions an agent can occupy during her movement belong to the set of points $\mathcal{R}$ and they are traced at discrete time steps. The set $\mathcal{R}$ might be a subset of $\mathbb{R}^{d}$, for some $d=1,2, \ldots$, or
it might be some other set. It is only assumed that $\mathcal{R}$ is a metric space.
A movement trace, or simply a trace, is any (finite) sequence $T=\left(u_{0}, u_{1}, \ldots, u_{k}\right)$ of at least two points. When we mention points we tacitly assume that they belong to $\mathcal{R}$. The points of a trace are not necessarily distinct. The length of a trace $T$ (i.e., the number of points of $T$ ) is denoted by $|T|$ and, for each $i=0,1, \ldots|T|-1$, let $T(i)$ denote the $i$-th point of the trace $T$. A trace $T$ can be interpreted as the recording of the movement of an agent starting from some initial time $t_{0}$ : for every $i=0,1, \ldots,|T|-1, T(i)$ is the traced position of the agent at time $t_{0}+i \cdot \tau$, where $\tau>0$ is the duration of a time step.
In our model, an agent can move along trajectories that are represented by traces. For any trace $T$, let $T_{\text {start }}$ and $T_{\text {end }}$ denote, respectively, the starting point and the ending point of the trace. Let $\mathcal{T}$ be any set (possibly infinite) of traces. We say that $\mathcal{T}$ is endless if for every trace $T \in \mathcal{T}$, there is a trace $T^{\prime} \in \mathcal{T}$ such that $T_{\text {start }}^{\prime}=T_{\text {end }}$.
For any point $u \in \mathcal{R}, \mathcal{T}^{\text {out }}(u)$ denotes the subset of traces of $\mathcal{T}$ whose starting point is $u$. Let

$$
P(\mathcal{T})=\left\{u \mid \mathcal{T}^{\text {out }}(u) \neq \emptyset\right\} \text { and } S(\mathcal{T})=\{\langle T, i\rangle|T \in \mathcal{T} \wedge 1 \leqslant i \leqslant|T|-1\}
$$

A Markov Trace $\operatorname{Model}(M T M)$ is a pair $\mathcal{D}=(\mathcal{T}, \Psi)$ such that:
i). $\mathcal{T}$ is an endless trace set such that $|P(\mathcal{T})|<\infty$;
ii). $\Psi$ is a Trace Selecting Rule for $\mathcal{T}(T S R)$, that is, $\Psi$ is a family of probability distributions $\left\{\psi_{u}\right\}_{u \in P(\mathcal{T})}$ such that for each point $u \in P(\mathcal{T}), \psi_{u}$ is a probability distribution over $\mathcal{T}^{\text {out }}(u)$. Similarly to the random trip model [14], any MTM $\mathcal{D}$ determines a Markov chain $\mathcal{M}_{\mathcal{D}}=$ $(S(\mathcal{T}), P[\mathcal{T}, \Psi])$ whose state space is $S(\mathcal{T})$ and the transition probabilities $P[\mathcal{T}, \Psi]$ are defined as follows: for every $T \in \mathcal{T}$,

- Deterministic-move Rule. For every $i$ with $1 \leqslant i<|T|-1, \operatorname{Pr}(\langle T, i\rangle \rightarrow\langle T, i+1\rangle)=1$;
- Next-trace Rule. For every $T^{\prime}$ in $\mathcal{T}^{\text {out }}\left(T_{\text {end }}\right), \operatorname{Pr}\left(\langle T| T,|-1\rangle \rightarrow\left\langle T^{\prime}, 1\right\rangle\right)=\psi_{T_{\text {end }}}\left(T^{\prime}\right)$;
all the other transition probabilities are 0 . It is immediate to verify that for any $s \in S(\mathcal{T})$ it holds that $\sum_{r \in S(\mathcal{T})} \operatorname{Pr}(s \rightarrow r)=1$.
- Stationary Properties. We first introduce some notions that are useful in studying the stationary distributions of a MTM. For any $u, v \in \mathcal{R}, \mathcal{T}(u, v)$ denotes the subset of traces of $\mathcal{T}$ whose starting point is $u$ and whose ending point is $v$. A crucial Markov chain, determined by a MTM $\mathcal{D}$, is its Kernel: its states are the cells of $\mathcal{D}$, considered as turn points (also known as way-points), and the transition probability $\operatorname{Pr}(u \rightarrow v)$ equals the probability that an agent, lying on $u$, chooses any trace ending in $v$. Given a MTM $\mathcal{D}=(\mathcal{T}, \Psi)$, the Kernel of $\mathcal{D}$ is the Markov chain $\operatorname{Ker}(\mathcal{D})=(P(\mathcal{T}), K[\mathcal{T}, \Psi])$ where the transition probabilities $K[\mathcal{T}, \Psi]$ are defined as follows: for every $u, v \in P(\mathcal{T})$,

$$
\operatorname{Pr}(u \rightarrow v)= \begin{cases}\sum_{T \in \mathcal{T}(u, v)} \psi_{u}(T) & \text { if } \mathcal{T}(u, v) \neq \emptyset \\ 0 & \text { otherwise }\end{cases}
$$

Observe that this definition is sound since, for every $u \in P(\mathcal{T})$, it holds that

$$
\sum_{v \in P(\mathcal{T})} \operatorname{Pr}(u \rightarrow v)=\sum_{v \in P(\mathcal{T})} \sum_{T \in \mathcal{T}(u, v)} \psi_{u}(T)=\sum_{T \in \mathcal{T} \text { out }(u)} \psi_{u}(T)=1
$$

where the second equality derives from the fact that $\mathcal{T}$ is endless. Along with a MTM $\mathcal{D}=$ $(\mathcal{T}, \Psi)$ we will use the following notations. For every $u \in P(\mathcal{T}), \Lambda_{\Psi}(u)=\sum_{T \in \mathcal{T} \text { out }(u)}(|T|-$ 1) $\psi_{u}(T)$. Observe that $\Lambda_{\Psi}(u)$ can be interpreted as the expected length of a trace starting from point $u$. Define also $\mathcal{T}^{\text {in }}(u)=\left\{T \in \mathcal{T} \mid T_{\text {end }}=u\right\}$. The stationary distributions of $\mathcal{D}$ and that of $\operatorname{Ker}(\mathcal{D})$ are strongly related as stated in the following

Theorem 1 Let $\mathcal{D}=(\mathcal{T}, \Psi)$ be any MTM. The following properties hold. a) A map $\pi$ : $S(\mathcal{T}) \rightarrow \mathbb{R}$ is a stationary distribution of $\mathcal{D}$ if and only if a stationary distribution $\sigma$ of $\operatorname{Ker}(\mathcal{D})$ exists such that

$$
\forall\langle T, i\rangle \in S(\mathcal{T}) \quad \pi(\langle T, i\rangle)=\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sigma\left(T_{\text {start }}\right) \psi_{T_{\text {start }}}(T)
$$

b) A map $\sigma: P(\mathcal{T}) \rightarrow \mathbb{R}$ is a stationary distribution of $\operatorname{Ker}(\mathcal{D})$ if and only if a stationary distribution $\pi$ of $\mathcal{D}$ exists such that

$$
\forall u \in P(\mathcal{T}) \quad \sigma(u)=\frac{1}{\sum_{T \in \mathcal{T}} \pi(\langle T, 1\rangle)} \sum_{T \in \mathcal{T} \text { out }(u)} \pi(\langle T, 1\rangle)
$$

## Stationary Distributions: Existence and Uniqueness.

Corollary 2 Let $\mathcal{D}=(\mathcal{T}, \Psi)$ be any MTM. The following properties hold. a) $\mathcal{D}$ has always a stationary distribution. b) $\mathcal{D}$ has a unique stationary distribution if and only if $\operatorname{Ker}(\mathcal{D})$ has a unique stationary distribution.

Next proposition shows that the Kernel of an MTM can be any finite Markov chain.
Proposition 3 Given any Markov chain $\mathcal{M}=(S, P)$ with $S \subseteq \mathcal{R}$, there exists a MTM $\mathcal{D}$ such that $\operatorname{Ker}(\mathcal{D})=\mathcal{M}$.

Let $\mathcal{D}=(\mathcal{T}, \Psi)$ be a MTM. For any two distinct points $u, v \in P(\mathcal{T})$, we say that $u$ is connected to $v$ in $\mathcal{D}$ if there exists a sequence of points of $P(\mathcal{T})\left(z_{0}, z_{1}, \ldots, z_{k}\right)$ such that $z_{0}=u, z_{k}=v$, and, for every $i=0,1, \ldots, k-1, \sum_{T \in \mathcal{T}\left(z_{i}, z_{i}+1\right)} \psi_{z_{i}}(T)>0$ Informally, this can be interpreted as saying that if an agent is in $u$ then, with positive probability, she will reach $v$. We say that $\mathcal{D}$ is strongly connected if, for every $u, v \in P(\mathcal{T}), u$ is connected to $v$. Observe that if $\mathcal{D}$ is not strongly connected then at least a pair of points $u, v \in P(\mathcal{T})$ exists such that $u$ is not connected to $v$.

Theorem 4 If $\mathcal{D}$ is a strongly connected MTM then $\mathcal{D}$ has a unique stationary distribution.
Stationary Distributions: Uniformity. Let $\mathcal{D}=(\mathcal{T}, \Psi)$ be any MTM. We say that $\mathcal{D}$ is uniformly selective if $\forall u \in P(\mathcal{T}), \psi_{u}$ is a uniform distribution. We say that $\mathcal{D}$ is balanced if $\forall u \in P(\mathcal{T}),\left|\mathcal{T}^{\text {in }}(u)\right|=\left|\mathcal{T}^{\text {out }}(u)\right|$ Observe that if a MTM $\mathcal{D}=(\mathcal{T}, \Psi)$ has a uniform stationary distribution then it must be the case that $|S(\mathcal{T})|<\infty$ or, equivalently, $|\mathcal{T}|<\infty$.

Theorem 5 A MTM $\mathcal{D}=(\mathcal{T}, \Psi)$ has a uniform stationary distribution if and only if is both uniformly selective and balanced.

Stationary Spatial and Destination Distributions. We use the following notations. For any trace $T \in \mathcal{T}$ and for any $u \in \mathcal{R}$, define

$$
\#_{T, u}=\left|\left\{i \in \mathbb{N}|1 \leqslant i<|T|-1 \wedge T(i)=u\} \mid \text { and } \mathcal{T}_{u}=\left\{T \in \mathcal{T} \mid \#_{T, u} \geqslant 1\right\}\right.\right.
$$

- We now derive the function $\mathfrak{s}(v)$ representing the probability that an agent lies in point $v \in \mathcal{R}$ w.r.t. the stationary distribution $\pi$. This is called Stationary (Agent) Spatial Distribution. By definition, for any point $u \in \mathcal{R}$, it holds that

$$
\mathfrak{s}(u)=\sum_{\langle T, i\rangle \in S(\mathcal{T}) \wedge T(i)=v} \pi(\langle T, i\rangle)=\sum_{T \in \mathcal{T}_{u}} \#_{T, u} \cdot \pi(\langle T, 1\rangle)
$$

If the stationary distribution $\pi$ is uniform, then $\mathfrak{s}(u)=(1 /|S(\mathcal{T})|) \cdot \sum_{T \in \mathcal{T}_{u}} \#_{T, u}$.
We say that an MTM is simple if, for any trace $T \in \mathcal{T}$ and $u \in \mathcal{R}, \#_{T, u} \leqslant 1$. Then, if the MTM is simple and $\pi$ is uniform, then it holds

$$
\begin{equation*}
\mathfrak{s}(u)=\frac{\left|\mathcal{T}_{u}\right|}{|S(\mathcal{T})|} \tag{1}
\end{equation*}
$$

- Another important distribution is given by function $\mathfrak{d}_{u}(v)$ representing the probability that an agent has destination $v$ under the condition she is in position $u$. This function will be called Stationary (Agent) Destination Distribution. By definition, it holds that

$$
\mathfrak{d}_{u}(v)=\frac{\sum_{\langle T, i\rangle \in S(\mathcal{T}) \wedge T(i)=u \wedge T_{\text {end }}=v} \pi(\langle T, i\rangle)}{\mathfrak{s}(u)}=\frac{\sum_{T \in \mathcal{T}_{u} \wedge T_{\text {end }}=v} \#_{T, u} \cdot \pi(\langle T, 1\rangle)}{\mathfrak{s}(u)}
$$

We define $\Gamma_{u}(v)=\left|\mathcal{T}_{u} \cap \mathcal{T}^{\mathrm{in}}(v)\right|$ and $\Gamma_{u}=\left|\mathcal{T}_{u}\right|$ and observe that, if the MTM is simple and $\pi$ is uniform, then

$$
\begin{equation*}
\mathfrak{d}_{u}(v)=\frac{\Gamma_{u}(v)}{\Gamma_{u}} \tag{2}
\end{equation*}
$$

## 3 The Manhattan Random-Way Point

In this section, we study a mobility model, called Manhattan Random-Way Point, an interesting variant of the Random-Way Point that has been recently studied in [10].
Consider a finite 2-dimensional square of edge length $L>0$. A set $A$ of $n$ independent Agents move over this square according to the following random rule. Starting from an initial position $\left(x_{0}, y_{0}\right)$, every agent selects a destination $(x, y)$ uniformly at random in the square (i.e. every point of the square has the same probability to be chosen). Then, the agent chooses (again uniformly at random) between the two feasible Manhattan (shortest) paths. Once the destination and the feasible path are randomly selected, the agents start following the chosen route with speed determined by the parameter V . In the sequel, we assume that all agents have the same speed $V$ that represents the travelled distance by an agent in the time unit. However, a variable agent speed can be easily modelled and analyzed by considering more traces for any source-destination pair. Once arrived at the selected destination, every agent independently re-applies the process described above again and again. This infinite process yields the Manhattan Random Way-Point.
We here consider a discrete version of the Manhattan Random-Way Point which is in fact a Markov Trace Model. Agents will act over a square cell grid of arbitrary-high resolution according to a global discrete clock. Every agent, within the next time step, can reach any grid point (that can be also considered as a square cell) which is adjacent to its current position. We emphasize that, as the grid resolution increases, the time unit decreases. This scalability allows to observe the process at arbitrary-small time unit and space resolution while preserving the ability to choice any possible agent speed.
In order to formally define the MTM, we introduce the following support graph $G_{\epsilon}\left(V_{\epsilon}, E_{\epsilon}\right)$ where $V_{\epsilon}=\{(i \epsilon, j \epsilon): i, j \in\{0,1, \ldots, N-1\}\}$ and $E_{\epsilon}=\left\{(u, v): u, v \in V_{\epsilon} \wedge d(u, v)=\epsilon\right\}$ where, here and in the sequel, $N=\lceil L / \epsilon\rceil$ and $d(\cdot, \cdot)$ is the Euclidean distance.
Now, given any point $v \in V_{\epsilon}$, we define a $\operatorname{set} \mathcal{C}(v)$ of feasible paths from $v$ as follows. For any point $u, \mathcal{C}(v)$ includes the (at most) two Manhattan paths having exactly one corner point. More precisely, let $v=(x, y)$ and $u=\left(x^{\prime}, y^{\prime}\right)$ we consider the path having first the horizontal


Figure 1: The spatial density function is shown by a gradation of gray (black corresponds to the maximum density and white corresponds to the minimum density). The destination probability over the cross of agent position $(L / 3, L / 4)$ is shown in gradation of blue.
segment from $(x, y)$ to $\left(x^{\prime}, y\right)$ and then the vertical segment to $\left(x^{\prime}, y^{\prime}\right)$. The second path is symmetrically formed by the vertical segment from $(x, y)$ to $\left(x, y^{\prime}\right)$ and the horizontal segment to $\left(x^{\prime}, y^{\prime}\right)$. Observe that if $x=x^{\prime}$ or $y=y^{\prime}$, then the two paths coincides. We are now able to define the Manhattan Markov Trace Model (in short Manhattan-mtm) $\left(\mathcal{T}_{\epsilon}, \Psi_{\epsilon}\right)$, where $\mathcal{T}_{\epsilon}=\left\{T \mid T\right.$ is the point sequence of a path in $\mathcal{C}(v)$ for some $\left.v \in V_{\epsilon}\right\}$, and $\Psi_{\epsilon}$ is the uniform $\operatorname{TSR}$ for $\mathcal{T}_{\epsilon}$. It is easy to verify the Manhattan-mtm enjoys the following properties.

Observation 6 The Manhattan-mtm is balanced, uniformly-selective and strongly-connected. So, from Theorems 5 and 4 , the Manhattan-mtm has a unique stationary distribution and it is the uniform one. Moreover, since the Manhattan-mtm is simple, the stationary spatial and the destination distributions are given by Eq.s $\mathbb{\square}$ and 园, respectively.

So, we just have to count the size of some subsets of traces (i.e paths in $G_{\epsilon}\left(V_{\epsilon}, E_{\epsilon}\right)$ ). Due to lack of space, all calculations are given in App. E. The point $(i \epsilon, j \epsilon)$ will be denoted by its grid coordinates $(i, j)$.
The stationary spatial distribution for $\left(\mathcal{T}_{\epsilon}, \Psi_{\epsilon}\right)$ is

$$
\begin{equation*}
\mathfrak{s}_{\epsilon}(i, j)=\frac{3\left(\left(4 N^{2}-6 N+2\right)(i+j)-(4 N-2)\left(i^{2}+j^{2}\right)+6 N^{2}-8 N+3\right)}{\left(N^{4}-N^{2}\right)(4 N-2)} \tag{3}
\end{equation*}
$$

We now study the Manhattan Random-Way Point over grids of arbitrarily high resolution, i.e. for $\epsilon \rightarrow 0$ in order to derive the probability densitiy functions of the stationary distributions. We first compute the probability that an agent lies into a square of center $(x, y)$ (where $x$ and $y$ are the Euclidean coordinates of a point in $V_{\epsilon}$ ) and side length $2 \delta$ w.r.t. the spatial distribution; then, we take the limits as $\delta, \epsilon \rightarrow 0$. We thus get the probability density function of the spatial distribution (see Fig. (1)

$$
\begin{equation*}
s(x, y)=\frac{3}{L^{3}}(x+y)-\frac{3}{L^{4}}\left(x^{2}+y^{2}\right) \tag{4}
\end{equation*}
$$

This is also the formula obtained in [10 for the classic (real time-space) MRWP model. The stationary destination density function can be computed very similarly by applying Eq. 2. This is described in Appendix $\mathbf{F}$. The probability $f_{\left(x_{0}, y_{0}\right)}(x, y)$ that an agent, conditioned
to stay in position $\left(x_{0}, y_{0}\right)$, has destination $(x, y)$ is

It is also possible to derive the probability that an agent, visiting point $\left(x_{0}, y_{0}\right)$, has destination in one of the last four cases (south, west, north, and east) (see Fig. (1)

$$
\phi_{\left(x_{0}, y_{0}\right)}^{\text {south }}=\phi_{\left(x_{0}, y_{0}\right)}^{\text {north }}=\frac{y_{0}\left(L-y_{0}\right)}{4 L\left(x_{0}+y_{0}\right)-4\left(x_{0}^{2}+y_{0}^{2}\right)}, \quad \phi_{\left(x_{0}, y_{0}\right)}^{\text {west }}=\phi_{\left(x_{0}, y_{0}\right)}^{\text {east }}=\frac{x_{0}\left(L-x_{0}\right)}{4 L\left(x_{0}+y_{0}\right)-4\left(x_{0}^{2}+y_{0}^{2}\right)}
$$

We observe that the resulting cross probability, (i.e. the probability an agent has destination over the cross centered on its current position), is equal to $1 / 2$ despite the fact that this region (i.e. the cross) has area 0 . This is crucial for getting an upper bound on flooding time 9 .

## 4 Modular Trace Models

Defining a MTM whose aim is the approximate representation of a concrete mobility scenario might be a very demanding task. We thus introduce a technique that makes the definition of MTMs easier when the mobility scenario is modular. For example, consider vehicular mobility in a city. Any mobility trace can be viewed as formed by the concatenation of trace segments each of which is the segment of the trace that lies on a suitable segment of a street (e.g., the segment of a street between two crossings). Moreover, given a street segment we can consider all the trace segments that lies on it. Then, it is reasonable to think that two alike street segments (e.g., two rectilinear segments approximately of the same length), have similar collections of trace segments. This leads us to the insight that all the traces can be defined by suitably combining the collection of trace segments relative to street segments. It works just like combining Lego blocks.
In the sequel, we use the term trace segment to mean a trace that is a part of longer traces. Given a trace (or a trace segment) $T$, the shadow of $T$, denoted by $S_{T}$, is the sequence of points obtained from $T$ by replacing each maximal run of repetitions of a point $u$ by a single occurrence of $u$. For example, the shadow of $(u, u, v, w, w, w, u)$ is $(u, v, w, u)$ (where $u, v$, $w$ are distinct points). Given any two sequences of points $T$ and $T^{\prime}$ (be them traces, trace segments, or shadows), the combination of $T$ and $T^{\prime}$, in symbols $T \cdot T^{\prime}$, is the concatenation of the two sequences of points. Moreover, we say that $T$ and $T^{\prime}$ are disjoint if no point occurs in both $T$ and $T^{\prime}$. Given any multiset $X$ we denote the cardinality of $X$ by $|X|$, including repeated memberships.
A bundle $B$ is any non-empty finite multiset of trace segments such that, for every $T, T^{\prime} \in B$, $S_{T}=S_{T^{\prime}}$. The common shadow of all the trace segments in $B$ is called the shadow of $B$ and it is denoted by $S_{B}$.
Two bundles $B$ and $B^{\prime}$ are non-overlapping if $S_{B}$ and $S_{B^{\prime}}$ are disjoint. Given two nonoverlapping bundles $B$ and $B^{\prime}$ the combination of $B$ and $B^{\prime}$, in symbols $B \cdot B^{\prime}$, is the bundle consisting of all the trace segments $T \cdot T^{\prime}$ for all the $T, T^{\prime}$ with $T \in B$ and $T^{\prime} \in B^{\prime}$. Notice that, since $B$ and $B^{\prime}$ are non-overlapping, it holds that $S_{B \cdot B^{\prime}}=S_{B} \cdot S_{B^{\prime}}$. Moreover, it holds that $\left|B \cdot B^{\prime}\right|=|B| \cdot\left|B^{\prime}\right|$.

A bundle-path is a sequence of bundles $P=\left(B_{1}, B_{2}, \ldots, B_{k}\right)$ such that any two consecutive bundles of $P$ are non-overlapping. A bundle-path $P$ determines a bundle $\operatorname{Bundle}(P)=B_{1}$. $B_{2} \cdots B_{k}$. Observe that $|\operatorname{Bundle}(P)|=\prod_{i=1}^{k}\left|B_{i}\right|$. Given a bundle-path $P$, let $P_{\text {start }}$ and $P_{\text {end }}$ denote, respectively, the starting point and the ending point of the traces belonging to Bundle $(P)$.
A route $R$ is a multiset of bundle-paths all having the same starting point $R_{\text {start }}$ and the same ending point $R_{\text {end }}$ (i.e. there exist points $R_{\text {start }}$ and $R_{\text {end }}$ such that for every bundle-path $P$ in $R$ it holds $P_{\text {start }}=R_{\text {start }}$ and $\left.P_{\text {end }}=R_{\text {end }}\right)$.
Informally speaking, a bundle-path is formed by traces having the same shadow; introducing such different traces allows to model agents travelling on the same path at different speeds (in this way, it is also possible to change speed around cross-ways and modeling other concrete events). Moreover, routes are introduced to allow different bundle-paths connecting two points. By introducing more copies of the same bundle-path into a route, it is possible to determine paths having more agent traffic. As described below, all such issues can be implemented without making the system analysis much harder: it still mainly concerns counting traces visiting a given bundle.
A Route System is a pair $\mathfrak{R}=(\mathcal{B}, \mathcal{R})$ where: (i) $\mathcal{B}$ is a set of bundles, and (ii) $\mathcal{R}$ is a multiset of routes over the bundles of $\mathcal{B}$ such that, for every $R \in \mathcal{R}$, there exists $R^{\prime} \in \mathcal{R}$ with $R_{\text {start }}^{\prime}=R_{\text {end }}$.
We need some further notations. Let $\mathcal{R}_{u}$ be the multiset $\left\{R \in \mathcal{R} \mid R_{\text {start }}=u\right\}$. $\#_{P, T}$ is the multiplicity of trace $T$ in Bundle $(P)$. $\#_{P, B}$ is the number of occurrences of bundle $B$ in the bundle-path $P$. Moreover $\#_{R, B}=\sum_{p \in R} \#_{P, B}$ and $\#_{B, u}=\sum_{T \in B} \#_{T, u}$, where the sums vary over all the elements, including repeated memberships. Let $\# B$ denote the total number of occurrences of points in all the trace segments of $B$, including repeated memberships, that is,

$$
\# B=\sum_{u \text { in } S_{B}} \#_{B, u}
$$

A Route System $\mathfrak{R}=(\mathcal{B}, \mathcal{R})$ defines a MTM $\mathcal{D}[\mathfrak{R}]=(\mathcal{T}[\mathfrak{R}], \Psi[\mathfrak{R}])$ where
(i) $\mathcal{T}[\mathfrak{R}]=\{T \mid \exists R \in \mathcal{R} \exists P \in R: T \in \operatorname{Bundle}(P)\}$ Notice that $\mathcal{T}[\mathfrak{R}]$ is a set not a multiset.
(ii) for every $u \in P(\mathcal{T}[\mathfrak{R}])$ and for every $T \in \mathcal{T}[\mathfrak{R}]^{\text {out }}(u)$,

$$
\psi[\mathfrak{R}]_{u}(T)=\frac{1}{\left|\mathcal{R}_{u}\right|} \sum_{R \in \mathcal{R}_{u}} \frac{1}{|R|} \sum_{P \in R} \frac{\#_{P, T}}{\mid \text { Bundle }(P) \mid}
$$

The above probability distribution assigns equal probability to routes starting from $u$, then it assigns equal probability to every bundle path of the same route and, finally, it assigns equal probability to every trace occurrence of the same bundle path.
The "stationary" formulas for general route systems are given in Appendix $G$. We here give the simpler formulas for balanced route systems. A Route System $\mathfrak{R}=(\mathcal{B}, \mathcal{R})$ is balanced if, for every $u \in \mathcal{S}$, it holds that
$\left|\left\{R \in \mathcal{R} \mid R_{\text {start }}=u\right\}\right|=\left|\left\{R \in \mathcal{R} \mid R_{\text {end }}=u\right\}\right|$.
We are now able to derive the explicit formulas for the spatial and the destination distributions; observe that such formulas can be computed by counting arguments or by computer calculations.

Proposition 7 Let $\mathfrak{R}=(\mathcal{B}, \mathcal{R})$ be a balanced Route System such that the associated MTM $\mathcal{D}[\mathfrak{R}]=(\mathcal{T}[\mathfrak{R}], \Psi[\mathfrak{R}])$ is strongly connected. Then, (i) The stationary spatial distribution $\mathfrak{s}$ of $\mathcal{D}[\mathfrak{R}]$ is, for every $u \in \mathcal{S}$,

$$
\mathfrak{s}(u)=\frac{1}{\Lambda_{\mathrm{b}}[\mathfrak{R}]} \sum_{B \in \mathcal{B}} \frac{\#_{B, u}}{|B|} \sum_{R \in \mathcal{R}} \frac{\#_{R, B}}{|R|} \text { with } \Lambda_{\mathrm{b}}[\mathfrak{R}]=\sum_{B \in \mathcal{B}} \frac{\# B}{|B|} \sum_{R \in \mathcal{R}} \frac{\#_{R, B}}{|R|}
$$

(ii) the stationary destination distributions $\mathfrak{d}$ of $\mathcal{D}[\mathfrak{R}]$ are, for every $u, v \in \mathcal{S}$,

$$
\mathfrak{d}_{u}(v)=\frac{1}{\mathfrak{s}(u) \Lambda_{\mathrm{b}}[\mathfrak{R}]} \sum_{B \in \mathcal{B}} \frac{\#_{B, u}}{|B|} \sum_{R \in \mathcal{R} \wedge R_{\text {end }}=v} \frac{\#_{R, B}}{|R|}
$$

Observation 8 In the formulas of the stationary distributions stated in Prop. 7, the factor $\frac{\#_{B, u}}{|B|}$ is the only one depending on trace segments. When points are homogeneous square cells of size length $d>0$, the following important interpretation hold. Given a trace segment $T$ and a cell $u$ of $T$, the ratio $v_{T, u}=d /\left(\#_{T, u} \tau\right)$ (where $\tau$ is the time unit) can be interpreted as the (agent) instanteneous speed at cell $u$ in trace $T$. We can thus define the instantaneous slowness as $\operatorname{SLOW}_{T, u}=1 / v_{T, u}=\left(\#_{T, u} \tau\right) /$ d. We thus get $\left(\#_{B, u}\right) /|B|=$ $(d / \tau) \cdot\left(\sum_{T \in B} \operatorname{SLOW}_{T, u}\right) /|B|$. Since $d / \tau$ is a constant, the factor $\#_{B, u} /|B|$ is proportional to the average of the instantaneous slowness in $u$ of bundle $B$ : this average will be denoted as SLOW $_{B, u}$. In order to compute the stationary distributions in Prop. 7 of a given balanced Route System, we can thus provide the values of $\mathrm{SLOW}_{B, u}$ rather than specifying all the trace segments.

### 4.1 Application: The DownTown Model

We now use the Modular Trace Model to describe vehicles that move over a squared city-like support. This support consists of a square of $(n+1) \times(n+1)$ crossing streets (horizontal and vertical) and buildings (where $n$ is an even number). Buildings are interdicted zones, while veichles move and park on the streets. Streets are in turn formed by parking and transit cells and every transit cells of a given street has its own direction (see Fig.s 2 and 3). Moreover, a parking cell has its natural direction given by the direction of its closest transit cells.
A vehicle (agent) moves from one parking cell (the start) to another parking one (the destination) by choosing at random one of the feasible paths. To every feasible path, a set of traces is uniquely associated that models the different ways a vehicle may run over that path: this will also allow to simulate traffic lights on the cross-ways.
Every street is an alternating sequence of cross-ways and blocks. We enumerate horizontal streets by an increasing even index $\{0,2, \ldots, n\}$, starting from the left-top corner. We do the same for vertical streets as well. In this way, every cross-way gets a pair of coordinates $(i, j)$. We say that a direction is positive over a horizontal street if it goes from left to right and while the opposite direction is said to be negative. As for vertical streets, the positive direction is the one going from top to bottom and the opposite is said to be negative (see Fig.s 2 and 3). Then, the blocks of a horizontal street $i$ will be indexed from left to right with coordinates $(i, 1),(i, 3),(i, 5), \ldots$ Similarly, the block of a vertical street $j$ will be indexed from top to bottom with coordinates $(1, j),(3, j), \ldots)$.

We now formally introduce the DownTown Route System $\mathfrak{R}^{D}=\left\langle\mathcal{B}^{D}, \mathcal{R}^{D}\right\rangle$; let's start with the bundle set $\mathcal{B}^{D}$.
[Blocks.] Each (street) block is formed by 4 stripes of $m$ cells each with indexing shown in Fig. 3. Two stripes are for transit while the two external ones are for parking use. The parking stripe adjacent to the transit stripe with positive direction is said positive parking stripe while the other one is said negative parking stripe.
For every $0 \leqslant i, j \leqslant n$ such that $(i$ odd $\wedge j$ even $) \vee(i$ even $\wedge j$ odd $)$, Block ( $i, j$ ) has the following bundles. Bundle $B_{T}^{+}(i, j)$ whose shadow is the stripe having positive direction; Bundle $B_{T}^{-}(i, j)$ is symmetric to $B_{T}^{+}(i, j)$ for the negative direction; For each parking cell


Figure 2: The DownTown model with $n=$ 10. Directions for each block are shown in gray. In blue are shown three possible routes. The starting cells are in green and the ending cells are in red.


Figure 3: The cross way at $(i, j)$ and its four adjacent blocks are shown, with $m=6$. The shadows of two transit bundles are shown blue. The shadows of start bundles, -- and -+ , are shown in green, the other two cases (i.e., ++ , and +- ) are symmetric. The shadows of end bundles, ++ and +- , are shown in red, the other two cases are symmetric. The shadow of $B_{C}^{H,-}(i, j)$ is shown in violet, the other three straight cross bundles are symmetric. The shadow of $B_{C}^{H,++}(i, j)$ is shown in orange, the other seven turn cross bundles are symmetric.
of index $k$, there are four start-Bundles $B_{S, k}^{++}(i, j), B_{S, k}^{+-}(i, j), B_{S, k}^{--}(i, j)$, and $B_{S, k}^{-+}(i, j)$; For each parking cell of index $k$, there are four end-Bundles $B_{E, k}^{++}(i, j), B_{E, k}^{+-}(i, j), B_{E, k}^{--}(i, j)$, and $B_{E, k}^{-+}(i, j)$. The shadows of the above bundles are shown in Fig. 3.
As for the trace segments, thanks to Obs. 区, we only describe the average slowness of the bundles. For the sake of simplicity, we assume the latter depends only on the cell positions. For any transit cell of index $k$, the average slowness in that cell is slow( $k$ ), where slow() is an arbitrary positive function. The slow() function is the same for all transit cells of any of the above bundles. Notice that, by choosing a suitable function slow(), we can implement variable agent speeds and simulate traffic lights at the cross-ways. For instance, assume there is a traffic light on a cross-way having two possible states (red and green): the 2 states alternate at regular fixed time (say 1 minute). Then the slowness of cells along every of the four adjacent transit blocks should be an increasing function of the distance between the cell (determined by index $k$ ) and the crossway. The exact form of such increasing function depends on the average traffic over that cell (notice that the ratio between the highest and the lowest slowness in the same transit block might be order of hundreds). As for the parking cells, we assume that the average slowness is equal to a positive constant pause.
[Cross-Ways.] A cross-way is formed by 12 cells as shown in Fig. 3. We have two types of associated bundles.
For every $0 \leqslant i, j \leqslant n$ such that ( $i$ even $\wedge j$ even ), we have: The 4 straight bundles $B_{C}^{H,+}(i, j)$, $B_{C}^{H,-}(i, j), B_{C}^{V,+}(i, j)$, and $B_{C}^{V,-}(i, j)$; The 8 turn bundles $B_{C}^{H,++}(i, j), B_{C}^{H,+-}(i, j), B_{C}^{H,--}(i, j)$ $B_{C}^{H,-+}(i, j)$; Moreover, $B_{C}^{V,++}(i, j), B_{C}^{V,+-}(i, j), B_{C}^{V,--}(i, j)$, and $B_{C}^{V,-+}(i, j)$. The relative shadows are shown in Fig. 3. Observe that the first sign indicates the sign of the in-direction and the other indicates the out-direction. We assume that the average slowness of cross-ways cells is set to a positive constant cross.
Let us now introduce the set of DownTown routes $\mathcal{R}^{D}$ formed by combining the bundles described above. First of all, every route contains exactly one bundle-path. So we can describe the bundle-path. For every pair $\left\langle c, c^{\prime}\right\rangle$ of parking cells not belonging to the same block, there is (only) one bundle-path that goes from $c$ to $c^{\prime}$. The structure of a bundle-path has a startbundle for $c$, followed by an alternating sequence of block and cross-way bundles, and finally an end-bundle for $c^{\prime}$, i.e., $S, C_{1}, B_{2}, C_{3}, \ldots, B_{k-1}, C_{k}, E$ with $k \geqslant 1$.
Notice that two consecutve bundles of a bundle-path sequence belong to two adjacent crossways and blocks, and the bundles of a bundle-path are all distinct. Let us describe the bundle-paths whose starting cells belong to horizontal blocks. The case of vertical blocks is fully symmetric. let $(i, j)$ be the coordinates of the starting horizontal block and let $(k, z)$ be the coordinates of the ending block.
[Vertical End Block.] The bundle-path is easily determined by the following driving directions: go straight down horizontal street $i$ toward cross-way $(i, z)$; turn to vertical street $z$ till block $(k, z)$.
[Horiz. End Block.] This case yields in turn two subcases.

- Case $j \neq z$. If $k=i$ then go straight down to horizontal street $i$ till block $(i, z)$; if $k \neq i$ then go to cross-way $(i, j+a)$ (where $a=1$ if $i<k$ and $a=-1$ otherwise) and turn to vertical street $j+a$ towards cross-way $(k, j+a)$, then turn to horizontal street $k$ towards block $(k, z)$. - Case $j=z$. The resulting bundle-path depends on whether the starting cell belongs to positive or negative parking stripe. Go to cross-way $(i, j+a)$ (where $a=1$ if we are in the positive case and $a=-1$ otherwise) and turn to vertical street $j+a$ towards cross-way $(k, j+a)$, then turn to horizontal street $k$ towards block $(k, z)$.
Typical examples of the above paths are shown in Fig. 2.


Figure 4: Asymptotical behaviour of the spatial probability distribution of horizontal positive transit cells $(0<\alpha<1)$. The role of vertical coordinate $i$ is almost negligible since the only transit direction of such cells is the positive horizontal one. This representation takes no care about the slowness of the cells. Clearly, in vertical positive cells, the roles of $i$ and $j$ interchange.

Observe there are some cross-way bundles on the border that do not belong to any route. For the sake of convenience, for every bundle $B \in \mathcal{B}^{D}$, we define

$$
\sigma-B=\sum_{R \in \mathcal{R}^{D}} \frac{\#_{R, B}}{|R|}
$$

Since every route contains exactly one bundle-path and a bundle occurs at most once in any bundle-path, then $\sigma-B$ equals the number of bundle-paths containing bundle $B$. In Appendix H, we compute $\sigma-B$ for each kind of bundle in order to get the stationary spatial distribution given by Prop. 7. Due to lack of space, below we give such formulas only for a cell in a horizontal transit block. As for parking cell and cross-way cells we provide explicit formulas in Appendix H,
Let $\Lambda=\Lambda_{\mathrm{b}}\left[\mathfrak{R}^{\mathfrak{D}}\right] / m^{2}$ be the normalization constant with $\Lambda_{\mathrm{b}}\left[\mathfrak{R}^{\mathfrak{D}}\right]$ defined in Prop 7 , Let $u$ be a transit cell of index $k$ (with any $k \in\{1, \ldots, m\}$ ) in the positive transit stripe of the horizontal transit block $(i, j)$ (i.e. $i$ even and $j$ odd). (with $i \notin\{0, n\})$. Then

$$
\begin{gathered}
\mathfrak{s}(u)=\frac{\operatorname{slow}(k)}{\Lambda}\left(a(i, j)+\frac{k}{m} b(j)+\frac{1}{m} c(j)\right) \text { where } \\
a(i, j)=(n-j)(2 n j+j+n-i-1)+(n-2) j-\frac{n}{2}+i-2, \\
b(j)=n(n+1)+\frac{n}{2}-2(n+1) j, \text { and } c(j)=(n+1)(n+j)+n-3
\end{gathered}
$$

An informal representation of the asymptotical behaviour of the above function is given in Fig. 4.

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## A Proof of Theorem 1

In order to prove Theorem 1, we need some preliminary lemmas.
Lemma 9 Let $\mathcal{D}=(\mathcal{T}, \Psi)$ be any MTM and let $\pi$ be a stationary distribution of $\mathcal{D}$. Then, the following properties hold.
(a) For every $\langle T, i\rangle \in S(\mathcal{T})$,

$$
\pi(\langle T, i\rangle)=\pi(\langle T, 1\rangle)
$$

(b) For every $T \in \mathcal{T}$,

$$
\pi(\langle T, 1\rangle)=\psi_{T_{\text {start }}}(T) \sum_{T^{\prime} \in \mathcal{T}^{\mathrm{in}}\left(T_{\text {start }}\right)} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right)
$$

(c) For every $u \in P(\mathcal{T})$,

$$
\sum_{T \in \mathcal{T}^{\text {out }}(u)} \pi(\langle T, 1\rangle)=\sum_{T \in \mathcal{T}^{\text {in }}(u)} \pi(\langle T, 1\rangle)
$$

Proof. Since $\pi$ is a stationary distribution of $\mathcal{D}$, it holds that, for every $\langle T, i\rangle \in S(\mathcal{T})$,

$$
\pi(\langle T, i\rangle)=\sum_{\left\langle T^{\prime}, j\right\rangle \in S(\mathcal{T})} \pi\left(\left\langle T^{\prime}, j\right\rangle\right) \operatorname{Pr}\left(\left\langle T^{\prime}, j\right\rangle \rightarrow\langle T, i\rangle\right)
$$

If $i>1$ then $\operatorname{Pr}\left(\left\langle T^{\prime}, j\right\rangle \rightarrow\langle T, i\rangle\right)=0$ for any $T^{\prime} \neq T$ or $j \neq i-1$. Thus, for every $1<i \leqslant|T|-1$,

$$
\pi(\langle T, i\rangle)=\pi(\langle T, i-1\rangle)
$$

This implies that, for every $\langle T, i\rangle \in S(\mathcal{T})$,

$$
\pi(\langle T, i\rangle)=\pi(\langle T, 1\rangle)
$$

and property (a) is proved.
If instead $i=1$ then $\operatorname{Pr}\left(\left\langle T^{\prime}, j\right\rangle \rightarrow\langle T, 1\rangle\right)=0$ whenever $T_{\text {end }}^{\prime} \neq T_{\text {start }}$ or $j \neq\left|T^{\prime}\right|-1$. Thus,

$$
\begin{aligned}
\pi(\langle T, 1\rangle) & =\sum_{T^{\prime} \in \mathcal{T}^{\text {in }}\left(T_{\text {start }}\right)} \pi\left(\left\langle T^{\prime},\right| T^{\prime}|-1\rangle\right) \psi_{T_{\text {start }}}(T) \\
& =\psi_{T_{\text {start }}}(T) \sum_{T^{\prime} \in \mathcal{T}^{\text {in }}\left(T_{\text {start }}\right)} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right) \quad(\text { from property }(\mathrm{a}))
\end{aligned}
$$

This proves property (b).
From property (b), it derives that

$$
\begin{aligned}
\sum_{T \in \mathcal{T}^{\text {out }}(u)} \pi(\langle T, 1\rangle) & =\sum_{T \in \mathcal{T}^{\text {out }}(u)} \sum_{T^{\prime} \in \mathcal{T}^{\text {in }}(u)} \pi\left(\left\langle T^{\prime},\right| T^{\prime}|-1\rangle\right) \psi_{u}(T) \\
& =\sum_{T \in \mathcal{T}^{\text {out }}(u)} \psi_{u}(T) \sum_{T^{\prime} \in \mathcal{T}^{\text {in }}(u)} \pi\left(\left\langle T^{\prime},\right| T^{\prime}|-1\rangle\right) \\
& =\sum_{T^{\prime} \in \mathcal{T}^{\text {in }}(u)} \pi\left(\left\langle T^{\prime},\right| T^{\prime}|-1\rangle\right) \\
& =\sum_{T^{\prime} \in \mathcal{T}^{\text {in }}\left(T_{\text {start }}\right)} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right)
\end{aligned}
$$

and property (c) is proved.

Lemma 10 Let $\mathcal{D}=(\mathcal{T}, \Psi)$ be any $M T M$ and let $\sigma$ be a stationary distribution of $\operatorname{Ker}(\mathcal{D})$. Let $\pi: S(\mathcal{T}) \rightarrow \mathbb{R}$ be defined as follows:

$$
\forall\langle T, i\rangle \in S(\mathcal{T}) \quad \pi(\langle T, i\rangle)=\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sigma\left(T_{\text {start }}\right) \psi_{T_{\text {start }}}(T)
$$

Then, $\pi$ is a stationary distribution of $\mathcal{D}$.
Proof. First of all, we verify that $\pi$ is a probability distribution over $S(\mathcal{T})$ :

$$
\begin{aligned}
\sum_{\langle T, i\rangle \in S(\mathcal{T})} \pi(\langle T, i\rangle) & =\sum_{\langle T, i\rangle \in S(\mathcal{T})} \frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sigma\left(T_{\text {start }}\right) \psi_{T_{\text {start }}}(T) \\
& =\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sum_{\langle T, i\rangle \in S(\mathcal{T})} \sigma\left(T_{\text {start }}\right) \psi_{T_{\text {start }}}(T) \\
& =\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sum_{u \in P(\mathcal{T})} \sum_{T \in \mathcal{T}^{\text {out }}(u)} \sum_{i=1}^{|T|-1} \sigma(u) \psi_{u}(T) \\
& =\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sum_{u \in P(\mathcal{T})} \sigma(u) \sum_{T \in \mathcal{T}_{\text {out }}(u)}(|T|-1) \psi_{u}(T) \\
& =\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u) \\
& =1
\end{aligned}
$$

For proving that $\pi$ is a stationary distribution of $\mathcal{D}$ it suffices to show that, for every $s \in S(\mathcal{T})$,

$$
\pi(s)=\sum_{r \in S(\mathcal{T})} \pi(r) \operatorname{Pr}(r \rightarrow s)
$$

For every $s \in S(\mathcal{T})$, let $\lambda(s)=\sum_{r \in S(\mathcal{T})} \pi(r) \operatorname{Pr}(r \rightarrow s)$. Let $\langle T, i\rangle$ be any state in $S(\mathcal{T})$, we distinguish two cases.
$(1<i \leqslant|T|-1)$ : In this case $\operatorname{Pr}\left(\left\langle T^{\prime}, j\right\rangle \rightarrow\langle T, i\rangle\right)=0$ for any $T^{\prime} \neq T$ and $\operatorname{Pr}(\langle T, j\rangle \rightarrow\langle T, i\rangle)=0$ for any $j \neq i-1$. Thus,

$$
\begin{aligned}
\lambda(\langle T, i\rangle) & =\sum_{T^{\prime} \in \mathcal{T}} \sum_{j=1}^{\left|T^{\prime}\right|-1} \pi\left(\left\langle T^{\prime}, j\right\rangle\right) \operatorname{Pr}\left(\left\langle T^{\prime}, j\right\rangle \rightarrow\langle T, i\rangle\right) \\
& =\sum_{j=1}^{|T|-1} \pi(\langle T, j\rangle) \operatorname{Pr}(\langle T, j\rangle \rightarrow\langle T, i\rangle) \\
& =\pi(\langle T, i-1\rangle) \operatorname{Pr}(\langle T, i-1\rangle \rightarrow\langle T, i\rangle) \\
& =\pi(\langle T, i-1\rangle)=\pi(\langle T, i\rangle)
\end{aligned}
$$

where the last equality derives from the definition of $\pi$.
$(i=1)$ : In this case $\operatorname{Pr}\left(\left\langle T^{\prime}, j\right\rangle \rightarrow\langle T, 1\rangle\right)=0$ for any $T^{\prime}$ and $j$ such that $T_{\text {end }}^{\prime} \neq T_{\text {start }}$ or $j \neq\left|T^{\prime}\right|-1$.

Thus,

$$
\begin{aligned}
& \lambda(\langle T, 1\rangle)=\sum_{T^{\prime} \in \mathcal{T}} \sum_{j=1}^{\left|T^{\prime}\right|-1} \pi\left(\left\langle T^{\prime}, j\right\rangle\right) \operatorname{Pr}\left(\left\langle T^{\prime}, j\right\rangle \rightarrow\langle T, 1\rangle\right) \\
& =\sum_{T^{\prime} \in \mathcal{T}^{\text {in }}\left(T_{\text {start }}\right)} \pi\left(\left\langle T^{\prime},\right| T^{\prime}|-1\rangle\right) \operatorname{Pr}\left(\left\langle T^{\prime},\right| T^{\prime}|-1\rangle \rightarrow\langle T, 1\rangle\right) \\
& =\sum_{T^{\prime} \in \mathcal{T}^{\text {in }}\left(T_{\text {start }}\right)} \pi\left(\left\langle T^{\prime},\right| T^{\prime}|-1\rangle\right) \psi_{T_{\text {start }}}(T) \\
& =\psi_{T_{\text {start }}}(T) \sum_{T^{\prime} \in \mathcal{T}^{\mathrm{in}}\left(T_{\text {start }}\right)} \pi\left(\left\langle T^{\prime},\right| T^{\prime}|-1\rangle\right) \\
& =\psi_{T_{\text {start }}}(T) \sum_{T^{\prime} \in \mathcal{T}^{\mathrm{in}}\left(T_{\text {start }}\right)} \frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sigma\left(T_{\text {start }}^{\prime}\right) \psi_{T_{\text {start }}^{\prime}}\left(T^{\prime}\right) \\
& =\frac{\psi_{T_{\text {start }}}(T)}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sum_{u \in P(\mathcal{T})} \sum_{T^{\prime} \in \mathcal{T}\left(u, T_{\text {start }}\right)} \sigma(u) \psi_{u}\left(T^{\prime}\right) \\
& =\frac{\psi_{T_{\text {start }}}(T)}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sum_{u \in P(\mathcal{T})} \sigma(u) \sum_{T^{\prime} \in \mathcal{T}\left(u, T_{\text {start }}\right)} \psi_{u}\left(T^{\prime}\right)
\end{aligned}
$$

Observe that $\sum_{T^{\prime} \in \mathcal{T}\left(u, T_{\text {start }}\right)} \psi_{u}\left(T^{\prime}\right)$ is equals to $\operatorname{Pr}\left(u \rightarrow T_{\text {start }}\right)$ (i.e., the transition probability from state $u$ to state $T_{\text {start }}$ of the Markov chain $\operatorname{Ker}(\mathcal{D})$ ). It follows that

$$
\lambda(\langle T, 1\rangle)=\frac{\psi_{T_{\text {start }}}(T)}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sum_{u \in P(\mathcal{T})} \sigma(u) \operatorname{Pr}\left(u \rightarrow T_{\text {start }}\right)
$$

Since $\sigma$ is a stationary probability distribution of $\operatorname{Ker}(\mathcal{D})$, it holds that

$$
\sum_{u \in P(\mathcal{T})} \sigma(u) \operatorname{Pr}\left(u \rightarrow T_{\text {start }}\right)=\sigma\left(T_{\text {start }}\right)
$$

Hence,

$$
\lambda(\langle T, 1\rangle)=\frac{\psi_{T_{\text {start }}}(T)}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sigma\left(T_{\text {start }}\right)=\pi(\langle T, 1\rangle)
$$

Lemma 11 Let $\mathcal{D}=(\mathcal{T}, \Psi)$ be any MTM and let $\pi$ be a stationary distribution of $\mathcal{D}$. Let $\sigma: P(\mathcal{T}) \rightarrow \mathbb{R}$ be defined as follows:

$$
\forall u \in P(\mathcal{T}) \quad \sigma(u)=\frac{1}{\sum_{T \in \mathcal{T}} \pi(\langle T, 1\rangle)} \sum_{T \in \mathcal{T}^{\text {out }}(u)} \pi(\langle T, 1\rangle)
$$

Then, $\sigma$ is a stationary distribution of $\operatorname{Ker}(\mathcal{D})$.
Proof. First of all we prove that $\sigma$ is a probability distribution over $P(\mathcal{T})$ :

$$
\begin{aligned}
\sum_{u \in P(\mathcal{T})} \sigma(u) & =\sum_{u \in P(\mathcal{T})} \frac{1}{\sum_{T \in \mathcal{T}} \pi(\langle T, 1\rangle)} \sum_{T \in \mathcal{T}^{\text {out }}(u)} \pi(\langle T, 1\rangle) \\
& =\frac{1}{\sum_{T \in \mathcal{T}} \pi(\langle T, 1\rangle)} \sum_{u \in P(\mathcal{T})} \sum_{T \in \mathcal{T}^{\text {out }}(u)} \pi(\langle T, 1\rangle) \\
& =\frac{1}{\sum_{T \in \mathcal{T}} \pi(\langle T, 1\rangle)} \sum_{T \in \mathcal{T}} \pi(\langle T, 1\rangle) \\
& =1
\end{aligned}
$$

For proving that $\sigma$ is a stationary distribution of $\operatorname{Ker}(\mathcal{D})$ it suffices to show that, for every $u \in P(\mathcal{T})$,

$$
\sigma(u)=\sum_{v \in P(\mathcal{T})} \sigma(v) \operatorname{Pr}(v \rightarrow u)
$$

We use the following abbreviations $\varsigma(u)=\sum_{v \in P(\mathcal{T})} \sigma(v) \operatorname{Pr}(v \rightarrow u)$ and $\alpha=\frac{1}{\sum_{T \in \mathcal{T}} \pi(\langle T, 1\rangle)}$. It holds that

$$
\begin{aligned}
\varsigma(u) & =\sum_{v \in P(\mathcal{T})} \alpha \sum_{T \in \mathcal{T}^{\text {out }}(v)} \pi(\langle T, 1\rangle) \operatorname{Pr}(v \rightarrow u) \\
& =\alpha \sum_{v \in P(\mathcal{T})} \sum_{T \in \mathcal{T}^{\text {out }}(v)} \pi(\langle T, 1\rangle) \operatorname{Pr}(v \rightarrow u) \\
& =\alpha \sum_{v \in P(\mathcal{T})} \operatorname{Pr}(v \rightarrow u) \sum_{T \in \mathcal{T}^{\text {out }}(v)} \pi(\langle T, 1\rangle) \\
& =\alpha \sum_{v \in P(\mathcal{T})}\left(\sum_{T \in \mathcal{T}(v, u)} \psi_{v}(T)\right)_{T \in \mathcal{T}^{\text {out }}(v)} \pi(\langle T, 1\rangle) \\
& =\alpha \sum_{v \in P(\mathcal{T})}\left(\sum_{T \in \mathcal{T}(v, u)} \psi_{v}(T) \sum_{T^{\prime} \in \mathcal{T}^{\text {in }}(v)} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right) \quad \text { (by Lemma } 9(\mathrm{c})\right) \\
& =\alpha \sum_{v \in P(\mathcal{T})} \sum_{T \in \mathcal{T}(v, u)} \sum_{T^{\prime} \in \mathcal{T}^{\text {in }}(v)} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right) \psi_{v}(T) \\
& =\alpha \sum_{v \in P(\mathcal{T})} \sum_{T \in \mathcal{T}(v, u)} \pi(\langle T, 1\rangle) \quad(\text { by Lemma } 9(\mathrm{~b})) \\
& =\alpha \sum_{T \in \mathcal{T}^{\text {in }}(u)} \pi(\langle T, 1\rangle) \\
& \left.=\alpha \sum_{T \in \mathcal{T}^{\text {out }}(u)} \pi(\langle T, 1\rangle) \quad(\text { by Lemma } 9](\mathrm{c})\right) \\
& =\sigma(u)
\end{aligned}
$$

Proof of Theorem 1, Consider property (a). The "if" part is equivalent to Lemma 10. Now we prove the "only if" part. Let $\pi$ be a stationary distribution of $\mathcal{D}$. From Lemma 11 the map $\sigma: P(\mathcal{T}) \rightarrow \mathbb{R}$ defined as

$$
\forall u \in P(\mathcal{T}) \quad \sigma(u)=\frac{1}{\sum_{T \in \mathcal{T}} \pi(\langle T, 1\rangle)} \sum_{T \in \mathcal{T} \text { out }(u)} \pi(\langle T, 1\rangle)
$$

is a stationary distribution of $\operatorname{Ker}(\mathcal{D})$. For every $\langle T, i\rangle \in S(\mathcal{T})$, let

$$
\lambda(\langle T, i\rangle)=\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sigma\left(T_{\text {start }}\right) \psi_{T_{\text {start }}}(T)
$$

From this definition and from Lemma 9 (a), for showing that $\lambda=\pi$ it suffices to prove that, for every $T \in \mathcal{T}, \lambda(\langle T, 1\rangle)=\pi(\langle T, 1\rangle)$. It holds that

$$
\begin{align*}
& \lambda(\langle T, 1\rangle)=\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sigma\left(T_{\text {start })}\right) \psi_{T_{\text {start }}}(T) \\
& =\frac{1}{\sum_{u \in P(\mathcal{T})} \frac{1}{\sum_{T^{\prime} \in \mathcal{T}} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right)} \sum_{T^{\prime} \in \mathcal{T}^{\text {out }}(u)} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right) \Lambda_{\Psi}(u)} \sigma\left(T_{\text {start })}\right) \psi_{T_{\text {start }}}(T) \\
& =\frac{\sum_{T^{\prime} \in \mathcal{T}} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right)}{\sum_{u \in P(\mathcal{T})} \sum_{T^{\prime} \in \mathcal{T} \text { out }(u)} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right) \Lambda_{\Psi}(u)} \sigma\left(T_{\text {start }}\right) \psi_{T_{\text {start }}}(T) \tag{6}
\end{align*}
$$

Now, observe that

$$
\begin{aligned}
\sum_{u \in P(\mathcal{T})} \sum_{T^{\prime} \in \mathcal{T}^{\text {out }}(u)} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right) \Lambda_{\Psi}(u) & =\sum_{u \in P(\mathcal{T})} \sum_{T^{\prime} \in \mathcal{T}^{\text {out }}(u)} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right) \sum_{T^{\prime \prime} \in \mathcal{T}^{\text {out }}(u)}\left(\left|T^{\prime \prime}\right|-1\right) \psi_{u}\left(T^{\prime \prime}\right) \\
& =\sum_{u \in P(\mathcal{T})} \sum_{T^{\prime \prime} \in \mathcal{T}^{\text {out }}(u)}\left(\left|T^{\prime \prime}\right|-1\right) \psi_{u}\left(T^{\prime \prime}\right) \sum_{T^{\prime} \in \mathcal{T}^{\text {out }}(u)} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right)
\end{aligned}
$$

Since $\pi$ is a stationary distribution of $\mathcal{D}$, from Lemma 9 (b) and (c), it holds that (since $T^{\prime \prime} \in \mathcal{T}^{\text {out }}(u)$ )

$$
\psi_{u}\left(T^{\prime \prime}\right) \sum_{T^{\prime} \in \mathcal{T}^{\text {out }}(u)} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right)=\pi\left(\left\langle T^{\prime \prime}, 1\right\rangle\right)
$$

Thus,

$$
\begin{aligned}
\sum_{u \in P(\mathcal{T})} \sum_{T^{\prime} \in \mathcal{T}^{\text {out }}(u)} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right) \Lambda_{\Psi}(u) & =\sum_{u \in P(\mathcal{T})} \sum_{T^{\prime \prime} \in \mathcal{T}^{\text {out }}(u)}\left(\left|T^{\prime \prime}\right|-1\right) \psi_{u}\left(T^{\prime \prime}\right) \sum_{T^{\prime} \in \mathcal{T}^{\text {out }}(u)} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right) \\
& =\sum_{u \in P(\mathcal{T})} \sum_{T^{\prime \prime} \in \mathcal{T}^{\text {out }}(u)}\left(\left|T^{\prime \prime}\right|-1\right) \pi\left(\left\langle T^{\prime \prime}, 1\right\rangle\right) \\
& \left.=\sum_{u \in P(\mathcal{T})} \sum_{T^{\prime \prime} \in \mathcal{T}^{\text {out }}(u)} \sum_{i=1}^{\left|T^{\prime \prime}\right|-1} \pi\left(\left\langle T^{\prime \prime}, i\right\rangle\right) \quad \text { (by Lemma } 9 \text { (a) }\right) \\
& =\sum_{\left\langle T^{\prime \prime}, i\right\rangle \in \mathcal{T}} \pi\left(\left\langle T^{\prime \prime}, i\right\rangle\right) \\
& =1
\end{aligned}
$$

By combining this with Eq. 6] we have that

$$
\begin{aligned}
\lambda(\langle T, 1\rangle) & =\sum_{T^{\prime} \in \mathcal{T}} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right) \sigma\left(T_{\text {start }}\right) \psi_{T_{\text {start }}}(T) \\
& =\psi_{T_{\text {start }}}(T) \sigma\left(T_{\text {start }}\right) \sum_{T^{\prime} \in \mathcal{T}} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right) \\
& =\psi_{T_{\text {start }}}(T) \frac{1}{\sum_{T^{\prime \prime} \in \mathcal{T}} \pi\left(\left\langle T^{\prime \prime}, 1\right\rangle\right)} \sum_{T^{\prime \prime} \in \mathcal{T}^{\text {out }}\left(T_{\text {start }}\right)} \pi\left(\left\langle T^{\prime \prime}, 1\right\rangle\right) \sum_{T^{\prime} \in \mathcal{T}} \pi\left(\left\langle T^{\prime}, 1\right\rangle\right) \\
& =\psi_{T_{\text {start }}(T) \sum_{T^{\prime \prime} \in \mathcal{T}^{\text {out }}\left(T_{\text {start }}\right)} \pi\left(\left\langle T^{\prime \prime}, 1\right\rangle\right)} \\
& =\pi(\langle T, 1\rangle) \quad(\text { by Lemma } 9(\mathrm{~b}) \text { and }(\mathrm{c}))
\end{aligned}
$$

Now consider property (b). The "if" part is equivalent to Lemma 11 It remains to prove the "only if" part. Let $\sigma$ be a stationary distribution of $\operatorname{Ker}(\mathcal{D})$. From Lemma 10 the map $\pi: S(\mathcal{T}) \rightarrow \mathbb{R}$ defined as

$$
\forall\langle T, i\rangle \in S(\mathcal{T}) \quad \pi(\langle T, i\rangle)=\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sigma\left(T_{\text {start }}\right) \psi_{T_{\text {start }}}(T)
$$

is a stationary distribution of $\mathcal{D}$. For every $u \in P(\mathcal{T})$, let

$$
\varsigma(u)=\frac{1}{\sum_{T \in \mathcal{T}} \pi(\langle T, 1\rangle)} \sum_{T \in \mathcal{T}^{\text {out }}(u)} \pi(\langle T, 1\rangle)
$$

We will prove that $\varsigma=\sigma$. First of all, observe that

$$
\begin{aligned}
\sum_{T \in \mathcal{T}} \pi(\langle T, 1\rangle) & =\sum_{T \in \mathcal{T}} \frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sigma\left(T_{\text {start }}\right) \psi_{T_{\text {start }}}(T) \\
& =\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sum_{T \in \mathcal{T}} \sigma\left(T_{\text {start }}\right) \psi_{T_{\text {start }}}(T) \\
& =\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sum_{u \in P(\mathcal{T})} \sum_{T \in \mathcal{T}^{\text {out }}(u)} \sigma(u) \psi_{u}(T) \\
& =\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sum_{u \in P(\mathcal{T})} \sigma(u) \sum_{T \in \mathcal{T}^{\text {out }}(u)} \psi_{u}(T) \\
& =\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sum_{u \in P(\mathcal{T})} \sigma(u) \quad\left(\text { since } \sum_{T \in \mathcal{T}^{\text {out }}(u)} \psi_{u}(T)=1\right) \\
& =\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \quad\left(\text { since } \sum_{u \in P(\mathcal{T})} \sigma(u)=1\right)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\varsigma(u) & =\frac{1}{\sum_{T \in \mathcal{T}^{\prime}} \pi(\langle T, 1\rangle)} \sum_{T \in \mathcal{T}^{\text {out }}(u)} \pi(\langle T, 1\rangle) \\
& =\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u) \sum_{T \in \mathcal{T}^{\text {out }}(u)} \pi(\langle T, 1\rangle) \\
& =\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u) \sum_{T \in \mathcal{T}^{\text {out }}(u)} \frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sigma\left(T_{\text {start })} \psi_{T_{\text {start }}}(T)\right. \\
& =\sum_{T \in \mathcal{T}^{\text {out }}(u)} \sigma\left(T_{\text {start }}\right) \psi_{T_{\text {start }}}(T) \\
& =\sum_{T \in \mathcal{T}_{\text {out }}(u)} \sigma(u) \psi_{u}(T) \\
& =\sigma(u) \sum_{T \in \mathcal{T}^{\text {out }}(u)} \psi_{u}(T) \\
& =\sigma(u) \quad\left(\text { since } \sum_{T \in \mathcal{T}^{\text {out }}(u)} \psi_{u}(T)=1\right)
\end{aligned}
$$

## B Proof of Corollary 2

Since $\operatorname{Ker}(\mathcal{D})$ is a finite Markov chain, property (a) is a consequence of Theorem 1 (a) and a standard result that says any finite Markov chain has a stationary distribution. Although it is a standard result, it is not so easy to find a clear reference for it. Thus, we outline a proof of this result. Let $\mathcal{M}=(S, P)$ be any Markov chain with $|S|<\infty$. Let $n=|S|$ and let $P=\left(p_{i j}\right)_{i, j \in S}$. Consider the subset of the $n$-dimensional Euclidean space $U=\left\{\left(x_{1}, x_{2} \ldots, x_{n}\right) \mid \forall i x_{i} \geqslant 0 \wedge \sum_{i=1}^{n} x_{i}=1\right\}$. Let $L: U \rightarrow U$ be the map defined as follows:

$$
\forall \bar{x}=\left(x_{1}, \ldots, x_{n}\right) \quad L(\bar{x})=\left(\sum_{j=1}^{n} p_{j 1} x_{j}, \sum_{j=1}^{n} p_{j 2} x_{j}, \ldots, \sum_{j=1}^{n} p_{j n} x_{j}\right)
$$

It is easy to verify that, $\forall \bar{x} \in U, L(\bar{x}) \in U$. Indeed, $\sum_{j=1}^{n} p_{j i} x_{j} \geqslant 0$ for all $i$. Moreover $\sum_{i=1}^{n} \sum_{j=1}^{n} p_{j i} x_{j}=$ $\sum_{j=1}^{n} x_{j} \sum_{i=1}^{n} p_{j i}=\sum_{j=1}^{n} x_{j}=1$, since $\sum_{i=1}^{n} p_{j i}=1$ for all $i$. Observe that any fixed point of $L$, that
is, a point $\bar{u} \in U$ such that $L(\bar{u})=\bar{u}$, is a stationary distribution for the Markov chain $\mathcal{M}$. The existence of at least a fixed point is guaranteed by Brouwer's fixed point theorem (see for instance [17]):

Every continuous map from a convex compact subset $K$ of a Euclidean space to $K$ itself has a fixed point.

Indeed, it is easy to see that $U$ is compact and convex and $L$ is a continuous map.
Property (b) immediately derives from Theorem 1 (a).

## C Proof of Proposition 3

Let $P=\left(p_{u v}\right)$. Define $\mathcal{T}=\left\{(u, v) \mid p_{u v}>0\right\}$. Observe that for every $(u, v) \in \mathcal{T}$, there exists at least a $w \in S$ such that $(v, w) \in \mathcal{T}$, since $\sum_{x \in S} p_{v x}=1$. This implies that $\mathcal{T}$ is endless. Also notice that $P(\mathcal{T})=S$. For each $u \in P(\mathcal{T})$, define $\psi_{u}(u, v)=p_{u v}$ for every $(u, v) \in \mathcal{T}^{\text {out }}(u)$. Clearly $\psi_{u}$ is a probability distribution over $\mathcal{T}^{\text {out }}(u)$. Thus, $\Psi=\left\{\psi_{u}\right\}_{u \in P(\mathcal{T})}$ is a TSR for $\mathcal{T}$. As a consequence $\mathcal{D}=(\mathcal{T}, \Psi)$ is a MTM. Since for every $u, v \in P(\mathcal{T}), \sum_{T \in \mathcal{T}(u, v)} \psi_{u}(T)=p_{u v}$, it holds that $\operatorname{Ker}(\mathcal{D})=\mathcal{M}$.

## D Proof of Theorem 4

Lemma 12 If $\mathcal{D}$ is a strongly connected MTM then $\operatorname{Ker}(\mathcal{D})$ is an irreducible Markov chain.
Proof. Recall that a Markov chain is said to be irreducible if it is possible to go from every state to every state (not necessarily in one step). Consider any two states $u, v \in P(\mathcal{T})$ of $\operatorname{Ker}(\mathcal{D})$. Since $\mathcal{D}$ is a strongly connected, there exists a sequence of points of $P(\mathcal{T})\left(z_{0}, z_{1}, \ldots, z_{k}\right)$ such that $z_{0}=u, z_{k}=v$, and, for every $i=0,1, \ldots, k-1$,

$$
\operatorname{Pr}\left(z_{i} \rightarrow z_{i+1}\right)=\sum_{T \in \mathcal{T}\left(z_{i}, z_{i}+1\right)} \psi_{z_{i}}(T)>0
$$

Thus,

$$
\operatorname{Pr}(u \rightarrow v)^{(k)} \geqslant \prod_{i=0}^{k-1} \operatorname{Pr}\left(z_{i} \rightarrow z_{i+1}\right)>0
$$

where $\operatorname{Pr}(u \rightarrow v)^{(k)}$ denotes the probability of going from $u$ to $v$ in $k$ steps. Thus, there is a positive probability of reaching $v$ from $u$.

Proof of Theorem 4. A standard result says that a finite irreducible Markov chain has a unique stationary distribution (see for instance [13] Chapter 11 - Theorem 11.10, here irreducible chains are called ergodic chains). From this result and Lemma 12 , it follows that $\operatorname{Ker}(\mathcal{D})$ has a unique stationary distribution. Then, from Corollary $2(\mathrm{~b})$, also $\mathcal{D}$ has a unique stationary distribution.

## E Proof of Theorem 5

We first need the following
Lemma 13 Let $\mathcal{D}=(\mathcal{T}, \Psi)$ be any MTM that is both uniformly selective and balanced, then $\operatorname{Ker}(\mathcal{D})$ has the following stationary distribution $\sigma: \forall u \in P(\mathcal{T}), \sigma(u)=\frac{\left|\mathcal{T}^{\mathrm{out}}(u)\right|}{|\mathcal{T}|}$

Proof. It holds that, for every $u \in P(\mathcal{T})$,

$$
\begin{aligned}
\sum_{v \in P(\mathcal{T})} \sigma(v) \operatorname{Pr}(v \rightarrow u) & =\sum_{v \in P(\mathcal{T})} \sigma(v) \sum_{T \in \mathcal{T}(v, u)} \psi_{v}(T) \\
& =\sum_{v \in P(\mathcal{T})} \sigma(v) \frac{|\mathcal{T}(v, u)|}{\left|\mathcal{T}^{\text {out }}(v)\right|} \quad \text { (since } \mathcal{D} \text { is uniformly selective) } \\
& =\sum_{v \in P(\mathcal{T})} \frac{\left|\mathcal{T}^{\text {out }}(v)\right|}{|\mathcal{T}|} \frac{|\mathcal{T}(v, u)|}{\left|\mathcal{T}^{\text {out }}(v)\right|} \\
& =\sum_{v \in P(\mathcal{T})} \frac{|\mathcal{T}(v, u)|}{|\mathcal{T}|} \\
& =\frac{\left|\mathcal{T}^{\text {in }}(u)\right|}{|\mathcal{T}|} \\
& =\frac{\left|\mathcal{T}^{\text {out }}(u)\right|}{|\mathcal{T}|} \quad \text { (since } \mathcal{D} \text { is balanced) } \\
& =\sigma(u)
\end{aligned}
$$

Proof of Theorem 5, Let $\pi$ be a uniform stationary distribution of $\mathcal{D}$. Then

$$
\forall\langle T, i\rangle \quad \pi(\langle T, i\rangle)=\frac{1}{|S(\mathcal{T})|}
$$

From Lemma 9 (c), it immediately derives that $\mathcal{D}$ is balanced. The uniformly selectiveity derives from Lemma 9 (b).

Conversely, assume that $\mathcal{D}$ is both uniformly selective and balanced. From Lemma 13 , the following probability distribution $\sigma$ over $P(\mathcal{T})$ :

$$
\forall u \in P(\mathcal{T}) \quad \sigma(u)=\frac{\left|\mathcal{T}^{\mathrm{out}}(u)\right|}{|\mathcal{T}|}
$$

is stationary for $\operatorname{Ker}(\mathcal{D})$. Thus, by Lemma 10, the following map $\pi$ is a stationary distribution of $\mathcal{D}$ :

$$
\forall\langle T, i\rangle \in S(\mathcal{T}) \quad \pi(\langle T, i\rangle)=\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sigma\left(T_{\text {start })} \psi_{T_{\text {start }}}(T)\right.
$$

Firstly, observe that the following holds:

$$
\begin{aligned}
\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u) & =\sum_{u \in P(\mathcal{T})} \frac{\left|\mathcal{T}^{\text {out }}(u)\right|}{|\mathcal{T}|} \sum_{T \in \mathcal{T}^{\text {out }}(u)}(|T|-1) \psi_{u}(T) \\
& =\sum_{u \in P(\mathcal{T})} \frac{\left|\mathcal{T}^{\text {out }}(u)\right|}{|\mathcal{T}|} \sum_{T \in \mathcal{T}^{\text {out }}(u)}(|T|-1) \frac{1}{\left|\mathcal{T}^{\text {out }}(u)\right|} \\
& =\frac{1}{|\mathcal{T}|} \sum_{u \in P(\mathcal{T})} \sum_{T \in \mathcal{T}^{\text {out }}(u)}(|T|-1) \\
& =\frac{|S(\mathcal{T})|}{|\mathcal{T}|}
\end{aligned}
$$

Then, it holds that, for every $\langle T, i\rangle \in S(\mathcal{T})$,

$$
\begin{aligned}
\pi(\langle T, i\rangle) & =\frac{1}{\sum_{u \in P(\mathcal{T})} \sigma(u) \Lambda_{\Psi}(u)} \sigma\left(T_{\text {start }}\right) \psi_{T_{\text {start }}}(T) \\
& =\frac{|\mathcal{T}|}{|S(\mathcal{T})|} \sigma\left(T_{\text {start }}\right) \psi_{T_{\text {start }}}(T) \\
& =\frac{|\mathcal{T}|}{|S(\mathcal{T})|} \frac{\left|\mathcal{T}^{\text {out }}\left(T_{\text {start }}\right)\right|}{|\mathcal{T}|} \frac{1}{\left|\mathcal{T}^{\text {out }}\left(T_{\text {start }}\right)\right|} \\
& =\frac{1}{|S(\mathcal{T})|}
\end{aligned}
$$

Thus, $\pi$ is uniform.

## F The Manhattan-mtm: Stationary Distributions

Let's remind the key-property that allows to compute such stationary distributions.
Observation 14 The Manhattan-mtm is balanced, uniformly-selective and strongly-connected. So, from Theorems 5 and 4 , the Manhattan-mtm has a unique stationary distribution and it is the uniform one. Moreover, since the MANHATTAN-MTM is simple, the spatial and the destinational stationary distributions are given by Eq.s $\mathbb{\square}$ and , respectively.

Since the length of a path from a point $(i, j)$ to a point $\left(i^{\prime}, j^{\prime}\right)$ is $\left|i-i^{\prime}\right|+\left|j-j^{\prime}\right|$, with some calculations, we obtain

$$
\begin{equation*}
\left|S\left(\mathcal{T}_{\epsilon}\right)\right|=\frac{3}{\left(N^{4}-N^{2}\right)(4 N-2)} \tag{7}
\end{equation*}
$$

Now let $\eta(u, v)$ be the number of paths starting from $u$ and visiting $v$. As for the Manhattan-MTM, we get

$$
\eta_{\epsilon}\left(\left(i^{\prime}, j^{\prime}\right),(i, j)\right)= \begin{cases}2 N-i-j & \text { if } i^{\prime}<i \text { and } j^{\prime}<j  \tag{8}\\ i+j+2 & \text { if } i^{\prime}>i \text { and } j^{\prime}>j \\ N+1-i+j & \text { if } i^{\prime}<i \text { and } j^{\prime}>j \\ N+1+i-j & \text { if } i^{\prime}>i \text { and } j^{\prime}<j \\ N^{2}-N j & \text { if } i^{\prime}=i \text { and } j^{\prime}<j \\ N^{2}-N i & \text { if } i^{\prime}<i \text { and } j^{\prime}=j \\ N+N j & \text { if } i^{\prime}=i \text { and } j^{\prime}>j \\ N+N i & \text { if } i^{\prime}>i \text { and } j^{\prime}=j \\ 2 N^{2}-2 N+1 & \text { if } i^{\prime}=i \text { and } j^{\prime}=j\end{cases}
$$

Thus

$$
\begin{align*}
\Gamma_{\epsilon}(i, j) & =\sum_{0 \leq i^{\prime} \leq N-1} \sum_{0 \leq j^{\prime} \leq N-1} \eta_{\epsilon}\left(\left(i^{\prime}, j^{\prime}\right),(i, j)\right) \\
& =\left(4 N^{2}-6 N+2\right)(i+j)-(4 N-2)\left(i^{2}+j^{2}\right)+6 N^{2}-8 N+3 \tag{9}
\end{align*}
$$

From Eq.s 1, 9 , 7 we get the spatial distribution

$$
\begin{equation*}
\mathfrak{s}_{\epsilon}(i, j)=\frac{3\left(\left(4 N^{2}-6 N+2\right)(i+j)-(4 N-2)\left(i^{2}+j^{2}\right)+6 N^{2}-8 N+3\right)}{\left(N^{4}-N^{2}\right)(4 N-2)} . \tag{10}
\end{equation*}
$$

Our next goal is to study the Manhattan Random-Way Point over grids of arbitrarily high resolution, i.e. for $\epsilon \rightarrow 0$. So, we will need to derive the probability-densitiy functions of the stationary distribution.

Let us compute the probability that an agent lies into a square of center $(x, y)$ (where $x$ and $y$ are the Euclidean coordinates of a point in $V_{\epsilon}$ ) and side length $2 \delta$ w.r.t. the spatial distribution.

$$
f_{\delta, \epsilon}(x, y)=\sum_{i \in\left\{\frac{N}{L}(x-\delta), \frac{N}{L}(x+\delta)\right\}} \sum_{j \in\left\{\frac{N}{L}(y-\delta), \frac{N}{L}(y+\delta)\right\}} \pi_{\epsilon}(i, j)
$$

Then the probability density function of the spatial distribution is given by

$$
\begin{equation*}
f(x, y)=\lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \frac{1}{4 \delta^{2}} f_{\delta, \epsilon}(x, y)=\frac{3}{L^{3}}(x+y)-\frac{3}{L^{4}}\left(x^{2}+y^{2}\right) \tag{11}
\end{equation*}
$$

We now compute the agent-destination distribution. This implies that the number of feasible paths visiting point $u$ and ending in point $v$ equals the number of feasible paths starting from $u$ and visiting $v$, i.e.

$$
\begin{equation*}
\Gamma_{\epsilon,\left(i_{0}, j_{0}\right)}(i, j)=\eta_{\epsilon}\left((i, j),\left(i_{0}, j_{0}\right)\right) \tag{12}
\end{equation*}
$$

We now replace Eq.s 12 and 9 into Eq. 2, and get the destination distribution

$$
\begin{equation*}
\mathfrak{d}_{\epsilon,\left(i_{0}, j_{0}\right)}(i, j)=\frac{\eta_{\epsilon}\left((i, j),\left(i_{0}, j_{0}\right)\right)}{\Gamma_{\epsilon}\left(i_{0}, j_{0}\right)} \tag{13}
\end{equation*}
$$

Let us now compute, the probability that an agent, visiting point ( $x_{0}, y_{0}$ ), has destination lying into the square of center $(x, y)$ and side length $2 \delta$ (where $\left(x_{0}, y_{0}\right)$ and $(x, y)$ are the Euclidean coordinates of points in $V_{\epsilon}$. By definition of $\mathfrak{d}_{\epsilon, u}(v)$, it follows that

$$
f_{\left(x_{0}, y_{0}\right), \delta, \epsilon}(x, y)=\sum_{i \in\left\{\frac{N}{L}(x-\delta), \frac{N}{L}(x+\delta)\right\}} \sum_{j \in\left\{\frac{N}{L}(y-\delta), \frac{N}{L}(y+\delta)\right\}} \mathfrak{d}_{\epsilon,\left(\frac{N}{L} x_{0}, \frac{N}{L} y_{0}\right)}(i, j)
$$

The probability density function of the destination distribution is

$$
f_{\left(x_{0}, y_{0}\right)}(x, y)=\lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \frac{1}{4 \delta^{2}} f_{\left(x_{0}, y_{0}\right), \delta, \epsilon}(x, y)
$$

By combining Eq.s 13 , 8, 9, and some calculations, we get

$$
f_{\left(x_{0}, y_{0}\right)}(x, y)= \begin{cases}\frac{2 L-x_{0}-y_{0}}{4 L\left(L\left(x_{0}+y_{0}+-\left(x_{0}^{2}+y_{0}^{2}\right)\right)\right.} & \text { if } x<x_{0} \text { and } y<y_{0}  \tag{14}\\ \frac{x_{0}+y_{0}}{4 L\left(L\left(x_{0}+y_{0}-\left(x_{0}^{2}+y_{0}^{2}\right)\right)\right.} & \text { if } x>x_{0} \text { and } y>y_{0} \\ \frac{L-x_{0}+y_{0}}{4 L\left(L\left(x_{0}+y_{0}\right)-\left(x_{0}^{2}+y_{0}^{2}\right)\right)} & \text { if } x<x_{0} \text { and } y>y_{0} \\ \frac{L+x_{0}-y_{0}}{4 L\left(L\left(x_{0}+y_{0}\right)-\left(x_{0}^{2}+y_{0}^{2}\right)\right)} & \text { if } x>x_{0} \text { and } y<y_{0} \\ +\infty & \text { if } x=x_{0} \text { and } y<y_{0} \text { (SOUTH Case) } \\ +\infty & \text { if } x<x_{0} \text { and } y=y_{0} \text { (WEST Case } \\ +\infty & \text { if } x=x_{0} \text { and } y>y_{0} \text { (NORTH Case) } \\ +\infty & \text { if } x>x_{0} \text { and } y=y_{0} \text { (EAST Case) } \\ +\infty & \text { if } x=x_{0} \text { and } y=y_{0}\end{cases}
$$

We now want to compute the stationary agent-destination distribution where the probability density function does not exist. To this aim, for every segment $s \in\{$ SOUTH, WEST, NORTH,EAST $\}$, we evaluate the probability that an agent, visting point $\left(x_{0}, y_{0}\right)$, has destination lying into the subsegment of $s$, having any given center and length $2 \delta$, conditioned to the event that the destination belongs to segment $s$. From Eq. 2 we observe that, for any segment $s$, it holds that

$$
\mathfrak{d}_{u}(v \mid v \in s)=\frac{\Gamma_{u}(v)}{\sum_{w \in s} \Gamma_{u}(w)}
$$

We apply the above equation to each of the four cases and get, respectively,

$$
f_{\left(x_{0}, y_{0}\right), \delta, \epsilon}^{\mathrm{South}}(y)=\frac{\sum_{j \in\{(N / L)(y-\delta),(N / L)(y+\delta)\}} \eta_{\epsilon}\left(\left((N / L) x_{0}, j\right),\left((N / L) x_{0},(N / L) y_{0}\right)\right)}{\sum_{j \in\left\{0,(N / L) y_{0}\right\}} \eta_{\epsilon}\left(\left((N / L) x_{0}, j\right),\left((N / L) x_{0},(N / L) y_{0}\right)\right)} \text { for } y<y_{0}
$$

$$
\begin{aligned}
& f_{\left(x_{0}, y_{0}\right), \delta, \epsilon}^{\mathrm{west}}(x)=\frac{\sum_{i \in\{(N / L)(x-\delta),(N / L)(x+\delta)\}} \eta_{\epsilon}\left(\left(i,(N / L) y_{0}\right),\left((N / L) x_{0},(N / L) y_{0}\right)\right)}{\sum_{i \in\left\{0,(N / L) x_{0}\right\}} \eta_{\epsilon}\left(\left(i,(N / L) y_{0}\right),\left((N / L) x_{0},(N / L) y_{0}\right)\right)} \text { for } x<x_{0} \\
& f_{\left(x_{0}, y_{0}\right), \delta, \epsilon}^{\text {north }}(y)=\frac{\sum_{j \in\{(N / L)(y-\delta),(N / L)(y+\delta)\}} \eta_{\epsilon}\left(\left((N / L) x_{0}, j\right),\left((N / L) x_{0},(N / L) y_{0}\right)\right)}{\sum_{j \in\left\{(N / L) y_{0}, N\right\}} \eta_{\epsilon}\left(\left((N / L) x_{0}, j\right),\left((N / L) x_{0},(N / L) y_{0}\right)\right)} \text { for } y>y_{0} \\
& f_{\left(x_{0}, y_{0}\right), \delta, \epsilon}^{\text {east }}(x)=\frac{\sum_{i \in\{(N / L)(x-\delta),(N / L)(x+\delta)\}} \eta_{\epsilon}\left(\left(i,(N / L) y_{0}\right),\left((N / L) x_{0},(N / L) y_{0}\right)\right)}{\sum_{i \in\left\{(N / L) x_{0}, N\right\}} \eta_{\epsilon}\left(\left(i,(N / L) y_{0}\right),\left((N / L) x_{0},(N / L) y_{0}\right)\right)} \text { for } x>x_{0}
\end{aligned}
$$

We consider the limits of the above four functions for $\epsilon \rightarrow 0$ and for $\delta \rightarrow 0$ and get the probability density functions of the destination distribution conditioned w.r.t. the four segments.

$$
\begin{gather*}
f_{\left(x_{0}, y_{0}\right)}^{\mathrm{SOuth}}(y)=\lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \frac{1}{2 \delta} f_{\left(x_{0}, y_{0}\right), \delta, \epsilon}^{\mathrm{South}}(y)=\frac{1}{y_{0}}  \tag{15}\\
f_{\left(x_{0}, y_{0}\right)}^{\mathrm{West}}(x)=\lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \frac{1}{2 \delta} f_{\left(x_{0}, y_{0}\right), \delta, \epsilon}^{\mathrm{West}}(x)=\frac{1}{x_{0}}  \tag{16}\\
f_{\left(x_{0}, y_{0}\right)}^{\mathrm{north}}(y)=\lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \frac{1}{2 \delta} f_{\left(x_{0}, y_{0}\right), \delta, \epsilon}^{\mathrm{north}}(y)=\frac{1}{L-y_{0}}  \tag{17}\\
f_{\left(x_{0}, y_{0}\right)}^{\mathrm{east}}(x)=\lim _{\delta \rightarrow 0} \lim _{\epsilon \rightarrow 0} \frac{1}{2 \delta} f_{\left(x_{0}, y_{0}\right), \delta, \epsilon}^{\mathrm{east}}(x)=\frac{1}{L-x_{0}} \tag{18}
\end{gather*}
$$

From the above equations, it easy to see that the stationary destination distribution is uniform over each of the four segments. We are now able to calculate the probability that an agent, visiting point $\left(x_{0}, y_{0}\right)$, has destination in one of the four segments.

$$
\begin{aligned}
& \phi_{\left(x_{0}, y_{0}\right), \epsilon}^{\text {South }}=\sum_{j \in\left\{0,(N / L) y_{0}\right\}} \mathfrak{d}_{\epsilon,\left((N / L) x_{0},(N / L) y_{0}\right)}\left((N / L) x_{0}, j\right) \\
& \phi_{\left(x_{0}, y_{0}\right), \epsilon}^{\mathrm{west}}=\sum_{i \in\left\{0,(N / L) x_{0}\right\}} \mathfrak{d}_{\epsilon,\left((N / L) x_{0},(N / L) y_{0}\right)}\left(i,(N / L) y_{0}\right) \\
& \phi_{\left(x_{0}, y_{0}\right), \epsilon}^{\text {north }}= \sum_{j \in\left\{(N / L) y_{0}, N\right\}} \mathfrak{d}_{\epsilon,\left((N / L) x_{0},(N / L) y_{0}\right)}\left((N / L) x_{0}, j\right) \\
& \phi_{\left(x_{0}, y_{0}\right), \epsilon}^{\text {east }}=\sum_{i \in\left\{(N / L) x_{0}, L\right\}} \mathfrak{d}_{\epsilon,\left((N / L) x_{0},(N / L) y_{0}\right)}\left(i,(N / L) y_{0}\right)
\end{aligned}
$$

By taking the limits for $\epsilon \rightarrow 0$, we get, respectively,

$$
\begin{align*}
& \phi_{\left(x_{0}, y_{0}\right)}^{\text {South }}=\lim _{\epsilon \rightarrow 0} \phi_{\left(x_{0}, y_{0}\right), \epsilon}^{\text {South }}=\frac{y_{0}\left(L-y_{0}\right)}{4 L\left(x_{0}+y_{0}\right)-4\left(x_{0}^{2}+y_{0}^{2}\right)}  \tag{19}\\
& \phi_{\left(x_{0}, y_{0}\right)}^{\mathrm{west}}=\lim _{\epsilon \rightarrow 0} \phi_{\left(x_{0}, y_{0}\right), \epsilon}^{\mathrm{west}}=\frac{x_{0}\left(L-x_{0}\right)}{4 L\left(x_{0}+y_{0}\right)-4\left(x_{0}^{2}+y_{0}^{2}\right)}  \tag{20}\\
& \phi_{\left(x_{0}, y_{0}\right)}^{\mathrm{north}}=\lim _{\epsilon \rightarrow 0} \phi_{\left(x_{0}, y_{0}\right), \epsilon}^{\text {north }}=\frac{y_{0}\left(L-y_{0}\right)}{4 L\left(x_{0}+y_{0}\right)-4\left(x_{0}^{2}+y_{0}^{2}\right)}  \tag{21}\\
& \phi_{\left(x_{0}, y_{0}\right)}^{\mathrm{east}}=\lim _{\epsilon \rightarrow 0} \phi_{\left(x_{0}, y_{0}\right), \epsilon}^{\text {east }}=\frac{x_{0}\left(L-x_{0}\right)}{4 L\left(x_{0}+y_{0}\right)-4\left(x_{0}^{2}+y_{0}^{2}\right)} \tag{22}
\end{align*}
$$

## G Modular MTM: Stationary Distributions

The next proposition provides the formulas for the stationary distributions of general modular MTM. Such formulas will be used to prove Proposition 7 .

Proposition 15 Let $\mathfrak{R}=(\mathcal{B}, \mathcal{R})$ be a Route System such that the associated $M T M \mathcal{D}[\mathfrak{R}]=(\mathcal{T}[\mathfrak{R}], \Psi[\mathfrak{R}])$ is strongly connected. Let $\sigma$ be the stationary distribution of the Kernel of $\mathcal{D}[\mathfrak{R}]$. Then, the stationary spatial distribution $\mathfrak{s}$ of $\mathcal{D}[\mathfrak{R}]$ is, for every $u \in \mathcal{S}$,
$\mathfrak{s}(u)=\frac{1}{\Lambda[\mathfrak{R}]} \sum_{B \in \mathcal{B}} \frac{\#_{B, u}}{|B|} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{\#_{R, B}}{|R|}$ with $\Lambda[\mathfrak{R}]=\sum_{B \in \mathcal{B}} \frac{\# B}{|B|} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{\#_{R, B}}{|R|}$
The stationary destination distributions $\mathfrak{d}$ of $\mathcal{D}[\mathfrak{R}]$ are, for every $u, v \in \mathcal{S}$,

$$
\mathfrak{d}_{u}(v)=\frac{1}{\mathfrak{s}(u) \Lambda[\mathfrak{R}]} \sum_{B \in \mathcal{B}} \frac{\#_{B, u}}{|B|} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w} \wedge R_{\text {end }}=v} \frac{\#_{R, B}}{|R|}
$$

Proof. Since $\mathcal{D}[\mathfrak{R}]$ is strongly connected, from Theorem $4 \mathcal{D}[\mathfrak{R}]$ has e unique stationary distribution. Let $\pi$ be the stationary distribution of $\mathcal{D}[\mathfrak{R}]$ and let $\sigma$ be the (unique) stationary distribution of its Kernel. Firstly, we prove (i). From the definitions and Theorem 1 (a), it holds that

$$
\begin{align*}
\mathfrak{s}(u) & =\sum_{T \in \mathcal{T}[\mathfrak{R}]_{u}} \#_{T, u} \pi(\langle T, 0\rangle) \\
& =\sum_{T \in \mathcal{T}[\mathfrak{R}]_{u}} \#_{T, u} \frac{1}{\sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \sigma(w) \Lambda_{\Psi[\mathfrak{R}]}(w)} \sigma\left(T_{\text {start }}\right) \psi[\mathfrak{R}]_{T_{\text {start }}}(T) \\
& =\frac{1}{\Lambda[\mathfrak{R}]} \sum_{T \in \mathcal{T}[\mathfrak{R}]_{u}} \#_{T, u} \sigma\left(T_{\text {start }}\right) \psi[\mathfrak{R}]_{T_{\text {start }}}(T) \tag{23}
\end{align*}
$$

To evaluate $\Lambda[\mathfrak{R}]$ we need the following claim
Claim $1 \sum_{T \in P}|T|=|\operatorname{Bundle}(P)| \sum_{B \in P} \frac{\# B}{|B|}$.
Proof. Let $P=\left(B_{1}, B_{2} \ldots B_{l}\right)$, then

$$
\begin{aligned}
\sum_{T \in P}|T| & =\sum_{T \in B_{1} \cdot B_{2} \ldots B_{l}}|T| \\
& =\sum_{s_{1} \in B_{1}} \sum_{s_{2} \in B_{2}} \ldots \sum_{s_{l} \in B_{l}}\left(\left|s_{1}\right|+\left|s_{2}\right|+\ldots\left|s_{l}\right|\right) \\
& =\sum_{i=1}^{l} \sum_{s \in B_{i}} \frac{|s| \prod_{j=1}^{l}\left|B_{j}\right|}{\left|B_{i}\right|} \\
& =\sum_{i=1}^{l} \sum_{s \in B_{i}} \frac{|s| \cdot|\operatorname{Bundle}(P)|}{\left|B_{i}\right|} \\
& =|\operatorname{Bundle}(P)| \sum_{i=1}^{l} \frac{1}{\left|B_{i}\right|} \sum_{s \in B_{i}}|s| \\
& =|\operatorname{Bundle}(P)| \sum_{i=1}^{l} \frac{1}{\left|B_{i}\right|} \# B_{i} \\
& =|\operatorname{Bundle}(P)| \sum_{B \in P} \frac{\# B}{|B|}
\end{aligned}
$$

$$
\begin{aligned}
& \Lambda[\mathfrak{R}]=\sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \sigma(w) \Lambda_{\Psi[\mathfrak{R}]}(w) \\
& =\sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \sigma(w) \sum_{T \in \mathcal{T}[\mathfrak{R}] \cot (w)}|T| \psi[\mathfrak{R}]_{w}(T) \\
& =\sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \sigma(w) \sum_{T \in \mathcal{T}[\{\mathfrak{R}] \text { jut }(w)} \frac{|T|}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{1}{|R|} \sum_{P \in R} \frac{\#_{P, T}}{|\operatorname{Bundle}(P)|} \\
& =\sum_{w \in P(\mathcal{T}\{\mathfrak{P}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{1}{|R|} \sum_{P \in R} \frac{1}{|\operatorname{Bundle}(P)|} \sum_{T \in \mathcal{T}[\mathfrak{R}\}] \operatorname{lout}(w)}|T| \cdot \#_{P, T} \\
& =\sum_{w \in P(\mathcal{T}\{\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{1}{|R|} \sum_{P \in R} \frac{1}{|\operatorname{Bundle}(P)|} \sum_{T \in P}|T| \\
& =\sum_{w \in P(\mathcal{T}[\mathfrak{P}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{1}{|R|} \sum_{P \in R} \frac{|\operatorname{Bundle}(P)|}{|\operatorname{Bundle}(P)|} \sum_{B \in P} \frac{\# B}{|B|} \quad \text { (by Claim【I) } \\
& =\sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{1}{|R|} \sum_{P \in R} \sum_{B \in \mathcal{B}} \frac{\# P, B \cdot \# B}{|B|} \\
& =\sum_{B \in \mathcal{B}} \frac{\# B}{|B|} \sum_{w \in P(\mathcal{T}[\mathfrak{R}\})} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{1}{|R|} \sum_{P \in R} \#_{P, B} \\
& =\sum_{B \in \mathcal{B}} \frac{\# B}{|B|} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{\#_{R, B}}{|R|}
\end{aligned}
$$

Returning to Eq. 23, we firstly observe that the following claim holds
Claim $2 \sum_{T \in P} \#_{T, u}=|\operatorname{Bundle}(P)| \sum_{B \in P} \frac{\#_{B, u}}{|B|}$.
Proof. Similar to the proof of Claim $\square$

Thus have that

$$
\begin{aligned}
& \mathfrak{s}(u)=\frac{1}{\Lambda[\mathfrak{R}]} \sum_{T \in \mathcal{T}[\mathfrak{R}]_{u}} \#_{T, u} \sigma\left(T_{\text {start }}\right) \psi[\mathfrak{R}]_{T_{\text {start }}}(T) \\
& =\frac{1}{\Lambda[\mathfrak{R}]} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \sum_{T \in \mathcal{T}[\mathfrak{R}]^{\text {out }}(w)_{u}} \#_{T, u} \sigma(w) \psi[\mathfrak{R}]_{w}(T) \\
& =\frac{1}{\Lambda[\mathfrak{R}]} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \sum_{T \in \mathcal{T}[\mathfrak{R}]^{\text {out }}(w)_{u}} \#_{T, u} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{1}{|R|} \sum_{P \in R} \frac{\#_{P, T}}{|\operatorname{Bundle}(P)|} \\
& =\frac{1}{\Lambda[\mathfrak{R}]} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{1}{|R|} \sum_{P \in R} \frac{1}{|\operatorname{Bundle}(P)|} \sum_{T \in \mathcal{T}[\mathfrak{R}]^{\text {out }}(w)_{u}} \#_{T, u} \cdot \#_{P, T} \\
& =\frac{1}{\Lambda[\mathfrak{R}]} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{1}{|R|} \sum_{P \in R} \frac{1}{|\operatorname{Bundle}(P)|} \sum_{T \in P} \#_{T, u} \\
& =\frac{1}{\Lambda[\mathfrak{R}]} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{1}{|R|} \sum_{P \in R} \frac{|\operatorname{Bundle}(P)|}{|\operatorname{Bundle}(P)|} \sum_{B \in P} \frac{\#_{B, u}}{|B|} \quad \text { (by Claim [2) } \\
& =\frac{1}{\Lambda[\mathfrak{R}]} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{1}{|R|} \sum_{B \in \mathcal{B}} \frac{\#_{R, B} \cdot \#_{B, u}}{|B|} \\
& =\frac{1}{\Lambda[\mathfrak{R}]} \sum_{B \in \mathcal{B}} \frac{\#_{B, u}}{|B|} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{\#_{R, B}}{|R|}
\end{aligned}
$$

This proves formula (i).
Now consider formula (ii) for the stationary destination distributions. From the definitions, Theorem 1 (a), and very similar calculations as those used for the stationary spatial distribution, we have

$$
\begin{align*}
\mathfrak{d}_{u}(v) & =\frac{1}{\mathfrak{s}(u)} \sum_{T \in \mathcal{T}[\mathfrak{R}]_{u} \wedge T_{\text {end }}=v} \#_{T, u} \cdot \pi(\langle T, 0\rangle) \\
& =\frac{1}{\mathfrak{s}(u) \Lambda[\mathfrak{R}]} \sum_{T \in \mathcal{T}[\mathfrak{R}]_{u} \wedge T_{\text {end }}=v} \#_{T, u} \sigma\left(T_{\text {start }}\right) \psi[\mathfrak{R}]_{T_{\text {start }}}(T) \\
& =\frac{1}{\mathfrak{s}(u) \Lambda[\mathfrak{R}]} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \sum_{T \in \mathcal{T}[\mathfrak{R}]^{\text {out }}(w)_{u} \wedge T_{\text {end }}=v} \#_{T, u} \sigma(w) \psi[\mathfrak{R}]_{w}(T) \\
& =\frac{1}{\mathfrak{s}(u) \Lambda[\mathfrak{R}]} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \sum_{T \in \mathcal{T}[\mathfrak{R}] \text { out }(w)_{u} \wedge T_{\text {end }}=v} \#_{T, u} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{1}{|R|} \sum_{P \in R} \frac{\#_{P, T}}{|\operatorname{Bundle}(P)|} \\
& =\frac{1}{\mathfrak{s}(u) \Lambda[\mathfrak{R}]} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{1}{|R|} \sum_{P \in R} \frac{1}{|\operatorname{Bundle}(P)|} \sum_{T \in \mathcal{T}[\mathfrak{R}] \text { out }(w)_{u} \wedge T_{\text {end }}=v} \#_{T, u} \cdot \# \#_{P, T} \\
& =\frac{1}{\mathfrak{s}(u) \Lambda[\mathfrak{R}]} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{1}{|R|} \sum_{P \in R} \frac{1}{|\operatorname{Bundle}(P)|} \sum_{T \in P \wedge T_{\text {end }}=v} \# \tag{24}
\end{align*}
$$

Notice that if $R \in \mathcal{R}_{w}$ is such that $R_{e n d} \neq v$ then

$$
\sum_{P \in R} \frac{1}{|\operatorname{Bundle}(P)|} \sum_{T \in P \wedge T_{\text {end }}=v} \#_{T, u}=0
$$

Thus, returning to Eq. 24 we have

$$
\begin{aligned}
\mathfrak{d}_{u}(v) & =\frac{1}{\mathfrak{s}(u) \Lambda[\mathfrak{R}]} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{1}{|R|} \sum_{P \in R} \frac{1}{|\operatorname{Bundle}(P)|} \sum_{T \in P \wedge T_{\text {end }}=v} \#_{T, u} \\
& =\frac{1}{\mathfrak{s}(u) \Lambda[\mathfrak{R}]} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w} \wedge R_{e n d}=v} \frac{1}{|R|} \sum_{P \in R} \frac{1}{|\operatorname{Bundle}(P)|} \sum_{T \in P} \#_{T, u} \\
& =\frac{1}{\mathfrak{s}(u) \Lambda[\mathfrak{R}]} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w} \wedge R_{e n d}=v} \frac{1}{|R|} \sum_{P \in R} \frac{|\operatorname{Bundle}(P)|}{|\operatorname{Bundle}(P)|} \sum_{B \in P} \frac{\#_{B, u}}{|B|} \quad \text { (by Claim [2) } \\
& =\frac{1}{\mathfrak{s}(u) \Lambda[\mathfrak{R}]} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w} \wedge R_{e n d}=v} \frac{1}{|R|} \sum_{B \in \mathcal{B}} \frac{\#_{R, B} \cdot \#_{B, u}}{|B|} \\
& =\frac{1}{\mathfrak{s}(u) \Lambda[\mathfrak{R}]} \sum_{B \in \mathcal{B}} \frac{\#_{B, u}}{|B|} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w} \wedge R_{\text {end }}=v} \frac{\#_{R, B}}{|R|}
\end{aligned}
$$

## G. 1 Proof of Proposition 7

We need the following preliminary lemmas.
Lemma 16 Let $\mathcal{D}=(\mathcal{T}, \Psi)$ be a MTM such that a function $f: P(\mathcal{T}) \rightarrow \mathbb{R}$ exists satisfying the following properties:
(i)

$$
\sum_{u \in P(\mathcal{T})} f(u)>0 \quad \text { and } \quad \forall u \in P(\mathcal{T}) \quad f(u) \geqslant 0
$$

(ii)

$$
\forall u \in P(\mathcal{T}) \quad \sum_{T \in \mathcal{T} \text { in }(u)} f\left(T_{\text {start }}\right) \psi_{T_{\text {start }}}(T)=f(u)
$$

Then, the following map

$$
\forall u \in P(\mathcal{T}) \quad \sigma(u)=\frac{f(u)}{\sum_{v \in P(\mathcal{T})} f(v)}
$$

is a stationary distribution of the Kernel of $\mathcal{D}$.
Proof. From Property (a), $\sigma$ is a probability distribution over $P(\mathcal{T})$. It holds that, for every $u \in P(\mathcal{T})$,

$$
\begin{aligned}
\sum_{v \in P(\mathcal{T})} \sigma(v) \operatorname{Pr}(v \rightarrow u) & =\sum_{v \in P(\mathcal{T})} \sigma(v) \sum_{T \in \mathcal{T}(v, u)} \psi_{v}(T) \\
& =\sum_{v \in P(\mathcal{T})} \frac{f(v)}{\sum_{w \in P(\mathcal{T})} f(w)} \sum_{T \in \mathcal{T}(v, u)} \psi_{v}(T) \\
& =\frac{1}{\sum_{w \in P(\mathcal{T})} f(w)} \sum_{v \in P(\mathcal{T})} \sum_{T \in \mathcal{T}(v, u)} f(v) \psi_{v}(T) \\
& =\frac{1}{\sum_{w \in P(\mathcal{T})} f(w)} \sum_{T \in \mathcal{T}^{\text {in }}(u)} f\left(T_{\text {start }}\right) \psi_{T_{\text {start }}}(T) \\
& =\frac{1}{\sum_{w \in P(\mathcal{T})} f(w)} f(u) \quad(\text { by Property }(\mathrm{b})) \\
& =\sigma(u)
\end{aligned}
$$

Lemma 17 Let $\mathfrak{R}=(\mathcal{B}, \mathcal{R})$ be a balanced Route System. Then, the map $\sigma$ so defined

$$
\forall u \in P(\mathcal{T}[\mathfrak{R}]) \quad \sigma(u)=\frac{\left|\mathcal{R}_{u}\right|}{|\mathcal{R}|}
$$

is a stationary distribution of the Kernel of the associated MTM $\mathcal{D}[\mathfrak{R}]$.
Proof. Consider the function $f: P(\mathcal{T}[\mathfrak{R}]) \rightarrow \mathbb{R}$ such that $f(u)=\left|\mathcal{R}_{u}\right|$, for every $u \in P(\mathcal{T}[\Re])$. Clearly, $f$ satisfies Property (a) of Lemma 16. To verify that it also satisfies Property (b) w.r.t. the MTM $\mathcal{D}[\mathfrak{R}]$, consider the following

$$
\begin{aligned}
\sum_{T \in \mathcal{T}[\mathfrak{R}]^{\mathrm{in}}(u)} f\left(T_{\text {start }}\right) \psi[\mathfrak{R}]_{T_{\text {start }}}(T) & =\sum_{T \in \mathcal{T}[\mathfrak{R}]^{\mathrm{in}}(u)} \frac{f\left(T_{\text {start }}\right)}{\left|\mathcal{R}_{T_{s t a r t}}\right|} \sum_{R \in \mathcal{R}_{T_{\text {start }}}} \frac{1}{|R|} \sum_{P \in R} \frac{\#_{P, T}}{|\operatorname{Bundle}(P)|} \\
& =\sum_{v \in P(\mathcal{T}[\mathfrak{R}])} \sum_{T \in \mathcal{T}[\mathfrak{R}]_{(v, u)}} \sum_{R \in \mathcal{R}_{v}} \frac{1}{|R|} \sum_{P \in R} \frac{\#_{P, T}}{|\operatorname{Bundle}(P)|} \\
& =\sum_{v \in P(\mathcal{T}[\mathfrak{R}])} \sum_{R \in \mathcal{R}_{v}} \frac{1}{|R|} \sum_{P \in R} \frac{1}{|\operatorname{Bundle}(P)|} \sum_{T \in \mathcal{T}[\mathfrak{R}]_{(v, u)}} \#_{P, T} \\
& =\sum_{v \in P(\mathcal{T}[\mathfrak{R}])} \sum_{R \in \mathcal{R}_{v, u}} \frac{1}{|R|} \sum_{P \in R} \frac{1}{|\operatorname{Bundle}(P)|} \sum_{T \in \mathcal{T}[\mathfrak{R}]_{(v, u)}} \#_{P, T} \\
& =\sum_{v \in P(\mathcal{T}[\mathfrak{R}])} \sum_{R \in \mathcal{R}_{v, u}} \frac{1}{|R|} \sum_{P \in R} \frac{1}{|\operatorname{Bundle}(P)|}|\operatorname{Bundle}(P)| \\
& =\sum_{v \in P(\mathcal{T}[\mathfrak{R}])}^{\left|\left\{R \in \mathcal{R}_{v} \mid \mathcal{R}_{\text {end }}=u\right\}\right|} \\
& =\left|\left\{R \in \mathcal{R} \mid R_{\text {end }}=u\right\}\right| \\
& =\left|\mathcal{R}_{u}\right|(\operatorname{since} \mathfrak{R} \text { is balanced }) \\
& =f(u)
\end{aligned}
$$

Hence, we can apply Lemma 16 and so we obtain that the map $\sigma$, defined as

$$
\sigma(u)=\frac{f(u)}{\sum_{w \in P(\mathcal{T})} f(w)}=\frac{\left|\mathcal{R}_{u}\right|}{|\mathcal{R}|}
$$

is a stationary distribution of the Kernel of the MTM $\mathcal{D}[\mathfrak{R}]$.

Proof of Proposition 7, Since $\mathfrak{R}$ is balanced and $\mathcal{D}[\mathfrak{R}]$ is strongly connected, from Lemma 17, the unique stationary distribution of the Kernel of $\mathcal{D}[\mathfrak{R}]$ is the following

$$
\forall u \in P(\mathcal{T}[\mathfrak{R}]) \quad \sigma(u)=\frac{\left|\mathcal{R}_{u}\right|}{|\mathcal{R}|}
$$

We firstly consider the spatial stationary distribution. From Proposition 15(i) and the above expression for $\sigma$ we obtain

$$
\begin{align*}
\mathfrak{s}(u) & =\frac{1}{\Lambda[\mathfrak{R}]} \sum_{B \in \mathcal{B}} \frac{\#_{B, u}}{|B|} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{\#_{R, B}}{|R|} \\
& =\frac{1}{\Lambda[\mathfrak{R}]} \sum_{B \in \mathcal{B}} \frac{\#_{B, u}}{|B|} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{1}{|\mathcal{R}|} \sum_{R \in \mathcal{R}_{w}} \frac{\#_{R, B}}{|R|} \\
& =\frac{1}{\Lambda[\mathfrak{R}]} \frac{1}{|\mathcal{R}|} \sum_{B \in \mathcal{B}} \frac{\#_{B, u}}{|B|} \sum_{R \in \mathcal{R}} \frac{\#_{R, B}}{|R|} \tag{25}
\end{align*}
$$

Moreover, it holds that

$$
\begin{align*}
\Lambda[\mathfrak{R}] & =\sum_{B \in \mathcal{B}} \frac{\# B}{|B|} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{\sigma(w)}{\left|\mathcal{R}_{w}\right|} \sum_{R \in \mathcal{R}_{w}} \frac{\#_{R, B}}{|R|} \\
& =\sum_{B \in \mathcal{B}} \frac{\# B}{|B|} \sum_{w \in P(\mathcal{T}[\mathfrak{R}])} \frac{1}{|\mathcal{R}|} \sum_{R \in \mathcal{R}_{w}} \frac{\#_{R, B}}{|R|} \\
& =\frac{1}{|\mathcal{R}|} \sum_{B \in \mathcal{B}} \frac{\# B}{|B|} \sum_{R \in \mathcal{R}} \frac{\#_{R, B}}{|R|} \\
& =\frac{1}{|\mathcal{R}|} \Lambda_{\mathrm{b}}[\mathfrak{R}] \tag{26}
\end{align*}
$$

By combining Equalities 25 and 26, we obtain

$$
\mathfrak{s}(u)=\frac{1}{\Lambda_{\mathrm{b}}[\mathfrak{R}]} \sum_{B \in \mathcal{B}} \frac{\#_{B, u}}{|B|} \sum_{R \in \mathcal{R}} \frac{\#_{R, B}}{|R|}
$$

The proof of formula (ii) is straightforward and it is very similar to the one of formula (i).

## H The DownTown Formulas

In this section, we give the spatial distribution formulas for some further kinds of cell of the DownTown Model.

- Start-bundles. We consider a starting-bundle $B_{S, k}^{+}(i, j)$ into a horizontal block (i.e. even $i$ and odd $j$ ). Observe that the number of bundle-paths containing this bundle equlas the number of parking cells located into blocks whose horizontal coordinate is not smaller than $j$ (but the cells into block $(i, j)$ ). Then,

$$
\begin{equation*}
\sigma-B_{S, k}^{++}(i, j)=m((n+1)(n-j+1)-2) \tag{27}
\end{equation*}
$$

Similarly, for a starting-bundle $B_{S, k}^{++}(i, j)$ into a horizontal block, we get

$$
\begin{equation*}
\sigma-B_{S, k}^{--}(i, j)=m((n+1)(j+1)-2) \tag{28}
\end{equation*}
$$

As for $\sigma-B_{S, k}^{+-}(i, j)$, it is equal to number of parking cells located into blocks whose horizontal coordinate is less than $j$. So, we get

$$
\begin{equation*}
\sigma-B_{S, k}^{+-}(i, j)=m((n+1) j-1) \tag{29}
\end{equation*}
$$

Similarly, for a starting-bundle $B_{S, k}^{-+}(i, j)$ into a horizontal block, we get

$$
\begin{equation*}
\sigma-B_{S, k}^{-+}(i, j)=m((n+1)(n-j)-1) \tag{30}
\end{equation*}
$$

As for starting-bundles into vertical blocks (i.e. odd $i$ and even $j$ ), we can get the correct formulas for $\sigma-B_{S, k}^{* *}(i, j)$ by replacing index $j$ with $i$.

- End-bundles. We consider an end-bundle $B_{E, k}^{++}(i, j)$ into a horizontal block (i.e. even $i$ and odd $j$ ). Observe that the number of bundle-paths containing this bundle equals the number of parking cells located into blocks whose horizontal coordinate is smaller than $j$ plus the number of positive parking stripes located into blocks whose horizontal coordinate is $j$ (but the positive cells into block $(i, j)$ ). Then,

$$
\begin{equation*}
\sigma-B_{E, k}^{++}(i, j)=m((n+1) j+n / 2-1) \tag{31}
\end{equation*}
$$

Similarly, for a starting-bundle $B_{E, k}^{--}(i, j)$ into a horizontal block we get

$$
\begin{equation*}
\sigma-B_{S, k}^{--}(i, j)=m((n+1)(n-j)+n / 2-1) \tag{32}
\end{equation*}
$$

Similarly, for the other two cases we get

$$
\begin{gather*}
\sigma-B_{E, k}^{+-}(i, j)=m((n+1)(n-j)+n / 2-1)  \tag{33}\\
\sigma-B_{E, k}^{-+}(i, j)=m((n+1) j+n / 2-1) \tag{34}
\end{gather*}
$$

As for ending-bundles into vertical blocks (i.e. odd $i$ and even $j$ ), we can get the correct formulas for $\sigma-B_{S, k}^{* *}(i, j)$ by replacing index $j$ with $i$.

- Transit-bundles. By applying counting arguments, we get the correct formulas for transit-bundles. Define function

$$
\eta(i, j)= \begin{cases}m^{2}(n-j+1)(n j-(j+1))+m^{2}(n-j-1)(n j+2 j+n-i) & i \text { even, } j \text { odd with } \\ m^{2}(n-j+1)\left(n j-\frac{3}{2}(j+1)\right)+m^{2}(n-j-1)(n j+2 j+n) & i \neq 0, n\} \\ m^{2}(n-j+1)\left(n j-\frac{n+1}{2}(j+1)\right)+m^{2}(n-j-1)(n j+2 j) & i=n \text { and } j \text { odd } \\ 0 & \text { otherwise }\end{cases}
$$

Then, we get

$$
\sigma-B_{T}^{+}(i, j)= \begin{cases}\eta(i, j) & i \text { even, } j \text { odd }  \tag{35}\\ \eta(j, i) & i \text { odd, } j \text { even } \\ 0 & \text { otherwise }\end{cases}
$$

By exploiting symmetric properties, from the above formula we get

$$
\begin{equation*}
\sigma-B_{T}^{-}(i, j)=\sigma-B_{T}^{+}(n-j, i) \tag{36}
\end{equation*}
$$

- Cross-bundles. By applying counting arguments, we get the correct formulas for cross-bundles.

$$
\sigma-B_{C}^{H,++}(i, j)= \begin{cases}m^{2}\left(n^{2}-n+(2 n-1) j-(n-1) i-2 i j\right) & i \text { and } j \text { even with }  \tag{37}\\ 0 & i \neq n \text { and } j \neq 0 \\ 0 & \text { otherwise }\end{cases}
$$

By exploiting symmetric properties, from the above formula we get

$$
\begin{align*}
\sigma-B_{C}^{V,+-}(i, j) & =\sigma-B_{C}^{H,++}(n-j, i)  \tag{38}\\
\sigma-B_{C}^{H,---}(i, j) & =\sigma-B_{C}^{H,++}(n-i, n-j)  \tag{39}\\
\sigma-B_{C}^{V,-+}(i, j) & =\sigma-B_{C}^{H,++}(j, n-i) \tag{40}
\end{align*}
$$

It holds that

$$
\sigma-B_{C}^{H,+}(i, j)= \begin{cases}m^{2}(n-j)(2 n j-i+j) & i \text { and } j \text { even with }  \tag{41}\\ & i, j \notin\{0, n\} \\ m^{2}(n-j)(2 n j+j / 2) & i=0 \text { and } j \text { even with } \\ & j \notin\{0, n\} \\ m^{2}(n-j)\left(\frac{3}{2} j(n+1)-n\right) & i=n \text { and } j \text { even with } \\ & j \notin\{0, n\} \\ 0 & \text { otherwise }\end{cases}
$$

By exploiting symmetric properties, from the above formula we get

$$
\begin{align*}
\sigma-B_{C}^{V,+}(i, j) & =\sigma-B_{C}^{H,+}(n-j, i)  \tag{42}\\
\sigma-B_{C}^{H,-}(i, j) & =\sigma-B_{C}^{H,+}(n-i, n-j)  \tag{43}\\
\sigma-B_{C}^{V,-}(i, j) & =\sigma-B_{C}^{H,+}(j, n-i) \tag{44}
\end{align*}
$$

It holds that

By exploiting symmetric properties, from the above formula we get

$$
\begin{align*}
\sigma-B_{C}^{V,++}(i, j) & =\sigma-B_{C}^{H,+-}(n-j, i)  \tag{46}\\
\sigma-B_{C}^{H,-+}(i, j) & =\sigma-B_{C}^{H,+-}(n-i, n-j)  \tag{47}\\
\sigma-B_{C}^{V,--}(i, j) & =\sigma-B_{C}^{H,+-}(j, n-i) \tag{48}
\end{align*}
$$

Computing the Downt-Town Spatial Distribution: Relevant locations. We now use the above values to compute the stationary spatial distribution given by Prop. 7. We give such formulas for some kind of cells (the others can be obtained by the same counting arguments). Let $\Lambda=\Lambda_{\mathrm{b}}\left[\mathfrak{R}^{\mathfrak{D}}\right] / m^{2}$ be the normalization constant with $\Lambda_{\mathrm{b}}\left[\mathfrak{R}^{\mathfrak{D}}\right]$ defined in Prop 7

- [Transit Blocks]. Let $u$ be a transit cell of index $k$ (with any $k \in\{1, \ldots, m\}$ ) in the positive transit stripe of the horizontal block $(i, j)$ (i.e. $i$ even and $j$ odd). (with $i \notin\{0, n\}$ ). Then

$$
\mathfrak{s}(u)=\frac{\operatorname{slow}(k)}{\Lambda_{\mathrm{b}}\left[\mathfrak{R}^{\mathfrak{D}}\right]}\left(\sigma-B_{T}^{+}(i, j)+\sum_{h=1}^{k} \sigma-B_{S, h}^{++}(i, j)+\sigma-B_{S, k}^{+-}(i, j)+\sum_{h=k}^{m} \sigma-B_{E, h}^{++}(i, j)+\sigma-B_{E, k}^{+-}(i, j)\right)
$$

Thus, since $\sigma-B_{S, h}^{*, *}(i, j)$ and $\sigma-B_{E, h}^{++}(i, j)$ do not depend on $h$, we have

$$
\begin{align*}
\mathfrak{s}(u)= & \frac{\operatorname{slow}(k)}{\Lambda_{\mathrm{b}}\left[\mathfrak{R}^{\mathfrak{D}}\right]}\left(\sigma-B_{T}^{+}(i, j)+k \cdot \sigma-B_{S, k}^{++}(i, j)+\sigma-B_{S, k}^{+-}(i, j)+(m-k+1) \sigma-B_{E, k}^{++}(i, j)+\sigma-B_{E, k}^{+-}(i, j)\right) \\
= & \frac{\operatorname{slow}(k)}{\Lambda_{\mathrm{b}}\left[\mathfrak{R}^{\mathfrak{D}}\right]} m(m(n-j+1)(n j-(j+1)+m(n-j-1)(n j+2 j+n-i)+k((n+1)(n-j+1)-2)+ \\
& \left.(n+1) j-1+(m-k+1)\left((n+1) j+\frac{n}{2}-1\right)+(n+1)(n-j)+\frac{n}{2}-1\right) \\
= & \frac{\operatorname{slow}(k)}{\Lambda}\left(a(i, j)+\frac{k}{m} b(j)+\frac{1}{m} c(j)\right) \tag{49}
\end{align*}
$$

where

$$
\begin{aligned}
a(i, j) & =(n-j)(2 n j+j+n-i-1)+(n-2) j-\frac{n}{2}+i-2 \\
b(j) & =n(n+1)+\frac{n}{2}-2(n+1) j \\
c(j) & =(n+1)(n+j)+n-3
\end{aligned}
$$

- [Parking Cells]. Let $u$ be a parking cell of index $k$ (with any $k \in\{1, \ldots, m\}$ ) in the positive
parking stripe of the horizontal block $(i, j)$ (i.e. $i$ even and $j$ odd). Then

$$
\begin{align*}
\mathfrak{s}(u)= & \frac{\text { pause }}{\Lambda_{\mathrm{b}}\left[\mathfrak{R}^{\mathfrak{D}}\right]}\left(\sigma-B_{S, k}^{++}(i, j)+\sigma-B_{S, k}^{+-}(i, j)+\sigma-B_{E, k}^{++}(i, j)+\sigma-B_{E, k}^{+-}(i, j)\right) \\
= & \frac{\text { pause }}{\Lambda_{\mathrm{b}}\left[\mathfrak{R}^{\mathfrak{D}}\right]} m((n+1)(n-j+1)-2+(n+1) j-1+(n+1) j+n / 2-1+ \\
& +(n+1)(n-j)+n / 2-1) \\
= & \frac{\text { pause }}{\Lambda} \frac{2 n^{2}+4 n-4}{m} \tag{50}
\end{align*}
$$

- [Cross-Ways]. Let $u$ be a crossing cell $(+,-)$ in the crossing block $(i, j)$ (i.e. $i, j$ even $)$ with $i, j \notin\{0, n\}$ Then

$$
\begin{align*}
\mathfrak{s}(u)= & \frac{\operatorname{cross}}{\Lambda_{\mathrm{b}}\left[\mathfrak{R}^{\mathfrak{D}}\right]}\left(\sigma-B_{C}^{H,+}(i, j)+\sigma-B_{C}^{H,+-}(i, j)+\sigma-B_{C}^{V,-}(i, j)+\sigma-B_{C}^{V,-+}(i, j)\right) \\
= & \frac{\operatorname{cross}}{\Lambda_{\mathrm{b}}\left[\mathfrak{R}^{\mathfrak{D}}\right]} m^{2}((n-j)(2 n j-i+j)+2 i j+i+j+i(2 n(n-i)-j+n-i)+ \\
& \left.+n^{2}-n+(2 n-2 j-1)(n-i)-(n-1) j\right) \\
= & \frac{\operatorname{cross}}{\Lambda}\left(n^{2}(2 j+2 i+3)-2 n\left(j^{2}+i^{2}+j+i+1\right)-(i-j)^{2}+2(i j+j+i)\right) \tag{51}
\end{align*}
$$


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