# ON THE ERGODIC PROPERTIES OF CERTAIN ADDITIVE CELLULAR AUTOMATA OVER $Z_{m}$ 

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#### Abstract

In this paper, we investigate some ergodic properties of $Z^{2}$-actions $T_{p, n}$ generated by an additive cellular automata and shift acting on the space of all doubly -infinitive sequences taking values in $Z_{m}$.


## 1. Introduction

Mathematical study of cellular automata was initiated by Hedlund late 1960s. Hedlund determined the properties of endomorphisms and automorphisms of the shift dynamical system[2]. Sato studied linear cellular automata with-dimensional cell space as well as higher-dimensional cell space[3]. The properties of endomorphisms of subshifts of finite type were studied by Coven et al. [1]. Sinai gave a formula for directional entropy[5]. Ergodic properties of cellular automata have been investigated from various aspects by Shereshevsky and proved that if the automata map is bipermutative then associated CA- action is strongly-mixing[4].

In this paper, we shall restrict our attention to additive cellular automata over $Z_{m}$. The organization of the paper is as follows: In section 2 we establish the basic formulation of problem necessary to state our main theorem. In section 3 we prove our main theorem and some results. Let us provide some notation and background.

## 2. Formulation of the problem

Let $Z_{m}=\{0,1, \ldots, m-1\}$ be a finite alphabet and $\Omega=Z_{m}^{Z}$ be the space of double-infinite sequences $x=\left(x_{n}\right)_{n=-\infty}^{\infty}, x_{n} \in Z_{m}, \sigma$ is the shift in $\Omega$, i.e. $\sigma x=x^{\prime}=\left\{x_{n}^{\prime}\right\}, x_{n}^{\prime}=x_{n+1}, x_{n} \in Z_{m}$. A continuous map $f_{\infty}: \Omega \rightarrow \Omega$ commuting with the shift (i.e. such that $f_{\infty} \circ \sigma=\sigma \circ f_{\infty}$ ) is called a cellular automaton. It is well known (see([2], Theorem 3.4)) that $f_{\infty}: \Omega \rightarrow \Omega$ is a cellular automaton if and only if there exist $l, r \in Z$ with $l \leq r$ and a mapping $f: Z_{m}^{r-l+1} \rightarrow Z_{m}$ such that

$$
f_{\infty}(x)=\left(y_{n}\right)_{n=-\infty}^{\infty}, y_{n}=f\left(x_{n+l}, \ldots, x_{n+r}\right)
$$

for all $x \in \Omega . n \in Z$. It is called the mapping $f$ the rule of $f_{\infty}$ and the interval $[l, r]$ the range of $f_{\infty}$. In [5], it was assumed that $\sigma$ and $f_{\infty}$ generate an action of the group $Z^{2}$ on $\Omega$ : for $(m, n) \in Z^{2}$ the corresponding transformation is $T_{p, n}=\sigma^{p} f_{\infty}^{n}$. Firstly, we consider additive cellular automata $f_{\infty}$ determined by an automation rule

$$
f\left(x_{n-k}, \ldots, x_{n+k}\right)=\left(\sum_{i=-k}^{k} \lambda_{i} x_{n+i}\right)(\operatorname{modm})\left(\lambda_{i} \in Z_{m}\right) .
$$

[^0]A cellular automaton (CA) defined on $\Omega$ is a map $F: \Omega \rightarrow \Omega$ such that for $x \in \Omega$ and $i \in Z,(F x)_{i}=f\left(x_{i-r}, \ldots, x_{i+r}\right.$ where $r \in N$ is radius and $f: \mathbb{Z}_{m}^{2 r+1} \rightarrow \mathbb{Z}_{m}$ is a given local rule. Generally, we take as $\left(\lambda_{i}=1\right)$. Let us consider a block $A={ }_{a-k}\left[i_{a-k}, \ldots, i_{a+k}\right]_{a+k}$. The first preimage of the block $A$ under $f_{\infty}$ is $\left\{y \in \Omega: y_{a-2 k}=j_{a-2 k}, \ldots, y_{a+2 k}=j_{a+2 k}, j_{a-2 k}, \ldots, j_{a+2 k} \in Z_{m}\right\}$ where
$y_{a-2 k}+\ldots+y_{a}=i_{a-k}(\operatorname{modm})$,
....
....
$y_{a-k}+\ldots+y_{a+k}=i_{a}(\operatorname{modm})$,
....
....
$y_{a}+\ldots+y_{a+2 k}=i_{a+k}(\bmod m)$.
It is easy to see from this system of equations that $\left(f_{\infty}\right)^{-1}(A)$ consists of $m^{2 k}$ following blocks $\left(j_{a-2 k}, \ldots, j_{a+2 k}\right)$. Now we calculate the measure

$$
\begin{aligned}
\mu\left(\left(f_{\infty}\right)^{-1}(A)\right) & =m^{2 k} \mu\left\{y \in \Omega: y_{a-2 k}=j_{a-2 k}, \ldots, y_{a+2 k}=j_{a+2 k}, j_{a-2 k}, j_{a+2 k} \in Z_{m}\right\} \\
& =m^{2 k} m^{-(4 k+1)}=m^{-(2 k+1)}
\end{aligned}
$$

Example. Let $A=\{0,1\}$ and $f\left(x_{-2}, x_{-1}, x_{0}, x_{1}, x_{2}\right)=\left(\sum_{i=-2}^{2} x_{i}\right)(\bmod 2)$.
Then

$$
\begin{aligned}
&\left(f_{\infty} \sigma\right)^{-1}\left({ }_{-2}[10101]_{2}\right)={ }_{-3}[111110000]_{5} \cup{ }_{-3}[100000111]_{5} \cup{ }_{-3}[010001011]_{5} \\
& \cup_{-3}[001001101]_{5} \cup{ }_{-3}[000101110]_{5} \cup{ }_{-3}[000011111]_{5} \\
& \cup_{-3}[111000001]_{5} \cup{ }_{-3}[011101000]_{5} \cup{ }_{-3}[001111100]_{5} \\
& \cup_{-3}[110100010]_{5} \cup{ }_{-3}[110010011]_{5} \cup{ }_{-3}[101100100]_{5} \\
& \cup_{-3}[100110110]_{5} \cup{ }_{-3}[101010101]_{5} \cup{ }_{-3}[010111010]_{5} \\
& \cup_{-3}[011011001]_{5} . \text { Thus we have } \\
& \mu\left(\left(f_{\infty} \sigma\right)^{-1}\left({ }_{-2}[10101]_{2}\right)\right)=16 \mu\left(-3\left[j_{-4}, \ldots, j_{4}\right]_{5}\right)=2^{4} 2^{-9}=2^{-5} .
\end{aligned}
$$

If we continue this operation, by the same way, we can determine the measure of (n-1)st preimage of the block $A=_{a-k}\left[i_{a-k}, \ldots, i_{a+k}\right]_{a+k}$ under $f_{\infty}$.

Evidently this (n-1)st preimage consist of such $\left(z_{n}\right)_{n=-\infty}^{\infty}$, for which we have following system of equations:

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za-nk}+\ldots+\mp@subsup{z}{a-(n-1)k}{}+\ldots+\mp@subsup{z}{a-(n-2)k}{}=\mp@subsup{h}{a-(n-1)k}{}(\operatorname{modm})
...
...
...
za-k}+\ldots+\mp@subsup{z}{a}{}+\ldots+\mp@subsup{z}{a+k}{}=\mp@subsup{h}{a}{}(\operatorname{modm})
....
....
```

$$
z_{a+(n-2) k}+\ldots+z_{a+(n-1) k}+\ldots+z_{a+n k}=h_{a+(n-1) k}(\bmod m),
$$

where $h_{a-(n-1) k}, \ldots, h_{a}, \ldots, h_{a+(n-1) k} \in Z_{m}$. So we can calculate the measure

$$
\mu\left(f_{\infty}^{-(n-1)}(A)\right)=m^{2(n-1) k} m^{-(2 n k+1)}=m^{-(2 k+1)}
$$

## 3. Results

Here we shall use the terminology of Sinai [5]. Let us consider as $Z^{2}$ - action $T_{p, n}=\sigma^{p} f_{\infty}^{n}$.

Proposition: Let $T_{p, n}=\sigma^{p} f_{\infty}^{n}$ be $Z^{2}$ - action as above and if $\mu$ is stationary Bernoulli measure on $\Omega$, that is, $\mu(i)=\frac{1}{m}, \forall i=0,1, \ldots, m-1$, then both $f_{\infty}$ and $T_{p, n}$ are Bernoulli measure preserving transformations.

Lemma: The surjective CA-map $f_{\infty}$ generated by the rule

$$
f\left(x_{n+l}, \ldots, x_{n+r}\right)=\left(\sum_{i=l}^{r} x_{n+i}\right)(\bmod m)
$$

is nonergodic with respect to the measure $\mu$, because the equality

$$
\mu\left({ }_{b}\left[e_{0}, \ldots, e_{s}\right]_{b+s} \cap f_{\infty}^{-n}\left({ }_{a}\left[d_{0}, \ldots, d_{k}\right]_{a+k}\right)\right)=\mu\left({ }_{b}\left[e_{0}, \ldots, e_{s}\right]_{b+s}\right) \mu\left({ }_{a}\left[d_{0}, \ldots, d_{k}\right]_{a+k}\right)
$$

can't be obtained sometimes. But we show that $Z^{2}-$ action $T_{p, n}=\sigma^{p} f_{\infty}^{n}$ defined by $(p, n) \mapsto T_{p, n}=\sigma^{p} f_{\infty}^{n}$ on $(\Omega, \mathcal{B}, \mu)$ is ergodic, weak-mixing and strong-mixing if $p>b+s+n \ell-a$.

Theorem 1: $[6$, Theorem 1.17] Let $(\mathrm{X}, \mathcal{B}, \mu)$ be a measure space and let $\mathcal{A}$ be a semi-algebra that generates $\mathcal{B}$. Let $\mathrm{T}: \mathrm{X} \rightarrow \mathrm{X}$ be a measure-preserving transformation. Then
(i) T is ergodic iff $\forall A, B \in \mathcal{A}$

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu\left(T^{-i} A \cap B\right)=\mu(A) \mu(B)
$$

(ii) T is weak-mixing iff $\forall \mathrm{A}, \mathrm{B} \in \mathcal{A}$

$$
\lim _{n \rightarrow \infty} \sum_{i=0}^{n-1}\left|\mu\left(T^{-i} A \cap B\right)-\mu(A) \mu(B)\right|=0
$$

and
(iii) T is strongly-mixing iff $\forall A, B \in \mathcal{A}$

$$
\lim _{n \rightarrow \infty} \mu\left(T^{-n} A \cap B\right)=\mu(A) \mu(B)
$$

Now we can give main theorem.
Theorem 2: Let $Z_{m}=\{0,1, \ldots, m-1\}$ be a finite alphabet and $\Omega=Z_{m}^{Z}$ be the space of double-infinite sequences $x=\left(x_{n}\right)_{n=-\infty}^{\infty}, x_{n} \in Z_{m}$. If additive cellular automata $f_{\infty}$ is given by the formula:

$$
f_{\infty}(x)=\left(y_{n}\right)_{n=-\infty}^{\infty}, y_{n}=f\left(x_{n+\ell}, \ldots, x_{n+r}\right)=\left(\sum_{i=\ell}^{r} x_{n+i}\right)(\bmod m)
$$

for all $x \in \Omega .(p, n) \in Z^{+} \times Z^{+}$, then $Z^{2}-$ action $T_{p, n}=\sigma^{p} f_{\infty}^{n}$ is ergodic, stronglymixing and weak-mixing.

Proof. To prove that $T_{p, n}$ is ergodic it is sufficient to verify (Theorem 1,ii)for any two cylinder sets $A={ }_{a}\left[d_{0}, \ldots, d_{k}\right]_{a+k}$ and $B={ }_{b}\left[e_{0}, \ldots, e_{s}\right]_{b+s}$, we have

$$
\begin{array}{r}
\lim _{p, n \rightarrow \infty} \frac{1}{p n} \sum_{(i, j) \in D} \mu\left({ }_{b}\left[e_{0}, \ldots, e_{s}\right]_{b+s} \cap T_{(-i,-j)}\left({ }_{a}\left[d_{0}, \ldots, d_{k}\right]_{a+k}\right)\right)= \\
\mu\left(b_{b}\left[e_{0}, \ldots, e_{s}\right]_{b+s}\right) \mu\left({ }_{a}\left[d_{0}, \ldots, d_{k}\right]_{a+k}\right),
\end{array}
$$

where $D=[0, p-1] \times[0, n-1] \cap Z^{2}$. For $\mathrm{i}>\mathrm{b}+\mathrm{s}+\mathrm{j} \ell$-a we have

$$
\mu\left({ }_{b}\left[e_{0}, \ldots, e_{s}\right]_{b+s} \cap T_{(-i,-j)}\left({ }_{a}\left[d_{0}, \ldots, d_{k}\right]_{a+k}\right)\right)=\mu\left({ }_{b}\left[e_{0}, \ldots, e_{s}\right]_{b+s}\right) \mu\left(a\left[d_{0}, \ldots, d_{k}\right]_{a+k}\right)
$$

On the other hand, we show that

$$
\begin{aligned}
& \lim _{p, n \rightarrow \infty} \frac{1}{p n} \sum_{(i, j) \in D} \mu\left({ }_{b}\left[e_{0}, \ldots, e_{s}\right]_{b+s} \cap T_{(-i,-j)}\left(a\left[d_{0}, \ldots, d_{k}\right]_{a+k}\right)\right) \\
= & \left.\lim _{p, n \rightarrow \infty} \frac{1}{p n} \mu\left({ }_{b}\left[e_{0}, \ldots, e_{s}\right]_{b+s}\right) \sum_{(i, j) \in D} f_{\infty}^{-j} \sigma^{-i}\left({ }_{a}\left[d_{0}, \ldots, d_{k}\right]_{a+k}\right)\right) \\
= & \left.\mu\left({ }_{b}\left[e_{0}, \ldots, e_{s}\right]_{b+s}\right) \lim _{p, n \rightarrow \infty} \frac{1}{p n} \sum_{(i, j) \in D} f_{\infty}^{-j}\left({ }_{a+i)}\left[d_{0}, \ldots, d_{k}\right]_{a+k+i}\right)\right) \\
= & \mu(B)_{p, n \rightarrow \infty} \frac{1}{p n} \sum_{j=0}^{n-1}\left(\mu \left(f_{\infty}^{-j}\left({ }_{a}\left[d_{0}, \ldots, d_{k}\right]_{a+k}\right)+\ldots+\mu\left(f_{\infty}^{-j}\left(a+p-1\left[d_{0}, \ldots, d_{k}\right]_{a+k+p-1}\right)\right)\right.\right. \\
= & \mu(B) \lim _{p, n \rightarrow \infty} \frac{1}{p n} \sum_{i=0}^{n-1}\left[p m^{-(k+1)}\right] \\
= & \mu(B) \mu(A) .
\end{aligned}
$$

So $Z^{2}$ - action $T_{p, n}=\sigma^{p} f_{\infty}^{n}$ is ergodic. Similarly for $i>b+s+j \ell-a$ we have
$\mu\left({ }_{b}\left[e_{0}, \ldots, e_{s}\right]_{b+s} \cap T_{(-i,-j)}\left(a\left[d_{0}, \ldots, d_{k}\right]_{a+k}\right)\right)=\mu\left({ }_{b}\left[e_{0}, \ldots, e_{s}\right]_{b+s}\right) \mu\left({ }_{a}\left[d_{0}, \ldots, d_{k}\right]_{a+k}\right)$.
Let $A={ }_{a}\left[d_{0}, \ldots, d_{k}\right]_{a+k}$ and $B={ }_{b}\left[e_{0}, \ldots, e_{s}\right]_{b+s}$ be any arbitrary cylinder sets. Then we have

$$
\begin{aligned}
\lim _{p, n \rightarrow \infty} \mu\left[T_{(-p,-n)}(A) \cap B\right] & =\lim _{p, n \rightarrow \infty} \mu\left[\left(f_{\infty}\right)^{-n}\left(a+p\left[d_{0}, \ldots, d_{k}\right]_{a+k+p} \cap B\right]\right. \\
& =\mu(B) \lim _{p, n \rightarrow \infty}\left(\mu f_{\infty}^{-n}\left(a+p\left[d_{0}, \ldots, d_{k}\right]_{a+k+p}\right)\right) \\
& =\mu(B) \mu(A) .
\end{aligned}
$$

Because every strongly-mixing transformation is weak-mixing, $T_{(p, n)}$ is weak-mixing.

One can prove that the natural extension of $T_{p, n}=\sigma^{p} f_{\infty}^{n}$ is ergodic and mixing.

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