ON THE ERGODIC PROPERTIES OF CERTAIN ADDITIVE CELLULAR AUTOMATA OVER \mathbb{Z}_m

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ABSTRACT. In this paper, we investigate some ergodic properties of Z^2 -actions $T_{p,n}$ generated by an additive cellular automata and shift acting on the space of all doubly -infinitive sequences taking values in Z_m .

1. INTRODUCTION

Mathematical study of cellular automata was initiated by Hedlund late 1960s. Hedlund determined the properties of endomorphisms and automorphisms of the shift dynamical system[2]. Sato studied linear cellular automata with-dimensional cell space as well as higher-dimensional cell space[3]. The properties of endomorphisms of subshifts of finite type were studied by Coven et al. [1]. Sinai gave a formula for directional entropy[5]. Ergodic properties of cellular automata have been investigated from various aspects by Shereshevsky and proved that if the automata map is bipermutative then associated CA- action is strongly-mixing[4].

In this paper, we shall restrict our attention to additive cellular automata over Z_m . The organization of the paper is as follows: In section 2 we establish the basic formulation of problem necessary to state our main theorem. In section 3 we prove our main theorem and some results. Let us provide some notation and background.

2. Formulation of the problem

Let $Z_m = \{0, 1, ..., m-1\}$ be a finite alphabet and $\Omega = Z_m^Z$ be the space of double-infinite sequences $x = (x_n)_{n=-\infty}^{\infty}$, $x_n \in Z_m$, σ is the shift in Ω , i.e. $\sigma x = x' = \{x'_n\}, x'_n = x_{n+1}, x_n \in Z_m$. A continuous map $f_{\infty} : \Omega \to \Omega$ commuting with the shift (i.e. such that $f_{\infty} \circ \sigma = \sigma \circ f_{\infty}$) is called a cellular automaton. It is well known (see([2], Theorem 3.4)) that $f_{\infty} : \Omega \to \Omega$ is a cellular automaton if and only if there exist $l, r \in Z$ with $l \leq r$ and a mapping $f : Z_m^{r-l+1} \to Z_m$ such that

$$f_{\infty}(x) = (y_n)_{n=-\infty}^{\infty}, y_n = f(x_{n+l}, ..., x_{n+r})$$

for all $x \in \Omega$. $n \in Z$. It is called the mapping f the rule of f_{∞} and the interval [l, r] the range of f_{∞} . In [5], it was assumed that σ and f_{∞} generate an action of the group Z^2 on Ω : for $(m, n) \in Z^2$ the corresponding transformation is $T_{p,n} = \sigma^p f_{\infty}^n$. Firstly, we consider additive cellular automata f_{∞} determined by an automation rule

$$f(x_{n-k},...,x_{n+k}) = (\sum_{i=-k}^{k} \lambda_i x_{n+i}) (modm) (\lambda_i \in Z_m).$$

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A cellular automaton (CA) defined on Ω is a map $F : \Omega \to \Omega$ such that for $x \in \Omega$ and $i \in Z$, $(Fx)_i = f(x_{i-r}, ..., x_{i+r})$ where $r \in N$ is radius and $f : \mathbb{Z}_m^{2r+1} \to \mathbb{Z}_m$ is a given local rule. Generally, we take as $(\lambda_i = 1)$. Let us consider a block $A =_{a-k} [i_{a-k}, ..., i_{a+k}]_{a+k}$. The first preimage of the block A under f_{∞} is $\{y \in \Omega : y_{a-2k} = j_{a-2k}, ..., y_{a+2k} = j_{a+2k}, j_{a-2k}, ..., j_{a+2k} \in \mathbb{Z}_m\}$ where $y_{a-2k} + ... + y_a = i_{a-k} (modm)$, $y_{a-k} + ... + y_{a+k} = i_a (modm)$, $y_a + ... + y_{a+2k} = i_{a+k} (modm)$.

It is easy to see from this system of equations that $(f_{\infty})^{-1}(A)$ consists of m^{2k} following blocks $(j_{a-2k}, ..., j_{a+2k})$. Now we calculate the measure

$$\mu((f_{\infty})^{-1}(A)) = m^{2k} \mu\{y \in \Omega : y_{a-2k} = j_{a-2k}, \dots, y_{a+2k} = j_{a+2k}, j_{a-2k}, j_{a+2k} \in Z_m\}$$

= $m^{2k} m^{-(4k+1)} = m^{-(2k+1)}.$

Example. Let $A = \{0,1\}$ and $f(x_{-2}, x_{-1}, x_0, x_1, x_2) = \left(\sum_{i=-2}^{2} x_i\right) \pmod{2}$. Then

$$(f_{\infty}\sigma)^{-1} (_{-2} [10101]_2) = _{-3} [111110000]_5 \cup _{-3} [100000111]_5 \cup _{-3} [010001011]_5 \cup \\ \cup _{-3} [001001101]_5 \cup _{-3} [000101110]_5 \cup _{-3} [000011111]_5 \cup \\ \cup _{-3} [111000001]_5 \cup _{-3} [011101000]_5 \cup _{-3} [001111100]_5 \cup \\ \cup _{-3} [110100010]_5 \cup _{-3} [110010011]_5 \cup _{-3} [101100100]_5 \cup \\ \cup _{-3} [100110110]_5 \cup _{-3} [101010101]_5 \cup _{-3} [010111010]_5 \cup \\ \cup _{-3} [011011001]_5. Thus we have$$

$$\mu((f_{\infty}\sigma)^{-1}(_{-2}[10101]_2)) = 16\mu(_{-3}[j_{-4},...,j_4]_5) = 2^42^{-9} = 2^{-5}.$$

If we continue this operation, by the same way, we can determine the measure of (n-1)st preimage of the block $A =_{a-k} [i_{a-k}, ..., i_{a+k}]_{a+k}$ under f_{∞} .

Evidently this (n-1)st preimage consist of such $(z_n)_{n=-\infty}^{\infty}$, for which we have following system of equations:

 $\begin{array}{l} z_{a-nk} + \ldots + z_{a-(n-1)k} + \ldots + z_{a-(n-2)k} = h_{a-(n-1)k}(modm), \\ \ldots \\ \ldots \\ z_{a-k} + \ldots + z_a + \ldots + z_{a+k} = h_a(modm), \\ \ldots \\ \ldots \\ \ldots \end{array}$

•••

 $z_{a+(n-2)k} + \ldots + z_{a+(n-1)k} + \ldots + z_{a+nk} = h_{a+(n-1)k}(modm),$ where $h_{a-(n-1)k}, \ldots, h_a, \ldots, h_{a+(n-1)k} \in Z_m$. So we can calculate the measure

$$\mu(f_{\infty}^{-(n-1)}(A)) = m^{2(n-1)k}m^{-(2nk+1)} = m^{-(2k+1)}.$$

3. Results

Here we shall use the terminology of Sinai [5]. Let us consider as $Z^2 - action$ $T_{p,n} = \sigma^p f_{\infty}^n$.

Proposition: Let $T_{p,n} = \sigma^p f_{\infty}^n$ be $Z^2 - action$ as above and if μ is stationary Bernoulli measure on Ω , that is, $\mu(i) = \frac{1}{m}, \forall i = 0, 1, ..., m-1$, then both f_{∞} and $T_{p,n}$ are Bernoulli measure preserving transformations.

Lemma: The surjective CA-map f_{∞} generated by the rule

$$f(x_{n+l}, ..., x_{n+r}) = (\sum_{i=l}^{r} x_{n+i})(modm)$$

is nonergodic with respect to the measure μ , because the equality

$$\begin{split} \mu({}_{b}[e_{0},...,e_{s}]_{b+s} \cap f_{\infty}^{-n}({}_{a}[d_{0},...,d_{k}]_{a+k})) &= \mu({}_{b}[e_{0},...,e_{s}]_{b+s})\mu({}_{a}[d_{0},...,d_{k}]_{a+k}) \\ \text{can't be obtained sometimes. But we show that } Z^{2} - action \ T_{p,n} &= \sigma^{p}f_{\infty}^{n} \text{ defined by } (p,n) \mapsto T_{p,n} &= \sigma^{p}f_{\infty}^{n} \text{ on } (\Omega,\mathcal{B},\mu) \text{ is ergodic, weak-mixing and strong-mixing if } \\ p > b + s + n\ell - a. \end{split}$$

Theorem 1: [6, Theorem 1.17] Let (X, \mathcal{B}, μ) be a measure space and let \mathcal{A} be a semi-algebra that generates \mathcal{B} . Let $T: X \to X$ be a measure-preserving transformation. Then

(i) T is ergodic iff $\forall A, B \in \mathcal{A}$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(T^{-i}A \cap B) = \mu(A)\mu(B),$$

(ii) T is weak-mixing iff $\forall A, B \in \mathcal{A}$

$$\lim_{n \to \infty} \sum_{i=0}^{n-1} \left| \mu(T^{-i}A \cap B) - \mu(A)\mu(B) \right| = 0$$

and

(iii) T is strongly-mixing iff $\forall A, B \in \mathcal{A}$

$$\lim_{n \to \infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

Now we can give main theorem.

Theorem 2: Let $Z_m = \{0, 1, ..., m-1\}$ be a finite alphabet and $\Omega = Z_m^Z$ be the space of double-infinite sequences $x = (x_n)_{n=-\infty}^{\infty}$, $x_n \in Z_m$. If additive cellular automata f_{∞} is given by the formula:

$$f_{\infty}(x) = (y_n)_{n=-\infty}^{\infty}, y_n = f(x_{n+\ell}, ..., x_{n+r}) = (\sum_{i=\ell}^{r} x_{n+i})(modm)$$

for all $x \in \Omega$. $(p, n) \in Z^+ \times Z^+$, then $Z^2 - action T_{p,n} = \sigma^p f_{\infty}^n$ is ergodic, stronglymixing and weak-mixing. *Proof.* To prove that $T_{p,n}$ is ergodic it is sufficient to verify (Theorem 1,ii)for any two cylinder sets $A =_a [d_0, ..., d_k]_{a+k}$ and $B =_b [e_0, ..., e_s]_{b+s}$, we have

$$\lim_{p,n\to\infty} \frac{1}{pn} \sum_{(i,j)\in D} \mu(b[e_0,...,e_s]_{b+s} \cap T_{(-i,-j)}(a[d_0,...,d_k]_{a+k})) = \mu(b[e_0,...,e_s]_{b+s})\mu(a[d_0,...,d_k]_{a+k}),$$

where $D = [0, p - 1] \times [0, n - 1] \cap Z^2$. For i>b+s+j ℓ -a we have $\mu(b[e_0, ..., e_s]_{b+s} \cap T_{(-i,-j)}(a[d_0, ..., d_k]_{a+k})) = \mu(b[e_0, ..., e_s]_{b+s})\mu(a[d_0, ..., d_k]_{a+k}).$ On the other hand, we show that

$$\lim_{p,n\to\infty} \frac{1}{pn} \sum_{(i,j)\in D} \mu(b[e_0,...,e_s]_{b+s} \cap T_{(-i,-j)}(a[d_0,...,d_k]_{a+k}))$$

$$= \lim_{p,n\to\infty} \frac{1}{pn} \mu(b[e_0,...,e_s]_{b+s}) \sum_{(i,j)\in D} f_{\infty}^{-j} \sigma^{-i}(a[d_0,...,d_k]_{a+k}))$$

$$= \mu(b[e_0,...,e_s]_{b+s}) \lim_{p,n\to\infty} \frac{1}{pn} \sum_{(i,j)\in D} f_{\infty}^{-j}(a_{i+1})[d_0,...,d_k]_{a+k+i}))$$

$$= \mu(B) \lim_{p,n\to\infty} \frac{1}{pn} \sum_{j=0}^{n-1} (\mu(f_{\infty}^{-j}(a[d_0,...,d_k]_{a+k}) + ... + \mu(f_{\infty}^{-j}(a_{i+p-1}[d_0,...,d_k]_{a+k+p-1})))$$

$$= \mu(B) \lim_{p,n\to\infty} \frac{1}{pn} \sum_{i=0}^{n-1} [pm^{-(k+1)}]$$

$$= \mu(B) \mu(A).$$

So $Z^2 - action T_{p,n} = \sigma^p f_{\infty}^n$ is ergodic. Similarly for $i > b + s + j\ell - a$ we have $\mu(b[e_0, ..., e_s]_{b+s} \cap T_{(-i,-j)}(a[d_0, ..., d_k]_{a+k})) = \mu(b[e_0, ..., e_s]_{b+s})\mu(a[d_0, ..., d_k]_{a+k}).$ Let $A =_a [d_0, ..., d_k]_{a+k}$ and $B =_b [e_0, ..., e_s]_{b+s}$ be any arbitrary cylinder sets. Then we have

$$\lim_{p,n\to\infty} \mu[T_{(-p,-n)}(A) \cap B] = \lim_{p,n\to\infty} \mu[(f_{\infty})^{-n}(_{a+p}[d_0,...,d_k]_{a+k+p} \cap B]$$
$$= \mu(B)\lim_{p,n\to\infty} (\mu f_{\infty}^{-n}(_{a+p}[d_0,...,d_k]_{a+k+p}))$$
$$= \mu(B)\mu(A).$$

Because every strongly-mixing transformation is weak-mixing, $T_{(p,n)}$ is weak-mixing.

One can prove that the natural extension of $T_{p,n}=\sigma^p f_\infty^n$ is ergodic and mixing. Acknowledgement

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References

- E. M. Coven and M. E. Paul, Endomorphisms of irreducible subshift of finite type, Math. Sys. Theory, 8 (1974), 167-175.
- [2] G. A. Hedlund, Endomorphisms and automorphisms of the shift dynamical system, Math. Sys. Theory, 3 (1969), 320-375.
- [3] T. Sato, Ergodicity of Linear Cellular Automata over Z_m , Inform.Processing Letters **61** (1997), 169-172.
- M. A. Shereshevsky, Ergodic properties of certain surjective cellular automata, Monatsh. Math. 114 (1992), 305-316.
- [5] Ya. G. Sinai, An answer to a question by J. Milnor, Comment. Math. Helvetici 60 (1985), 173-178.
- [6] P. Walters, An Introduction to Ergodic Theory, Springer-Verlag (1982).

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